MAT-395: Linear Algebra Exercise Book

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Chapter 1

Vector Spaces

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1. Let V be a vector space over F. Prove that $0v = 0$ and $r0 = 0$ for all $v \in V$ and $r \in F$. Describe the different 0's in these equations. Prove that if $rv = 0$, then $r = 0$ or $v = 0$. Prove that $rv = v$ implies that $v = 0$ or $r = 1$.

3. a. Find an abelian group V and a field F for which V is a vector space over F in at least two different ways, that is, there are two different definitions of scalar multiplication making V a vector space over F.

   b. Find a vector space V over F and a subset S of V that is (1) a subspace of V and (2) a vector space using operations that differ from those of V.

4. Suppose that V is a vector space with basis $\mathscr{B} = \{b_i \mid i \in I\}$ and S is a subspace of V. Let $\{B_1, B_2, \ldots, B_k\}$ be a partition of $\mathscr{B}$. Then is it true that

$$S = \bigoplus_{i=1}^{k} (S \cap \langle B_i \rangle)?$$

   What if $S \cap \langle B_i \rangle \supseteq \{0\}$ for all i?

9. Let M be an $m \times n$ matrix whose rows are linearly independent. Suppose that the k columns $c_{i_1}, \ldots, c_{i_k}$ of M span the columns space of M. Let C be the matrix obtained from M by deleting all columns except $c_{i_1}, \ldots, c_{i_k}$. Show that the rows of C are also linearly independent.

11. Show that if S is a subspace of a vector space V, then $\dim(S) \le \dim(V)$. Furthermore, if $\dim(S) = \dim(V) < \infty$, then $S = V$. Give an example to show that the finiteness is required in the second statement.

13. What is the relationship between $S \oplus T$ and $T \oplus S$? Is the direct sum operation commutative? Formulate and prove a similar statement concerning associativity. Is there an “identity" for direct sum? What about “negatives"?

15. Prove that the vector space $\mathscr{C}$ of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ is infinite-dimensional.

17. Let S be a subspace of V. The set $v+S=\{v+s \mid s\in S\}$ is called an **affine subspace** of V.

    a) Under what conditions is an affine subspace of V a subspace of V?

    b) Show that any two affine subspaces of the form $v+S$ and $w+S$ are either equal or disjoint.

26. Let V be a real vector space with complexification $V^{\mathbb{C}}$ and let U be a subspace of $V^{\mathbb{C}}$. Prove that there is a subspace S of V for which

$$U=S^{\mathbb{C}}=\{s+ti \mid s,t\in S\}$$

if and only if U is closed under complex conjugation $\chi\colon V^{\mathbb{C}}\to V^{\mathbb{C}}$ defined by $\chi(u+iv)=u-iv$.

1.3 Proofs

1.3.1 Proof of Exercise 1

Let V be a vector space over F. Prove that $0v = 0$ and $r0 = 0$ for all $v \in V$ and $r \in F$. Describe the different 0's in these equations. Prove that if $rv = 0$, then $r = 0$ or $v = 0$. Prove that $rv = v$ implies that $v = 0$ or $r = 1$.

Proof. For this proof, let $v \in V$ and $r \in R$ both be arbitrary. We begin by showing that $0v = 0$. Using the properties of the field F and the distributive property,

$$0v = (0+0)v = 0v + 0v.$$

Since $0v \in V$, we use the additive inverse of $0v$ to conclude that $0v = 0$. Note that 0 in this instance corresponds to $0 \in F$, the additive identity of F.

From here, we will show that $r0 = 0$. Using the properties of the abelian group V and the distributive property,

$$r0 = r(0+0) = r0 + r0.$$

Since $r0 \in V$, we use the additive inverse of $r0$ to conclude that $r0 = 0$. Note that 0 in this instance corresponds to $0 \in V$, the zero vector of V.

Next, we will show that if $rv = 0$, then $r = 0$ or $v = 0$. Assume $r \neq 0$ (because $r = 0$ is silly). Notice that $rv = 0 = r0$ since $r0 = 0$. Then $\frac{1}{r}(rv) = \frac{1}{r}(r0)$ if and only if $v = 0$.

