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Introduction to Mathematical Logic for Computer Science

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Preface

The idea of mathematical logic is to study logical problems while using mathematical methods.

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Introduction to Mathematical Logic

The idea of mathematical logic is to study logical problems while using mathematical methods.

To be able to use these mathematical methods, we first need to describe our problems in a more concise way. This is done by identifying propositions, also known as statements. These propositions are sentences which can be evaluated in a logical manner and as a whole can be either true or false. Some examples of propositions would be the following

- It is sunny outside
- $x > 0$
- I will pass this exam only if I study

Examples of what are not propositions are as follows

- Go get the salt for me
- Are you cold out there?

This is an example of an imperative and a question respectively. It is a sentence representing a command, not a statement alone which can be verified for truthfulness. In the case of the question, if we look at the context of only that statement sentence alone there is no truth or falseness to the question.

It is important to be able to identify these propositions, as they are the core of what is analysed using mathematical methods.

Chapter 1

Requirements

1.1 Sets

Sets are groups of objects, represented by a method of generator the set or a strictly defined list of these elements.
Example. *Note* $\mathbb{N}$ *contains 0*

- S = $\{x \in N | x < 3\} = \{0, 1, 2\}$

Chapter 2

Propositional Logic

2.1 Introduction to Propositional Logic

2.2 Syntax

2.3 Semantics

2.4 Formal Proofs

Here on out is messy notes typed in class that will be parsed through into book content

2.5 Intro

About logic and it's use in software engineering. Particularly used for program verification.

2.5.1 Logic Arguments

Example 2.5.1.1. If the train arrives late and there are no taxis then John will not be late.
Premise: John was not late. The train arrived late
$\implies$ There were taxis

Symbolized:
p. Train is late
q. Taxis available
r. John is late
$p \wedge \ q \implies r$
$r \wedge p \implies q$

Example 2.5.1.2. If it is raining and Jane does not have her umbrella, Jane will get wet.
Premise: Jane was not wet. It is raining
$\implies$ Jane had her umbrella

Propositional Logic
smallest building block is a statement without connectives, such as "and". The example with the trains is reasoning with propositional logic.
Predicate Logic (First Order Logic)
includes a means to describe relationships between objects, for example "every course has at least one object"
Program Verification
ways to describe what a system is required to do and reasoning about its correctness.

2.5.2 Review

example to do: sum from 1 to n for $\frac{1}{i(i+1)} = \frac{n}{n+1}$
ex. p(n): 3 divides $(4^n + 5)$

2.6 Propositional Logic

What is a proposition?
A declarative sentence which is either true or false.

A simple (atomic) proposition is the basic building blocks to create compound propositions using connectives.

Proving this property of R(A): A has the length ¿ 0
Prove $R(A) \forall A \in Form(\mathcal{L}^p)$

1. Assume $A \in Atom(\mathcal{L}^p)$
   therefore $A = p$
   Then $length(A) = 1 > 0$

2. Let $A \in Form(\mathcal{L}^p)$
   Assume $R(A)$. Prove $R(\neg A)$
   As $R(A) \implies length(A) > 0$
   $length(\neg A) = length(A) + 3 > 3 > 0$

3. Let $A, B \in Form(\mathcal{L}^p)$
   Assume $R(A), R(B)$
   Prove $R(A * B)$
   The length of $(A * B) > 0$ then clearly

Parse Tree The leaves of a parse tree of a well formed formula are labeled with atoms The nodes of a parse tree of a well formed formula labeled have one children the node labeled with a binary connective have exactly two children

Prove m(n) for every n in natural let M(n): A of height n has the same number of left and right brackets prove M(n) for any formula of height n, for every n in natural b.c. smallest formula, n=1 (single atom) therefore a in atom '(' = 0 ')' = 0 Inductive hypothesis n ¿ 1 assume for any formula A of height ¡= n (n) prove M(n + 1)

1. If I study for exams, then I get good grades

2. I do not eat healthy food whether or not I study for exams

3. I will pass the class only if I get good grades

4. If I do not study for exams, then I get good grades only if I eat healthy food

5. I will either pass the class or eat healthy food, but not both

All the atoms
s: I study for exams
g: I get good grades
h: I eat healthy food
p: I will pass the class

1. $(s \implies g)$
2. $(\neg h)$ [whether or not does not add anything]
3. $(p \implies g)$
4. $((\neg s) \implies (g \implies h))$
5. $(p \vee h) \wedge (\neg (p \wedge h))$

Well formed formula
1) $Atom(\mathcal{L}^p) \subset Form(\mathcal{L}^p)$
2) If $A \in Form(\mathcal{L}^p) \implies (\neg A) \in Form(\mathcal{L}^p)$
3) If $A, B \in Form(\mathcal{L}^p) \implies (A * B) \in Form(\mathcal{L}^p)$, where * is a binary connective.

