

CS 245 NOTES

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Disclaimer: These notes are not meant to be a replacement for lecture. They are meant to be a method of learning for myself in the ways of L^AT_EX, and are meant to be used for reference or review of content.

Huge Forewarning: Majority of these notes have been directly typeset in class and have not been proofread, so they are prone to extreme amounts of formatting mistakes and errors. Much of these notes also will diverge from the content covered in lecture itself as these notes turn into an independent introduction to logic for computer science - meant to be a full replacement for lecture but cover the same material.

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2.1 Introduction

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3 Intro

About logic and its use in software engineering. Particularly used for program verification.

3.1 Logic Arguments

Example 3.1.1. If the train arrives late and there are no taxis then John will not be late.

Premise: John was not late. The train arrived late

\implies There were taxis

Symbolized:

p. Train is late

q. Taxis available

r. John is late

$p \wedge q \implies r$

$r \wedge p \implies q$

Example 3.1.2. If it is raining and Jane does not have her umbrella, Jane will get wet.

Premise: Jane was not wet. It is raining

\implies Jane had her umbrella

Propositional Logic

smallest building block is a statement without connectives, such as "and". The example with the trains is reasoning with propositional logic.

Predicate Logic (First Order Logic)

includes a means to describe relationships between objects, for example "every course has at least one object"

Program Verification

ways to describe what a system is required to do and reasoning about its correctness.

3.2 Review

example to do: sum from 1 to n for $\frac{1}{i(i+1)} = \frac{n}{n+1}$

ex. $p(n)$: 3 divides $(4^n + 5)$

4 Propositional Logic

What is a proposition?

A declarative sentence which is either true or false.

A simple (atomic) proposition is the basic building blocks to create compound propositions using connectives.

Proving this property of $R(A)$: A has the length ≥ 0

Prove $R(A) \forall A \in Form(\mathcal{L}^p)$

1. Assume $A \in Atom(\mathcal{L}^p)$
therefore $A = p$
Then $length(A) = 1 > 0$
2. Let $A \in Form(\mathcal{L}^p)$
Assume $R(A)$. Prove $R(\neg A)$
As $R(A) \implies length(A) > 0$
 $length(\neg A) = length(A) + 3 > 3 > 0$
3. Let $A, B \in Form(\mathcal{L}^p)$
Assume $R(A), R(B)$
Prove $R(A * B)$
The length of $(A * B) > 0$ then clearly

Parse Tree The leaves of a parse tree of a well formed formula are labeled with atoms The nodes of a parse tree of a well formed formula labeled have one children the node labeled with a binary connective have exactly two children

Prove $m(n)$ for every n in natural let $M(n)$: A of height n has the same number of left and right brackets prove $M(n)$ for any formula of height n , for every n in natural b.c. smallest formula, $n=1$ (single atom) therefore a in atom $'(' = 0 \text{ ')}' = 0$ Inductive hypothesis $n \geq 1$ assume for any formula A of height $\leq n$ (n) prove $M(n + 1)$

1. If I study for exams, then I get good grades
2. I do not eat healthy food whether or not I study for exams

3. I will pass the class only if I get good grades
4. If I do not study for exams, then I get good grades only if I eat healthy food
5. I will either pass the class or eat healthy food, but not both

All the atoms

s: I study for exams

g: I get good grades

h: I eat healthy food

p: I will pass the class

1. $(s \implies g)$
2. $(\neg h)$ [whether or not does not add anything]
3. $(p \implies g)$
4. $((\neg s) \implies (g \implies h))$
5. $(p \vee h) \wedge (\neg(p \wedge h))$

Well formed formula

- 1) $Atom(\mathcal{L}^p) \subset Form(\mathcal{L}^p)$
- 2) If $A \in Form(\mathcal{L}^p) \implies (\neg A) \in Form(\mathcal{L}^p)$
- 3) If $A, B \in Form(\mathcal{L}^p) \implies (A * B) \in Form(\mathcal{L}^p)$, where $*$ is a binary connective.

Structural Induction. Recursively proving that a concatenation or similar of a bunch of elements holds the same property as those elements \implies Base case the lowest level element

Suppose that if the number of occurrences of atoms in a formula is m and the number of occurrences of any binary connective is n. Then prove that $m = n + 1$ proof:

Base Case: For an atom, $m = 1$ and $n = 0$. So the statement is true.

Inductive hypothesis: Assume it is true for formula ϕ and ψ ,

Here m_ϕ = number of occurrences of atoms in ϕ

m_ψ = number of occurrences of atoms in ψ

n_ϕ = number of occurrences of binary connectives in ϕ

n_ψ = number of occurrences of binary connectives in ψ

so $m_\phi = n_\phi + 1$

$m_\psi = n_\psi + 1$

Inductive step:

Consider the formula $\Omega = (\neg\phi)$, $m_\Omega = m_\phi$, $n_\Omega = n_\phi$

Thus true for \neg

Consider the formula $\Omega = (\phi * \psi)$, $m_\Omega = m_\phi + m_\psi$, $n_\Omega = n_\phi + n_\psi + 1$ by IH

$m_\Omega = n_\phi + n_\psi + 2$

$\implies m_\Omega = n_\Omega + 1$.

