Symanzik Effective Field Theory

Provides a description of discretization effects of the underlying lattice QFT.

https://doi.org/10.1016/0550-3213(83)90468-6
https://doi.org/10.1016/0550-3213(83)90469-8
https://inspirehep.net/literature/191662
https://inspirehep.net/literature/183956
https://inspirehep.net/literature/168237
https://doi.org/10.1142/9789812777270_0004
https://doi.org/10.1142/9789814678766_0004

→ functional form for continuum extrapolation of lattice data

→ procedure for building improved actions

Describe lattice $\mathcal{L}$ by a local effective $\mathcal{L}$ (LE$\mathcal{L}$) which which is written in terms of continuum fields: $\mathcal{L}_{Sym}$
ie.

$$\mathcal{L}_{lat} \doteq \mathcal{L}_{Sym}$$

where $\doteq$ means equality of on-shell quantities (matrix elements) each evaluated in the theories on the LHS & RHS

The EFT construction on the RHS takes advantage of the usual separation of scales, where short-distance effects are expressed in the coefficients and long-distance effects in the operators.

The LHS corresponds to the lattice QFT used for the numerical simulations, while the RHS is an expansion in terms of continuum fields and matrix elements of physical continuum states. For QCD, we have:

$$\mathcal{L}_{Sym} = \mathcal{L}_{QCD} + \mathcal{L}_{I}$$

with

$$\mathcal{L}_{QCD} = \frac{1}{2g^2} \, tr\left[G_{\mu\nu}^{\,2}\right] + \sum_f \bar\psi_f (\not{D} + m_f)\psi_f$$

where g^2, m are the renormalized couplings or masses that depend on the bare parameters of the lattice theory:

$$g^2 = g^2(g_0^2; m_0 a, c_i; \mu a) \qquad m = m_0 Z_m(g_0^2; m_0 a, c_i; \mu a)$$

and the c_i are adjustable parameters (coefficients) of higher dimensional terms $(d > 4)$ in $\mathcal{L}_{lat}$.

The 2nd term on RHS describes discretization effects in $\mathcal{L}_{lat}$ in terms of higher dimensional operators ($\dim \mathcal{O}_n > 4$)

$$\mathcal{L}_I = \sum_n a^{\dim \mathcal{O}_n - 4} \cdot k_n \cdot \mathcal{O}_n^R(\mu)$$

with

$$k_n = k_n(g, ma, c_i; \mu a)$$

where matrix-elements of the $\mathcal{O}_n$ capture long-distance effects and scale with the typical momenta Λ of the participating particles:

$$\langle \mathcal{O}_n \rangle \sim \Lambda^{\dim \mathcal{O}_n - 4}$$

hence describe discretization effects of $(a\Lambda)^{\dim \mathcal{O}_n - 4}$

If a is small enough so that $a\Lambda \ll 1$, we can treat effects from $\mathcal{L}_I$ as perturbations, so that we can use the physical states of $\mathcal{L}_{QCD}$.

- It is enough to consider only <u>on-shell</u> quantities $\Rightarrow$ fewer operators in $\mathcal{L}_I$

https://inspirehep.net/literature/201371 https://doi.org/10.1016/0550-3213(85)90002-1

- Operators that don't affect on-shell quantities are <u>redundant</u>.
- matrix elements as functional integrals: redundant operators are obtained from field redefinitions (a.k.a. spectrum-conserving transformations)
- can also use EoMs

<u>Examples</u>:

(1) <u>Wilson gauge action</u>: start gauge-link definition

$$U_\mu(x) \equiv P \exp\left(-i \int_x^{x+a\hat{\mu}} A_\mu(y)\, dy\right)$$

plaquette

$$P_{\mu\nu} \equiv \frac{1}{3} \operatorname{Re} \operatorname{Tr}\left[U_\mu(x)\, U_\nu(x+a\hat{\mu})\, U_\mu^\dagger(x+a\hat{\nu})\, U_\nu^\dagger(x)\right]$$

and the action $S_g^W = \sum_x \mathcal{L}_g^W(x)$ with

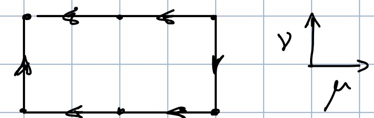
$$\mathcal{L}_g^W = \beta \sum_{\mu>\nu} \left[1 - P_{\mu\nu}(x)\right] \qquad \beta = \frac{6}{g_0^2}$$

