# Renormalization and Scale Setting

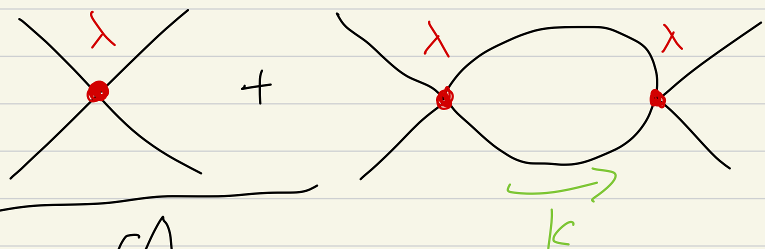
**Renormalization/RG**: bare parameters $\to$ physics!

$N$ couplings $\to$ $N$ experiments to fix theory.

"Scale setting": choice of which inputs to use.

**Continuum ex:** $\phi^4$ theory, one-loop.

Scattering: $\phi\phi \to \phi\phi$

$i\mathcal{M} =$ 



$$\sim \int^{\Lambda} d^4k \, \frac{1}{k^2} \cdot \frac{1}{(p+k)^2}$$

**Log divergent**: $-\mathcal{M} = \lambda + \frac{\lambda^2}{32\pi^2} \log \frac{s}{\Lambda^2}.$

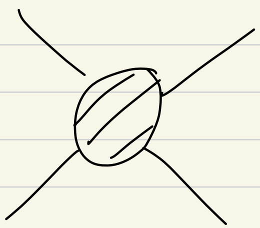
$\Lambda \to \infty$; $\infty$ amplitude! Is the theory "sick"?

($s$: Mandelstam $E_{CM}^2$)

Theory is okay! Only cross-section differences.

$$\mathcal{M}(s_1) - \mathcal{M}(s_2) \sim \left(\log\frac{s_1}{\Lambda^2} - \log\frac{s_2}{\Lambda^2}\right) = \log\frac{s_1}{s_2}$$

Define renormalized coupling at ref. energy $s_0$.



$$\Big|_{s=s_0} = -i\lambda_R \Rightarrow \lambda_R = \lambda + \frac{\lambda^2}{32\pi^2}\log\frac{s_0}{\Lambda^2} + \dots$$

Scattering in terms of $\lambda_R$ is finite:

$$\mathcal{M}(s) = -\lambda_R - \frac{\lambda_R^2}{32\pi^2}\log\frac{s}{s_0} + \dots$$

Definition of $\lambda_R$ is our renormalization condition.

No divergences!*

*(except as $s_0 \to \infty$. No continuum limit on lattice!)

If we change $s_0 \to s_0'$, $\lambda_R$ changes too.

calculate dependence $\to$ renormalization group.

$$\frac{d\lambda_R}{ds_0} = \frac{1}{s_0}\frac{\lambda^2}{32\pi^2}.$$

Define the beta function:

$$\beta(\lambda_R) \equiv s_0 \frac{d\lambda_R}{ds_0} = +(\dots)\lambda_R^2$$


Arrows show UV flow (to $s_0 \to \infty$.)

Opposite is IR flow (to $s_0 \to 0$.) ($\lambda_R \to 0$.)

"Continuum limit" $s_0 \to \infty$ flows into $\lambda_R \to \infty$ (perturbative control lost.)

Alternatively, hold $\lambda_R(s_0)$ fixed, look at $\lambda_R(s_1)$ w/ $s_1 \ll s_0$. then $\boxed{\lambda_R(s_1) \to 0}$ "triviality".

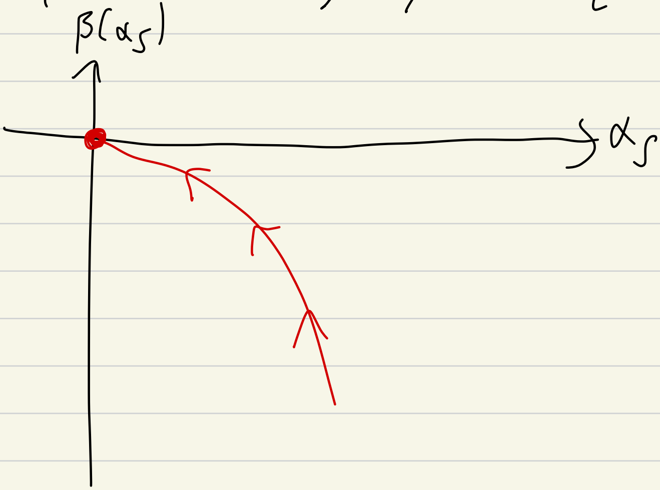
gauge coupling

For QCD first term in $\beta(\alpha_s)$ $(\alpha_s \equiv g_s^2/4\pi)$ has crucial minus sign.

(Gross, Wilczek '73; Politzer '73)

$$\Lambda \frac{d\alpha_s}{d\Lambda} \equiv \beta(\alpha_s) = -\frac{1}{2\pi}\left(11 - \frac{2}{3}N_f\right)\alpha_s^2 + \ldots$$

($N_f$ = # of light, active quarks.)