Therefore proven true for binary connective

Therefore we have proven this true by structural induction.

example of this property

$$\phi = (((\neg p) \implies s) \wedge r) m_\phi = 3n_\phi - 2$$

$$\psi = (p \implies (\neg q)) m_\psi = 2n_\psi = 1$$

1. Give the truth table for the following formulas.

a) $(p \implies q) \wedge r$

b) $(p \implies \neg p)$

c) $((p \implies q) \wedge (q \implies p))$

d) $(p \wedge r) \vee (\neg r \implies q)$

e) $(\neg r \wedge p) \vee (r \vee \neg p)$

2) answer the following questions about semantic implication a) let Σ be the set of well formed propositional formulas and let A be a well formed propositional formula. What does it mean for $\Sigma \models A$. What does it mean for $\Sigma \not\models A$.

b) Consider the set Σ and formula A .

$$\Sigma = p, (p \implies q) A = q$$

Does $\Sigma \models A$?

c) Consider $\Sigma = q, (p \implies q), A = p$

Does $\Sigma \models A$?

b) tt:

$$p \ q \ p \implies q$$

$$0 \ 0 \ 1$$

$$0 \ 1 \ 1$$

$$1 \ 0 \ 0$$

$$1 \ 1 \ 1$$

As the only time when Σ^t is true is the last line, where A^t is also true, we have $\Sigma \models A$ is true.

c) As $\Sigma \models A$ is not true, there is one truth valuation then where $\Sigma^t \implies A^t$. Therefore $\Sigma \not\models A$.

3. Let C be the set \wedge, \vee, \implies of propositional connectives and F be the set α, β, γ of well-formed propositional formulas.

a) Suppose that, θ is a well formed formula that uses only connectives C and the formulas in F .

The formulas α, β, γ are all tautologies.

Prove that θ is a tautology.

Base case:

α, β, γ are tautologies

Induction hyp:

let ϕ and ψ be two well formed formulas and they use only connectives in C and formulas in F .

ϕ and ψ are both tautologies.

Inductive step:

$$\theta = \phi \wedge \psi$$

$$\phi \vee \psi$$

$$\phi \implies \psi$$

truth table:

$\phi^t \psi^t \wedge^t \text{or}^t \text{implies}^t$ (of ϕ and ψ)

0 0 0 0 1

0 1 0 1 1

1 0 0 1 0

1 1 1 1 1 j- this is the only formula we get as $\phi^t = \psi^t = 1$ for any truth valuation

Thus proved true.

b)

Suppose that θ is a well formed formula that as well only used those connectives and formulas and also the \neg connective.

Find a counter example to show that θ is not necessarily a tautology.

$A^t \in 0, 1$

$t : \text{Atom}\mathcal{L}^p \rightarrow 0, 1$

Having atom values

$A_1 \dots A_n \models A$ means that we can deduce the value of A (being true) when we know the premise or propositional atoms.

thrm 2.5.2.

proof = j direction

$a_1 \dots a_n \models a$

prove $a_1 \text{and} \dots a_n \implies a$ is a tautology.

$a + 1 \dots a_n \models a$ for every t for which $a_1^t = \dots a_n^t = 1$ we have $a^t = 1$

prove that for any truth valuation

$t^1, a_1^{t^1} \text{and} \dots a_n^{t^1} \implies a^t = 1$

1) let t be any truth valuation which

$a_1^t = \dots a_n^t = 1$ (of course $a^t = 1$)

$a^t \text{and} \dots a_n^t \implies a^t = 1$

2) let t' be any truth valuation for which

which $a_i^{t'} = 0$ for some i between 1 and n

$a_1^{t'} \text{and} \dots a_n^{t'} \implies a^t$

as some $a_i = 0$ we have

$0 \implies a^t = 1$

now j= direction

$a_1 \text{and} \dots a_n \implies a$ is a tautology

prove $a_1 \dots a_n$ has a tautological consequence of a

i.e. prove $a_1 \dots a_n \models a$

for every t $a_1^t \text{ and } \dots \text{ and } a_n^t \implies a^t = 1$
let t' be a truth valuation such that
 $a_1^{t'} = \dots = a_n^{t'} = 1$
prove $a^{t'} = 1$
 $a_1^{t'} \text{ and } \dots \text{ and } a_n^{t'} \implies a^{t'} = 1$
becomes
 $1 \implies a^{t'} = 1$
thus $a^{t'} = 1$

Adequate set - set of connectives C , such that any other connective can be represented using the connectives from C

Th. $\{\neg, \wedge, \vee\}$ is an adequate set.

