LGT unit 3 lecture 2

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1 Introduction

Lecture 2 of unit 3 for the LGT class. This lecture we talk about the path integral for fermions, fermion correlation functions, The hopping parameter expansion, and an intro to chiral symmetry in the continuum and on the lattice.

References:

- Christoph Lehner's lecture notes, ch 10 and 12
- Gattringer and Lang, section 5.1,5.3,7.1,7.2
- Montvay and Munster, section 4.4
- Coleman, Aspects of Symmetry, section 4.1

2 Fermionic path integral

Now that we have a sense of the Dirac operator on the lattice, we can construct the fermionic path integral in order to define fermionic correlation functions in the proper way. The reason we need to define the path integral differently for fermions is because the fields we wish to integrate over respect Fermi statistics. To see this clearly, we look at a general n-point correlation function

$$\left\langle \psi^{(f_1)}(n)_{\alpha_1,a_1}\psi^{(f_2)}(n)_{\alpha_2,a_2}...\psi^{(f_k)}(n)_{\alpha_k,a_k}\overline{\psi}^{(g_1)}(n)_{\beta_1,b_1}...\overline{\psi}^{(g_k)}(n)_{\beta_k,b_k}\right\rangle \quad (1)$$

This quantity needs to be antisymmetric under exchange of all indices to respect Fermi statistics. If we exchange all indices of the first field with all indices of the second field, it is effectively like picking up an extra minus sign when we commute the first two fields. Thus we can define the path integral and correlation functions in terms of fully anticommuting numbers, which is how our Fermi fields behave,

$$\{\psi_a,\psi_b\} = \overline{\psi}_a,\overline{\psi}_b = 0, \quad \{\psi_a,\overline{\psi_b}\} = i\delta_{ab} \quad (2)$$

Thus to define the fermionic path integral, we need to understand how to differentiate and integrate over fully anticommuting variables, called Grassmann variables. I will quickly review Grassmann variable integrals, in particular Gaussian integrals, and differentiation with respect to these Grassmann variables.

We begin by considering a set of two Grassmann variables, $\eta_1, \eta_2, \overline{\eta}_1, \overline{\eta_2}$, with the condition that any two of them anticommute, $\{\eta_1, \eta_2\} = \eta_1\eta_2 + \eta_2\eta_1 = 0$. This implies that the square of any of these Grassmann variables vanishes, $\eta_1^2 = 0$. This means that a power series in terms of any of these variables terminates, for example, $e^{\eta} = 1 + \eta + \frac{\eta^2}{2} + \frac{\eta^3}{6} + ... = 1 + \eta$. We can also define polynomials in terms of these variables with a finite number of terms, for example, $A = a + a_1\eta_1 + a_2\eta_2 + a_{12}\eta_1\eta_2$

For the case of N Grassmann variables, we can write the most general polynomial as

$$A = a + \sum_i a_i\eta_i + \sum_{i<j} a_{ij}\eta_i\eta_j + \sum_{i<j<k} a_{ijk}\eta_i\eta_j\eta_k + ... + a_{12...N}\eta_1\eta_2...\eta_N \quad (3)$$

with complex coefficients $a, a_i, a_{ij}, ..., a_{12...N}$. These polynomials can be added and multiplied to form a Grassmann algebra, where the Grassmann variables are the generators of this algebra.

We can differentiate elements of the Grassmann algebra with respect to the generators. We return to the $N = 2$ case, where we define the left derivative of A with respect to η_1 by

$$\partial_{\eta_1} A = a_1 + a_{12}\eta_2 \quad (4)$$

However, we can also write our polynomial as $A = a + a_1\eta_1 - a_{12}\eta_2\eta_1$, which implies that the derivatives with respect to the Grassmann variables also anticommute with the Grassmann variables as well. We can also do $\partial_{\eta_2}(\partial_{\eta_1} A)$ to show that the derivatives need to anticommute with eachother as well. All together, we get the following rules for our derivatives.