(UV flow: $\Lambda \to \infty$)

asymptotic freedom

($\Lambda \to \infty$ is $\alpha_s \to 0$.)

In QCD, $\alpha_s$ is inconvenient to renormalize the theory (we can't scatter quarks/gluons)

Instead, rely on dimensional transmutation

$$\boxed{\alpha_s(\Lambda_0) = \alpha_{ref} \longleftrightarrow M_H = C_{ref}\Lambda_0}$$

(roughly.)

(R. Sommer, 1401.3270)

## Scale setting – focus on QCD.

Gauge coupling $\alpha_s$ + quark masses $m_u, m_d, m_s, m_c, m_b$.

(Top quark is too heavy/unstable; $\theta = 0$.)

Six experimental inputs (w/ dependence on parameters; proton mass $\not\to m_b$.)

can be any six processes ($\pi\pi \to \pi\pi$ scattering?)

We want highest precision → hadron masses
→ decay constants

Quark masses ⟷ direct meson ($\bar{q}q$) masses.

Heavy quarks:
$$\begin{cases} M_{D_s} \sim m_c & (\bar{c}s) \\ M_{B_s} \sim m_b & (\bar{b}s) \end{cases}$$

Light quarks: chiral perturbation theory

(chiral symmetry restored at $m_q = 0$ ...)

Implies the Gell-Mann-Oakes-Renner relation:

(GMOR): $$M^2_{q_1 q_2} = B(m_{q_1} + m_{q_2})$$
(pseudoscalar)

Works well for light, strange. Pions, kaons!

$$\begin{cases} \pi^+ \sim \bar{d}u & K^+ \sim \bar{s}u \\ \pi^0 \sim (\bar{u}u + \bar{d}d) & K^0 \sim \bar{s}d \end{cases}$$

(Flavor Lattice Averaging Group)

FLAG 2024 $\longrightarrow$ "Edinburgh consensus" (no QED)

$$\begin{cases} M_{\pi^+} = 135.0 \text{ MeV} \\ M_{K^+} = 491.6 \text{ MeV} \\ M_{K^0} = 497.6 \text{ MeV} \\ M_{D_s^+} = 1967 \text{ MeV} \\ M_{B_s^0} = 5367 \text{ MeV} \end{cases}$$

One more input to fix overall scale.

Simulation produces $aM_{\pi^+}$; need to set $a$ to fix $M_{\pi^+} = 135$ MeV!

Three classes of scale setting:

A) Hadron masses;

B) Decay constants;

C) Theory scales.

Precision is essential! $S$ to set scale.

$(aM_H)$ from calculation, precision $\delta(aM_H)$.

Then: $$M_H = \frac{\cancel{a} M_H}{\cancel{a} S} \cdot S_{exp}$$ Frac error:

$$\frac{\delta M_H}{M_H} = \sqrt{\left(\frac{\delta(aM_H)}{aM_H}\right)^2 + \left(\frac{\delta(aS)}{aS}\right)^2 + \left(\frac{\delta S_{exp}}{S_{exp}}\right)^2}$$

lattice $aM_H$ error — lattice $aS$ error — experimental.

(https://pdg.lbl.gov)

$\delta S_{exp}/S_{exp}$ provides ultimate limit on error!

$\delta(aS)/aS$ also important.

A) Hadron masses - what are good options?

- Proton: $\frac{\delta m_p^{exp}}{m_p} \sim 3\times10^{-8}$

But: exponential signal-to-noise degradation (see exercises)

- $\pi, K$: very precise, but we used them!

- $\eta$: also very precise, hard on lattice (disconnected diagrams...)

- $\Omega$ baryon: $\frac{\delta m_\Omega^{exp}}{m_\Omega^{exp}} \sim 0.02\%$

(sss)

Also exp. S/N problem, better than proton.

Lattice calcs ~ sub-percent $aM_\Omega$.

B) Decay constants – more details next time.

$$\Gamma(\pi \to \mu\nu) \sim f_\pi^2 ; \quad \frac{\delta\Gamma^{exp}}{\Gamma^{exp}} \sim 10^{-3}$$

But disentangle EW physics. Systematics?
(f$_\pi$ still commonly used.)

C) Theory scales: $S_{exp}$ vs. $aS$ precision;

(systematic effects, N/S, or expensive.)

Theory scale $1/T$ can be a proxy.

Easy to compute $a/T$ precisely; relate to some physical $S_{exp}$ once.

$$\lim_{a \to 0} \left( \frac{aS}{a/T} \right) = \boxed{TS} ; \quad S_{exp} \to T_{ref}.$$

Future calculations use $TM_H$ vs. $\left(\frac{aM_H}{aS}\right) S_{exp}$.