$p \implies q \equiv \neg p \vee q$
 $p \iff q \equiv (p \implies q) \wedge (q \implies p)$

Corollary:

$\{\neg, \wedge\}$
 $\{\neg, \vee\}$
 $\{\neg, \implies\}$

Hilbert system axioms:

- 1) $A \implies (B \implies A)$
- 2) $((A \implies (B \implies C)) \implies ((A \implies B) \implies (A \implies C)))$
- 3) $((\neg A \implies \neg B) \implies (B \implies A))$

Modus Ponens (MP)

$\phi, \phi \implies \psi \vdash_H \psi$
 $\frac{\phi, \phi \implies \psi}{\psi}$

- 1) α_1 (premise)

Proving $\vdash_H A \implies A$

- 1) $A \implies ((A \implies A) \implies A)$ (axiom 1)
- 2) $A \implies ((A \implies A) \implies A) \implies ((A \implies (A \implies A)) \implies (A \implies A))$ (axiom 2)
- 3) $((A \implies (A \implies A)) \implies (A \implies A))$ (modus ponens 1,2)
- 4) $A \implies (A \implies A)$ (axiom 1)
- 5) $A \implies A$ (modus ponens 3,4)

TH transitivity

$A \implies B, B \implies C \vdash_H A \implies C$

- 1) $A \implies B$ (premise)
- 2) $B \implies C$ (premise)

- 3) $(A \implies (B \implies C)) \implies ((A \implies B) \implies (A \implies C))$ (axiom 3)
- 4) $(B \implies C) \implies (A \implies (B \implies C))$ (axiom 1)
- 5) $B \implies C$ (modus ponens 2, 4)
- 6) $(A \implies B) \implies (A \implies C)$ (modus ponens 3,5)
- 7) $A \implies C$ (modus ponens 1,6)

Th. (Deduction Theorem)

$\Sigma \vdash_H (A \implies B) \text{ iff } A_1 A \vdash_H B \quad \alpha_1, \dots, \alpha_n \vdash_H \alpha$
 $\vdash_H (\alpha_1 \implies \dots (\alpha_{n-1} (\alpha_n \implies \alpha)) \dots)$

ex.3

- $\vdash_H (\neg A \implies (A \implies B))$
- 1) $\neg A \implies (\neg B \implies \neg A)$ (axiom 1)
 - 2) $(\neg B \implies \neg A) \implies (A \implies B)$
 - 3) $\neg A \implies (A \implies B)$ (ex. 2 transitivity)

ex. 4

$\vdash_H \neg\neg A \implies A$
 By Ded. Thrm $\neg\neg A \vdash_H A$

- 1) $\neg\neg A$ (premise)
- 2) $\neg\neg A \implies (\neg\neg\neg\neg A \implies \neg\neg A)$ (axiom 1)
- 3) $\neg\neg\neg\neg A \implies \neg\neg A$ (modus ponens 1,2)
- 4) $(\neg\neg\neg\neg A \implies \neg\neg A)(\neg A \implies \neg\neg\neg A)$ (axiom 3)
- 5) $\neg A \implies \neg\neg\neg A$ (modus ponens 3, 4)
- 6) $(\neg A \implies \neg\neg\neg A) \implies (\neg\neg A \implies A)$ (axiom 3)
- 7) $\neg\neg A \implies A$ (modus ponens 5, 6)
- 8) A (modus ponens 1, 7)

Any set of connectives with the capability to express all truth-tables is said to be adequate

$\{\wedge, \vee, \neg\}$ is adequate

$\neg p$

$p \wedge q$

$p \vee q$

$p \implies q$

$p \iff q$

$p \implies q \equiv \neg p \vee q$

$p \iff q \equiv (p \wedge q) \vee (\neg p \wedge \neg q) \equiv (p \implies q) \wedge (q \implies p) \equiv (\neg p \vee q) \wedge (\neg q \vee p)$

Exercise:- Show $\neg, \wedge, \vee, \implies$ and adequate.

NAND (\uparrow / $\bar{\wedge}$) / Sheffer Stroke (\neg), and NOR (\downarrow) / shroder connective are adequate.

Proof

NOR \vdash

$\neg p \equiv p \vdash$

$p \vee q \equiv \neg \neg(p \vee q) \equiv \neg(p \vdash q) \equiv (p \vdash q) \vdash (p \vdash q)$

$p \wedge q \equiv \neg(\neg A \vee \neg B) \equiv (\neg A \vdash \neg B)$, by De Morgan's Law.