$$\partial_{\eta_i} 1 = 0, \quad \partial_{\eta_i}\eta_i = 1, \quad \{\partial_{\eta_i}, \partial_{\eta_j}\} = 0, \quad \{\partial_{\eta_i}, \eta_j\} = 0 \quad (5)$$

We also would like to define integration over Grassmann variables. We require three properties of Grassmann variables, the first is linearity,

$$\int d^N\eta A \in \mathbb{C}, \int d^N\eta(\alpha A_1 + \beta A_2) = \alpha \int d^N\eta A_1 + \beta \int d^N\eta A_2 \quad (6)$$

Second, we require our Grassmann variable polynomial to vasnish at the boundary,

$$\int d^N\eta\partial_{\eta_i} A = 0 \quad (7)$$

This implies that whenever a polynomial in Grassmann variables can be written as a derivative of some other polynomial in Grassmann variables A', the integral over A vanishes. Thus the integral over A must be proportional to the coefficient $a_1 2...N$ for the highest powers of the Grassmann variables in the polynomial. This leads to our normalization condition

$$\int d^N\eta\eta_1\eta_2...\eta_N = 1 \quad \rightarrow \quad \int d^N\eta A = a_{12...N} \quad (8)$$

Also, we can do integrals over any number of Grassmann variables, not just the product of all of them. We let $d^N\eta = d\eta_N d\eta_{N-1}...d\eta_1$, and we have the following conditions

$$\int d\eta_i 1 = 0, \quad \int d\eta_i \eta_i = 1, \quad \{d\eta_i, d\eta_j\} = 0 \tag{9}$$

Arguably the most important property of the integration over Grassmann variables is how the integration measure transforms under a linear change of variables. We define our transformation as

$$\eta'_i = \sum_{j=1}^{N} M_{ij}\eta_j \tag{10}$$

where M is a complex $N \times N$ matrix. We can see how this affects the measure by applying this transformation to the normalization integral

$$\begin{aligned}\int d^N\eta\eta_1...\eta_N &= \int d^N\eta'\eta'_i...\eta'_N = \int d^N\eta' \sum_{i_1,...,i_N} M_{1i_1}...M_{Ni_N}\eta_{i_1}...\eta_{i_N} \\ &= \int d^N\eta' \sum_{i_1,...,i_N} M_{1i_1}...M_{Ni_N}\epsilon_{i_1 i_2 ... i_N}\eta_1...\eta_N = \det[M]\int d^N\eta'\eta_1...\eta_N\end{aligned} \tag{11}$$

we reordered the product of Grassmann variables, which vanishes if any of the indices are equal, so we write it in terms of the antisymmetric tensor. The summation of this tensor with and the matrix elements gives the determinant. Due to the fact that both sets of Grassmann varaibles of normalized we find how the measure transforms;

$$d^N\eta = \det[M]d^N\eta' \tag{12}$$

Note this is the inverse of the bosonic case, where the determinant would be inverted.

This property of the measure under linear transformations allows us to show that if we shift any one Grassmann variable within the integral, the measure is invariant to the shift - that is Grassmann integrals are translationally invariant.

The final thing we must establish is how to do a Gaussian integral over Grassmann variables. We consider a Grassmann algebra with $2N$ generators, $\eta_i, \overline{\eta}_i, i = 1, 2, ..., N$. All of these generators commute with all of the other generators. The first important formula is the Matthews-Salam formula

$$Z_F = \int d\eta_N d\overline{\eta}_N...d\eta_1\overline{\eta}_1 \exp\left(\sum_{i,j=1}^{N} \overline{\eta_i}M_{ij}\eta_j\right) = \det[M] \tag{13}$$

This can be proven by the same transformation we defined in the previous section, and by the transformation of the measure we pick up a determinant of the transformation matrix M.

$$Z_F = \det[M] \int \prod_{i=1} d\eta_i' d\overline{\eta}_i \exp\left(\sum_{j=1}^{N} \overline{\eta}_j \eta_j'\right) = \det[M] \prod_{i=1}^{N} \int d\eta_i' d\overline{\eta}_i \exp\left(\overline{\eta}_j \eta_j'\right)$$
$$= \det[M] \prod_{i=1}^{N} \int d\eta_i' d\overline{\eta}_i (1 + \overline{\eta}_i \eta_i') = \det[M] \tag{14}$$

expanding the exponential in a power series, which terminates after the first two terms because these are Grassmann variables.