Static potential: $r_0$, $r_1$

Gradient flow: $\sqrt{t_0}$, $w_0$

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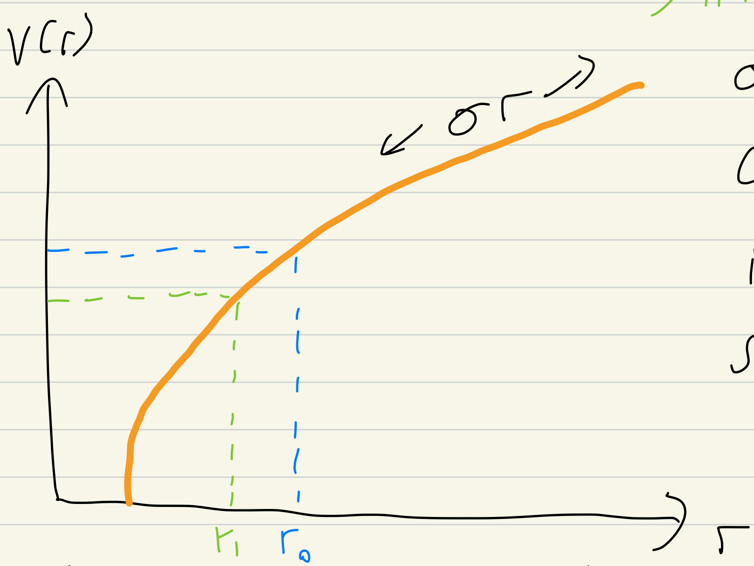
Static potential → $r_0$, $r_1$ "Sommer parameters"

$$V(r) = A - \frac{\alpha}{r} + \sigma r$$ → linear - confinement!




$\sigma$: string tension

Can give overall scale, used in pure-gauge.

Subject to:
- Volume sensitivity
- String breaking

Avoid by going to "knee" region.

$$F(r) = -\frac{dV}{dr} = -\frac{\alpha}{r^2} - \sigma\,, \text{ then:}$$

(FLAG'24, 2+1+1 = $N_f$) (1.6%)

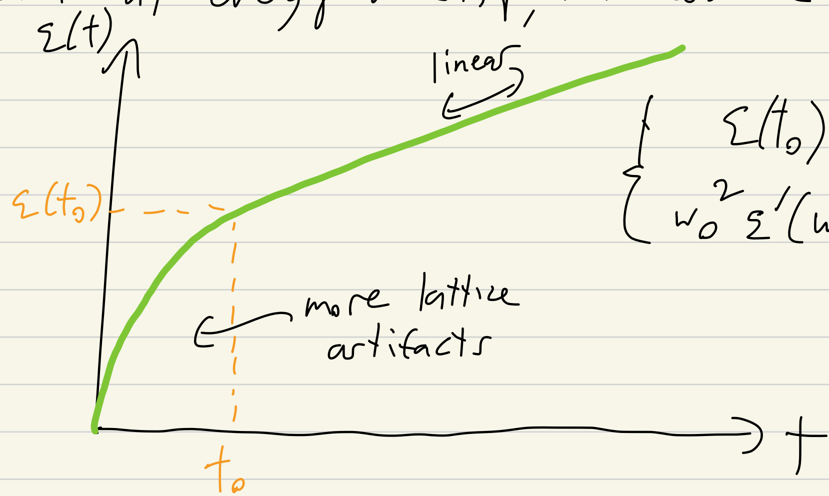
$$r_0^2 |F(r_0)| = 1.65 \Rightarrow r_0 = 0.4580(73)\text{ fm}$$

$r_1^2 |F(r_1)| = 1 \Rightarrow r_1 = 0.3068(37)$ fm
(1.2%)

Gradient flow $\rightarrow \sqrt{t_0}$, $w_0$
(M. Lüscher, 1006.4518)

More abstract; gradient flow $\rightarrow$ "smeared out" gauge fields, radius $r_{smear} \approx \sqrt{8t}$ in "flow time" $t$.

Look at energy density, dimless: $\mathcal{E}(t) \equiv t^2 \langle E(t) \rangle$




$$\begin{cases} \mathcal{E}(t_0) \equiv 0.3 \\ w_0^2 \mathcal{E}'(w_0^2) \equiv 0.3 \end{cases}$$

(FLAG '24, 2+1+1:)

$$\begin{cases} \sqrt{t_0} = 0.14292(104) \text{ fm} & (0.7\%) \\ w_0 = 0.17256(103) \text{ fm} & (0.6\%) \end{cases}$$

Current "gold standard" theory scales!

## perturbative renormalization

Why PT picture? Adding electroweak!

We don't usually include weak, QED, leptons... in our simulations. (Scale separation!)

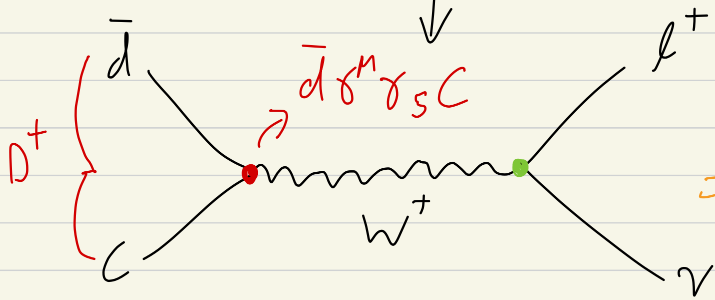
Rely on factorization! in strong + EW process, we can separate out QCD matrix elements.