$p \implies q \equiv (\neg p \vee q) \equiv \neg(\neg p \vdash q) \equiv ((p \vdash p) \vdash q)$

$p \iff q \equiv (p \implies q) \wedge (q \implies p)$

Hilbert System

$\Sigma \vdash A$ means, A is provable from Σ

Axioms

Ax. 1 $\phi \implies (\psi \implies \phi)$

Ax. 2 $(\phi \implies (\psi \implies \gamma)) \implies ((\phi \implies \psi) \implies (\phi \implies \gamma))$

Ax. 3 $(\neg \phi \implies \neg \psi) \implies (\psi \implies \phi)$

Modus Ponens (MP)

$\frac{\phi, \phi \implies \psi}{\psi}$

Is the following an axiom?

$(p \implies q) \implies (((\neg p) \implies q) \implies (p \implies q))$

Ex. 1

$\alpha \implies \beta, \beta \implies \gamma \vdash \alpha \implies \gamma$

This formula is equivalent to,

$\alpha \implies \beta, \beta \implies \gamma, \alpha \vdash \gamma$

1. $\alpha \implies \beta$ (premise)

2. $\beta \implies \gamma$ (premise)

3. α (premise)

4. β (mp 1,3)

5. γ (mp 2,4)

PROVED

Ex. 2

$(\alpha \implies \beta) \implies \gamma \vdash \alpha \implies (\beta \implies \gamma)$

1. $(\alpha \implies \beta) \implies \gamma$ (premise)

2. $((\alpha \implies \beta) \implies \gamma) \implies (\beta \implies ((\alpha \implies \beta) \implies \gamma))$ (Ax. 1)

3. $(\beta \implies ((\alpha \implies \beta) \implies \gamma))$ (mp 1,2)

4. $(\beta \implies ((\alpha \implies \beta) \implies \gamma)) \implies ((\beta \implies (\alpha \implies \beta)) \implies (\beta \implies \gamma))$ (Ax. 2)

5. $((\beta \implies (\alpha \implies \beta)) \implies (\beta \implies \gamma))$ (mp 3,4)

6. $(\beta \implies (\alpha \implies \beta))$ (Ax. 1)

7. $(\beta \implies \gamma)$ (mp 5,6)

8. $(\beta \implies \gamma) \implies (\alpha \implies (\beta \implies \gamma))$ (ax. 1)

9. $(\alpha \implies (\beta \implies \gamma))$ (mp 7,8)

Ex. 3

$\phi \implies (\psi \implies \theta), \psi \vdash (\phi \implies \theta)$

1. $(\phi \implies (\psi \implies \theta))$ (premise)

2. ψ (premise)

3. $(\phi \implies (\psi \implies \theta)) \implies ((\phi \implies \psi) \implies (\phi \implies \theta))$ (ax. 2)

4. $((\phi \implies \psi) \implies (\phi \implies \theta))$ (mp 1,3)

5. $\psi \implies (\phi \implies \psi)$ (ax. 1)

6. $\phi \implies \psi$ (mp 2, 5)

7. $\phi \implies \theta$ (mp 4,6)

1) Reflexivity (REF)

$\psi \vdash \psi, \frac{\phi}{\phi}$

2) Addition of premises (+)

$\Sigma \vdash \phi$

then $\Sigma' \cup \Sigma \vdash \phi, \frac{\omega}{\frac{\phi}{\omega\omega'}}$

3) \neg elimination

$\Sigma, \neg\psi \vdash \omega$

$\Sigma, \neg\psi \vdash \neg\omega$

then

$\Sigma \vdash \phi$

prove hilbert axioms using natural deduction

axiom 1:

$\phi \implies (\psi \implies \phi)$

axiom 2:

$(\phi \implies (\psi \implies \omega)) \implies ((\phi \implies \psi \implies \phi \implies \omega))$

axiom 3:

$(\neg\phi \implies \neg\psi) \implies (\psi \implies \phi)$

axiom2:

1. $(\phi \implies \psi \implies \zeta)$ assumption

2. $\phi \implies \psi$ assumption

3. ϕ assumption

4. $\psi \implies -$

5. $\psi \implies \zeta \implies -$

6. $\zeta \implies -$

7. $\phi \implies \zeta \implies +$

8. $\phi \implies \psi \implies \phi \implies \zeta \implies +$

Q5 a) hint

$\Sigma, \phi \vdash_H \psi$

1. α_1

\cdot
 \cdot
 \cdot
 $\alpha_n = \psi$
 $\Sigma \vdash \phi \implies \psi$
1. $\phi \implies \alpha_1$
 \cdot
 \cdot
 \cdot
n. $\phi \implies \alpha_n(\phi \implies \psi)$
not sure if steps are actual step counts
Case $\alpha_i \in \Sigma$ (3 steps)
Case $\alpha_i \in Axioms$ (8 steps)
Case $\alpha_i = \phi$ (7 steps)
Case α_i - MP (10 steps)
Soundness thrm

$\Sigma \vdash \alpha \implies \Sigma \models \alpha$
contrapositive
 $\Sigma \not\models \alpha \implies \Sigma \not\vdash \alpha$
e.g. make a counter example (truth table, \models) and show that it is not valid.