Structural Induction. Recursively proving that a concatenation or similar of a bunch of elements holds the same property as those elements $\implies$ Base case the lowest level element
Suppose that if the number of occurrences of atoms in a formula is m and the number of occurrences of any binary connective is n. Than prove that m = n + 1 proof:
Base Case: For an atom, m = 1 and n = 1. So the statement is true.
Inductive hypothesis: Assume it is true for formula ϕ and ψ,
Here m_ϕ = number of occurrences of atoms in ϕ
m_ψ = number of occurrences of atoms in ψ
n_ϕ = number of occurrences of binary connectives in ϕ
n_ψ = number of occurrences of binary connectives in ψ
so $m_\phi = n_\phi + 1$
$m_\psi = b_\psi + 1$
Inductive step:
Consider the formula $\Omega = (\neg\phi), m_\Omega = m_\phi, n_\Omega = n_\phi$
Thus true for $\neg$
Consider the formula $\Omega = (\phi * \psi), m_\Omega = m_\phi + m_p si, n_\Omega = n_\phi + n_\psi + 1$ by IH
$m_\Omega = n_\phi + n_\psi + 2$
$\implies m_\Omega = n_\Omega + 1.$
Therefore proven true for binary connective
Therefore we have proven this true by structural induction.

example of this property
$\phi = (((\neg p) \implies s) \wedge r) m_\phi = 3 n_\phi - 2$
$\psi = (p \implies (\neg q)) m_\psi = 2 n_\psi = 1$

1. Give the truth table for the following formulas.
a) $(p \implies q) \wedge r$
b) $(p \implies \neg p)$

c) $((p \implies q) \wedge (q \implies p))$
d) $(p \wedge r) \vee (\neg r \implies q)$
e) $(\neg r \wedge p) \vee (r \vee \neg p)$

2) answer the following questions about semantic implication a) let Σ be the set of well formed propositional formulas and let A be a well formed propositional formula. What does it mean for $\Sigma \vDash A$. What does it mean for $\Sigma \nvDash A$.

b) Consider the set Σ and formula A.
$\Sigma = p, (p \implies q) A = q$
Does $\Sigma \models A$?

c) Consider $\Sigma = q, (p \implies q)$, A = p
Does $\Sigma \not\models A$?

b) tt:
p q $p \implies q$
0 0 1
0 1 1
1 0 0
1 1 1
As the only time when Σ^t is true is the last line, whereA^t is also true, we have $\Sigma \models A$ is true.

c) As $\Sigma \models A$is not true, there is one truth valuation then where $\Sigma^t \implies A^t$. Therefore $\Sigma \not\models A$.

3. Let C be the set $\wedge, \vee, \implies$ of propositional connectives and F be the set α, β, γ of well-formed propositional formulas.
a) Suppose that, θ is a well formed formula that uses only connecties C and the formulas in F.
The formulas α, β, γ are all tautologies.
Prove that θ is a tautology.

Base case:
$\alpha\beta\gamma$ are tautologies
Induction hyp:
let ϕ and ψ be two well formed formulas and they use only connectives in C and forumulas in F.
ϕ and ψ are both tautologies.
Inductive step:
$\theta = \phi \wedge \psi$
$\phi \vee \psi$
$\phi \implies \psi$

truth table:
$\phi^t \psi^t \wedge^t or^t implies^t$ (of ϕ and ψ)
0 0 0 0 1
0 1 0 1 1

1 0 0 1 0
1 1 1 1 1 ¡- this is the only formula we get as $\phi^t = \psi^t = 1$ for any truth valuation
Thus proved true.

b)
Suppose that $\theta$is a well formed formula that as well only used those connectives and formulas and also the $\neg$ connective.
Find a counter example to show that θ is not necessarily a tautology.
$A^t \in 0, 1$
$t : Atom\mathcal{L}^p - > 0, 1$

Having atom values
$A_1...A_n \models A$ means that we can deduce the value of A (being true) when we know the premise or propositional atoms.

thrm 2.5.2.
proof =¿ direction
$a1...an \models a$
prove $a1and....an \implies a$ is a tautology.

$a + 1...a_n \vDash$ a for every t for which $a1^t = ...an^t = 1$ we have $a^t = 1$

prove that for any truth valuation
$t^1, a1^{t^1} and...an^{t^1} \implies a^t = 1$

1) let t be any truth valuation which
$a1^t = ...an^t = 1$(of course $a^t = 1$)
$a^t and...an^t \implies a^t = 1$
2) let t' be any truth valuation for which
which $a_i^{t'} = 0$ for some i between 1 and n
$a_1^{t'} and...a_n^{t'} \implies a^t$
as some ai = 0 we have
$0 \implies a^t = 1$

now ¡= direction
$a1and...an \implies a$ is a tautology
prove $a1...an$ has a tautological consequence of a
i.e. prove $a1...an \vDash a$

for every $ta1^t and...an^t \implies a^t = 1$
let t' be a truth valuation such that
$a1^{t'} = ... = an^{t'} = 1$

prove $a^{t'} = 1$
$a_1^{t'} and ... a_n^{t'} \implies a^t = 1$
becomes
$1 \implies a^{t'} = 1$
thus $a^{t'} = 1$

Adequate set - set of connectives C, such that any other connective can be represented using the connectives from C