The Fermi fields behave like Grassmann variables in the path integral, so we can generalize our discussion of integrals over Grassmann variables very simply by setting the M in the Gaussian form to the Dirac operator, and we can write the fermionic partition function as a determinant,

$$Z_F[U] = \int \mathcal{D}[\psi, \overline{\psi}] e^{-S_E} = \int \mathcal{D}[\psi, \overline{\psi}] e^{\sum \overline{\psi}(x)_i D_{i,x;j,y}(U) \psi(y)_j} = \det(D(U)) \tag{15}$$

$$S_E = \sum_{x,y,i,j} \overline{\psi}(x)_i D_{i,x;j,y}(U) \psi(y)_j \tag{16}$$

With the partition function defined, we can turn to full correlation functions of fermionic fields.

3 Fermion correlation functions

3.1 The generating functional

With the path integral defined, we can now develop the method of evaluating fermionic correlation functions. We start by defining the fermionic generating functional, including Grassmannian sources.

We generalize to a $4N$-dimensional algebra generated by $\eta_i, \overline{\eta}_i, \theta_i, \overline{\theta}_i, i = 1, 2, ..., N$. All $4N$ Grassmann variables anticommute with every other one. Let's write an integral over the $\eta, \overline{\eta}$ and use the other generators as source terms.

$$W[\theta, \overline{\theta}] = \int \prod_{i=1}^{N} d\eta_i d\overline{\eta}_i \exp\left(\sum_{k,l=1}^{N} \overline{\eta_k} M_{kl} \eta_l + \sum_{k=1}^{N} \overline{\theta}_k \eta_k + \sum_{k=1}^{N} \overline{\eta}_k \theta_k\right) \tag{17}$$

To evaluate this, we must complete the square in the exponential, and then shift our variables. First we rewrite the exponential as

$$\left(\overline{\eta}_i + \overline{\theta}_j (M^{-1})_{ji}\right) M_{ik} \left(\eta_k + (M^{-1})_{kl} \theta_l\right) - \overline{\theta}_n (M^{-1})_{nm} \theta_m \tag{18}$$

Grassmann integrals are invariant to shifts, $\int d\psi f(\psi) = \int d\psi f(\psi + \alpha)$, so we shift our integration variables

$$\eta_k' = \eta_k + (M^{-1})_{kl} \theta_l, \quad \overline{\eta}_i' = \overline{\eta}_i + \overline{\theta}_j (M^{-1})_{ji} \tag{19}$$

Putting this into W,

$$W[\theta,\overline{\theta}] = \exp\left(-\sum_{n,m=1}^{N}\overline{\theta}_n(M^{-1})_{nm}\theta_m\right)\int\prod_{i=1}^{N}d\eta_i d\overline{\eta}_i \exp\left(\sum_{k,l=1}^{N}\overline{\eta_k}M_{kl}\eta_l\right)$$
$$= \det[M]\exp\left(-\sum_{n,m=1}^{N}\overline{\theta}_n(M^{-1})_{nm}\theta_m\right) \tag{20}$$

Once again we readily generalize to Fermi fields, setting $M = -D$. The generating functional is key in evaluating correlation functions. For example, a general correlation function of operator O can be written as

$$\langle O(U,\psi(x)_i\overline{\psi}(y)_j\rangle = \frac{1}{Z}\int\mathcal{D}[U,\psi,\overline{\psi}]O(U,\psi(x)_i\overline{\psi}(y)_j)e^{-S_F(U)-S_g(U)} \tag{21}$$

