Exercises for lectures I and II

1. QED Hamiltonian

   (a) Perform a Legendre transform on the QED Lagrangian in $3+1$ dimensions to obtain the QED Hamiltonian.

   (b) Does A_0 have any dynamics? Verify that it arises as the Lagrange multiplier for the Gauss-law constraint. There is a gauge, called the Hamiltonian or temporal gauge, in which $A_0 = 0$. In such a gauge, how do you ensure the Gauss law still holds?

   (c) As an optional exercise, you could repeat the above parts for a general non-Abelian Yang-Mills theory coupled to matter.

2. Staggered fermions

   (a) Consider the naive Hamiltonian for Dirac fermions on a d-dimensional spatial lattice with lattice spacing a:

   $$H = m \sum_{\mathbf{n}} \bar{\psi}_{\mathbf{n}} \psi_{\mathbf{n}} - \frac{i}{2a} \sum_{\mathbf{n},j} \left[\bar{\psi}_{\mathbf{n}} \gamma^j \psi_{\mathbf{n}+\hat{j}} - \bar{\psi}_{\mathbf{n}+\hat{j}} \gamma^j \psi_{\mathbf{n}} \right], \tag{1}$$

   where $\psi_{\mathbf{n}}$ is the Dirac spinor at spatial lattice site $\mathbf{n}$ and γ^μ are the gamma matrices for $d+1$ dimensional Minkowski space. Perform a Fourier transform of the fermion fields to momentum space and derive the dispersion relation $E(\mathbf{p})$ for the naive fermions. How many fermion doublers do you find?

   (b) Define the staggering transformation as:

   $$\chi_{\mathbf{n}} = \Omega_{\mathbf{n}} \psi_{\mathbf{n}}, \tag{2}$$

   where the position-dependent matrix $\Omega_{\mathbf{n}}$ is given by:

   $$\Omega_{\mathbf{n}} = (\gamma^d)^{n_d} \dots (\gamma^1)^{n_1}. \tag{3}$$

   Implement the staggering transformation on the naive fermion Hamiltonian, and show that the transformed Hamiltonian is diagonal in spinor space.

   (c) If we construct a theory consisting of only one of the decoupled components of the staggered fermion, we are left with 2^D copies of single-component fermions. These single-component fermions can in fact be bunched together into Dirac fermions (see section 10.1.2 in Gattringer and Lang for more details of this construction). Thus, in 3+1 dimensions, the $2^3 = 8$ single component fermions are bunched into 2 four-component Dirac fermions. Likewise, in 2+1-D, the $2^2 = 4$ single component fermions are grouped into 2 two-component Dirac fermions. Finally, in 1+1-D, the 2 single component fermions are grouped into one Dirac fermion. These copies of Dirac fermions are referred to as tastes - thus staggering reduces but does not completely eliminate doubling except in 1+1 dimensions.

Let us inspect how a Dirac spinor can be constructed from the single-component spinor $\chi_{\vec{n}}$ (note that we use the same notation for the full spinor as well as its individual components) in the 1+1 dimensional case. By discretizing the continuum Dirac Hamiltonian, verify that you obtain the following discrete staggered Hamiltonian:

$$H = m \sum_n (-1)^n \chi_n^\dagger \chi_n - \frac{i}{2a} \sum_n \left(\chi_n^\dagger \chi_{n+1} - \chi_{n+1}^\dagger \chi_n \right) . \tag{4}$$

As an optional exercise, you could think about how this correspondence works for higher dimensions. (Check out Appendix D in this work on QED quantum simulations in 2+1 dimensions for an illuminating discussion on this.)

(d) Staggered fermions are very popular in quantum simulations. Can you guess why that is the case?

3. Gauss law on the staggered lattice

   Recall that the Kogut-Susskind Hamiltonian for 1+1-D QED is:

$$H = \sum_n \left[\frac{1}{2a} \left(\chi_n^\dagger U_n \chi_{n+1} + \chi_{n+1}^\dagger U_n^\dagger \chi_n \right) + m(-1)^n \chi_n^\dagger \chi_n + \frac{g^2 a}{2} E_n^2 \right] . \tag{5}$$

   The Gauss law operator at site n is:

$$G_n = E_n - E_{n-1} + \chi_n^\dagger \chi_n - \frac{1}{2} \left(1 - (-1)^n\right) \tag{6}$$

   Verify that the Gauss law operator commutes with the Hamiltonian. Even though it does not affect the commutation, the scalar staggering term is needed to recover the correct Gauss law in the continuum limit.

4. Spectrum of the lattice Schwinger model