Th. $\{\neg, \wedge, \vee\}$ is an adequate set.
$p \implies q \equiv \neg p \vee q$
$p \iff q \equiv (p \implies q) \wedge (q \implies p)$
Corollary:
$\{\neg, \wedge\}$
$\{\neg, \vee\}$
$\{\neg, \implies\}$

Hilbert system axioms:
1) $A \implies (B \implies A)$
2) $((A \implies (B \implies C)) \implies ((A \implies B) \implies (A \implies C)))$
3) $((\neg A \implies \neg B) \implies (B \implies A))$

Modus Ponens (MP)
$\phi, \phi \implies \psi \vdash_H \psi$
$\frac{\phi, \psi \implies \psi}{\psi}$
1) α_1 (premise)

Proving $\vdash_H A \implies A$
1) $A \implies ((A implies A) \implies A)$ (axiom 1)
2) $A \implies ((A \implies A) \implies A) \implies ((A \implies (A \implies A)) \implies (A \implies A))$ (axiom 2)
3) $((A \implies (A \implies)) \implies (A \implies A))$ (modus ponens 1,2)
4) $A \implies (A \implies A)$ (axiom 1)
5) $A \implies A$ (modus ponens 3,4)

TH transitivity
$A \implies B, B \implies C \vdash_H A \implies C$
1) $A \implies B$ (premise)
2) $B \implies C$ (premise)
3) $(A \implies (B \implies C)) \implies ((A \implies B) \implies (A \implies C))$ (axiom 3)
4) $(B \implies C) \implies (A \implies (B \implies C)$ (axiom 1)
5) $B \implies C)$ (modus ponens 2, 4)
6) $(A \implies B) \implies (A \implies C)$ (modus ponens 3,5)
7) $A \implies C)$ (modus ponens 1,6)

Th. (Deduction Theorem)
$\Sigma \vdash_H (A \implies B) iff A_1 A \vdash_H B$ $\alpha_1, ..., \alpha_n \vdash_H \alpha$
$\vdash_H (\alpha_1 \implies ...(\alpha_{n-1}(\alpha_n \implies \alpha))...)$

ex.3
$\vdash_H (\neg A \implies (A \implies B))$
1) $\neg A \implies (\neg B \implies \neg A)$ (axiom 1)
2) $(\neg B \implies \neg A) \implies (A \implies B)$
3) $\neg A \implies (A \implies B)$ (ex. 2 transitivity)

ex. 4
$\vdash_H \neg\neg A \implies A$
By Ded. Thrm $\neg\neg A \vdash_H A$

1) $\neg\neg A (premise)$
2) $\neg\neg A \implies (\neg\neg\neg\neg A A \implies \neg\neg A)(axiom 1)$
3) $\neg\neg\neg\neg A \implies \neg\neg A$(modus ponens 1,2)
4) $(\neg\neg\neg\neg A \implies \neg\neg A)(\neg A \implies \neg\neg\neg A)$ (axiom 3)
5) $\neg A \implies \neg\neg\neg A)$(modus ponens 3, 4)
6) $(\neg A \implies \neg\neg\neg A) \implies (\neg\neg A \implies A)$ (axiom 3)
7) $\neg\neg A \implies A$ (modus ponens 5, 6)
8) A (modus ponens 1, 7)

Any set of connectives with the capability to express all truth-tables is said to be adequate
$\{\wedge, \vee, \neg\}$ is adequate
$\neg p$
$p \wedge q$
$p \vee q$
$p \implies q$
$p \iff q$
$p \implies q \equiv \neg p \vee q$
$p \iff q \equiv (p \wedge q) \vee (\neg p \wedge \neg q) \equiv (p \implies q) \wedge (q \implies p) \equiv (\neg p \vee q) \wedge (\neg q \vee p)$

Exercise:- Show $\neg, \wedge, \neg, \vee, \wedge, \implies$ and adequate.
NAND ($\uparrow$ /$\overline{\wedge}$)/Sheffer Stroke (—), and NOR($\downarrow$)/shroder connective are adequate.

Proof
NOR $\vdash$
$\neg p \equiv p \vdash$
$p \vee q \equiv \neg\neg(p \vee q) \equiv \neg(p \vdash q) \equiv (p \vdash q) \vdash (p \vdash q)$
$p \wedge q \equiv \neg(\neg A \vee \neg B) \equiv (\neg A \vdash \neg B)$, by De Morgan's Law.

$p \implies q \equiv (\neg p \vee q) \equiv \neg(\neg p \vdash q) \equiv ((p \vdash p) \vdash q)$
$p \iff q \equiv (p \implies q) \wedge (q \implies p)$

Hilbert System
$\Sigma \vdash A$ means, A is provable from Σ
Axioms
Ax. 1 $\phi \implies (\psi \implies \phi)$
Ax. 2 $(\phi \implies (\psi \implies \gamma)) \implies ((\phi \implies \psi) \implies (\phi \implies \gamma))$
Ax. 3 $(\neg\phi \implies \neg\phi) \implies (\phi \implies \psi)$

Modus Ponens (MP)
$\frac{\phi, \phi \implies \psi}{\phi}$

Is the following an axiom?
$(p \implies q) \implies (((\neg p) \implies q) \implies (p \implies q))$