Writing the full partition function as

$$Z = \int\mathcal{D}[U,\psi,\overline{\psi}]e^{-S_F(U)-S_G(U)} = \int\mathcal{D}[U]\det(D(U))e^{-S_G(U)} \tag{22}$$

Writing our correlation function with additional source terms $\theta,\overline{\theta}$,

$$\begin{aligned}\langle O(U,\psi(x)_i\overline{\psi}(y)_j\rangle &= \frac{1}{Z_F}\int\mathcal{D}[U,\psi,\overline{\psi}]O(U,\psi(x)_i\overline{\psi}(y)_j)e^{-S_F(U)-S_g(U)-\overline{\theta}\psi-\overline{\psi}\theta}\\
&= \frac{1}{Z}\int\mathcal{D}[U,\psi,\overline{\psi}]O(U,\psi(x)_i\partial_{\theta(y)_j})e^{-S_F(U)-S_g(U)-\overline{\theta}\psi-\overline{\psi}\theta}\\
&= \frac{1}{Z}\int\mathcal{D}[U,\psi,\overline{\psi}]O(U,-\partial_{\theta(y)_j}\psi(x)_i)e^{-S_F(U)-S_g(U)-\overline{\theta}\psi-\overline{\psi}\theta}\\
&= \frac{1}{Z}\int\mathcal{D}[U,\psi,\overline{\psi}]O(U,\partial_{\theta(y)_j}\partial_{\overline{\theta}(x)_i})e^{-S_F(U)-S_g(U)-\overline{\theta}\psi-\overline{\psi}\theta}\\
&= \frac{1}{Z}\int\mathcal{D}[U]O(U,-\partial_{\overline{\theta}(x)_i}\partial_{\theta(y)_j})e^{-S_g(U)}\int\mathcal{D}[U,\psi,\overline{\psi}]e^{-S_F(U)-\overline{\theta}\psi-\overline{\psi}\theta}\\
&= \frac{1}{Z}\int\mathcal{D}[U]O(U,-\partial_{\overline{\theta}(x)_i}\partial_{\theta(y)_j})e^{-S_g(U)}\times W[\theta,\overline{\theta}]\\
&= \frac{1}{Z}\int\mathcal{D}[U]\det(D(U))e^{-S_g(U)}(O(U,-\partial_{\overline{\theta}(x)_i}\partial_{\theta(y)_j})e^{\overline{\theta}D^{-1}(U)\theta})\end{aligned} \tag{23}$$

As a quick example, consider the two point function,

$$\langle\psi(x)_i\overline{\psi}(y)_j\rangle = \langle-\partial_{\overline{\theta}(x)_i}\partial_{\theta(y)_j})\times W[\theta,\overline{\theta}]\rangle = \langle D^{-1}(U)_{i,x;j,y}\rangle \tag{24}$$

When we want to define an n-point function, or any type of Green's function higher than a two point function, we need to be careful about permutations of fields - the calculation is exactly the same as the two point case, but we also pick up a sum over permutations P

$$\langle\psi(x_1)_{i_1}\overline{\psi}(y_1)_{j_1}...\psi(x_n)_{i_n}\overline{\psi}(y_n)_{j_n}\rangle = \sum_P(-1)^P\langle D^{-1}(U)_{i_1,x_1;j_{P_1},y_{P_1}}...D^{-1}(U)_{i_n,x_n;j_{P_n},y_{P_n}}\rangle \tag{25}$$

Note we can rewrite this using the generating functional,

$$\langle\psi(x_1)_{i_1}\overline{\psi}(y_1)_{j_1}...\psi(x_n)_{i_n}\overline{\psi}(y_n)_{j_n}\rangle = \frac{1}{Z}\int D[U]e^{-S_g(U)}\left(\frac{\partial}{\partial\theta_{j_1}}\frac{\partial}{\partial\overline{\theta}_{i_1}}...\frac{\partial}{\partial\theta_{j_n}}\frac{\partial}{\partial\overline{\theta}_{i_n}}\times W[\theta,\overline{\theta}]\right) \tag{26}$$

