

6.1 Subset Selection

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Linear Model Selection and Regularization

- ▶ Chapter 6 extends the linear model framework.
- ▶ Chapters 7 and 8 generalize the linear model and consider some general non-linear models.
- ▶ In this section, we focus on methods for selecting a subset of predictors.

Why Linear Models?

- ▶ Linear models may seem too simple, but...
 - ▶ they are highly interpretable.
 - ▶ they often have good predictive capabilities.
- ▶ In Chapter 6, we will discuss ways to improve the basic least squares fit by using some alternate fitting procedures.

Why consider alternatives to least squares?

- ▶ Prediction accuracy
 - ▶ especially important when $p > n$
- ▶ Model interpretability
 - ▶ Removing irrelevant factors to simplify models can help with interpretability.
 - ▶ We will consider some approaches for automating *feature selection*

A Preview of Ch 6

- ▶ Subset selection (6.1)
 - ▶ Identify a subset of the p predictors to be used in the final model, which we fit using least squares.
- ▶ Shrinkage (6.2)
 - ▶ Fit a model using all p predictors, with some of the coefficients shrunk toward zero. This can help reduce variance and perform variable selection.
- ▶ Dimension Reduction (6.3)
 - ▶ Project the p predictors into an M -dimensional subspace, $M < p$ and use these as predictors in a linear regression model.

Best Subset Selection

To access the *best* subset, we need to fit a separate linear model to each possible combination of the predictors.

Best Subset Selection

Basic algorithm:

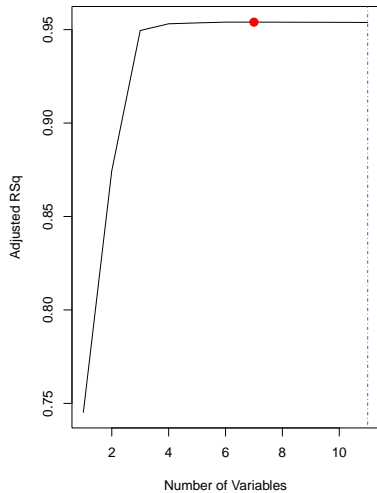
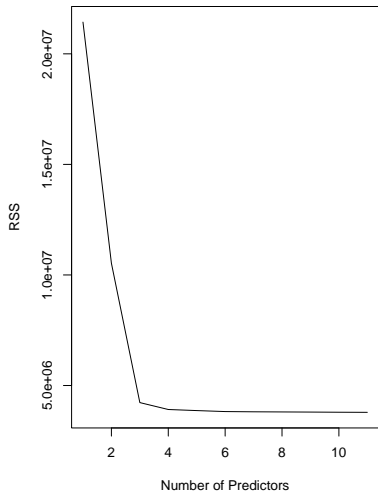
1. Let Ω_0 denote the *null model*, which contains no predictors:
 $\hat{y} = \bar{y}$.
2. For $k = 1, 2, \dots, p$:
Fit all $\binom{p}{k}$ models that contain exactly k predictors.
Select the best among these $\binom{p}{k}$ models and call it Ω_k . Here, “best” means smallest RSS (or largest R^2).
3. Select a single best model from among $\Omega_0, \dots, \Omega_p$ using cross-validated prediction error, AIC, BIC, or Adjusted R^2 .

Example: Credit Data

```
data("Credit")
library(leaps)
bestset <- regsubsets(Balance ~ ., Credit, nvmax=11)
reg.sumry <- summary(bestset)

par(mfrow = c(1, 2))
mx = which.max(reg.sumry$adjr2)
plot(reg.sumry$rss, xlab = "Number of Predictors",
     ylab = "RSS", type = "l")
plot(reg.sumry$adjr2, xlab = "Number of Variables",
     ylab = "Adjusted RSq", type = "l")
points(mx, reg.sumry$adjr2[mx], col = "red", cex = 2, pch = 1)
abline(v=11, lty=4, col="purple")
```


Example: Credit Data



Extensions to Other Models

- ▶ The best subsets approach can be used for other types of models.
- ▶ Where RSS is not available, we instead use the *deviance*
 - ▶ $\text{Deviance} = -2 \times \text{maximized log-likelihood}$

Stepwise Selection

Best subset selection has some significant drawbacks

- ▶ Cannot be applied with very large p .
- ▶ Larger search spaces increase the chances of finding models that look good on training data, but perform poorly on future data.
 - ▶ Huge search space can lead to *overfitting*.
- ▶ These can also be computationally intensive.

Forward Stepwise Selection

- ▶ We begin with a null model and then add predictors to the model one at a time, until all predictors are included in the model.
- ▶ The predictor added at each step is selected based on which one results in the greatest additional improvement to the existing model.

Forward Stepwise Selection

The basic algorithm

1. Let Ω_0 denote the *null model*, which contains no predictors:
 $\hat{y} = \bar{y}$.
2. For $k = 1, 2, \dots, p - 1$:
Consider all $p - k$ models that augment the predictors in Ω_k with one additional predictor.
Select the best among these models and call it Ω_{k+1} .
3. Select a single best model from among $\Omega_0, \dots, \Omega_p$ using cross-validated prediction error, AIC, BIC, or Adjusted R^2 .