Ex. 1
$\alpha \implies \beta, \beta \implies \gamma \vdash \alpha \implies \gamma$
This formula is equivalent to,
$\alpha \implies \beta, \beta \implies \gamma, \alpha \vdash \gamma$
1. $\alpha \implies \beta$ (premise)
2. $\beta \implies \gamma$(premise)
3. α(premise)
4. β(mp 1,3)
5. γ(mp 2,4)

PROVED

Ex. 2
$(\alpha \implies \beta) \implies \gamma \vdash \alpha \implies (\beta \implies gamma)$
1. $(\alpha \implies \beta) \implies \gamma$(premise)
2. $((\alpha \implies \beta) \implies \gamma) \implies (\beta \implies ((\alpha \implies \beta) \implies \gamma))$ (Ax. 1)
3. $(\beta \implies ((\alpha\beta) \implies gamma))$ (mp 1,2)
4. $(\beta \implies ((\alpha\beta) \implies gamma)) \implies ((\beta \implies (\alpha \implies beta)) \implies (\beta \implies \gamma))$ (Ax. 2)
5. $((\beta \implies (\alpha \implies \beta)) \implies (\beta \implies \gamma))$ (mp 4,5)
6. $(\beta \implies (\alpha \implies \beta))$ (Ax. 1)
7. $(\beta \implies \gamma)$ (mp 5,6)
8. $(\beta \implies \gamma) \implies (\alpha \implies (\beta \implies \gamma))$ (ax. 1)
9. $(\alpha \implies (\beta \implies \gamma))$ (mp 7,8)

Ex. 3
$\phi \implies (\psi \implies \theta), \psi \vdash (\phi \implies \theta)$
1. $(\phi \implies (\psi \implies \theta)$ (premise)
2. ψ (premise)
3. $(\phi \implies (\psi \implies \theta)) \implies ((\phi \implies \psi) \implies (\phi \implies \theta))$ (ax. 2)

4. $((\phi \implies \psi) \implies (\phi \implies \theta)$ (mp 1,3)
5. $\psi \implies (\phi \implies \psi)$ (ax. 1)
6. $\phi \implies \psi$ (mp 2, 5)
7 $\phi \implies \theta$ (mp 4,6)

1) Reflexivity (REF)
$\psi \vdash \psi, \frac{\phi}{\phi}$
2) Addition of premises (+)
$\Sigma \vdash \phi$
then $\Sigma' \cup \Sigma \vdash \phi, \frac{\frac{\omega}{\phi}}{\frac{\omega\omega'}{\phi}}$
3) $\neg$ elimination
$\Sigma, \neg\psi \vdash \omega$
$\Sigma, \neg\psi \vdash \neg\omega$
then
$\Sigma \vdash \phi$

prove hilbert axioms using natural deduction

axiom 1:
$\phi \implies (\psi \implies \phi)$
axiom 2:
$(\phi \implies (\psi \implies \omega)) \implies ((\phi \implies \psi \implies \phi \implies \omega))$
axiom 3:
$(\neg\phi \implies \neg\psi) \implies (\psi \implies \phi)$

axiom2:
1. $(\phi \implies \psi \implies \zeta)$ assumption
2. $\phi \implies \psi$ assumption
3. ϕ assumption
4. $\psi \implies -$
5. $\psi \implies \zeta \implies -$
6 $\zeta \implies -$
7. $\phi \implies \zeta \implies +$
8.$\phi \implies \psi \implies \phi \implies \zeta \implies +$

Q5 a) hint
$\Sigma, \phi \vdash_H \psi$
1. α_1
.
.
.
$\alpha_n = \psi$
$\Sigma \vdash \phi \implies \psi$

1. $\phi \implies \alpha_1$
.
.
.
n.$\phi \implies \alpha_n(\phi \implies \psi)$
not sure if steps are actual step counts
Case $\alpha_i \in \Sigma$ (3 steps)
Case $\alpha_i \in Axioms$ (8 steps)
Case $\alpha_i = \phi$ (7 steps)
Case α_i - MP (10 steps)
Soundness thrm

$\Sigma \vdash \alpha \implies \Sigma \models \alpha$
contrapositive
$\Sigma \not\models \alpha \implies \Sigma \not\vdash \alpha$
e.g. make a counter example (truth table, $\models$) and show that it is not valid.

Th 2.6.2 $\Sigma \vdash \phi$then exists $\Sigma^o - > finite \Sigma^o \subset \Sigma$
Proof: Structural induction on the size of Σ
Bc. $\Sigma = \phi \phi \vdash \phi$
$\Sigma^o := \Sigma and \Sigma^o - > finite$
Ind. hyp.
Assume that for $\Sigma \vdash \phi$ where $\Sigma is of size \leq k$, there exists a $\Sigma^o \subset \Sigma$ such that Σ^o-¿ finite and $\Sigma^o \vdash \phi$
Ind. setp.
$\Sigma \vdash \phi$
Σ is of size k+1
Consider 10 cases
+) $\Sigma \vdash \phi size \leq k$
then $\Sigma \cup \Sigma' \vdash \phi$ size k_1
By ind. hyp $\exists \Sigma^o \subset \Sigma, \Sigma^o$ -¿ finite
$\Sigma^o \vdash \phi$
$\Sigma^o \subset \Sigma \cup \Sigma'$
$\Sigma^o \vdash \phi, \Sigma^o$ -¿ finite
$\neg$ -)
1 $\Sigma, \neg\phi \vdash \psi$
2 $\Sigma, \neg\phi \vdash \neg\psi$
3 (?)$\Sigma \vdash \phi$
$\neg$) by ind hype
$\exists \Sigma_1 \subset \Sigma$
Σ_1 -¿ finite
$\Sigma_1 \vdash \psi$
$\Sigma_1 \subset \Sigma \cup \neg\psi$