Finally, we show that if $rv = v$, then $v = 0$ or $r = 1$. Using our scalar multiplication axioms, we arrive at the conclusion that $rv = v = 1v$ if and only if $(r-1)v = 0$. Since $r - 1 \in F$, we may use the fact that if $(r-1)v = 0$, then $r - 1 = 0$ or $v = 0$. $\square$

1.3.2 Proof of Exercise 3

a. Find an abelian group V and a field F for which V is a vector space over F in at least two different ways, that is, there are two different definitions of scalar multiplication making V a vector space over F.

b. Find a vector space V over F and a subset S of V that is (1) a subspace of V and (2) a vector space using operations that differ from those of V.

Proof. We begin with part a.

a. $\mathbb{C}$ over $\mathbb{C}$ is already a vector space when equipped with ordinary addition and scalar multiplication of complex numbers.

   Therefore, define the function $g\colon \mathbb{C}\times\mathbb{C}\to\mathbb{C}\times\mathbb{C}$ given by $g((x,v)) = (\overline{x}, v)$ where $\overline{x}$ is the complex conjugate of x. When composed with the ordinary multiplication map $\mathbf{mul}\colon \mathbb{C}\times\mathbb{C}\to\mathbb{C}$, $\mathbf{mul}\circ g$, defines a scalar multiplication function. Since multiplication of complex numbers is both associative and distributive, we need only check that $1v = v$ for any $v\in\mathbb{C}$. Since $1\in\mathbb{R}$ however, $\overline{1} = 1$, so $1v = v$ for any $v\in\mathbb{C}$. Hence $\mathbb{C}$ over $\mathbb{C}$ with operations of ordinary addition and $\mathbf{mul}\circ g$ for scalar multiplication forms a vector space.

b. $\mathbb{C}$ over $\mathbb{C}$ is a vector space with ordinary addition and scalar multiplication as its operations. The subset $\mathbb{R}\subseteq\mathbb{C}$ is a subspace of $\mathbb{C}$ since it's closed under ordinary addition and scalar multiplication.

   Now using $\mathbb{R}$ as a vector space over $\mathbb{R}$, let $+'\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a vector addition function defined as $x +' y = e^x e^y$. From the properties of exponent rules, $+'$ satisfies the properties of vector addition. Therefore, $\mathbb{R}$ over $\mathbb{R}$ using $+'$ and with ordinary addition as scalar multiplication is a vector space with different operations from $\mathbb{C}$ over $\mathbb{C}$.

$\square$

1.3.3 Proof of Exercise 4

Suppose that V is a vector space with basis $\mathscr{B} = \{b_i \mid i \in I\}$ and S is a subspace of V. Let $\{B_1, B_2, \ldots, B_k\}$ be a partition of $\mathscr{B}$. Then is it true that

$$S = \bigoplus_{i=1}^{k} (S \cap \langle B_i \rangle)?$$

What if $S \cap \langle B_i \rangle \supseteq \{0\}$ for all i?

Proof. Let $V = \mathbb{R}^2$ and $S = \{(x, y) \mid x + 2y = 0\}$. Let $\mathscr{B}$ be the standard basis and $\mathbf{B} = \{\{e_1\}, \{e_2\}\}$. Then notice that

$$S \cap \langle \{e_1\} \rangle = \{0\} \quad \text{and} \quad S \cap \langle \{e_2\} \rangle = \{0\}$$

Therefore, $\sum_{k=1}^{2} (S \cap \langle \{e_i\} \rangle) = \{0\}$, so S is not the direct sum of spans of blocks of basis vectors intersected with S.

Alternatively, fix $k, n \in \mathbb{N}$ with $k < n$, let $V = \mathbb{R}^\omega$, and let $S = \mathbb{R}^n$ be a subspace of V. We will assume that $S \cap \langle B_i \rangle \supseteq \{0\}$. For each block $B_i \in \mathbf{B}$, let $S \cap \{B_i\} = \{b_i, 0\}$. Since S is closed under linear combination, $\langle \{b_i\} \rangle \subseteq S \cap \{B_i\}$. Then there are k basis vector of $\mathscr{B}$ in $\mathbb{R}^n$ which, together, span a subspace $\mathbb{R}^k$ of $\mathbb{R}^n$. But since $k < n$, $\mathbb{R}^k$ does not span $\mathbb{R}^n$ by Theorem 1.7. Thus S is not the direct sum of S intersected with finitely many spans of blocks in partition $\mathbf{B}$. □

1.3.4 Proof of Exercise 9

Let M be an $m \times n$ matrix whose rows are linearly independent. Suppose that the k columns $c_{i_1}, \ldots, c_{i_k}$ of M span the columns space of M. Let C be the matrix obtained from M by deleting all columns except $c_{i_1}, \ldots, c_{i_k}$. Show that the rows of C are also linearly independent.

