Consider external currents:

For example: axial vector current

$$A_\mu^{lat} = \bar{\psi}_f \gamma_\mu \gamma_5 \psi_{f'}$$

can be described in Symanzik EFT by

$$A_\mu^{lat} \doteq \bar{Z}_A^{-1} A_\mu + a\, k_A\, \partial_\mu \bar{\psi}_f \gamma_5 \psi_{f'} + \ldots$$

$$A_\mu = \bar{\psi}_f \gamma_\mu \gamma_5 \psi_{f'} \qquad \text{continuum current}$$

Then we have

$$\langle H' | \bar{Z}_A A_\mu^{lat} | H \rangle_{lat} = \langle H' | A_\mu | H \rangle_{cont}$$

$$+ a\, \bar{Z}_A k_A\, \partial_\mu \langle H' | \bar{\psi} \gamma_5 \psi | H \rangle_{cont}$$

$$+ a\, k_{\sigma \cdot F} \int d^4y\, \langle H' | T\, O_5\, A_\mu | H \rangle_{cont}$$

$$+ O(a^2)$$

As before, if $k_A \neq 0$, add correction operators to lattice current:

$$A_\mu^{lat} = \bar{\psi}_f \gamma_\mu \gamma_5 \psi_{f'} + a\, c_A\, \partial_\mu^{lat} \bar{\psi}_f \gamma_5 \psi_{f'}$$

adjust c_A so that $k_A = 0$. At tree-level, one finds $c_A = 0$. [@ 1-loop : $c_A = -g^2 C_F \cdot 0.00568$]

Note, Z_A, k_A, $K_{O.F}$ $(g^2, ma; c_i's; \mu a)$

If $ma \ll 1$, then we can expand the coefficients in powers of ma:
For example:

$$Z_A(ma) = Z_A\left(1 + b_A \tfrac{1}{2}(m_f + m_{f'})a\right) + O(ma)^2$$

At tree-level : $Z_A = 1$, $b_A = 1$

In the usual presentation of Symanzik EFT one expands in ma from the start and the improvement coefficients are obtained at $ma = 0$ i.e. for the current the dim 4 operators in the Symanzik EFT are:

$$m\, A_\mu \,, \quad \partial_\mu \bar{\psi} \gamma_5 \psi$$

the results are the same.
So far we have assumed that $am \ll 1$ as well as $a\Lambda_{QCD} \ll 1$.
For the charm quark $m_c \gtrsim \Lambda_{QCD}$, but as long as $am_c \ll 1$ the above program works.
However since $m_c > |\vec{p}|$ (typical 3-mom of charm quark inside bound state) $(am_c)^n$ terms dominate discretization effects.

We now consider heavy quarks w/ $ma \not\ll 1$:

To see what happens when $ma \sim \mathcal{O}(1)$ consider the quark propagator as function of Euclidean time and $\vec{p}$:

$$\langle \psi(\vec{p}\,', t') \bar{\psi}(\vec{p}, t) \rangle = (2\pi)^3 \delta^3(p-p') \, C(\vec{p}, t'-t)$$

$$\psi(\vec{p}, t) = a^3 \sum_x e^{-i\vec{p}\cdot\vec{x}} \psi(\vec{x}, t)$$

We have

$$C(\vec{p}, t) = \int_{-\pi}^{\pi} \frac{dp_0}{2\pi} e^{ip_0 t} \tilde{D}^{-1}(\vec{p}, p_0)$$

Exercise I:

It is instructive to see the appearance of the doublers for the naive action and for Wilson action how the choice of $r=1$ removes them.

notation: $\quad S_\mu = \frac{1}{a} \sin(p_\mu a) \qquad \hat{p}_\mu = \frac{2}{a} \sin(p_\mu a/2)$

Then for Wilson propagator (in lattice units $a=1$)

$$\begin{aligned} \tilde{D}_W(p) &= m_0 + i\not{s} + \tfrac{1}{2} r \sum_\mu \hat{p}_\mu^2 \\ &= m_0 + i\gamma_0 \sin p_0 + 1 - \cos p_0 + i\vec{\gamma}\cdot\vec{s} + \tfrac{1}{2} r \sum_i \hat{p}_i^2 \end{aligned}$$