thrm 2.6.3 transistivity of deduciblity)
$\Sigma \vdash \Sigma'$ and $\Sigma' \vdash \phi \implies \Sigma \vdash \phi$

Note A $\vdash$B means A $\vdash$ B_i for all formulas in B.

1) $A_1, ..., A_n \vdash \phi$ (premise)
2) $A1, ..., , A_{n-1} \vdash A_n \implies \phi$ ($\implies$ +)
.
.
.
n) EMPTY FUCKING SET $\vdash A_1 \implies (A_2 \implies ...(A_n \implies \phi)..)$
$\Sigma \vdash A_1 \implies (A_2 \implies ...(A_n \implies \phi)..)$ introduction of premises (already true with current premises)
$\Sigma \vdash A_1$ (premise)
$\Sigma \vdash (A_2 \implies ...(A_n \implies \phi)..)$ ($\implies$ -)
.
.
.
$\Sigma \vdash A_n \implies \phi$
$\Sigma \vdash A_n$
$\Sigma \vdash \phi$ ($\implies$ - N TIMES .....)

ex: $\Sigma, \phi \vdash \psi, \Sigma, \phi \vdash \neg\psi \implies \vdash \neg\phi$ (reduction ad absurdum) ($\neg$ +) -¿ not a basic rule
Obtained from $\neg$ -
Box notation:
$\neg\neg\phi$ (assumption
$\neg\phi$ (assumption)
$\neg\neg\phi$ (copy)
ϕ ($\neg$- 2,3)
ψ(premise OF ABOVE $\vdash$shit)
$\neg\psi$ (premise AGAIN)
$\neg\phi$ ($\neg$- 1,4)

YOU MUST PROVE NEGATION ADDITION INLINE IF YOU USE IT

$\Sigma, \phi \vdash \psi$
$\Sigma \vdash \Sigma$
$\Sigma, \neg\neg\phi \vdash \Sigma$
$\neg\neg\phi \vdash \phi$ (EXAMPLE YOU MUST PROVE IN FUCKING LINE)
$\Sigma, \neg\neg\phi \vdash \phi$
$\Sigma, \neg\neg\phi \vdash \Sigma$
$\Sigma, \neg\neg\phi \vdash \phi$
$\Sigma, \phi \vdash \psi$
$\Sigma, \neg\neg\phi \vdash \psi$

DOING ALL THIS FUCKING SHIT AGAIN FOR $\neg\psi$
WE GET A CONTRADICTION

$\Sigma \vdash \neg\phi$

1. $\phi \implies \psi, \phi \vdash$

Proof of soundness theorem
structural induction on the length of the proof n.
bc: n = 1
$\phi \vdash \phi$ (ref)
n = 2
$\Sigma, \phi \vdash \phi$
$\phi \vdash \phi$ (ref)
$\Sigma, \phi \vdash \phi$ (ref, +)

$\phi \vDash \phi$ (trivial truth table)
$\Sigma, \phi \vDash \phi$ (also trivial)

Inductive Hypothesis:
Assume that for every proof $\Sigma \vdash \phi$ of size $\leq$ k
we have $\Sigma \vDash \phi$

Inductive Conclusion:
1. ϕ_1(premise)
.
.
.
n) ϕ_n(premise)
n+1) ϕ_{n+1} (ND rules)
,
.
.
k) ϕ_k (ND rules)
k+1) ϕ_{k+1} (ND rules) [valid?]

Case 1:
ϕ obtained by writing
$(\wedge +)$ rule
.
.
.
i) ϕ_i
.

.
.
j)$\phi_j i \leq j \leq k$
.
.
.
k+1) $\phi = \phi_i \wedge \phi_j (\wedge +)$
i $\leq k$
$\Sigma \vdash \phi_i$
by ind. hyp
$\Sigma \models \phi_i$
Similarly for j
For every t,$\Sigma^t \implies \phi_i^t = 1$ and $\phi_j^t = 1 \implies (\phi_i \wedge \phi_j)^t = 1$

Case 2: $\implies$ -
.
.
.
i) ϕ i
.
.
j) $\phi_i \implies \phi_j i, j \leq k$
.
.
k) $\phi = \phi_j$

We have $\Sigma^t = 1 \implies \phi_i^t = 1$and $(\phi_i \implies \phi_j)^t = 1 \implies \phi_j^t = 1$