As another quick concrete example, the fermionic four point function is given by

$$\begin{aligned}\langle\psi(x_1)_{i_1}\overline{\psi}(y_1)_{j_1}\psi(x_2)_{i_2}\overline{\psi}(y_2)_{j_2}\rangle &= \langle D^{-1}(U)_{i_1,x_1;j_1,y_1}D^{-1}(U)_{i_2,x_2;j_2,y_2}\rangle \\ &\quad -\langle D^{-1}(U)_{i_1,x_1;j_2,y_2}D^{-1}(U)_{i_2,x_2;j_1,y_1}\rangle\end{aligned} \tag{27}$$

3.2 The pion correlator

We can use this machinery we just developed to analyze the pion, which can be created by using the operator

$$\hat{O}_\pi(t) = i\sum_x \hat{\overline{u}}(\vec{x},t)\gamma_5\hat{d}(\vec{x},t) \tag{28}$$

where u and d are two different quenched degrees of freedom. Since the definition of the conjugate momenta for such a degree of freedom is $\hat{\overline{u}} = \hat{u}^\dagger\gamma_4$,

$$\hat{O}_\pi(t)^\dagger = \sum_x(i\hat{u}^\dagger(\vec{x},t)\gamma_4\gamma_5\hat{d}(\vec{x},t))^\dagger = -\sum_x i\hat{d}^\dagger(\vec{x},t)\gamma_5\gamma_4\hat{u}(\vec{x},t) = \sum_x i\hat{\overline{d}}(\vec{x},t)\gamma_5\hat{u}(\vec{x},t) \tag{29}$$

We define the pion correlator and use Wick's theorem,

$$\langle\hat{O}_\pi(t)\hat{O}_\pi^\dagger(0)\rangle = \sum_{x,y}\langle\mathrm{Tr}[D^{-1}(U)_{\vec{x},t;0,0}(D^{-1}(U)_{\vec{x},t;0,0})^\dagger]\rangle \tag{30}$$

Using γ_5-Hermiticity to write the second Dirac operator as its adjoint. In our exercise we will analyze this correlation function on the lattice.

4 Hopping parameter expansion

We showed in the previous section how to evaluate correlation functions involving fermions. In particular, the two point function

$$\left\langle\psi(n)_i\overline{\psi}(m)_j\right\rangle = a^{-4}D^{-1}_{i,n;j,m} \tag{31}$$

where index i includes Dirac and color indices. We can evaluate the right hand side of this equation with the hopping parameter expansion, that is for large quark mass m. We have an explicit form of a Dirac operator as well, the Wilson Dirac operator. We can rewrite the operator as

$$D = C(1-\kappa h) \quad \text{with} \quad \kappa = \frac{1}{2(am+4)}, \quad C = m + \frac{4}{a} \tag{32}$$

$$H = \sum_{\mu} (\mathbb{I} - \gamma_\mu) U_\mu(n) \delta_{n+\hat{\mu},m} \tag{33}$$

H is called the hopping matrix, and κ the hopping parameter. We can absorb the constant C into the definition of the fields, allowing us to rewrite $D = \mathbb{I} - \kappa H$. The idea is that κ becomes small for large mass, so we expand D^{-1} and $\det[D]$ in powers of the hopping parameter. We can rewrite the quark propagator in terms of a geometric series,

$$D^{-1} = (\mathbb{I} - \kappa H)^{-1} = \sum_{j=0}^{\infty} \kappa^j H^j \tag{34}$$

This series converges for $\kappa < 1/8$

Looking more closely at H and displaying indices explicitly,

$$D^{-1}_{\alpha,a;\beta,b} = \sum_{j=0}^{\infty} \kappa_j H^j_{\alpha,a;\beta,b} \tag{35}$$