Th 2.6.2 $\Sigma \vdash \phi$ then exists $\Sigma^o - > finite \Sigma^o \subset \Sigma$
Proof: Structural induction on the size of Σ
Bc. $\Sigma = \phi \vdash \phi$
 $\Sigma^o := \Sigma$ and $\Sigma^o - > finite$
Ind. hyp.
Assume that for $\Sigma \vdash \phi$ where Σ is of size $\leq k$, there exists a $\Sigma^o \subset \Sigma$ such that Σ^o is finite and $\Sigma^o \vdash \phi$
Ind. setp.
 $\Sigma \vdash \phi$
 Σ is of size $k+1$
Consider 10 cases
+) $\Sigma \vdash \phi$ size $\leq k$
then $\Sigma \cup \Sigma' \vdash \phi$ size k_1
By ind. hyp $\exists \Sigma^o \subset \Sigma, \Sigma^o$ is finite
 $\Sigma^o \vdash \phi$
 $\Sigma^o \subset \Sigma \cup \Sigma'$
 $\Sigma^o \vdash \phi, \Sigma^o$ is finite
 \neg -)
1 $\Sigma, \neg \phi \vdash \psi$
2 $\Sigma, \neg \phi \vdash \neg \psi$
3 (?) $\Sigma \vdash \phi$
 \neg) by ind hyp
 $\exists \Sigma_1 \subset \Sigma$
 Σ_1 is finite
 $\Sigma_1 \vdash \psi$

$$\Sigma_1 \subset \Sigma \cup \neg\psi$$

thrm 2.6.3 transitivity of deduciblity)

$$\Sigma \vdash \Sigma' \text{ and } \Sigma' \vdash \phi \implies \Sigma \vdash \phi$$

Note $A \vdash B$ means $A \vdash B_i$ for all formulas in B .

$$1) A_1, \dots, A_n \vdash \phi \text{ (premise)}$$

$$2) A_1, \dots, A_{n-1} \vdash A_n \implies \phi \text{ (} \implies + \text{)}$$

.

.

.

$$n) \text{ EMPTY FUCKING SET } \vdash A_1 \implies (A_2 \implies \dots (A_n \implies \phi) \dots)$$

$\Sigma \vdash A_1 \implies (A_2 \implies \dots (A_n \implies \phi) \dots)$ introduction of premises (already true with current premises)

$$\Sigma \vdash A_1 \text{ (premise)}$$

$$\Sigma \vdash (A_2 \implies \dots (A_n \implies \phi) \dots) \text{ (} \implies - \text{)}$$

.

.

.

$$\Sigma \vdash A_n \implies \phi$$

$$\Sigma \vdash A_n$$

$$\Sigma \vdash \phi \text{ (} \implies - \text{ N TIMES)}$$

ex: $\Sigma, \phi \vdash \psi, \Sigma, \phi \vdash \neg\psi \implies \vdash \neg\phi$ (reduction ad absurdum) $(\neg +)$ -i, not a basic rule

Obtained from $\neg -$

Box notation:

$\neg\neg\phi$ (assumption)

$\neg\phi$ (assumption)

$\neg\neg\phi$ (copy)

ϕ ($\neg - 2,3$)

ψ (premise OF ABOVE \vdash shit)

$\neg\psi$ (premise AGAIN)

$\neg\phi$ ($\neg - 1,4$)

YOU MUST PROVE NEGATION ADDITION INLINE IF YOU USE IT

$$\Sigma, \phi \vdash \psi$$

$$\Sigma \vdash \Sigma$$

$$\Sigma, \neg\neg\phi \vdash \Sigma$$

$$\neg\neg\phi \vdash \phi \text{ (EXAMPLE YOU MUST PROVE IN FUCKING LINE)}$$

$$\Sigma, \neg\neg\phi \vdash \phi$$

$$\Sigma, \neg\neg\phi \vdash \Sigma$$

$$\Sigma, \neg\neg\phi \vdash \phi$$

$$\Sigma, \phi \vdash \psi$$

$\Sigma, \neg\neg\phi \vdash \psi$

DOING ALL THIS FUCKING SHIT AGAIN FOR $\neg\psi$
WE GET A CONTRADICTION

$\Sigma \vdash \neg\phi$

1. $\phi \implies \psi, \phi \vdash$

Proof of soundness theorem

structural induction on the length of the proof n.

bc: $n = 1$

$\phi \vdash \phi$ (ref)

$n = 2$

$\Sigma, \phi \vdash \phi$

$\phi \vdash \phi$ (ref)

$\Sigma, \phi \vdash \phi$ (ref, +)

$\phi \models \phi$ (trivial truth table)

$\Sigma, \phi \models \phi$ (also trivial)

Inductive Hypothesis:

Assume that for every proof $\Sigma \vdash \phi$ of size $\leq k$

we have $\Sigma \models \phi$

Inductive Conclusion:

1. ϕ_1 (premise)

.

