LGT4HEP traineeship

Lecture notes on finite-volume formalism for lattice QCD

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lecture II: QCD in a finite volume

To be covered in today's lecture:

- ☐ General features of QCD in a finite volume with PBCs
  arXiv: 1409.1966
- ☐ Finite-volume corrections to single-hadron observables:
  An early paper: Luescher 1986
  - o Example 1: Nucleon mass
    arXiv: 0403015
  - o Example 2: Nucleon axial charge
    arXiv: 0403015

□ General features of QCD in a finite volume with PBCs

Quantum chromodynamics (QCD) is an interesting theory. Despite QED where we had the issue with the propagation of massless photons, in QCD the states that go on shell and propagate are not the massless gluons, but instead confined massive particles: mesons, baryons, or even glueballs! This is the statement that QCD has a mass gap. This means that we don't really have to reformulate the theory in a FV to make it well-defined, since there is no Gauss' law to satisfy: there is no charged (under SU(3) charged) quarks or gluons at confinement scale or lower, which we are interested in. Since the lightest hadronic states in the theory are pions, they are the ones setting the size of finite-volume (FV) corrections to a range of quantities.

Further, since the FV corrections concern effects at the boundaries, these are considered IR effects and hence to determine them, we don't need to know all details of short-distance physics. This is great, since the reason we perform LQCD calculations is that we don't have the analytic dependence of quantities such as masses on the parameters of the short-distance theory, in this case QCD. The fact that FV effects are IR physics, means that we can use effective descriptions of the theory at low energy's, with which we can analytically calculate observables (not everything!) and estimate leading FV effects. Here, we work out an example of this method to calculate volume corrections to the mass of the nucleon, and in an exercise you'd be applying the same techniques to the nucleon's axial charge.

□ Finite-volume corrections to single-hadron observables:

Here we focus on nucleon's properties, but with proper EFTs, one can look at other single-hadron observables too.

○ Example 1: Nucleon mass

In order to estimate the FV corrections to the mass of the nucleon, a nice framework is chiral perturbation theory (χPT). I simply state the form of Lagrangian at leading order in the expansion parameter of the theory p/Λ_χ, m_π/Λ_χ, where p is a typical momenta, m_π is the mass of the pion, and Λ_χ is the scale of chiral symmetry breaking. Since nucleons are heavy, a more convenient formulation of the

Lagrangian is what is called "heavy baryon χPT". In this formulation, the mass of the nucleon is subtrated from the dynamics, leaving a NR two-component nucleon field. Expressing nucleon momentum as:

$$P_\mu = M_N^{(0)} v_\mu + l_\mu$$

where v_μ is a velocity four-vector, which is $v_\mu = (1,0,0,0)$ in the rest frame of the nucleon. l_μ is a residual momentum that carries the rest of momentum not associated with the mass. The kinetic energy Lagrangian is:

$$\mathcal{L}_N = \bar{N}\,[\,i\,\partial\cdot v\,]\,N + M_N^{(0)}\bar{N}N + \cdots$$

Here, N is a two-component vector in both spin and iso-spin spaces: $N = \begin{pmatrix} N_\downarrow \\ N_\uparrow \end{pmatrix}$, $N = \begin{pmatrix} n \\ p \end{pmatrix}$.

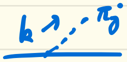
The relevant interations of the nucleons for our purpose are those with the pions, and can be described at LO by:

$$\mathcal{L}_{\pi N} = -\frac{g_A}{f_\pi}\,\bar{N}\,(\vec{\sigma}\cdot\vec{\partial})(\vec{\tau}\cdot\vec{\pi})\,N$$

Here $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are pauli matrices in spin space and $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ are pauli matrices in isospin space. $g_A = 1.27$ is the nucleon axial charge and $f_\pi \simeq 130$ MeV is the pion decay constant. The Feynman rules relevant for us are:

$$= \frac{i}{p\cdot v - M_N^{(0)} + i\epsilon}$$

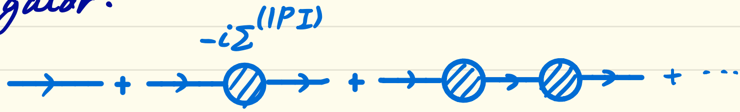
$$= -4c_1 m_\pi^2 \quad \text{with } c_1 = -0.93 \pm 0.10 \text{ GeV}^{-1}$$



$$= \frac{g_A}{f_\pi} k_i \sigma_i \tau_j$$

From insertions of quark mass matrix.