Proof. By Theorem 1.15, elementary column operations do not affect the rank of a matrix. Thus $rk(M) = rk(C) = n$, so the rows of C are linearly independent. $\square$

1.3.5 Proof of Exercise 11

Show that if S is a subspace of a vector space V, then $\dim(S) \leq \dim(V)$. Furthermore, if $\dim(S) = \dim(V) < \infty$, then $S = V$. Give an example to show that the finiteness is required in the second statement.

Proof. Let S be a subspace of V. Let $\mathscr{B} = \{b_i \mid i \in I\}$ and $\mathscr{C}$ be bases and V and S respectively. Then if $v \in S$, v can be written as a finite linear combination of vectors in $\mathscr{B}$, where all of the coefficients are nonzero, say

$$v = \sum_{j \in B_v} r_j b_j$$

where $B_v \subseteq \mathscr{B}$ is the index set of basis vectors whose span contains v. Because $\mathscr{C}$ is a basis for the subspace S,

$$\bigcup_{v \in S} B_v \subseteq I$$

for if the vectors in S can be expressed as finite linear combinations of the vectors in a proper superset $\mathscr{B}'$ of $\mathscr{B}$, then $\mathscr{B}'$ is a maximal linearly independent set, violating the maximality of $\mathscr{B}$.

Since $|B_v| < \aleph_0$ for all $v \in S$, Theorem 0.17 allows us to conclude that

$$|\mathscr{C}| \leq |I| \leq \aleph_0 |\mathscr{B}|.$$

Therefore $\dim(S) \leq \dim(V)$ as desired.
(DO REST OF THIS EXERCISE)

$\square$

1.3.6 Proof of Exercise 13

What is the relationship between $S \oplus T$ and $T \oplus S$? Is the direct sum operation commutative? Formulate and prove a similar statement concerning associativity. Is there an “identity" for direct sum? What about “negatives"?

Proof. I claim that $(S(V), \{0\}, \oplus)$ is a commutative monoid. Let $S, T, R \in S(V)$ be independent subspaces of V. Note that $S \oplus T$ is itself a subspace in $S(V)$, so $S(V)$ is closed under $\oplus$. To show that $\oplus$ is associative, notice the following.

$$\begin{aligned}(S \oplus T) \oplus R &= \{u + v \mid u \in S, v \in T\} \oplus R \\ &= \{(u + v) + w \mid u \in S, v \in T, w \in R\}\end{aligned}$$

Using the associativity of $+$,

$$\begin{aligned}&= \{u + (v + w) \mid u \in S, v \in T, w \in R\} \\ &= U \oplus \{v + w \mid v \in T, w \in R\} \\ &= U \oplus (V \oplus R)\end{aligned}$$

Hence $\oplus$ is associative. $\oplus$ is also commutative.

$$S \oplus T = \{u + v \mid u \in S, v \in T\}$$

Using the commutativity of $+$,

$$\begin{aligned}&= \{v + u \mid u \in S, v \in T\} \\ &= T \oplus S\end{aligned}$$

To show that $\{0\}$ is the identity element of $S(V)$, simply note that

$$S \oplus \{0\} = \{u + 0 \mid u \in S\}$$

Since 0 is the additive identity of V,

$$\begin{aligned}&= \{u \mid u \in S\} = S \\ &= \{0 + u \mid u \in S\} \\ &= \{0\} \oplus S\end{aligned}$$

Despite being a commutative monoid, $(S(V), \{0\}, \oplus)$ does not have inverses. Suppose for contradiction $S \oplus T = \{0\}$. Then if $x \in S \oplus T$, then $x = u + v$ for $u \in S$ and $v \in T$. But by set equality, $x = 0$, so $u = -v$, violating the independence of S and T.

Hence $(S(V), \{0\}, \oplus)$ is a commutative monoid. $\square$

1.3.7 Proof of Exercise 15

Prove that the vector space $\mathscr{C}$ of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ is infinite-dimensional. We begin with a Lemma about the linear independence of monomials.

Lemma 1.3.1. *Fix $n \in \mathbb{N}$. The set of vectors $\{1, x, x^2, \dots, x^n\}$ is a linearly independent subset of $\mathscr{C}$.*

Proof. Suppose the linear combination

$$0 = r_0 x^0 + r_1 x^1 + \cdots + r_n x^n.$$

has a nontrivial solution. Since each x^i is distinct and there is a nontrivial solution, $n > 1$, so

$$x^0 = 1 = -\frac{1}{r_0}\left(r_1 x^1 + \cdots + r_n x^n\right)$$

Since there is no $y \in \mathbb{R}$ with $xy = 1$, we've arrived at a contradiction since 1 cannot be written as the sum of monomials. Hence $\{1, x, x^2, \dots, x^n\}$ is linearly independent. □

Now we proceed with the proof.