.

.

n) ϕ_n (premise)

n+1) ϕ_{n+1} (ND rules)

,

.

.

k) ϕ_k (ND rules)

k+1) ϕ_{k+1} (ND rules) [valid?]

Case 1:

ϕ obtained by writing

$(\wedge +)$ rule

.
 .
 .
 i) ϕ_i
 .
 .
 .
 j) $\phi_j, i \leq j \leq k$
 .
 .
 .
 k+1) $\phi = \phi_i \wedge \phi_j (\wedge +)$
 $i \leq k$
 $\Sigma \vdash \phi_i$
 by ind. hyp
 $\Sigma \models \phi_i$
 Similarly for j
 For every $t, \Sigma^t \implies \phi_i^t = 1$ and $\phi_j^t = 1 \implies (\phi_i \wedge \phi_j)^t = 1$

Case 2: $\implies -$

.
 .
 .
 i) ϕ
 .
 .
 j) $\phi_i \implies \phi_j, i, j \leq k$
 .
 .
 k) $\phi = \phi_j$

We have $\Sigma^t = 1 \implies \phi_i^t = 1$ and $(\phi_i \implies \phi_j)^t = 1 \implies \phi_j^t = 1$

Completeness thm proof:

$\Sigma \models \phi \implies \Sigma \vdash \phi$

Contrapositive:

$\Sigma \not\models \phi \implies \Sigma \not\vdash \phi$

$\Sigma \not\models \phi \implies \Sigma \cup \{\neg\phi\}$ consistent $\implies \Sigma \cup \{\neg\phi\}$ has a model (satisfiable) (model is a truth valuation that satisfies it) $\implies \Sigma \not\models \phi$

ASIDE:

Σ consistent iff there is no formula $\phi \in \text{Form}(\mathcal{L}^p)$

$\Sigma \vdash \phi$
 $\Sigma \models \phi$

Ex. $p, \neg p \vdash p$
 $\vdash \neg p$
inconsistent
BACK TO PROOF

Σ - maximal constant if
1) Σ - consistent
2) $\forall \phi \notin \Sigma, \Sigma \cup \phi$ is consistent

Soundness thm (2):
if Σ satisfiable then Σ consistent

Soundness thm (1) \iff Soundness thm (2)

\implies direction
 $\Sigma \vdash \phi \implies \Sigma \models \phi$
Prove that if Σ satisfiable then Σ consistent
Assume the contrary Σ satisfiable and Σ inconsistent

ASSIDE
 Σ inconsistent: $\exists \phi : \Sigma \vdash \phi$
 $\Sigma \vdash \neg \phi$

Doing Soundness thm (1). on the contrary
 Σ
 $\models \phi$
 $\Sigma \models \neg \phi$
Therefore we get a contradiction
 $\forall t$ truth valuations $\exists \Sigma^t = 1 \implies \phi^t = 1$ and $\neg \phi^t = 1$ (contradiction)
 $\implies \Sigma$ consistent

\Leftarrow DIRECTION
 Σ -sat $\implies \Sigma$ consistent
Prove that $\Sigma \vdash \phi \implies \Sigma \models \phi$
Assume the contrary
 $\Sigma \not\models \phi \implies \Sigma \not\vdash \phi$
 $\Sigma \not\models \phi \implies \Sigma \cup \neg \phi$ satisfiable

$\Sigma \vdash \phi(\text{hyp})$
 $\Sigma \cup \neg\phi \vdash \phi$
 -i coontradiction
 so assumption is invalid $\Sigma \models \phi$
 so $\text{ST1} \iff \text{ST2}$
 $\Sigma \not\models \iff (\text{UNFINISHED FUCKING BULLSHIT})$

Completeness thmr(2)
 $\Sigma \text{consist} \implies \Sigma \text{ satisfiable}$

Lemma
 $\Sigma \text{ max consist}$
 $\phi \in \Sigma \iff \Sigma \vdash \phi$
 \implies
 $\Sigma \text{ max const and } \phi \in \Sigma$
 $\Sigma' := \Sigma - \phi$
 1) $\phi \vdash \phi(\text{ref})$
 2) $\Sigma', \phi \vdash \phi(+,1)$
 $\Sigma \vdash \phi$

i=
 $\Sigma \text{ max const and } \Sigma \vdash \phi$
 Prove $\phi \in \Sigma$
 Assume the contrary: $\phi \notin \Sigma$