Now we have all the ingredients to carry out the calculation of FV effects to the nucleon mass. Note first that the "radiative" corrections to the mass of the nucleon arise from interactions with pions. The nucleon mass can then by obtained from fully dressed nucleon propagator:

$D_N =$ 

$$= \frac{i}{p\cdot v - M_N^{(0)} + i\epsilon}\left[1 - i\Sigma^{(1PI)}\frac{i}{p\cdot v - M_N^{(0)} + i\epsilon} + \left(-i\Sigma^{(1PI)}\frac{i}{p\cdot v - M_N^{(0)} + i\epsilon}\right)^2 + \cdots\right]$$

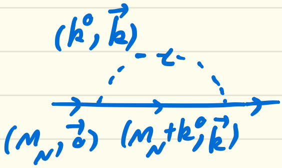
$$= \frac{i}{p\cdot v - M_N^{(0)} - \Sigma^{(1PI)} + i\epsilon} = \frac{iZ_N}{p\cdot v - M_N + i\epsilon}$$

Here, $M_N^{(0)}$ is the bare nucleon mass and $\Sigma^{(1PI)}$ is the one-particle irreducible self-energy function. The pole of the fully dressed propagator gives the mass of the nucleon:

$$p\cdot v = M_N \Rightarrow \left[p\cdot v - M_N^{(0)} - \Sigma^{(1PI)}\right]_{p\cdot v = M_N} = 0 \Rightarrow M_N = M_N^{(0)} + \Sigma^{(1PI)}\Big|_{p\cdot v = M_N}$$

Now we have to identify in the theory the leading diagrams contributing to $\Sigma^{(1PI)}$. They are:

$$= -4c_1 m_\pi^2 \equiv -i\Sigma_{LO}$$



$$= \left(\frac{g_A}{f_\pi}\right)^2 \times \frac{3}{2}\int \frac{d^4k}{(2\pi)^4}\,\vec{k}^2\,\frac{i}{k^0 - i\epsilon}\,\frac{i}{k^{0\,2} - \vec{k}^2 - m_\pi^2 + i\epsilon} \equiv -i\Sigma_{NLO}$$

where we have chosen to work at rest frame of the nucleon since we are interested in the corrections to the mass.

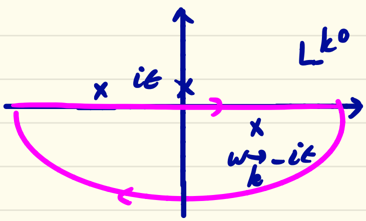
> **Exercise 7:** Derive/justify the factor of $3/2\,\vec{k}^2$ in the loop expression above. Note that: $\vec{\tau}\cdot\vec{\pi} = \begin{bmatrix} \pi^0/\sqrt{2} & \pi^+ \\ \pi^- & -\pi^0/\sqrt{2} \end{bmatrix}$.

To get one step closer to evaluating this integral, we can perform integration over k^0:

$$\left[\omega_{\vec{k}} \equiv \sqrt{\vec{k}^2 + m_\pi^2}\right]$$

$$-i\Sigma_{NLO} = -\frac{3 g_A^2}{2 f_\pi^2}\int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2\pi}\,\frac{(-2\pi i)}{\omega_{\vec{k}}\,(2\omega_{\vec{k}})}\,\vec{k}^2$$

$$= \frac{3 i g_A^2}{4 f_\pi^2}\int \frac{d^3k}{(2\pi)^3}\,\frac{\vec{k}^2}{\vec{k}^2 + m_\pi^2}$$



First, you may note that this integral is UV divergent, introducing a dependence on the scale that must be renormalized.

However, we are not interested in deriving this known result here. What we are interested in is to obtain the finite and infinite volume difference in this quantity. Such a UV divergence is present in a finite volume and cancels out in the difference. Note that for this discussion, we have assumed that the time extent of the spacetime volume is infinity, so that the only finite-size corrections are in the spatial directions, again in a finite cubic volume with periodic boundary conditions, such that: $\begin{cases} k^0 \text{ continuous} \\ \vec{k} = \frac{2\pi}{L}\vec{n},\ n \in \mathbb{Z}^3 \end{cases}$

Therefore:
$$\delta\Sigma \equiv \Sigma(L) - \Sigma(\infty)$$
$$= \underbrace{\left(\Sigma_{LO}(L) - \Sigma_{LO}(\infty)\right)}_{0} + \left(\Sigma_{NLO}(L) - \Sigma_{NLO}(\infty)\right) + \cdots$$
$$= -\frac{3g_A^2}{4f_\pi^2}\left[\frac{1}{L^3}\sum_{\vec{k}=\frac{2\pi\vec{n}}{L}} - \int\frac{d^3k}{(2\pi)^3}\right]\frac{\vec{k}^2}{\vec{k}^2+m_\pi^2} + \underbrace{\cdots}_{\text{higher orders}}$$

A powerful relation is the Poisson resummation formula:
$$\frac{1}{L^3}\sum_{\substack{\vec{k} \\ k = 2\pi\vec{n}/L,\ n\in\mathbb{Z}^3}} f(\vec{k}) = \sum_{\substack{\vec{m} \\ m\in\mathbb{Z}^3}}\int\frac{d^3k}{(2\pi)^3} f(k)\, e^{i\vec{k}\cdot\vec{m}L}$$