Proof. Suppose that $\mathscr{C}$ is finite dimensional with $n = \dim(\mathscr{C})$. Choose basis β with $|\beta| = n$. Notice that the set $\{1, x, x^2, x^3, \dots, x^n\}$ is a linearly independent set by Lemma 1.3.2 of cardinality $n+1$. This violates the maximality of β. Since $n \in \mathbb{N}$ was arbitrary, $\mathscr{C}$ is infinite-dimensional. □

1.3.8 Proof of Exercise 17

Let S be a subspace of V. The set $v+S=\{v+s \mid s\in S\}$ is called an **affine subspace** of V.

a) Under what conditions is an affine subspace of V a subspace of V?

b) Show that any two affine subspaces of the form $v+S$ and $w+S$ are either equal or disjoint.

Proof. Let S be a subspace of the vector space V.

a. I claim that $s+V$ is a subspace if and only if $v\in S$. Assume the contrary that $v+S$ is a subspace with $v\notin S$; it should then be closed under linear combination. Let $u_1,u_2\in v+S$ and fix $a,b\in F$. Then $au_1+bu_2\in v+S$. However,

$$au_1+bu_2=a(v+s_1)+b(v+s_2)=(a+b)v+(as_1+bs_2).$$

This new vector is not in $v+S$, so S is not a subspace. On the other hand, v in the subspace S means that $v+S$ is a subspace since S is closed under linear combination.

b. Let $v,w\in V$ be vectors and let S be an arbitrary subspace. Let $v+S$ and $w+S$ be affine subspaces of V. For argument's sake, $(v+S)\cap(w+S)$ is empty or nonempty. Suppose there is some $u\in(v+S)\cap(w+S)$. Fix $s_1,s_2\in S$ with $u=v+s_1=w+s_2$. Then

$$u=v+s_1=w+s_2\Leftrightarrow v-w=s_1-s_2\Leftrightarrow(v-w)\in S$$

Now suppose that $x\in v+S$. Fix $t\in S$ with $x=v+t$. Since S is closed under linear combination,

$$v+t=v+(t+0)=v+t-(v-w)+(s_1-s_2)=w+(t+s_1-s_2).$$

Because $s_1-s_2\in S$, $x\in w+S$. If $x\in w+S$, then a similar argument will show that $x\in v+S$. Hence $v+S=w+S$.

$\square$

1.3.9 Proof of Exercise 26

Let V be a real vector space with complexification $V^{\mathbb{C}}$ and let U be a subspace of $V^{\mathbb{C}}$. Prove that there is a subspace S of V for which

$$U = S^{\mathbb{C}} = \{s + ti \mid s, t \in S\}$$

if and only if U is closed under complex conjugation $\chi\colon V^{\mathbb{C}} \to V^{\mathbb{C}}$ defined by $\chi(u + iv) = u - iv$.

Proof. Let U be a subspace of $V^{\mathbb{C}}$. We will show that there is a subspace S of V such that $U = S^{\mathbb{C}}$ if and only if U is closed under the complex conjugate map.
Suppose that S is a subspace of V such that $U = S^{\mathbb{C}}$. Let $(v, w) \in U$. Since $(v, w) \in U$, v and w are in S. Since S is a subspace, $0 \in S$ and $-w \in S$ as well. Therefore, $(v, -w) \in U$, and thus U is closed under χ.

Alternatively, assume that U is closed under χ. Let $U = S \times S$. Let $(a + bi) \in \mathbb{C}$ and $(v, w) \in U$ both be arbitrary. Then

$$(a + bi)(v, w) = (av - bw, aw + bv) \in U$$

Since U is closed under the complex conjugate map, $(av - bw, -aw - bv) \in U$ as well. Furthermore, since U is a subspace, U is closed under linear combination. Thus,

$$(av - bw, aw + bv) + (av - bw, -aw - bv) = (2av - 2bw, 0) \in U.$$

By definition of U, $U = S \times S$, so $av - bw \in S$. Hence S is closed under linear combination, and thus S is a subspace of V. $\square$

Chapter 2

Linear Transformations