Here we will show a few entries of the jth power of H

$$\begin{aligned}
H^0_{n,\alpha,a;m,\beta,b} &= \delta_{\alpha\beta}\delta_{ab}\delta_{nm} \\
H^1 n,\alpha,a;m,\beta,b &= \sum_{\mu} (\mathbb{I} - \gamma_\mu)_{\alpha\beta}) U_\mu(n)_{ab} \delta_{n+\hat{\mu},m} \\
H^2_{n,\alpha,a;m,\beta,b} &= \sum_{l,\rho,c} H(n,l)_{\alpha,a;\rho,c} H(l,m)_{\rho,c;\beta,b} \\
H^j_{n,\alpha,a;m,\beta,b} &= \sum_{\mu} \left(\prod_{i=1}^{j} (\mathbb{I} - \gamma_{\mu_i} \right)_{\alpha\beta} P_{\mu_1 \ldots \mu_j}(n)_{ab} \delta_{n+\hat{\mu_1}+\ldots+\hat{\mu_j},m}
\end{aligned} \tag{36}$$

Where P in the last line is shorthand for a string of gauge links $P_{\mu_1 \ldots \mu_j} = (U_{\mu_1}(n) U_{\mu_2}(n+\mu_1) \ldots U_{\mu_j}(n+\mu_1+\ldots+\mu_{j-1}))_{ab}$ This expansion has a clear interpretation due to the delta function at the end of each power of H. The contribution from that term in the series is only non-vanishing if the starting position is related to the ending position by $m = n+\mu_1+\ldots+\mu_j$ for a combination of $\mu_i \in \pm1, \pm2, \pm3, \pm4$. If this condition is obeyed, the two sites n and m are connected by a product of link variables on the path as given by P. In Dirac space we have products of $(\mathbb{I} - \gamma_\mu)$, where we note that $(\mathbb{I} - \gamma_\mu)(\mathbb{I} + \gamma_\mu) = 0$, meaning we cannot have any back tracking paths, no paths containing 180 degree turns.

In summary, we find that the quark propagator can be expanded in powers of κ, which allows us to write it as a sum over non back tracking paths. A path of length j comes with a factor κ_j, so the leading term is the shortest path between points n and m. Higher powers of the expansion will contribute longer and longer paths connecting the two points. In partcular, in the limit of infinite quark mass, which defined the static quark potential, only the shortest possible path will contribute, i.e. the straight Wilson line.

Next, we want to expand on this slighly and expand the fermion determinant in the hopping parameter expansion.

$$\det[D] = \det[\mathbb{I} - \kappa H] = \exp(\mathrm{Tr}[\ln(\mathbb{I} - \kappa H)]) = \exp\left(-\sum_{j=1}^{\infty} \frac{1}{j}\kappa^j \mathrm{Tr}[H^j]\right) \quad (37)$$

From this expansion we see that we need to compute traces of powers of the hopping matrix. We have the general form for the jth power of the hopping matrix above, and taking the trace means that on top of tracing color and Dirac indices, we identify the spacetime arguments n and m. This means that the fermion determinant is the exponential of a sum over closed fermion loops, closed loops of link variables. For each loop the trace over color is taken, so these loops are gauge invariant, as is the determinant. A loop of length j comes with a factor κ^j/j. We also have a product of Dirac factors in the same way as before, meaning back tracking is not allowed here either.

5 Chiral symmetry

I will take the rest of this lecture to introduce some of the continuum theory that surrounds the content in the remainder of this unit, chiral symmetry. Spontaneous chiral symmetry breaking is a physically important phenomena in the continuum theory, so we have plenty motivation to try to accurately represent this phenomena on the lattice. However as we saw with Wilson fermions, to get rid of the doublers we needed to break Chiral symmetry explicitly in the Dirac operator. There is a deep reason for this, the Nielsen-Ninomiya theorem, which in turn requires us to generalize chiral symmetry on the lattice using the Ginsparg-Wilson equation. But first, we look at chiral symmetry in the continuum to get some idea of why it's so important to properly model on the lattice.