Completeness thrm proof:
$\Sigma \models \phi \implies \Sigma \vdash \phi$
Contrapositive:
$\Sigma \nvdash \phi \implies \Sigma \not\models \phi$

$\Sigma \nvdash \phi \implies \Sigma \cup \{\neg\phi\}$consistent $\implies \Sigma \cup \{\neg\phi\}$has a model (satisfiable) (model is a truth valuation that satisfies it) $\implies \Sigma \not\models \phi$

ASIDE:
Σconsistant iff there is no formula $\phi \in Form(\mathcal{L}^p)$

$\Sigma \vdash \phi$
$\Sigma \models \phi$

Ex. $p, \neg p \vdash p$

$\vdash \neg p$
inconsistent
BACK TO PROOF

Σ- maximal constent if
1) Σ- consistent
2) $\forall \phi \notin \Sigma, \Sigma \cup \phi$is consistnet

Soundness thrm (2):
if Σsatisfiable then Σ consistent

Soundness thrm (1) $\iff$ Soundness thrm (2)

$\implies$ direction
$\Sigma \vdash \phi \implies \Sigma \vDash \phi$
Prove that if Σsatisfiable then Σ consistent
Assuem the contrary Σsatifiable and Σinconsistent

ASSIDE
Σ inconsistent: $\exists \phi : \Sigma \vdash \phi$
$\Sigma \vdash \neg\phi$

Doing Soundness thrm (1). on the contrary
Σ
$\vDash \phi$
$\Sigma \vDash \neg\phi$
Therefore we get a contradiction
$\forall$ t truth valuations $\ni \Sigma^t = 1 \implies \phi^t = 1$and $\neg\phi^t = 1$(contradiction)
$\implies \Sigma$ consistent

¡= DIRECTION
Σ-sat $\implies \Sigma$ consistnent
Prove that $\Sigma \vdash \phi \implies \Sigma \vDash \phi$
Assume the contrary
$\Sigma \nvdash \phi \implies \Sigma \nvDash \phi$
$\Sigma \nvDash \phi \implies \Sigma \cup \neg\phi$ satisfiable
$\Sigma \vdash \phi$(hyp)
$\Sigma \cup \neg\phi \vdash \phi$
-¿ coontradiction
so assumption is invalid $\Sigma \vDash \phi$
so ST1 $\iff$ ST2
$\Sigma \nvDash \iff$ (UNFINISHED FUCKING BULLSHIT)

Completeness thmr(2)
Σconsist $\implies$ Σ satisfiable

Lemma
Σ max consist
$\phi \in \Sigma \iff \Sigma \vdash \phi$
$\implies$
Σ max const and $\phi \in \Sigma$
$\Sigma' := \Sigma - \phi$
1) $\phi \vdash \phi$(ref)
2) $\Sigma', \phi \vdash \phi$(+,1)
$\Sigma \vdash \phi$

¡=
Σ max const and $\Sigma \vdash \phi$
Prove $\phi \in \Sigma$
Assume the contrary: $\phi \notin \Sigma$

ASIDE DEFN OF MAX CONST
Σ max const:$\forall \phi \notin \Sigma, \Sigma \cup \phi$
$\Sigma \cup \phi$- inconsistn
$\exists \alpha$:
$\Sigma \cup \phi \vdash \alpha$
$\Sigma \cup \phi \vdash \neg\alpha$ ($\neg$ +)

(NEGATION ELIM)
$\Sigma, \neg\phi \vdash \phi$
$\Sigma, \neg\phi \vdash \neg\phi$
$\Sigma \vdash \phi$ ($\neg$ -)

Σ - inconsistent (contradiction)
$\phi \in \Sigma$.

MIDTERM LEVEL PROOF (remember/write down)
$\Sigma \nvdash \phi \implies \Sigma \cup \neg\phi$ consistent
Proof:
Assume the contrary $\Sigma \cup \neg\phi$inconssitent
$\exists \alpha$
$\Sigma \cup \neg\phi \vdash \alpha$
$\Sigma \cup \neg\phi \vdash \neg\alpha$
$\Sigma \vdash \phi$ ($\neg$ -)

(contradiction with hyp)
$\implies \Sigma \cup \neg\phi$ const.

Lindervaum lemma:
Σ-consistent
$\exists \Sigma' : \Sigma \cup \Sigma'$ -¿ maximum consistent

3)
$\Sigma \cup \neg\phi$ - satisfiable
$then \Sigma \not\models \phi$

$\exists t : \Sigma^t = 1$ and $(\neg\phi)^t = 1$
Assume the contrary: $\Sigma \models \phi$
$\forall t : \Sigma^t = 1 \implies$
$\phi^t = 1$
$(\neg\phi)^t = 0$
But that contradicts the $\exists$ line just above.
Contradicts the conclusion, thus assumption is not correct
$\Sigma \not\models \phi$

Soundness
Formally if $\Sigma \vdash \alpha$ is a premise then $\Sigma \models \alpha$
Informally if a formula is provable then it is true

Completeness
FOrmally if $\Sigma \models \alpha$ then $\Sigma \vdash \alpha$ is provable
Informally, if a formula is true, then it is provable