Exercise I:

(a) Show that in the classical continuum limit

$$\mathcal{L}_g^W \longrightarrow \frac{1}{2g_0^2} \operatorname{Tr}[G_{\mu\nu}^2] + \frac{a^2}{24} \operatorname{Tr}[G_{\mu\nu}(D_\mu^2 + D_\nu^2) G_{\mu\nu}] + \dots$$

(b) Show that the leading discretization effects can be removed by adding rectangular Wilson loops to the action, e.g.

$$R_{\mu\nu} = \frac{1}{3} \operatorname{Re} \operatorname{Tr}$$ 

Expand this operator in the small a limit and show that the improved action

$$\mathcal{L}_g^{imp} = -\beta \sum_{\mu>0} \left[\frac{5}{3} P_{\mu\nu} - \frac{1}{12} (R_{\mu\nu} + R_{\nu\mu}) \right]$$

yields:

$$\mathcal{L}_g^{imp} \longrightarrow \frac{1}{2g_0^2} \operatorname{Tr}[G_{\mu\nu}^2] + \mathcal{O}(a^4)$$

This is called Symanzik improvement.
This analysis can be extended to include 1-loop corrections,
see: M. Lüscher + P. Weisz in https://doi.org/10.1016/0370-2693(85)90966-9

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(2) Wilson fermion action

$$\mathcal{L}_f^W = m_0 \bar\psi \psi + \bar\psi \gamma_\mu D_\mu^{lat} \psi - \frac{ar}{2} \bar\psi \Delta^2 \psi$$

with $$D_\mu^{lat} \psi(x) = \frac{1}{2a} \left[U_\mu(x) \psi(x+a\hat\mu) - U_\mu^\dagger(x-a\hat\mu) \psi(x-a\hat\mu) \right]$$

and
$$\bar\psi \Delta^2 \psi = \frac{1}{a^2} \bar\psi \left[U_\mu(x) \psi(x+a\hat\mu) + U_\mu^\dagger(x-a\hat\mu) \psi(x-a\hat\mu) - 2\psi(x) \right]$$

where the usual choice is $r=1$. The operator $\bar\psi \Delta^2 \psi$ is dimension 5 and therefore carries an explicit factor of a. Hence it doesn't affect the classical continuum limit of the action; it however gives rise to discretization effects that are linear in a. This term is called the Wilson term and was added to the action to remove the doublers.

Exercise II:

Starting with the naive fermion action (with $m_0 = 0$)

$$\mathcal{L}_f^{\text{naive}} = \bar{\psi} \gamma_\mu D_\mu^{lat} \psi$$

show that the free-field quark propagator in momentum space takes the form:

$$\tilde{D}^{\text{naive}}(p) = \frac{i}{a} \gamma_\mu \sin(p_\mu a)$$

(omitting color indices)

which means that the naive action contains 15 unwanted quark states.

Then show that the free Wilson propagator takes the form

$$\tilde{D}^{W}(p) = \frac{i}{a} \gamma_\mu \sin(p_\mu a) + \frac{1}{a} \sum_\mu [1 - \cos(p_\mu a)]$$

which removes the poles at $p_\mu = \pi/a$, by giving the doublers a mass term $\sim 2/a$.