ASIDE DEFN OF MAX CONST
 $\Sigma \text{ max const: } \forall \phi \notin \Sigma, \Sigma \cup \phi$
 $\Sigma \cup \phi$ -inconsistn
 $\exists \alpha :$
 $\Sigma \cup \phi \vdash \alpha$
 $\Sigma \cup \phi \vdash \neg \alpha (\neg +)$

(NEGATION ELIM)
 $\Sigma, \neg\phi \vdash \phi$
 $\Sigma, \neg\phi \vdash \neg\phi$
 $\Sigma \vdash \phi (\neg -)$

Σ - inconsistent (contradiction)
 $\phi \in \Sigma$.

MIDTERM LEVEL PROOF (remember/write down)
 $\Sigma \not\models \phi \implies \Sigma \cup \neg\phi \text{ consistent}$

Proof:

Assume the contrary $\Sigma \cup \neg\phi$ inconsistent

$\exists \alpha$

$\Sigma \cup \neg\phi \vdash \alpha$

$\Sigma \cup \neg\phi \vdash \neg\alpha$

$\Sigma \vdash \phi (\neg -)$

(contradiction with hyp)

$\implies \Sigma \cup \neg\phi$ const.

Lindervbaum lemma:

Σ -consistent

$\exists \Sigma' : \Sigma \cup \Sigma' \text{ - } \perp$ maximum consistent

3)

$\Sigma \cup \neg\phi$ - satisfiable

then $\Sigma \not\models \phi$

$\exists t : \Sigma^t = 1$ and $(\neg\phi)^t = 1$

Assume the contrary: $\Sigma \models \phi$

$\forall t : \Sigma^t = 1 \implies$

$\phi^t = 1$

$(\neg\phi)^t = 0$

But that contradicts the \exists line just above.

Contradicts the conclusion, thus assumption is not correct

$\Sigma \not\models \phi$

Soundness

Formally if $\Sigma \vdash \alpha$ is a premise then $\Sigma \models \alpha$

Informally if a formula is provable then it is true

Completeness

Formally if $\Sigma \models \alpha$ then $\Sigma \vdash \alpha$ is provable

Informally, if a formula is true, then it is provable

Definition of consistent

Σ is consistent if for no $\alpha, \Sigma \vdash \alpha$ and $\Sigma \vdash (\neg\alpha)$

A set of Σ is consistent if it does not allow us to prove a contradiction

Question :-

Is the following set consistent?

$\Sigma = ((p \implies q) \implies q), (q \implies p)$

Claim :-

If Σ is satisfiable, then Σ is consistent

Proof :-

Given a satisfying assignment v of Σ , we choose a variable p from Σ and define α as $(\neg p)$ if $v(p) = T$ and p if $v(p) = F$

We then have the scenario where v satisfied Σ but does not satisfy our formula α . Therefore, $\Sigma \not\models \alpha$ and, from soundness, $\Sigma \not\vdash \alpha$

We are using that technique to prove the question.

Find an α such that $\Sigma \not\models \alpha$

Let $v(p) = T$ and $v(q) = T$

Now let $\alpha = (\neg P)$

We now have $\Sigma \not\models \alpha$ and $\Sigma \not\vdash \alpha$ and therefore Σ is consistent by our definition. (because it is consistent if it does not allow us to prove a contradiction, and that is what we have shown)

Question :-

Is the following set consistent?

$\Sigma = (p \implies q), p, (\neg q)$

Answer :-

NO! lul not telling you why, jk.

$p \implies q$

p

q (mp)

$\neg q$ (premise)

We can prove a contradiction from this!

Question :-

Is the following set consistent?

$\Sigma = (p \implies (q \implies r)), (p \implies q), (\neg(p \implies r))$

Question :-

If Σ_a is consistent and Σ_b is consistent, is $\Sigma_a \cup \Sigma_b$ always consistent?

Answer :- no

$\Sigma_a = p$

$\Sigma_b = \neg p$

$\Sigma_a \cup \Sigma_b = p, \neg p$

Prove a contradiction

How about $\Sigma_a - \Sigma_b$

We can make the claim A intersection B is less than A or less than B

If Σ is consistent and $\Sigma' \subset \Sigma$ is consistent then Σ' is consistnet

Soundness Thrm:

If $\Sigma \vdash \phi \implies \Sigma \models \phi$

Completeness Thrm:

If $\Sigma \models \phi \implies \Sigma \vdash \phi$

Soundness Thrm (2);

If Σ -satisfiable $\implies \Sigma$ -consistent

Completeness Thrm (2):

If Σ -consistent $\implies \Sigma$ -satisfiable

Proof of competeness (1):

We will prove by contrapositive:

1)

$\Sigma \not\models \phi \implies \Sigma \not\vdash \phi$

$\Sigma \not\models \phi \implies$

$\Sigma \neg \phi$ consistent

2)

$\Sigma \neg \phi$ consistent \implies

$\Sigma \neg \phi$ -satisfiable

3)

$\Sigma \neg \phi$ -satisfiable $\implies \Sigma \models \phi$

Prove ???????