$(c\bar{d})$

ex: $D^+ \longrightarrow \ell^+\nu.$ $\quad (\ell = e, \mu.)$

$$\boxed{\langle \ell^+\nu|\mathcal{H}_{SM}|D^+\rangle}$$

$$\approx \langle \ell^+\nu|\mathcal{H}_{EW}|c\bar{d}\rangle \cdot \langle c\bar{d}|\mathcal{H}_{QCD}|D^+\rangle$$




QCD vacuum

$$= \langle 0|\bar{d}\gamma^\mu\gamma_5 c|D^+\rangle$$

$$\Rightarrow \mathcal{M} \sim \frac{g^2}{M_W^2}\langle 0|\bar{d}\gamma^\mu\gamma_5 c|D^+\rangle.$$

(kinematics)

The QCD ME defines a decay constant:

$$\langle 0 | \bar{d} \gamma^\mu \gamma_5 c | D^+(p) \rangle = i p^\mu f_D$$

Calculate on lattice: define currents

$$\begin{cases} A_\mu = \bar{d} \gamma_5 \gamma^\mu c \\ P = \bar{d} \gamma_5 c \end{cases} \qquad \langle 0 | A_\mu \quad P | 0 \rangle$$

Compute:

$$C(t) = \sum_x \langle 0 | A_\mu(x,t) P(0,0) | 0 \rangle$$

$$\xrightarrow[t \to \infty]{} \frac{e^{-m_D T}}{2 m_D} \langle 0 | A_\mu | D^+ \rangle \langle D^+ | P | 0 \rangle$$

$\sim f_D$

Add correlator $\langle PP \rangle$ to determine $\langle D^+ | P | 0 \rangle$.

But, we need one more piece.

EW part of decay is perturbative.

Calculated w/ e.g. $\overline{MS}$ renormalization; depends on RG scale, $\mu$.

Must account for RG dependence ($\mu$) in $f_D$ calculation $\longrightarrow$ matching.

Consider bare operator $\mathcal{O}_0$.

Continuum: $\mathcal{O}^{\overline{MS}}(\mu) = Z_C(\mu) \cdot \mathcal{O}_0$

Lattice: $\mathcal{O}^{lat}(1/a) = Z_L(1/a) \cdot \mathcal{O}_0$

Either regulator handles UV divergences. But details are different. Consider matrix elements:

$$\langle \alpha | \mathcal{O}_0 | \beta \rangle = \langle \alpha | \mathcal{O}_0 | \beta \rangle$$

$$\Rightarrow \boxed{\langle \alpha | \mathcal{O}^{\overline{MS}}(\mu) | \beta \rangle = \frac{Z_C(\mu)}{Z_L(1/a)} \cdot \langle \alpha | \mathcal{O}^{lat}(1/a) | \beta \rangle}$$

"matching factors"

$\mu$-dependent – always partly perturbative. ($Z_C(\mu)$)

We can get $Z_L(1/a)$ using PT or not.

PT matching factors are widely available in literature.

Focus on methods, tricks, phenomena.

## How to get Z-factors:

Back to example: $D^+ \to \ell^+ \nu$.

$$\langle 0 | \bar{d} \gamma_5 \gamma^\mu c | D^+(p) \rangle = i p^\mu f_D$$

want $\overline{MS}$ renormalized ME, Lattice current $A^\mu$:

$\overline{MS}$ current.

$$\langle 0 | A^\mu | D^+ \rangle = \langle 0 | Z_A A^\mu | D^+ \rangle$$

matching factor.

Axial current satisfies partial conservation:

"PCAC"

$$\boxed{\langle \partial_\mu A^\mu \mathcal{O} \rangle = 2m \langle P \mathcal{O} \rangle + \mathcal{O}(a)}$$

Renormalize: $Z_A \langle \partial_\mu A^\mu \mathcal{O} \rangle = 2m Z_m(\mu) Z_P(\mu) \langle P \mathcal{O} \rangle$

Scale-dependence cancels on RHS:

$$Z_m(\mu) Z_P(\mu) = 1$$

$\langle 0 | A^\mu P | 0 \rangle$

Choose zero component:

$$\langle 0 | Z_A A^0 | D^+(p) \rangle = M_{D^+} f_{D^+}$$

Use PCAC:

$$2m \langle 0 | P | D^+(p) \rangle = (p_\mu p^\mu) f_{D^+} = M_{D^+}^2 f_{D^+}$$

$$Z_A \langle 0 | A^0 | D^+ \rangle = \frac{2m}{M_{D^+}} \langle 0 | P | D^+ \rangle$$

$$\Rightarrow \boxed{Z_A = \frac{2m \langle 0 | P | D^+ \rangle}{\langle 0 | A^0 | D^+ \rangle}}$$

"ratio method"