We first look at a theory with only a single flavor of fermion. The action for a massless fermion is

$$S_F[\psi, \overline{\psi}, A] = \int d^4x \overline{\psi}\gamma_\mu(\partial_\mu + iA_\mu)\psi = \int d^4x \overline{\psi} D\psi \quad (38)$$

Where D is our massless Dirac operator. We define the flavor singlet chiral transformation as

$$\psi \to \psi' = e^{i\alpha\gamma_5}\psi, \quad \overline{\psi} \to \overline{\psi}' = \overline{\psi}e^{i\alpha\gamma_5} \quad (39)$$

The Lagrangian density is invariant under this chiral transformation

$$\begin{aligned} \overline{\psi}'\gamma_\mu(\partial_\mu + iA_\mu)\psi' &= \overline{\psi}e^{i\alpha\gamma_5}\gamma_\mu(\partial_\mu + iA_\mu)e^{i\alpha\gamma_5}\psi \\ &= \overline{\psi}e^{i\alpha\gamma_5}e^{-i\alpha\gamma_5}\gamma_\mu(\partial_\mu + iA_\mu)\psi = \overline{\psi}\gamma_\mu(\partial_\mu + iA_\mu)\psi \end{aligned} \quad (40)$$

Using the fact that $\gamma_\mu, \gamma_5 = 0$. Note that a mass term breaks this symmetry, and the sign in the exponential does not change,

$$m\overline{\psi}'\psi' = m\overline{\psi}e^{i2\alpha\gamma_5}\psi \quad (41)$$

Chiral symmetry allows us to treat left handed(lh) and right handed (rh) massless fermions separately, it decouples them. We can introduce the lh and rh projection operators,

$$P_R = \frac{\mathbb{I} + \gamma_5}{2}, \quad P_L = \frac{\mathbb{I} - \gamma_5}{2} \tag{42}$$

These projection operators obey

$$\begin{gathered} P_R^2 = P_R, \quad P_L^2 = P_L, \quad P_R P_L = P_L P_R = 0, \quad P_R + P_L = \mathbb{I} \\ \gamma_\mu P_L = P_R \gamma_\mu, \quad \gamma_\mu P_R = P_L \gamma_\mu \end{gathered} \tag{43}$$

These projectors allow us to define right and left handed fermion fields

$$\psi_R = P_R \psi, \quad \psi_L = P_L \psi, \quad \overline{\psi}_R = \overline{\psi} P_L, \quad \overline{\psi}_L = \overline{\psi} P_R \tag{44}$$

with these left and right handed fermion fields we can write the lagrangian as a sum of a left handed part and a right handed part

$$\mathcal{L}[\psi, \overline{\psi}, A] = \overline{\psi}_L D \psi_L + \overline{\psi}_R D \psi_R \tag{45}$$

The left and right handed components are completely decoupled. A mass term, which breaks this flavor singlet chiral symmetry, also mixes the left and right handed components,

$$m\overline{\psi}\psi = m(\overline{\psi}_R \psi_L + \overline{\psi}_L \psi_R) \tag{46}$$

When we refer to the chiral limit, this is the limit of vanishing quark mass, since the action is only invariant under chiral symmetry with massless quarks.

The key equation for chiral symmetry is

$$\{\gamma_5, D\} = 0 \tag{47}$$

Now we generalize to N_f quark flavors. The fields now carry a flavor index, but we supress it by writing the action in matrix/vector notation

$$S_F[\psi, \overline{\psi}, A] = \int d^4x \overline{\psi}(\gamma_\mu(\partial_\mu + iA_\mu) + M)\psi \tag{48}$$

Where M is a mass matrix, which is diagonal and whose eigenvalues in flavor space are the masses of each flavor of fermion.