1)

$\Sigma \not\models \phi$

Prove $\Sigma \cup \neg \phi$ consistnet

Assume $\Sigma \cup \neg \phi$ inconsist

$\exists \alpha : \Sigma \cup \neg \phi \vdash \phi$

$\Sigma \cup \neg \phi \vdash \neg \phi$

$\Sigma \vdash \phi(\neg -)$

(contradiction)

2)

$\Sigma \cup \neg \phi$ - consistent then $\Sigma \cup \neg \phi$ - satisfiable

Lemma Lindenbaum:

For every Σ -consistent, $\exists \Sigma' \ni \Sigma \cup \Sigma'$ - maximal consistent

Lemma (5.3.6)

Σ^* -maximal consistent and $\forall p \in \text{Atom}(\mathcal{L}^p), p^t = 1 \iff p \in \Sigma^*$

Then

$\forall A \in \text{Form}(\mathcal{L}^p), A^t = 1 \iff A \in \Sigma^*$

$\Sigma \cup \neg\phi$ is consistent (hypothesis)

Prove $\Sigma \cup \neg\phi$ is satisfiable.

By lindebaum lemma, we can extend this set

$\exists \Sigma^*$ -max const \ni

$\Sigma \cup \neg\phi \subset \Sigma^*$

By second lemma

$\neg\phi \in \Sigma^*$

$(\neg\phi)^t = 1$

$\Sigma \subset \Sigma^*$

By second lemma

$\forall A \in \Sigma \subset \Sigma^* \implies A^t = 1$

$\implies \Sigma^t = 1$

$\forall t : (\neg\phi)^t = 1$ and $\Sigma^t = 1$

$\implies \Sigma \cup \neg\phi$ -satisfiable

LAST STEP? PROOF WAS JUST PROVEN WTF

$\Sigma \cup \neg\phi$ -satisfiable

Prove $\Sigma \not\models \phi$

Let us assume $\Sigma \models \phi$

$\exists t \ni \Sigma^t = 1$ and $(\neg\phi)^t = 1$ by defn of satisfiable.

$\implies \phi^t = 0$

$\Sigma \models \phi$

$\forall t' \ni \Sigma^{t'} = 1 \implies \phi^{t'} = 1$

But this contradicts, so our assumption is not correct so

$\Sigma \not\models \phi$, THUS PROVEN COMPLETENESS?

SPRING 2013 MIDTERM (for review)

while the waves are high i will go surfing

$p \wedge q$

swimming even though waves high

$p \wedge q$

i will make sand castles only if i dont go swimming or diving

(PROPER $p \implies \neg(q \vee r)$)

i wont go swimming unless i remember to bring my towel and bathing suit

$p \implies (q \wedge r)$

$\vdash_h (A \implies B) \implies (\neg B \implies \neg A)$
 Deduction thrm. $A \implies B \vdash_H \neg B \implies \neg A$
 1) $A \implies B$
 2) $\neg\neg A \implies A$ (ex. 5)
 3) $\neg\neg A \implies B$ (transitivity ex. 2)
 4) $B \implies (\neg\neg B)$ (ex. *)

MINI PROOF (*)

$\vdash B \implies \neg\neg B$
 1) $\neg\neg\neg B \implies \neg B$ (ex. 5)
 2) $(\neg\neg\neg B \implies \neg B) \implies (B \implies \neg\neg B)$ (ax. 3)

5) $\neg\neg A \implies \neg\neg B$ (trans ex.2)
 6) $(\neg\neg A \implies \neg\neg B) \implies (\neg B \implies \neg A)$ (ax. 3)
 7) $(\neg B \implies \neg A)$ (mp)

Deduction theorem OUTTA THIS BITCH

$\vdash_h (A \implies B) \implies (\neg B \implies \neg A)$

ALSO SHIT LIKE

$\vdash_h (\neg A \implies B) \implies (\neg B \implies A)$
 Ded thrm $\neg A \implies B \vdash_H \neg B \implies A$

1) $\neg A \implies B$
 2) $B \implies \neg\neg B$ (ex. *)
 3) $\neg A \implies \neg\neg B$ (transitivity ex. 2)
 4) $(\neg A \implies \neg\neg B) \implies (\neg B \implies A)$ (ax. 3)
 5) $\neg B \implies A$ (mp)

DDEDUCTION THRM OUT OF THIS FUCKING BITCH

$\vdash_H (\neg A \implies B) \implies (\neg B \implies A)$