Other "ratio tricks" can be useful, e.g., for Wilson fermions,

Two vector currents. Simple, local current: (two-flavor)

$$V_\mu^a(x) = \bar{\psi}(x) \frac{\tau^a}{2} \gamma_\mu \psi(x)$$

Conserved isovector current:

$$V^a_{\mu,c}(x) = \frac{1}{2}\Big[\bar{\psi}(x)\,\frac{\tau^a}{2}(\gamma_\mu - 1)\,U(x,\mu)\,\psi(x+a\hat{\mu}) + \bar{\psi}(x+a\hat{\mu})\,\frac{\tau^a}{2}(\gamma_\mu + 1)\,U^\dagger(x,\mu)\,\psi(x)\Big]$$

"Point-split", but <u>doesn't renormalize</u>: $\boxed{Z_{V,c} = 1.}$

Simple current <u>does</u> renormalize:

$$Z_V = 1 + \mathcal{O}(a).$$

<u>Ratio trick</u>: $\boxed{Z_V = \frac{\langle H | V_\mu^c | H \rangle}{\langle H | V_\mu | H \rangle}}$

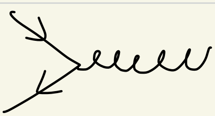
compute once, work w/ simpler local current.

<u>Another approach</u>: <u>lattice perturbation theory</u>.

Calculations are available for most things; just show interesting features.

<u>Quark-gluon vertex</u>: in continuum QCD,

$$\mathcal{L} \supset \bar{\psi} i D^\mu \psi \sim g_s \bar{\psi} A^\mu_a T^a \psi$$

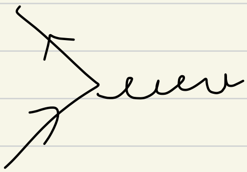
 $\sim g_s$.

$\exp(igA_\mu^a T^a)$

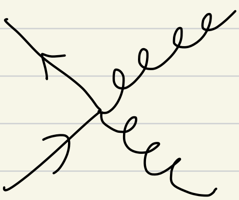
On the lattice:

$$\mathcal{L} \supset \bar\psi(x)\,\Gamma\dots U_\mu(x+y)\dots\psi(x+y)$$

$$\Rightarrow \bar\psi A_\mu \psi + \bar\psi A_\mu A_\nu \psi + \dots$$



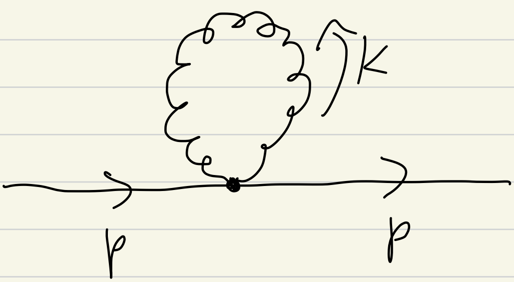
$\sim g$ ← continuum vertex, $a \to 0$.



$\sim g^2 a$ ← lattice artifact vertex.

Gives a lattice-only contribution to quark prop:

"tadpole diagram"




$$\sim (g^2 a)\int^{\pi/a} d^4k \cdot \frac{1}{k^2}$$

$$\sim (g^2 a)\int^{\pi/a} k\,dk \sim \frac{1}{a}.$$

UV divergence! Not a disaster (we have cutoff)

$$\delta m_0 \sim \frac{1}{a} \Rightarrow \delta(am_0) \sim a^0.$$

$\Rightarrow$ additive renormalization of quark mass.

$(am_0 = 0 \not\Rightarrow m_R = 0.)$

Need to deal with this...

- Chiral symmetry - prevents this contribution (for mass.) Staggered, overlay, DWF.
- Calculate $m_R$, shift bare mass $am_0$ as needed. (Wilson.)

For Wilson, axial Ward identity (AWI) helps:

$$\langle \partial_\mu A^\mu \mathcal{O} \rangle = 2m \langle P \mathcal{O} \rangle$$

$$\Rightarrow \boxed{m_{AWI} \equiv \frac{\langle 0 | \partial_\mu A^\mu | \pi \rangle}{2 \langle 0 | P | \pi \rangle}}$$

Caveat: need $Z_A$, $Z_P$; currents must be renormalized.

But, those are multiplicative. Guaranteed

$$\boxed{m_R \propto m_{AWI}.} \qquad m_{AWI} = 0 \to m_R = 0.$$

For other quantities in lattice PT, "tadpoles" (  ) lead to strong cutoff dependence.