Definition of consistent
Σ is consistent if for no $\alpha, \Sigma \vdash \alpha$ and $\Sigma \vdash (\neg\alpha)$
A set of Σ is consistent if it does not allow us to prove a contradiction

Question :-
Is the following set consistent?
$\Sigma = ((p \implies q) \implies q), (q \implies p)$

Claim :-
If Σ is satisfiable, then Σ is consistent
Proof :-
Given a satisfying assignment v of Σ, we choose a variable p from Σ and define α as $(\neg p)$ if v(p) = T and p if v(p) = F

We then have the scenario where v satisfiedΣ but does not satisfy our formula α. Therefore, $\Sigma \not\models \alpha$and, from soundness, $\Sigma \not\vdash \alpha$

We are using that technique to prove the question.
Find an α such that $\Sigma \not\vdash \alpha$
Let v(p) = T and v(q) = T
Now let $\alpha = (\neg P)$

We now have $\Sigma \not\models \alpha and \Sigma \not\vdash \alpha$ and therefore $\Sigma$is consistent by our definition. (because it is consistent if it does not allow us to prove a contradiction, and that is what we have shown)

Question :-
Is the following set consistent?
$\Sigma = (p \implies q), p, (\neg q)$

Answer :-
NO! lul not telling you why, jk.
p $\implies$ q
p
q (mp)
$\neg$q (premise)
We can prove a contradiction from this!

Question :-
Is the following set consistent?
$\Sigma = (p \implies (q \implies r)), (p \implies q), (\neg(p \implies r))$

Question :-
If $\Sigma_a$is consistent and Σ_bi consistent, is$\Sigma_a \cup \Sigma_b$ always consistent?
Answer :- no
$\Sigma_a = p$
$\Sigma_b = \neg p$
$\Sigma_a \cup \Sigma_b = p, \neg p$
Prove a contradiction

How about $\Sigma_a - \Sigma_b$
We can make the claim A intersection B is less than A or less than B
If Σ is consistent and $\Sigma' \subset \Sigma$ is consistent then Σ' is consistnet

Soundness Thrm:
If $\Sigma \vdash \phi \implies \Sigma \vDash \phi$

Completeness Thrm:
If $\Sigma \vDash \phi \implies \Sigma \vdash \phi$

Soundness Thrm (2);
If Σ-satisfiable $\implies$ Σ-consistent

Completeness Thrm (2):
If Σ-consistent $\implies$ Σ-satisfiable

Proof of competeness (1):
We will prove by contrapositive:
1)
$\Sigma \nvdash \phi \implies \Sigma \vDash \phi$
$\Sigma \nvdash \phi \implies$
$\Sigma \neg\phi$ consistent
2)
$\Sigma \neg\phi$ consistent $\implies$
$\Sigma \neg\phi$-satisfiable
3)
$\Sigma \neg\phi$-satisfiable $\implies \Sigma \vDash \phi$

Prove ???????
1)
$\Sigma \nvdash \phi$
Prove $\Sigma \cup \neg\phi$ consistnet
Assume $\Sigma \cup \neg\phi$ inconsist
$\exists \alpha : \Sigma \cup \neg\phi \vdash \phi$
$\Sigma \cup \neg\phi \vdash \neg\phi$
$\Sigma \vdash \phi(\neg -)$
(contradiction)
2)
$\Sigma \cup \neg\phi$ - consistent then $\Sigma \cup \neg\phi$ - satisfiable

Lemma Lindenbaum:
For every Σ-consistent, $\exists \Sigma' \ni \Sigma \cup \Sigma'$ - maximal consistent

Lemma (5.3.6)
Σ^*-maximal consistent and $\forall p \in Atom(\mathcal{L}^p)$, $p^t = 1 \iff p \in \Sigma^*$
Then

$\forall A \in Form(\mathcal{L}^p), A^t = 1 \iff A \in \Sigma^*$

$\Sigma \cup \neg\phi$is consistent (hypothesis)
Prove $\Sigma \cup \neg\phi$ is satisfiable.
By lindebaum lemma, we can extend this set
$\exists \Sigma^*$-max const $\ni$
$\Sigma \cup \neg\phi \subset \Sigma^*$
By second lemma
$\neg\phi \in \Sigma^*$
$(\neg\phi)^t = 1$
$\Sigma \subset \Sigma^*$
By second lemma
$\forall A \in \Sigma \subset \Sigma^* \implies A^t = 1$
$\implies \Sigma^t = 1$
$\forall t : (\neg\phi)^t = 1$ and $\Sigma^t = 1$
$\implies \Sigma \cup \neg\phi$ -satisfiable
LAST STEP? PROOF WAS JUST PROVEN WTF
$\Sigma \cup \neg\phi$-satisfiable
Prove $\Sigma \not\models \phi$
Let us assume $\Sigma \models \phi$
$\exists t \ni \Sigma^t = 1$ and $(\neg\phi)^t = 1$ by defn of satisfiale.
$\implies \phi^t = 0$
$\Sigma \models \phi$
$\forall t' \ni \Sigma^{t'} = 1 \implies \phi^{t'} = 1$
But this contradicts, so our assumption is not correct so
$\Sigma \not\models \phi$, THUS PROVEN COMPLETENESS?

