Numerical simulation of two-dimensional heat convection-diffusion with ellipsoid geometries

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Abstract

This is a test.

1 Introduction

2 Model overview

Throughout this work we will denote the spatial region under scrutiny by $\Omega \subseteq \mathbb{R}^2$, and we write $\partial\Omega$ to indicate the boundary of the region. The function $T : \Omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ will denote the temperature $T = T(x, y, t)$, and the functions $v_x : \Omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $v_y : \Omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ will denote $v_x = v_x(x, y, t)$ the horizontal and $v_y = v_y(x, y, t)$ the vertical components of the velocity of the fluid flow.

2.1 Two-dimensional Navier-Stokes formulation

The Navier-Stokes equations for an incompressible fluid are given by

$$\nabla \cdot v = 0 \tag{1}$$

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\frac{1}{\rho}\nabla p + \alpha \nabla^2 v \tag{2}$$

where $v = (v_x, v_y)$ is the fluid velocity, ρ denotes mass density of the fluid, and p denotes pressure. Incompressibility is incorporated through Equation (), since nonzero divergence implies the presence of either sources or sinks of fluid. Now because v is two-dimensional, Equation () actually implies a pair of partial differential equations for both the x and y components. The pressure p is determined through the Poisson equation

$$\nabla^2 p = b, \tag{3}$$

where b is taken as a constant term for the purposes of this study. Many wonderful resources exist online for understanding the formulation of these equations, specifically Barba's work [3], but we do not include a derivation of these relations here.

2.2 Heat convection-diffusion equation

The heat convection-diffusion equation found in the literature [2],

$$\frac{\partial T}{\partial t} - \alpha \nabla^2 T + v \cdot \nabla T = 0, \tag{4}$$

is a combination of parabolic and hyperbolic partial differential equations relating the temporal change in temperature with its spatial diffusion and the flow of heat packets elsewhere in the fluid. Here, ∇ is the gradient operator $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$, ∇^2 denotes the Laplacian $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $v = (v_x, v_y)$ denotes the fluid velocity vector, and α is a diffusivity constant.

The ubiquity of parabolic PDEs in modeling scientific phenomenon is well known, since it captures the essence of diffusive processes. Elliptic PDEs often arise out of physical environments with a present potential field (e.g. gravitational or electrostatic), or in the cases of incompressible fluid flow [2, 1]. We use the latter for motivating this model, as bathtub water exhibits the standard characteristics of incompressible fluid.

2.3 Numerical methods

We proceed by providing a brief overview of the numerical methods used in this study. To approximate the solution to a PDE, we discretize the region Ω under consideration into a coordinate mesh. For convention, we write $T^n_{i,j}$ to denote the temperature at the (i, j) mesh coordinate at the time interval $t = n$.

Modifying the notation employed by Ames [1], we establish the following nomenclature for the numerical operations in the model:

$$\begin{array}{ll} \overrightarrow{\Delta}_i T^n_{i,j} = T^n_{i+1,j} - T^n_{i,j} & \text{Forward differencing} \\ \overleftarrow{\Delta}_i T^n_{i,j} = T^n_{i,j} - T^n_{i-1,j} & \text{Backward differencing} \\ \overleftrightarrow{\Delta}_i T^n_{i,j} = T^n_{i+1/2,j} - T^n_{i-1/2,j} & \text{Central differencing} \end{array}$$

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2.4 Bathtub optimization problem

The goal of this study is to identify a process through which the avid bather may seek to achieve uniform water temperature, ideally wasting as little water as possible. Using the aforementioned numerical models to simulate bathtub dynamics, we propose an optimization problem by which to approach this goal. Let t_{total} indicate the period in which the faucet is turned on, let T_{f} denote the temperature of the water in the faucet, and let $V(t)$ denote the total volume of water that has left the faucet. We assume that $\frac{dV}{dt} = c$, i.e., the strength of the faucet is fixed at a chosen c throughout the duration of the bath. Now, consider the surface integral

$$f_{\text{cont}}(t) = \int_\Omega |T(x, y, t) - T_{\text{f}}| \ d\omega. \tag{5}$$

We propose the minimization problem

$$\arg\min_{c} V(t_{\text{total}}) \quad \text{such that} \quad \int_0^{t_{\text{total}}} f_{\text{cont}}(t)\ dt < \gamma, \tag{6}$$

where γ is some tolerance for variation. This is minimization problem is subject to a total variation constraint, which has been subject to considerable study, especially within the context of medical imaging [5].

This optimization leads to a natural discretization. We rewrite:

$$V(t) \longrightarrow V^t \tag{7}$$

$$\frac{dV}{dt} = c \longrightarrow V^{t+1} - V^t = c \tag{8}$$

$$f_{\text{cont}}(t) = \int_{\Omega} |T(x,y,t) - T_{\text{f}}|\ d\omega \longrightarrow f_{\text{disc}}(t) = \sum_{x,y} |T(x,y,t) - T_{\text{f}}|, \tag{9}$$

which translate to the problem

$$\arg\min_{c} V^{t_{\text{total}}} \quad \text{such that} \quad \sum_{t=0}^{t_{\text{total}}} f_{\text{disc}}(t) < \gamma. \tag{10}$$

3 Empirical results

4 Analysis

5 Discussion

References

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