Forward Stepwise Selection

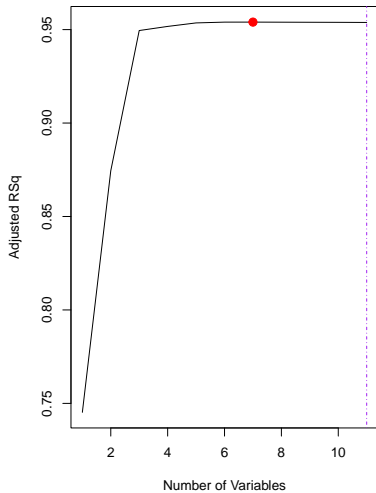
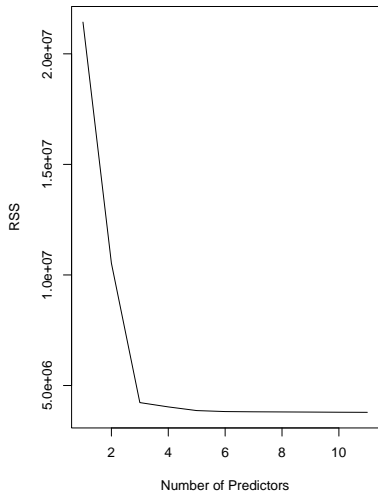
Pro: computational advantage (Searches through only $1 + p(p + 1)/2$ models)

Con: not guaranteed to result in the best possible model

Example: Credit Data

```
regfit.fwd <- regsubsets(Balance ~ ., Credit, nvmax = 11,  
                        method = "forward")  
regfit.smry <- summary(regfit.fwd)  
  
par(mfrow = c(1, 2))  
mx = which.max(regfit.smry$adjr2)  
plot(regfit.smry$rss, xlab = "Number of Predictors",  
     ylab = "RSS", type = "l")  
plot(regfit.smry$adjr2, xlab = "Number of Variables",  
     ylab = "Adjusted RSq", type = "l")  
points(mx, regfit.smry$adjr2[mx], col = "red", cex = 2,  
       pch = 20)  
abline(v=11, lty=4, col="purple")
```

Example: Credit Data



Example: Credit Data

# Variables	Best subset	Forward stepwise
One	rating	rating
Two	rating, income	rating, income
Three	rating, income, student	rating, income, student
Four	cards , income, student, limit	rating , income, student, limit

Backward Stepwise Selection

- ▶ Similar to forward stepwise selection, but starts with the full model.
- ▶ Iteratively removes one predictor at a time.

Backward Stepwise Selection

The basic algorithm

1. Let Ω_p denote the *full model*, which contains all p predictors.
2. For $k = p, p - 1, \dots, 1$:
Consider all k models contain all but one of the predictors in Ω_k .
Select the best among these models and call it Ω_{k-1} .
3. Select a single best model from among $\Omega_0, \dots, \Omega_p$ using cross-validated prediction error, AIC, BIC, or Adjusted R^2 .

Backward Stepwise Selection

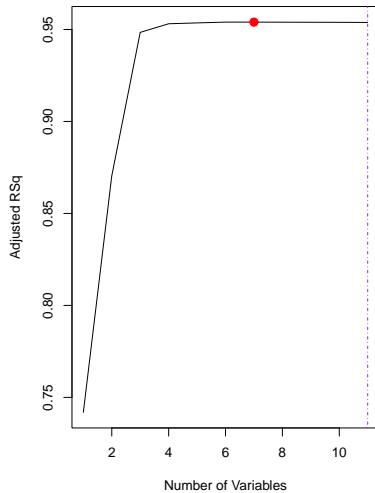
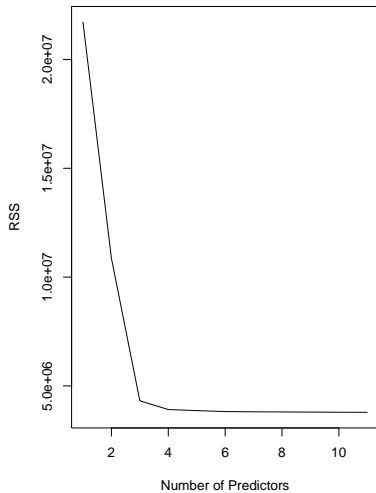
Pro: computational advantage (Searches through only $1 + p(p + 1)/2$ models)

Cons: - not guaranteed to result in the best possible model -
requires $n > p$ to run

Example Code: Credit Data

```
regfit.bwd <- regsubsets(Balance ~ ., Credit, nvmax = 11,  
                        method = "backward")  
regfit.smry <- summary(regfit.bwd)  
  
par(mfrow = c(1, 2))  
mx = which.max(regfit.smry$adjr2)  
plot(regfit.smry$rss, xlab = "Number of Predictors",  
     ylab = "RSS", type = "l")  
plot(regfit.smry$adjr2, xlab = "Number of Variables",  
     ylab = "Adjusted RSq", type = "l")  
points(mx, regfit.smry$adjr2[mx], col = "red", cex = 2, pch = 1)  
abline(v=11, lty=4, col="purple")
```

