Lattice BSM in-class exercises: Lecture 3

Instructions: Before doing the longer in-class exercises, take a minute to think about and answer the warm-up questions below, then discuss with your classmates.

Warm-Up 3.1: In the Banks-Zaks limit $N_f \to N_f^{\mathrm{AF}}$ from below, gauge-fermion theories are asymptotically free with a weakly-coupled infrared fixed point. Based on the two-loop beta function, describe the theory when $N_f > N_f^{\mathrm{AF}}$.

Exercise 6: As promised in the lecture video, let's start by getting a better understanding of mass deformation. To do so, we'll derive a quantitative relationship between m_f and the induced confinement scale [2] Λ_C.

a) We begin by assuming that we are studying a theory in the conformal window, with fixed-point coupling $\alpha^\star$. The mass anomalous dimension $\gamma_m(\mu)$ approaches a constant value $\gamma^\star$ in the infrared ($\mu \to 0$) limit. As shown in the lecture, if $\gamma_m(\mu)$ is approximately constant, then the solution for the running mass is

$$m(\mu) = m_0 \mu^{\gamma^\star}. \tag{10}$$

The induced confinement scale is identified using a decoupling argument as

$$\Lambda_C \approx M, \quad m(M) \equiv M. \tag{11}$$

We assume that a "seed mass" $m_0 \ll M$ has been defined at the ultraviolet scale Λ, which satisfies $\Lambda \gg M$ but also $\alpha(\Lambda) \approx \alpha^\star$. (In other words, we work only in the "plateau" regime where the coupling is approximately constant.)

Make sketches of α vs. μ and $m(\mu)$ vs. μ based on the setup described above - convince yourself that the basic argument makes sense. Label the scales Λ, m_0, and M. There are differences in convention as to whether $\gamma^\star$ should be positive or negative, which I haven't been very careful about so far: **which sign must $\gamma^\star$ have** in this case for mass deformation to trigger confinement in the infrared limit $\mu \to 0$? *(Hint: I already made the α vs. μ sketch in the pre-lecture video.)*

b) Now **solve the equations defined in part A to find $\Lambda_C(m_0)$.** Show that you recover *power-law* dependence of the induced confinement scale on the input mass, with the power dependent on $\gamma^\star$. (This relationship is also known as *mass hyperscaling* in the literature.)

c) Mass hyperscaling gives us some practical ways to use lattice results to learn about theories in the conformal window. Suppose your research group is studying a gauge

[2] You may recall that I complained in lecture 1 about phenomenologists using "the confinement scale", and now here I am using it myself. This is to remind us that the following derivation gives only qualitative results: it will capture the parametric dependence of things like hadron masses, but does not predict them quantitatively. If you don't like Λ_C, frame the derivation in terms of the mass of some hadron M_H.

theory that you think might be infrared conformal. You have measured the pion, baryon, and vector meson masses M_P, M_B, M_V at several different bare fermion masses. You plan to make three different plots of your results:

- A plot of all three hadron masses vs. m_f;
- A plot of the ratio M_B/M_V vs. m_f;
- A plot of the ratio M_B/M_V vs. M_P/M_V. (This is called an "Edinburgh plot", and has been used in lattice QCD for a while: see https://arxiv.org/abs/hep-lat/0209024.)

Make a sketch of what you expect each of these plots to look like if the theory you have simulated is indeed IR conformal and shows mass hyperscaling. *(Bonus questions if you have time: how would you extract $\gamma^\star$ from fitting to the data in one or more of these plots? What would they look like in QCD instead?)*

Exercise 7: Finally, a few small exercises involving beta functions and higher representations. For your convenience, I'll reproduce the first two perturbative coefficients for SU(N_c) here:

$$\beta_0 = \frac{1}{4\pi}\left(\frac{11}{3}N_c - \frac{4}{3}T(R)N_f\right), \tag{12}$$

$$\beta_1 = \frac{1}{(4\pi)^2}\left(\frac{34}{3}N_c^2 - \left[\frac{20}{3}N_c + 4C_2(R)\right]T(R)N_f\right). \tag{13}$$

As a reminder, the quadratic Casimir invariant $C_2(R)$ can be obtained using the formula

$$C_2(R) = \frac{T(R)d_G}{d_R} = \frac{T(R)}{d_R}(N_c^2 - 1). \tag{14}$$

a) The four-index antisymmetric representation A_4 of SU(N_c) has dimension and trace Casimir

$$d_{A_4} = \frac{N(N-1)(N-2)(N-3)}{24}, \tag{15}$$

$$T(A_4) = \frac{(N-2)(N-3)(N-4)}{12}. \tag{16}$$

(Note that this representation is only unique for $N_c \geq 8$; for example, in $N_c = 5$ this is just the conjugate of the F irrep. So we are only interested in it for $N_c \geq 8$.)

The big table of irreps from lecture 2 (courtesy of Dan Hackett) claims that a theory with $N_f \geq 1$ fermions charged under A_4 will be asymptotically free only if $N_c \leq 9$. **Confirm that the $N_c = 10$ case is *not* asymptotically free.** Is it ever possible to have a pair of these fermions in an asymptotically-free theory, i.e. $N_f = 2$ for any N_c?

b) **Find a theory** (N_c, N_f, R) for which β_0 exactly vanishes. (This is simplest to do for $R = F$, but that's not the only option.) Is the theory you found asymptotically free?

c) One of the unusual bound states which can arise in certain hypercolor (i.e. non-QCD) theories are *gluequarks*, bound states of the form gQ, where g is a gluon. For this state to be a hypercolor singlet, the fermion Q must be in the adjoint representation of the hypercolor group. In the context of composite Higgs models, the *partial compositeness* mechanism requires the existence of "top partner" T bound states of the hypercolor sector that have the same Standard Model quantum numbers as the top quark.

If we want T to be created as a gluequark gQ, then Q must be charged as a fundamental of $SU(3)_c$ (real-world QCD color). But this can't possibly work as the basis for a composite Higgs model unless the hypercolor theory is asymptotically free. So, is it?