After integrating over p_0 we have

$$C(\vec{p},t) = Z_2 \; e^{-E|t|} \qquad (t \neq 0)$$

$$\times \; \frac{\gamma_0 \, \mathrm{sign}\, t \, \sinh E + m_0 + 1 - \cosh E - i \vec{\gamma}\cdot\vec{s} + \frac{1}{2} r \sum_i \hat{p}_i^2}{2 \sinh E}$$

with $$\cosh E = 1 + \frac{[m_0 + \frac{1}{2} r \sum_i \hat{p}_i^2]^2 + \vec{s}^{\,2}}{2[1 + m_0 + \frac{1}{2} r \sum_i \hat{p}_i^2]}$$

$$Z_2(p) = \frac{1}{1 + m_0 + \frac{1}{2} r \sum_i \hat{p}_i^2}$$

Expand in powers of $(\vec{p}a)$ [note $m_0 \to m_0 a$, $\vec{s}^{\,2} \to \vec{s}^{\,2} a^2$, $\hat{p}_i^2 \to a^2 \hat{p}_i^2$, ...]

$$E^2 = m_1^2 + \frac{m_1}{m_2} \vec{p}^{\,2} + \ldots$$

where $$m_1 = E(\vec{p}=0) \qquad 1/m_2 = \left.\frac{\partial^2 E}{\partial p_i^2}\right|_{\vec{p}=0}$$

$$m_1 = \ln(1+m_0) \qquad \frac{1}{m_2} = \frac{2}{m_0(2+m_0)} + \frac{r}{1+m_0}$$

$$m_1/m_2 = 1 - \frac{2}{3} m_1^2 + \frac{1}{2} m_1^3 + \ldots \xrightarrow[m_0 a \to 0]{} 1$$

Also expand Z_2 in powers of $(\vec{p}a)$

$$Z_2(\vec{p}) = \frac{1}{1+m_0} + O(\vec{p}^{\,2}) \qquad \frac{1}{1+m_0} = e^{-m_1}$$

$$Z_2^{-1} \psi(x) = e^{m_1/2} \psi(x) \quad \text{canonical normalization}$$

In summary:

When $m_0 a \sim 1$ $\quad a^2E^2 \mp a^2m^2 + (a\vec{p})^2 + \mathcal{O}(a\vec{p})^n$

Consider non relativistic expansion of $E(\vec{p})$:

$$E = m_1 + \frac{\vec{p}^2}{2m_2} + \mathcal{O}(\vec{p}^4)$$

same m_1 & m_2 as before, but different interpretation:

instead of Symanzik EFT with $\mathcal{L}_{QCD}$

match to HQET (or NRQCD) + corrections

The Symanzik EFT breaks down, because when $ma \sim 1$, $\mathcal{L}_k$ contains terms $\sim (ma)^n$

In particular,

$$\mathcal{L}_I = \ldots + a^n \sum_{n=3}' K_n \sum_{\mu=0}^{3} \bar{\psi} (\gamma_\mu D_\mu)^n \psi$$

$(a\gamma_0 D_0)^n \sim (am)^n$. Use EoM

$$(-\gamma_0 D_0)\psi = (\vec{\gamma}\cdot\vec{D} + m)\psi$$

repeat n times $\sim a^n(\vec{\gamma}\cdot\vec{D} + m)^n$

$+ \sim [D_0, \vec{D}] \sim F_{0i}$, etc... $\leftarrow$ no "large" terms

expand $a^n(\vec{\gamma}\cdot\vec{D} + m)^n \sim (ma)^{n-\ell}\, \bar{\psi}(\vec{\gamma}\cdot\vec{D})^\ell \psi$

$\ell = 0$ term modifies $\bar{\psi}\psi$ term in $\mathcal{L}_{QCD}$

and $\ell = 1$ modifies $\bar{\psi}\vec{\gamma}\cdot\vec{D}\psi$

summing it all up:

$$\mathcal{L}_{sym} = \underbrace{\mathcal{L}_{gauge} + \bar{\psi}\left(m_1 + \gamma_0 D_0 + \sqrt{\frac{m_1}{m_2}}\,\vec{\gamma}\cdot\vec{D}\right)\psi}_{\neq \mathcal{L}_{QCD}} + \mathcal{L}_I'$$

