Renormalization and scale setting in-class exercises

Exercise 1: To extract hadron masses in lattice QCD, we look at the two-point correlation function

$$C_H(t) = \langle \Gamma_H(t) \Gamma_H(0) \rangle, \tag{1}$$

where Γ is some local operator which couples to the hadronic state H. At very large time separation, the behavior of this correlation function is exponential decay, dominated by the lightest state which couples to the operator Γ_H:

$$C_H(t) \xrightarrow{t\to\infty} Ae^{-M_H t}. \tag{2}$$

However, since we use Monte Carlo methods, it's also important to compute the variance of this correlation function, which by definition takes the form

$$N\sigma_H^2 = \langle \Gamma_H(t)^2 \Gamma_H(0)^2 \rangle - \langle \Gamma_H(t) \Gamma_H(0) \rangle^2, \tag{3}$$

where N is our statistical sample size. The second term here is just $C_H(t)^2$, but the first term is a new two-point correlator with the *square* of the original operator at the source and sink.

a) For the pion two-point correlator, the local operator Γ_π takes the form $\bar{q}\gamma^5 q$, so that the state overlapping with $C_\pi(t)$ at long times is one propagating quark and anti-quark. When we square this operator, we get a propagating state with two quarks and two anti-quarks, so the lowest-energy state is *two* pions, with total energy $E = 2M_\pi$.

   Compute the signal-to-noise ratio $S/N = C_\pi(t)/\sigma_\pi(t)$, and show that it becomes constant in the large-t limit. (This allows us to determine pion masses to arbitrarily good precision by looking at larger t.)

b) Now consider the proton two-point correlator, where Γ_p contains three light quarks qqq. If we square this operator, the propagating state is three quarks with three anti-quarks, so the lowest-energy state is three pions.

   Again compute the signal-to-noise ratio $S/N = C_p(t)/\sigma_p(t)$. This time, show that it becomes *exponentially* small in the large-t limit, scaling as $e^{-\alpha t}$. What is the decay constant α?

Look up numbers for the proton and pion mass in the PDG (https://pdg.lbl.gov). How large can t be in femtometers (fm) before the signal-to-noise degrades by a factor of more than 10^3?

c) The Ω baryon is instead made of three strange quarks, while the squared correlator overlaps with three (fictitious) η_s mesons, which are like pions made from purely strange quarks. Using chiral perturbation theory arguments, the mass of the η_s meson can be approximated as

$$M_{\eta_s}^2 \approx M_{K^+}^2 + M_{K^0}^2 - M_\pi^2. \tag{4}$$

Go to the PDG again and re-do your calculation from the end of the last part. How much further out in t can we go in the Ω baryon two-point correlator before we find the same 10^3 signal-to-noise degradation?

(For more details here, see https://inspirehep.net/literature/287173 by G.P. Lepage and https://arxiv.org/abs/1401.3270 by R. Sommer.)

Exercise 2: In lecture, the QCD β-function was given as

$$\beta(\alpha_s) = \frac{\mu}{2}\frac{d\alpha_s}{d\mu} = -\beta_0 \alpha_s^2 - \beta_1 \alpha_s^3, \tag{5}$$

where the first two coefficients ("two loops") are given by

$$\beta_0 = \frac{1}{4\pi}\left(11 - \frac{2}{3}N_f\right), \tag{6}$$

$$\beta_1 = \frac{1}{(4\pi)^2}\left(102 - \frac{38}{3}N_f\right). \tag{7}$$

Here N_f is the number of active light quarks, where "active" (roughly) means that the quark is light compared to the renormalization scale, $m_q \lesssim \mu$.

a) Although we don't include it in lattice calculations since it is too heavy and unstable, we should include the top quark if we try to study the ultraviolet limit $\mu \to \infty$. With the top quark included, the Standard Model has $N_f = 6$. Is asymptotic freedom maintained? If not, how many quarks would there have to be in nature for QCD to no longer be asymptotically free?

b) Using your favorite numerical tool, solve for and plot $\alpha(\mu)$, first using only the first term in $\beta(\alpha)$ and setting $\beta_1 = 0$ (this is the "one-loop" result), and then keeping both terms (the "two-loop" result.) The strong coupling is often given in terms of its value at the Z-boson pole,

$$\alpha_s(M_Z) = 0.1180(9), \tag{8}$$

where $M_Z = 91.2$ GeV; use this as a common initial condition for your one-loop and two-loop solutions. How close are the two solutions in predicting $\alpha_s(m_b)$ at the scale of the bottom quark ($m_b \approx 4.2$ GeV)? What about at the charm mass ($m_c \approx 1.3$ GeV)? What does this tell you about where in terms of μ perturbation theory (i.e. series expansion in $\alpha_s(\mu)$) will be reliable?

(If you're feeling very ambitious, try to account for changes in N_f in your numerical solution as you cross the μ thresholds corresponding to quark masses; removing one quark from the theoretical calculation as we move below its mass threshold and it is no longer active is known as *decoupling*.)