Don't spoil $a \to 0$, but "uncomfortably large"

Problematic modes are gluon high modes (UV)

Idea: (Lepage + Mackenzie) $\Rightarrow$ split apart.

$$U_\mu(x) = e^{iagA_\mu(x)} = 1 + iagA_\mu(x) - a^2 g^2 A_\mu A_\nu + \dots$$

$$\boxed{\text{(integrate)}} \longrightarrow \; = u_0\left(1 + iagA_\mu(x) + \dots\right)$$

Rescale out "mean-field value":

$$\tilde{U}_\mu(x) = U_\mu(x) / u_0 . \qquad (0 < u_0 < 1)$$

Compute on lattice:

$$u_0 = \left\langle \frac{1}{N_c} \, \mathrm{Re}\, \mathrm{Tr}\, U_\mu(x) \right\rangle$$

$$u_0 \simeq u_p = \left\langle \frac{1}{N_c} \mathrm{Re\,Tr}\, P_{\mu\nu}(x) \right\rangle^{1/4}$$

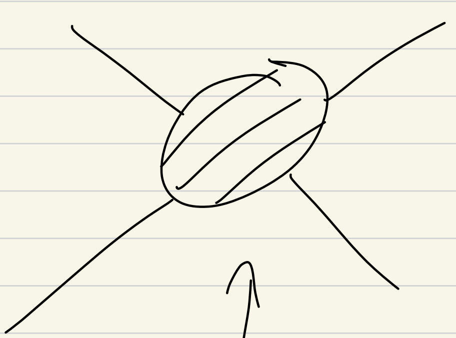
Calculate and rescale coupling for PT:

$$\boxed{\tilde{g}_0^2 = \frac{g_0^2}{u_0^4}}$$

Better convergent PT!

## "Non-perturbative" renormalization

RI-MOM scheme. Similar to perturbative condition, define renorm. using vertex function.



$$\Big|_{s=s_0} = -i\lambda.$$

momentum subtraction.

This is a $\boxed{\text{MOM}}$ scheme

"Full" vertex function, $\Gamma_4(s)$.

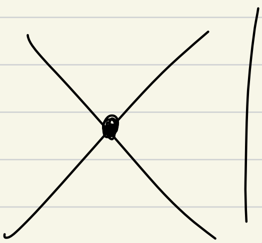
In terms of $Z$-factors, bare theory:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_0)^2 - \frac{\lambda_0}{4!}\phi_0^4$$

$$\to \mathcal{L} = \frac{1}{2} Z_\phi (\partial_\mu \phi)^2 - Z_\phi^2 Z_\lambda \lambda \frac{\phi^4}{4!}$$

$$Z_\phi \phi = \phi_0 \, , \quad Z_\lambda \lambda = \lambda_0 .$$

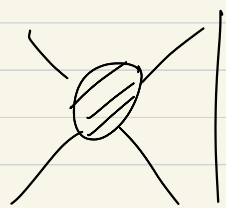
Compare to bare vertex function,



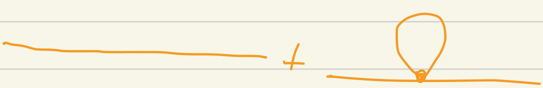
$$= -i\lambda_0 = -iZ_\lambda \lambda$$

$s=s_0$

$$= Z_\lambda Z_\phi^2 \cdot$$

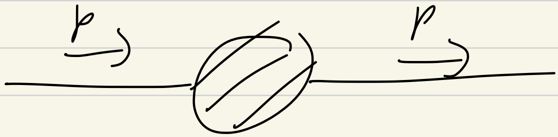


$s=s_0$



(need to include $Z_\phi$ as well; set by a separate renorm. condition.)

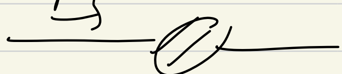
Need field renorm; from propagator.

$S(p) =$ 

$\sim Z_\phi p^2 + (\dots) m^2 \to 0$

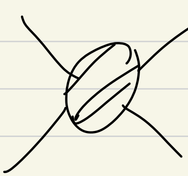
In continuum, $\boxed{Z_\phi = \dfrac{dS(p)}{dp^2}}$.

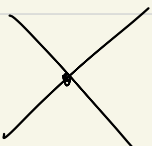
Formally, to renormalize $\lambda$ in continuum $\phi^4$:

1) Compute propagator:  $= S(p)$

$$\Rightarrow Z_\phi = \frac{d}{dp^2} S(p).$$

2) Compute vertex functions:

 $\Big|_{p^2} = -i\Gamma(p)$

 $\Big|_{p^2} = -i\Gamma_0(p)$

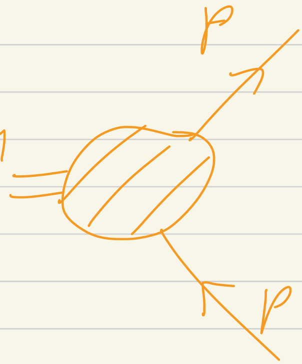
3) $\boxed{Z_\chi^{-1}(p) = Z_\phi^2 \cdot \Gamma(p) \cdot \Gamma_0^{-1}(p)}$

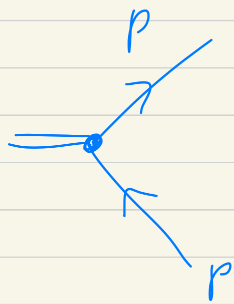
To lattice QCD: $\mathcal{O}_\Gamma = \bar{\psi}\, \Gamma\, \psi \Rightarrow Z_\Gamma$

spin matrix   quark

vertex functions:

quark state

$\langle p | \mathcal{O}_\Gamma | p \rangle = \Gamma$ 


$\langle p | \mathcal{O}_\Gamma | p \rangle_0 = \Gamma$ 


(note: these are amputated diagrams, vs. what we compute on lattice)

$\psi_0 = \sqrt{Z_\psi}\,\psi$

Bare vs. renormalized:

$$\langle p|\mathcal{O}_\Gamma|p\rangle_0 = Z_\Gamma^{RI}(\mu)\, Z_\psi^{RI}(\mu) \langle p|\mathcal{O}_\Gamma|p\rangle.$$

RG scale, $\boxed{\mu^2 \equiv p^2}$ in RI/MOM.