Note: same m_1, m_2 as before
$\mathcal{L}_k'$: now small corrections again

solutions:

1) a) add an asymmetry to (improved) Wilson action:

$$\mathcal{L}^F = m_0 \bar{\psi}\psi + \bar{\psi}\gamma_0 D_0^{lat}\psi - \frac{ar}{2}\bar{\psi}\Delta_0^2\psi$$
$$+ \zeta\,\bar{\psi}\,\vec{\gamma}\cdot\vec{D}^{lat}\psi - \frac{ar}{2}\zeta\,\bar{\psi}\,\vec{\Delta}^2\psi$$

Can adjust ζ so that $m_1 = m_2$. See this for example in new $E(\vec{p})$ derived from $C(\vec{p},t)$ from $\mathcal{L}^F$

b) For improvement terms also use EoM to eliminate operators which contain $(\gamma_0 D_0^{lat})^n$

$$\rightarrow \quad \mathcal{L}_{sym} = \mathcal{L}_{QCD} + \mathcal{L}_I$$

$\uparrow$ small, i.e. no $(am)^n$

Note: $k_n = k_n(g^2, ma, \ldots)$ bounded functions of ma

works for all ma, i.e. charm and bottom
"modified Symanzik EFT"

2) Start again with (improved) Wilson action, but instead of using cont. QCD in the EFT matching, use HQET (or NRQCD):

$$\mathcal{L}_{lat} \doteq \mathcal{L}_{HQ}$$

RHS: written in terms of cont. HQET (NRQCD) fields

$$\mathcal{L}_{HQ} = \sum_n C_n^{lat}(m_Q, g^2; m_Q a; \mu/m_Q)\, O_n(\mu)$$

a, m_Q dependence only in coefficients.

$$\mathcal{L}_{HQ} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \ldots$$

$$\mathcal{L}^{(0)}_{HQET} = \bar{h}_v (i v\cdot D - m_1) h_v$$

↑ benign

$$\mathcal{L}^{(1)}_{HQET} = C^{lat}_{kin}\, O_{kin} + C^{lat}_B\, O_B$$

↑ kinetic energy ↑ chromo-magnetic

also have

$$\mathcal{L}_{QCD} \doteq \mathcal{L}^{cont}_{HQ}$$

same operators as $\mathcal{L}_{HQ}$ but different coefficients

HQ discretization effects in mismatch of coefficients

For example:

$$\text{if } m = m_2 \qquad c^{lat}_{kin} = c^{con}_{kin}$$

$$\text{if } c_{SW} = 1 \qquad c^{lat}_{B} = c^{con}_{B} \qquad \text{, etc...}$$

Now in $a \to 0$ limit, eventually $m_Q a \ll 1$ and we recover the Symanzik LEL

$$\Rightarrow \lim_{a \to 0} c^{lat}_n = c^{con}_n$$

→ Fermilab method (W.F. with Fermilab interpretation)

3) Start with EFT (HQET or NRQCD) and discretize:

$$\mathcal{L}^{lat}_{HQET} = \bar{\psi}_h D_0 \psi_h - c'_{kin} O_{kin} - c'_B O_B$$

adjust c's so that $c^{lat}_{kin} = \frac{1}{2m}$ $\quad c^{lat}_B = \frac{1}{2m}$
(at tree-level)

$c'_{kin} = 0 = c'_B$: static limit

In HQET dim ≥ 4 ops are treated as corrections

For HQET through $\frac{1}{m}$ matching/renorm. is performed non perturbatively

ensures well defined continuum limit

NRQCD similar, but O_{kin} included in action as a leading term

must keep $\frac{1}{a} \sim m_Q$ (power-law divergences in coefficients)

corrections through $\sim \frac{1}{m^2}$ (tree-level) included in action

→ explicit $a \to 0$ limit not possible. Instead take "continuum limit" at finite a by adding a sufficient number of higher order corrections such that remaining errors are small

4) Start with an improved light quark action (HISQ, tm Wilson, ... see below) that can be used for charm

Calculate physical quantity of interest for range of HQ masses

$$am_c \le am_Q < 1 \quad (\text{i.e.} < am_b)$$

- extrapolate to physical b quark mass using guidance from HQET