Example Code: Credit Data



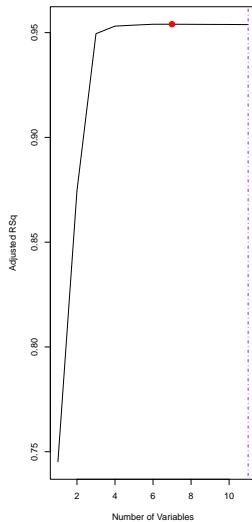
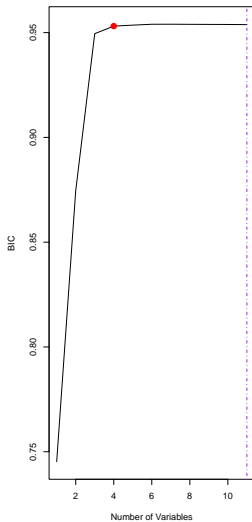
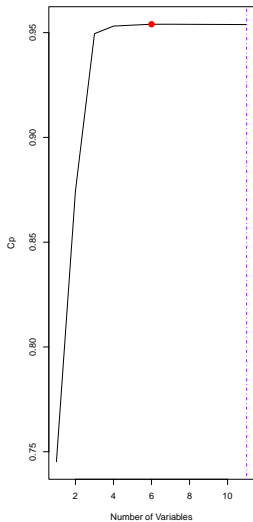
Choosing the Optimal Model

- ▶ The model containing all the predictors will always have the smallest RSS and the largest R^2
- ▶ We want to choose a model with low test error, not low training error
- ▶ For all three approaches, we ended by selecting a single best model from among $\Omega_0, \dots, \Omega_p$ using cross-validated prediction error, AIC, BIC, or Adjusted R^2 .
 - ▶ These allow us to either *directly* estimate test error or to *adjust* test error to account for bias.

C_p , AIC, BIC, Adjusted R^2

- ▶ These techniques adjust the training error for the model size (penalize for complexity) and can be used to select among models with different numbers of variables.
 - ▶ These models do not need to be subsets of each other.

Credit Data Example (Best Subsets)



Some Details: Adjusted R^2

We saw these last semester, but we'll recap.

- ▶ R^2 is strictly decreasing as the number of variables in the model increases.
- ▶ Adjusted R^2 basically penalizes the R^2 statistic directly.

$$R_{adj}^2 = 1 - \frac{\text{RSS}/(n - d - 1)}{\text{TSS}/(n - 1)}$$

- ▶ Maximizing R_{adj}^2 is equivalent to minimizing $\text{RSS}/(n - d - 1)$
- ▶ We select the model with the greatest R_{adj}^2

Some Details: Mallow's C_p

- ▶ Mallow's C_p

$$C_p = \frac{1}{n}(\text{RSS} + 2d\hat{\sigma}^2)$$

where d is the total number of parameters in the model and $\hat{\sigma}^2$ is the variance of the error associated with each response

- ▶ We select the model with the lowest C_p

Some Details: AIC

- ▶ AIC is defined for any models fit by maximum likelihood

$$\text{AIC} = -2 \log L + 2d$$

where L is the maximized likelihood for the estimated model

- ▶ For the linear model with Gaussian errors, maximum likelihood and least squares are the same (so C_p and AIC are equivalent)
- ▶ We select the model with the lowest AIC

Some Details: BIC

$$\text{BIC} = -2 \log L + d \log(n)$$

- ▶ Similar to Mallow's C_p and AIC, but uses a different penalty term.
 - ▶ Replaces the 2 in AIC by $\log n$
 - ▶ Since $\log n > 2$ for $n > 7$, BIC tends to penalize largest variables more heavily than AIC and Mallow's C_p .

Validation and Cross-Validation

- ▶ Each of the procedures (best subset, stepwise) returns a sequence of models Ω_k , $k = 0, 1, 2, \dots$
- ▶ We need to select k to determine our final model

Validation and Cross-Validation

- ▶ Compute the (cross) validation set error for each model under consideration, then select the k for which the resulting estimated error is smallest.
- ▶ Pros:
 - ▶ provides a direct estimate of test error
 - ▶ does not require estimation of σ
 - ▶ can be used in a wider range of model selection tasks, since we do not need to pinpoint degrees of freedom or calculate a likelihood
- ▶ Con: computationally intensive
- ▶ You will do this in the Chapter 6 Lab.

Selecting a Final Model

- ▶ It is often the case that each approach gives a slightly different result.
- ▶ We will select a model using the *one standard error rule*.
 - ▶ Calculate the standard error of the estimated test MSE for each model size.
 - ▶ Select the smallest model for which the estimated test error is within one standard error of the lowest point on the curve.