Complication: matrix elements are matrices in spin/color. Worry about gauge dependence!

$\Rightarrow$ fix gauge: typical choice is Landau gauge,

$$\partial_\mu A^\mu = 0 \Rightarrow \max_g \left[ \sum_\mu \mathrm{Re}\,\mathrm{Tr}\left(g U_\mu g^\dagger\right) \right]$$

(see hep-lat/0309184.)

Then, Z-factor from vertex functions,

$$\frac{1}{Z_\Gamma^{RI}(\mu)} = \frac{1}{4N_c} \mathrm{Tr}\left[ \langle p|\mathcal{O}_\Gamma|p\rangle \cdot \langle p|\mathcal{O}_\Gamma|p\rangle_0^{-1} \right]$$

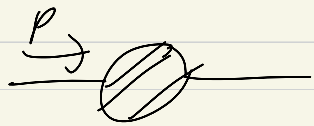
$$= \frac{1}{4N_c} Z_\psi^{RI}(\mu) \, \mathrm{Tr}\left[ \Lambda_\Gamma(p) \cdot \Gamma_0^{-1} \right]$$

where we define amputated vertex function,

$$\Lambda_\Gamma(p) \equiv S^{-1}(p) \cdot G_\Gamma(p) \cdot S^{-1}(p)$$

$$G_\Gamma(x,y) = \langle \bar{\psi}(x) \, O_\Gamma(0) \, \psi(y) \rangle$$

$\Rightarrow G_\Gamma(p)$ via Fourier transform.

$S(p)$ is the quark propagator, 

Still need field renorm. $Z_\psi(\mu)$; couple of options.

1) Look at propagator:

$$Z_\psi^{RI}(\mu) = \frac{1}{4N_c} \mathrm{Tr}\left[ S(p) S_0^{-1}(p) \right]$$

free propagator. (set $U$ to 1.)

2) Use a conserved current: e.g. if $Z_V = 1$,

$$1 = \frac{1}{Z_V^{RI}(\mu)} = \frac{1}{4N_c} \cdot Z_\psi^{RI}(\mu) \cdot \mathrm{Tr}\left[\Lambda_V(p) \cdot \Gamma_V^{-1}(p)\right]$$

<u>Almost done!</u> "RI" = "regularization independent."

Same perturbation theory as continuum w/ different regulator. (But, not same as $\overline{MS}$.)

Use continuum-only PT to do matching.

$$Z_\Gamma(\mu, a) = Z_\Gamma^{RI}(\mu, a) \cdot \frac{Z_\Gamma^{\overline{MS}}(\mu)}{Z_\Gamma^{RI}(\mu)}$$

lattice, NP

continuum, PT,

Practicalities: main issue in practice w/ RI/MOM is choosing $p^2$ $(=\mu^2)$

Two competing effects:

1) $p$ too large (near $\frac{\pi}{a}$) $\Longrightarrow$ lattice artifacts!

2) $p$ too small → finite-volume (usually OK...)

$\Longrightarrow$ $\boxed{\text{PT converges too slowly!}}$

As usual, look for plateau of stability; most operators have some mild $p$-dependence.

Final technical note: $\boxed{\text{RI/SMOM}}$ is a modification w/ better convergence for some $Z_P$.

"Symmetrization" of momentum in vertex defs.

(arXiv:0901.2599)

Other NP renormalization schemes:

- Schrödinger functional (BG field)
- Gradient flow (very new, work TBD for Z-factor computation)

(see e.g. 2311.16799
2201.09740.)

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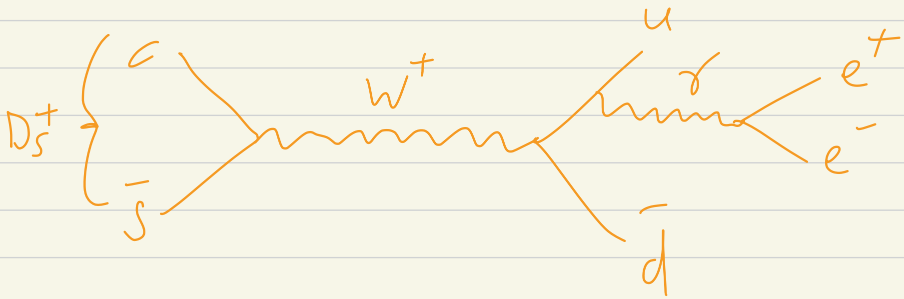
Look at another example → challenges.

