Lattice BSM in-class exercises: Lecture 2

Instructions: Before doing the longer in-class exercises, take a minute to think about and answer the warm-up questions below, then discuss with your classmates.

Warm-Up 2.1: Suppose that QCD was actually a two-color theory. How could we tell? (One answer is "because we use 3 colors for lattice QCD and it works", but suppose we didn't know that and were just deciding from experimental information.)

(There is no exercise 3!)

Exercise 4: Some exercises with confinement and bound states:

a) Compute the tensor product of a pair of two-index antisymmetric irreps of $SU(N)$, $A_2 \otimes A_2$. To check your answer, compute the dimensions of each irrep in your result for $N = 5$; they should add up to $[\dim(A_2)]^2 = [N(N-1)/2]^2 = 100$.

   For what value of N does this product contain a color singlet state?

b) Show that baryons in an (N_c, N_f, F) theory are fermions if N_c is odd, and bosons if N_c is even. If $N_f = 1$, there are strong restrictions on the allowed baryon states - explain them. *(Hint: for $N_c = 3$, $N_f = 1$ is just the subset of QCD states that only contain a single quark flavor. You can look those states up in the PDG and see if you notice a pattern...)*

c) Consider a theory with $N_c = 4$ and both fundamental (F) fermions f and two-index symmetric (S_2) fermions F. What are the possible color-singlet bound states which cannot be decomposed into smaller singlets? (For QCD, the answer is just mesons $\bar{q}q$ and baryons qqq. A tetraquark state $\bar{q}q\bar{q}q$ is decomposable into two mesons, although there are other color contractions possible.)

Exercise 5:

a) Let's start with some counting. The dimension of the group $SU(N)$ is $N^2 - 1$; there are N^2 independent entries possible in an $N \times N$ matrix X_{ij}, minus 1 for the single constraint that $\det X = 1$. Two other groups have shown up in our study of fermion bilinears: $SO(2N)$ and $Sp(2N)$.

   - Recall that $SO(2N)$ is defined to be the group of $2N \times 2N$ matrices which "preserve the dot product", i.e. $X^{ik}\delta^{k\ell}X^{\ell j} = \delta^{ij}$. Find the number of independent equations of constraint that this condition gives, and **show that the dimension of the group is** $\dim(SO(2N)) = N(2N-1)$.
   - Similarly, $Sp(2N)$ is defined to be the group of $2N \times 2N$ matrices which "preserve the symplectic form", i.e. $X^{ik}E^{k\ell}X^{\ell j} = E^{ij}$, where E is the symplectic form that we saw in lecture which has the important property $E_{ij} = -E_{ji}$. **Show that the dimension of the symplectic group is** $\dim(Sp(2N)) = N(2N+1)$.

b) One of the simplest theories that exhibits the $Sp(2N)$ remnant chiral symmetry is $(N_c, N_f, R) = (2, 2, F)$ - the two-color variant of QCD. (You can easily convince yourself from Young diagrams that F is real or pseudo-real for $N_c = 2$; it turns out to be pseudo-real.) The chiral symmetry breaking pattern is $SU(4) \rightarrow Sp(4)$. Using the group dimensions above, **how many pions are generated** by this symmetry breaking? The three usual pions $\pi^a = \bar{q}\gamma_5\sigma^a q$ are included in this set: what would you guess the extra pions look like in terms of color and quark flavor?

c) Similar to part B, one of the simplest theories that should exhibit a remnant $SO(2N)$ remnant chiral symmetry is $(N_c, N_f, R) = (4, 2, A_2)$. For this theory, **how many pions are generated?** What do the extra pions look like in terms of quark color and flavor in this theory, compared to the symplectic case above?

d) **Bonus challenge:** as with the baryon states, the $N_f = 1$ case here has some distinctive properties. How many pions are there in the complex, real, and pseudo-real cases for an $N_f = 1$ theory? (For N small there are isomorphisms between the SU, SO and Sp groups that you can look up, or in some cases they become trivial.) Are any of these theories appealing as a potential basis for composite dark matter, based on what you know about the spectrum from this exercise?