We now consider the Symanzik LE $\mathcal{L}$ for the Wilson fermion action:

$$\mathcal{L}_{Sym}^{W} = m_f \bar{\psi}_f \not{D} \psi_f + \mathcal{L}_I^{W}$$

We already know that $\mathcal{L}_I^W$ starts at dimension 5. There are two possible operators (see S + W):

$$\mathcal{O}_5 = i \bar{\psi}_f \sigma_{\mu\nu} G_{\mu\nu} \psi_f \qquad \mathcal{O}_5' = \bar{\psi}_f \not{D}^2 \psi_f$$

The two operators are related: $\mathcal{O}_5' = \bar{\psi}_f D^2 \psi_f - \frac{1}{2} \mathcal{O}_5$
and $\mathcal{O}_5'$ can be generated by a field redefinition of the form:

$$\psi_f \to e^{\varepsilon a \not{D}} \psi_f \qquad \bar{\psi}_f \to \bar{\psi}_f e^{\bar{\varepsilon} a \not{D}}$$

[see: El-Khadra, Kronfeld, Mackenzie, https://doi.org/10.1103/PhysRevD.55.3933]

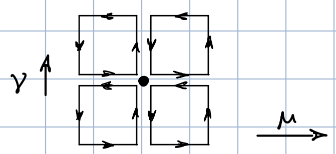
This means that $\mathcal{O}_5'$ is redundant and the Symanzik LE $\mathcal{L}$ for the Wilson fermion action takes the form

$$\mathcal{L}_I^W = a k_5^W \mathcal{O}_5 + \sum_{\dim \mathcal{O}_n > 5} a^{\dim \mathcal{O}_n - 4} k_n^W \mathcal{O}_n$$

We can now use this analysis to improve the Wilson action, by adding a discretized version of $\mathcal{O}_5$ with a coefficient adjusted so that the resulting Symanzik LE $\mathcal{L}$ has $k_5 = 0$:

$$\mathcal{L}_f^{SW} = \mathcal{L}_f^{W} + \frac{i}{4} c_{SW}\, \bar{\psi}\, \sigma_{\mu\nu}\, G_{\mu\nu}^{lat}\, \psi$$

where $G_{\mu\nu}^{lat}$ is constructed from a "clover-leaf" of plaquettes:



$$- (\mu \leftrightarrow \nu)$$

To determine the coefficient c_{SW} for tree-level improvement, a perturbative treatment using on-shell matrix elements of quark states is sufficient. For non-perturbative improvement, the improvement conditions must be formulated in terms of hadronic external states.
We note that once the improvement coefficients are determined to a given order, all on-shell quantities are automatically improved to that order.
However, if we want to compute processes involving external currents, such as weak matrix elements, a separate matching calculation is needed to obtain improved currents.

For the tree-level $\mathcal{O}(a)$ improvement of the Wilson action, we could, for example compute forward scattering matrix elements of the gauge-current. It turns out that when $m_0 a \ll 1$, the discretization effects in the external state spinors start at $\mathcal{O}(a^2)$ [see El-Khadra, Kronfeld, Mackenzie].

In short, we can obtain the tree-level $\mathcal{O}(a)$ improvement condition by simply considering the quark-gluon vertex.

Exercise III:

(a) Derive the quark-gluon vertex for the naive fermion action and show that:

$$\Gamma_\mu^{naive}(p, p') = -i g_0 t^a \gamma_\mu \cos\left[\tfrac{1}{2}(p+p')_\mu a\right]$$

(b) Same for the Wilson action to show that:

$$\Gamma_\mu^{W}(p, p') = -i g_0 t^a \gamma_\mu \left\{ \cos\left[\tfrac{1}{2}(p+p')_\mu a\right] - i \sin\left[\tfrac{1}{2}(p+p')_\mu a\right] \right\}$$

(c) Same for the improved Wilson (SW action):

$$\Gamma_\mu^{SW}(p, p') = \Gamma_\mu^{W}(p, p') - i g_0 t^a \tfrac{1}{2} c_{SW}\, \sigma_{\mu\nu} \cos\left[\tfrac{1}{2}(p+p')_\mu a\right] \sin\left[\tfrac{1}{2}(p+p')_\nu a\right]$$

Use the Gordon Identity (sandwiching the vertex between quark spinors) to see that with $c_{SW} = 1$ the $O(a)$ terms cancel.