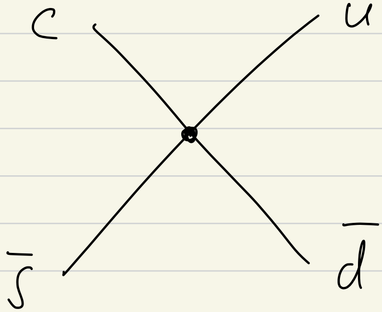
<u>ex:</u> $D_s^+ \to \pi^+ e^+ e^-$




<u>Rare decay</u> ($m_e$ suppression), from factorization

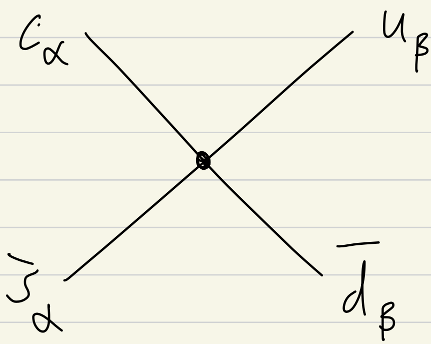
$$\langle \pi^+ | (c\bar{s})_L (u\bar{d})_L | D^+ \rangle \Big|_{\mathcal{H}_{QCD}}$$

Ignoring color, one 4-fermi operator:




$\sim \mathcal{O}$

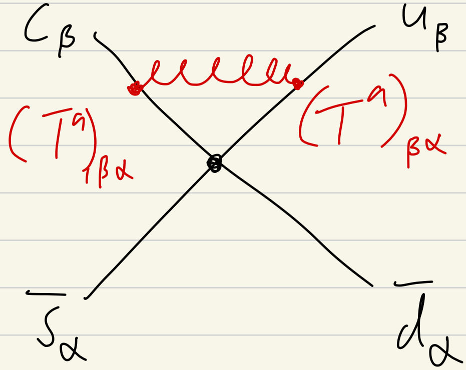
With color, there are two:




$\sim \mathcal{O}_2$

In QCD, $\mathcal{O}_1 \leftrightarrow \mathcal{O}_2$.

Must include both!




$\sim \mathcal{O}_1$

Perturbative renorm:

$$\mathcal{H}_{EW} = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} \left[ C_1(\mu) \mathcal{O}_1 + C_2(\mu) \mathcal{O}_2 \right]$$

These operators have $\boxed{\underline{\text{anomalous}}\ \underline{\text{dimensions}}}$:

$$\frac{d C_\pm(\mu)}{d \ln \mu} = \gamma_\pm C_\pm(\mu)$$

$$\left( \mathcal{O}_\pm = \frac{1}{2} \left( \mathcal{O}_2 \pm \mathcal{O}_1 \right) \right)$$

Calculate in PT, $\gamma_\pm$ have different signs:

$$\boxed{\begin{aligned} C_+(\mu) &\sim \alpha_s(\mu)^{-6/23} ; \\ C_-(\mu) &\sim \alpha_s(\mu)^{12/23} . \end{aligned}}$$

$\Rightarrow$ As $\mu$ decreases ($\alpha_s(\mu)$ increases), $\mathcal{O}_-$ will dominate. (But, perturbative...)

Slight complication... but, on lattice operators of different dimension can mix!

ex: isovector axial current for Wilson fermions.

$$A_\mu^a = \bar{\psi}(x) \gamma^\mu \gamma_5 \frac{\tau^a}{2} \psi(x)$$

vs. continuum current, extra terms appear...

$$A_\mu^a = (A_\mu^a)_{cont.} + c \frac{a}{2} (\overrightarrow{\partial}_\mu + \overleftarrow{\partial}_\mu) P^a + \ldots$$

$$(P^a = \bar{\psi}(x) \gamma_5 \frac{\tau^a}{2} \psi(x).)$$

Mixing w/ higher-dim operator: lattice artifact!

Remove by taking $a \to 0$.

Improve extrapolation w/ perturbative subtraction:

$$(A_\mu^a)_{imp} \equiv A_\mu^a + c_A \frac{a}{2} (\overrightarrow{\partial}_\mu + \overleftarrow{\partial}_\mu) P^a$$

w/ $c_A$ cancelling.

However mixing can also happen w/ lower-dim operators.

DeGrand + DeTar (Ch 16) talk about this effect for $K\bar{K}$ mixing; artifact operators that grow w/ $1/a$.

Must remove (e.g. chiral symmetry).

Another famous example is trace anomaly:

$$O_G = \langle G_{\mu\nu} G^{\mu\nu} \rangle$$ (gluon condensate.)

On lattice, access from plaquette $P$.

(1807.09518)

$$\langle P \rangle = a^4 C_G \langle G_{\mu\nu} G^{\mu\nu} \rangle + C_1 \langle 1 \rangle$$

mixes w/ identity, 4 dim. lower!

$\Rightarrow$ $a^{-4}$ UV divergence, unless first term can be subtracted.

Finite T: cancel vs. $T=0$.

Zero T: some approaches w/ gradient flow.