Before we talk about chiral transformations in this case, we note the action is invariant under vector transformations

$$\begin{gathered} \psi' = e^{i\alpha T_i}\psi, \quad \overline{\psi}' = \overline{\psi} e^{-i\alpha T_i} \\ \psi' = e^{i\alpha \mathbb{I}}\psi, \quad \overline{\psi}' = \overline{\psi} e^{-i\alpha \mathbb{I}} \end{gathered} \tag{49}$$

Where the T_i are the generators of SU(N_f). The action is clearly invariant under these vector transformations, but this symmetry also extends to the case where every flavor has the same mass. This is isospin symmetry for N_f flavors. The

U(1) vector symmetry holds for arbitrary masses as well, and its corresponding conserved quantity is baryon number.

Now for chiral or axial transformations

$$\begin{aligned}\psi' = e^{i\alpha\gamma_5 T_i}\psi, \quad \overline{\psi}' = \overline{\psi}e^{-i\alpha\gamma_5 T_i}\\ \psi' = e^{i\alpha\gamma_5 \mathbb{I}}\psi, \quad \overline{\psi}' = \overline{\psi}e^{-i\alpha\gamma_5 \mathbb{I}}\end{aligned} \tag{50}$$

Where now the left handed and right handed components of the different flavors mix. Once again, like the flavor singlet case, we find the action is invariant under axial transformations when $M = 0$.

Altogether, the massless action has the symmetry

$$\mathrm{U}(N_f)_V \otimes \mathrm{U}(N_f)_A = \mathrm{SU}(N_f)_L \otimes \mathrm{SU}(N_f)_R \otimes \mathrm{U}(1)_V \otimes \mathrm{U}(1)_A \tag{51}$$

Where we use a notation for the two factors of $\mathrm{SU}(N_f)$ to stress the fact that the left and right handed components are independently symmetric under $\mathrm{SU}(N_f)$ transformations.

In the full quantized theory, the fermion determinant is not invariant under the chiral U(1) transformations, the symmetry is explicitly broken by the fermion integration measure, which leads to the axial anomaly, that is the divergence of the flavor singlet axial current is not zero. Introducing degenerate masses reduces the total symmetry from two independent left and right handed $\mathrm{SU}(N_f)$ factors to a subgroup of vector transformations $\mathrm{SU}(N_f)_V$. If we allow for non-degenerate masses, this symmetry further breaks down into a string of N_f U(1) vector transformations.

In terms of the measured quark masses, the up and down quark are nearly degenerate, giving us an approximate vector SU(2) symmetry in the QCD action. If we include the strange quark mass, we have a much more approximate vector SU(3) flavor symmetry. On the typical QCD scale of several hundred MeV, the approximate SU(2) symmetry is quite good.

If the u and d quarks were massless, we would have an exact $\mathrm{SU}(2)_L\otimes$ $\mathrm{SU}(2)_R\otimes$ $\mathrm{U}(1)_V$ symmetry of the action. In this case we would expect the nucleon and its parity conjugate partner N^* to have the same mass. However, no such mass degeneracy is observed, the mass difference in nature is around 600 MeV, which is far too large a difference to be explained by explicit chiral symmetry breaking from the masses of the quarks, so there must be another mechanism at work. The action is invariant, so the strong chiral symmetry breaking effect must be from spontaneous breaking of chiral symmetry. The ground state is not invariant under chiral symmetry, even though the action is. An order parameter for chiral symmetry breaking is the so called chiral condensate,

$$\langle \overline{u}(x)u(x)\rangle \tag{52}$$

The chiral condensate transforms like a mass term, so it is not invariant under chiral transformations. Due to this, when the chiral condensate is non-zero, this implies that chiral symmetry is broken spontaneously.

Another result of spontaneously broken symmetry is the appearance of Nambu-Goldstone bosons by Goldstone's theorem. Since in nature the SU(2) symmetry is approximate, these bosons for spontaneous chiral SU(2) symmetry breaking are approximately massless, very light, and are referred to as pseudo Goldstone bosons. For this particular case, these are the three light pions, which have mass 140MeV.

With all of this in mind, we can begin to investigate chiral symmetry on the lattice.