Using this, we already see that the first term in the sum is the infinite-volume value, leaving a purely finite volume contributions in the difference:

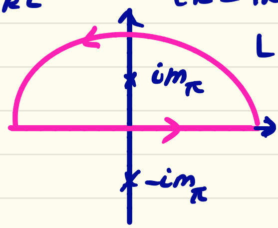
$$\delta\Sigma = -\frac{3g_A^2}{4f_\pi^2}\sum_{\vec{m}\neq 0}\int\frac{d^3k}{(2\pi)^3}\,\frac{\vec{k}^2}{\vec{k}^2+m_\pi^2}\,e^{i\vec{m}\cdot\vec{k}L}$$

$$= -\frac{3g_A^2}{4f_\pi^2}\sum_{\vec{m}\neq 0}\frac{(2\pi)}{(2\pi)^3}\int_0^\infty dk\,\vec{k}^2\,\frac{\vec{k}^2}{\vec{k}^2+m_\pi^2}\int_{-1}^{1}d(\cos\theta)\,e^{i|\vec{m}||\vec{k}|L\cos\theta}$$

$$= -\frac{3g_A^2}{16\pi^2 f_\pi^2}\sum_{\vec{m}\neq 0}\int_0^\infty dk\,\frac{k^4}{k^2+m_\pi^2}\,\frac{1}{i|\vec{m}|kL}\left(e^{i|\vec{m}|kL}-e^{-i|\vec{m}|kL}\right)$$

$$= i\,\frac{3g_A^2}{16\pi^2 f_\pi^2}\sum_{\vec{m}\neq 0}\frac{1}{|\vec{m}|L}\int_{-\infty}^{\infty}dk\,\frac{k^3}{k^2+m_\pi^2}\,e^{i|\vec{m}|kL}$$

$[k\equiv|\vec{k}|]$



$$= \frac{3g_A^2}{16\pi f_\pi^2}\sum_{\vec{m}\neq 0}\frac{e^{-|\vec{m}|m_\pi L}}{|\vec{m}|L}$$

$$= \frac{3g_A^2}{16\pi f_\pi^2}\,m_\pi^2\left[\frac{e^{-m_\pi L}}{L}\times 6+\cdots\right] = \delta M_N \equiv M_N(L)-M_N(\infty)$$

Exercise 8: Write down explicitly the contributions from terms up to and including $O(e^{-2m_\pi L})$ in δM_N. Plot as a function of $m_\pi L$ contributions from each of these terms, and compare with the exact evaluation. Use $0\lesssim m_\pi L\lesssim 6$ for the range of plot.

This example clearly demonstrates that FV corrections to the mass of stable hadrons, such as nucleons, are exponentially suppressed in volume. It also serves as an example on how the knowledge of a low-energy effective field theory allows to determine leading volume effects. It turned out that volume corrections to other properties of single hadrons are also exponentially suppressed in volume.

Exercise 9: Consider the following chiral perturbation theory lagrangean:

$$\mathcal{L}_{N\gamma} = \frac{e}{4M_N} F^{\mu\nu}\left[\mu_0 \bar{N}\sigma_{\mu\nu}N + \mu_1 \bar{N}\sigma_{\mu\nu}\tau_3^{(\xi^\dagger)}N\right]$$

that describes the magnetic moment of the nucleon through coupling to the electromagnetic field strength tensor at lowest orders.

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Here, $\sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]$ and nucleons are in relativistic four-component spinor representation. μ_0 and μ_1 are two low-energy const. and $\tau_3^{(\xi^\dagger)} = \frac{1}{2}(\xi^\dagger\tau^3\xi + \xi^\dagger\tau^3\xi)$ with: $\xi = e^{\frac{i\vec{\pi}\cdot\vec{\tau}}{f_\pi}}$.

Show that at $O(g_A^2/f_\pi^2)$, the FV corrections to the magnetic moment of nucleon is:

$$\delta\mu \equiv \mu(L) - \mu(\infty) = -\frac{g_A^2}{12\pi f_\pi^2} M_N m_\pi \sum_{\vec{m}\neq 0} \left(1 - \frac{2}{m_\pi |\vec{m}| L}\right) e^{-m_\pi |\vec{m}| L}$$

BONUS

An interesting point regarding the expressions we just derived is that these can be used to simultaneously extrapolate both in volume and in the pion mass. If a lattice-QCD calculation is performed at a larger quark mass, then as long as the associated pion mass is not too high such that χPT can be still applicable, one can use the expressions obtained to extrapolate to $m_\pi^{\text{phys.}}$. This is another example of the benefits of an EFT study of observables.