SPRING 2013 MIDTERM (for review)

while the waves are high i will go surfing
$p \wedge q$
swimming even though waves high

$p \wedge q$
i will make sand castles only if i dont go swimming or diving
(PROPER $p \implies \neg(q \vee q)$)
i wont go swimming unless i remember to bring my towel and bathing suit
$p \implies (q \wedge r)$

$\vdash_h (A \implies B) \implies (\neg B \implies \neg A)$
Deduction thrm. $A \implies B \vdash_H \neg B \implies \neg A)$
1) $A \implies B$
2) $\neg\neg A \implies A$ (ex. 5)

3) $\neg\neg A \implies B$ (transetivity ex. 2)
4) $B \implies (\neg\neg B)$(ex. *)

MINI PROOF (*)
$\vdash B \implies \neg\neg B$
1) $\neg\neg\neg B \implies \neg B$(ex. 5)
2) $(\neg\neg\neg B \implies \neg B) \implies (B \implies \neg\neg B)$ (ax. 3)

5) $\neg\neg A \implies \neg\neg B$ (trans ex.2)
6) $(\neg\neg A \implies \neg\neg B) \implies (\neg B \implies \neg A)$(ax. 3)
7 $(\neg B \implies \neg A)$ (mp)
Deduction theorem OUTTA THIS BITCH
$\vdash_h (A \implies B) \implies (\neg B \implies \neg A)$

ALSO SHIT LIKE
$\vdash_h (\neg A \implies B) \implies (\neg B \implies A)$
Ded thrm $\neg A \implies B \vdash_H \neg B \implies A$
1) $\neg A \implies B$
2) $B \implies \neg\neg B$ (ex. *)
3) $\neg A \implies \neg\neg B$(transitivity ex. 2)
4) $(\neg A \implies \neg\neg B) \implies (\neg B \implies A)$(ax. 3)
5) $\neg B \implies A$ (mp)
DDEDUCTION THRM OUT OF THIS FUCKING BITCH
$\vdash_H (\neg A \implies B) \implies (\neg B \implies A)$

Chapter 3

Prepositional Logic *(First Order Logic)*

Involves quantifiers (∀, ∃) Objects: constants variables

Example statement $\forall x.\exists y.(P(x, f(a), z) \rightarrow Q(u))$ Ordered pairs of objects are written as
$< x, y >$
$< x, y > \neq < y, x >$
$< x_1, y_1 > = < x_2, y_2 > \iff x_1 = x_2, y_1 = y_2$

Ordered n-tuple
$< x_1, ..., x_n >$.
$S = \{< m, n > | m, n \in \mathbb{N}, m < n\} = \{< 0, 1 >, < 0, 2 >, < 0, 3 >, ...\}$
Cartesian Product of Sets
SxT = $\{< s, t > | s \in S, t \in T\}$
TxS = $\{< t, s > | s \in S, t \in T\}$
Example. S = $\{1, 2\}$, T = $\{3\}$.
SxT = $\{< 1, 3 >, < 2, 3 >\}$
TxS = $\{< 3, 1 >, < 3, 2 >\}$

S^n = SxSxSx....xS n times
Ordered pair - binary relation ¡x,y¿
Ordered n-tuple - n-ary relation $< x_1, ..., x_n >$

A n-ary relation R: R = $\{< x_1, ..., x_n >\} | x_i \in S, i \in [1, n], x_1, ...x_n$ satisfy the relation R $\} \subset S^n$

$\leq$ - binary relation over $\mathbb{N}$R = $\{< m, n > | m, n \in \mathbb{N}, m \leq n\} \subset \mathbb{N}^2$

Ex. R = $\{< x, y, z > | x + y < z, x, y, z \in \mathbb{N}\}$

R = $\{< x, y > | x, yaresibilings\}$ S is the set of all peopl. R $\subset S^2$.

Reflexive $< x, x > \in R \forall x \in S$ (xRx)

Symmetric if $< x, y >\in Rthen < y, x >\in R$

Transitive $< x, y >\in R$ and $< y, z >\in R$ then $< x, z >\in R$

If a relation satisfies all of these then it is considered an equivalence relation. Equivilance Relation $forall x \in S$
$\overline{x} = \{y|y \in S \ni< x, y >\in R\}$

An n-ary function (mappting) f: $S \to T$
is a (n+1) ary relation $R_f \ni$
$R_f = \{< x_1, ..., x_n, y > |x_1, ..., x_n \in S, y \in T, f(x_1, ...x_n) = y\}$

f - is a function $f : S \to T$ if $\forall s \in S$ there exists a unique element $y \in T \ni f(x) = y$ $(< x, y >\in R_f, xR_fy)$

$f : S^n \to T$
$f(x_1, ..., x_n) = y$
$R_f \in S^n xT$
$\{< x_1, ..., x_n, y > |x_1, ..., x_n \in S, y \in T, f(x_1, ..., x_n) = y\}$

$S_1 \subset S$
$f_{|S_1} : S_1^n \to T$
$R_{f_{|S_1}} \subset R_f \cap (S_1^n xT)$