YM partition function and correlation function

- The path integral formalism is similar to scalar fields

$$
\begin{aligned}
Z &= \int \mathcal{D}U \exp\left[-S_G[U]\right] \\
\langle \Omega | \mathcal{O}_a \mathcal{O}_b | \Omega \rangle &= \frac{1}{Z} \int \mathcal{D}U \mathcal{O}_a[U] \mathcal{O}_b[U] e^{-S_G[U]}
\end{aligned}
\tag{1}
$$

where the integration measure is

$$
\mathcal{D}U = \prod_{n \in \Lambda} \prod_{\mu=1}^{D} \mathrm{d}U_\mu(n) \tag{2}
$$

- Since $U_\mu(n) = e^{iaA_\mu(n)}$ is a group valued object, the measure $\mathrm{d}U_\mu(n)$ involves integration over all elements of the continuous compact Lie group G
  - Since Z and $S_G[U]$ are gauge invariant, then the measure must be invariant under gauge transformations: the Haar measure which is defined by requiring that for $U, V \in G$

$$
\underbrace{\mathrm{d}U = \mathrm{d}(VU) = \mathrm{d}(UV)}_{\text{L, R invariance}}, \quad \underbrace{\int \mathrm{d}U(\alpha f(U) + \beta g(U)) = \alpha \int \mathrm{d}U f(U) + \beta \int \mathrm{d}U g(U)}_{\text{linearity}}, \quad \text{and} \underbrace{\int \mathrm{d}U = 1}_{\text{normalization}} \tag{3}
$$

  - Concretely for $U = U(\omega) = \exp(i\omega_a T_a)$, the metric in G is $\mathrm{d}s^2 = g(\omega)_{mn} \mathrm{d}\omega_n \mathrm{d}\omega_m$, where

$$
g(w)_{mn} = \mathrm{tr}\left[\frac{\partial U(\omega)}{\partial \omega_n} \frac{\partial U(\omega)^\dagger}{\partial \omega_m}\right], \tag{4}
$$

and the measure is

$$
\mathrm{d}U = c\sqrt{\det g(\omega)} \prod_a \mathrm{d}\omega_a \quad \text{with constant } c \text{ such that } \int \mathrm{d}U = 1. \tag{5}
$$

For the case of $SU(N)$, see J Broznan Phys Rev D 38 (1988)1994 for explicit constructions.

- The group integration forces the expectation of gauge variant quantities to vanish (Elitzur's theorem, see Elitzur 1975); that is

$$
\langle \mathcal{O} \rangle = 0 \quad \forall \mathcal{O} \underset{G}{\to} \mathcal{O}' \neq \mathcal{O}. \tag{6}
$$

  - To see this, consider a single group integration over a group element $U \in G$:

$$
\int \mathrm{d}U U = \int \mathrm{D}(UV) U = \int \mathrm{d}U U V^\dagger = \int \mathrm{d}(W^\dagger U') U' V^\dagger = \int \mathrm{d}U'' W U'' V^\dagger = W \left[\int \mathrm{d}U U\right] V^\dagger \tag{7}
$$

which holds for all W and $V^\dagger$, implying $\int \mathrm{d}UU = 0$.
  - Note that any group integral $\int dg f(g) = 0$ unless $f(g)$ has a singlet component
  - Now consider a gauge variant quantity such as a single gauge link (or a product of gauge links)

$$
\begin{aligned}
\langle U_\mu(n) \rangle &= \frac{1}{Z} \int \mathcal{D}U e^{-S_G[U]} U_\mu(n) \\
&= \frac{1}{Z} \int D\hat{U} e^{-\hat{S}[\hat{U}]} \prod_{\nu=\pm 1}^{\pm d} \mathrm{d}U_\nu(n) e^{-\tilde{S}\left[\hat{U}, \{U_\nu(n)\}\right]} U_\mu(n),
\end{aligned}
\tag{8}
$$

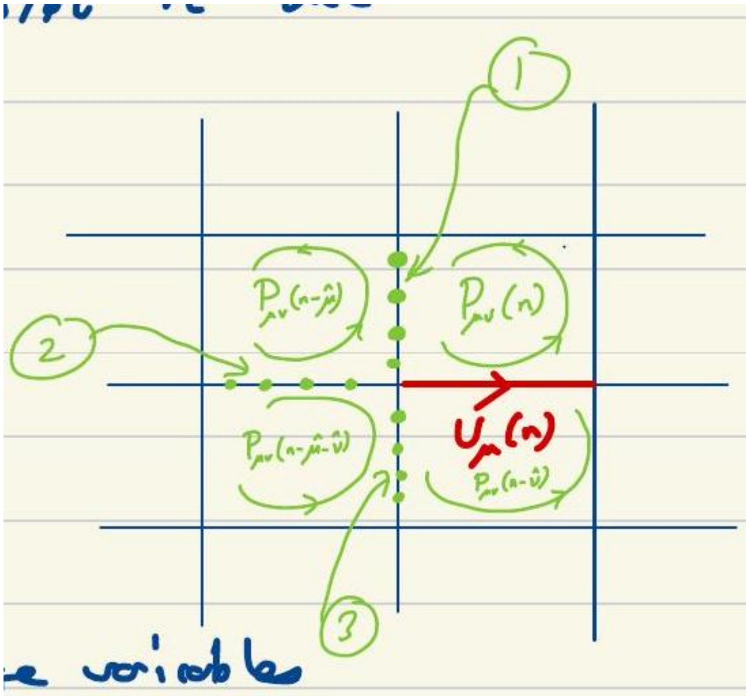

Figure 1: An illustration of the links that we will use in our various changes of variables, as well as their associated plaquettes.

where $\mathcal{D}\hat{U}$ is the measure for all links not connected to site n and

$$S_G[U] = \hat{S}[\hat{U}] + \underbrace{\tilde{S}[\hat{U}, \{U_\nu(n)\}]}_{\text{all terms involving links at site } n}$$

There are $2(D-1)$ plaquettes directly involving $U_\mu(n)$:

$$P_{\mu\nu}(n), P_{\mu\nu}(n-\hat{\nu}) \quad \forall \nu \neq \mu \tag{9}$$

For

$$P_{\mu\nu}(n) = U_\mu(n)U_\nu(n+\hat{\mu})U_\mu^\dagger(n+\hat{v})U_\nu^\dagger(n) \tag{10}$$

We can now use the invariance of the Haar measure to change variables for the particular link:

$$U_\nu(n) \to U_\mu(n)U_\nu(n) \quad 1 \tag{11}$$

$$\begin{aligned} \Rightarrow \operatorname{tr}\left(P_{\mu\nu}(n)\right) &\to \operatorname{tr}\left(U_\mu(n)U_\nu(n+\hat{\mu})U_\mu^\dagger(n+\hat{\nu})U_\nu^\dagger(n)U_\mu^\dagger(n)\right) \\ &= \operatorname{tr}\left[U_\nu(n+\hat{\mu})U_\mu^\dagger(n+\hat{\nu})U_\nu^\dagger(n)\right] \quad \text{independent of } U_\mu(n) \end{aligned} \tag{12}$$

which is independent of $U_\mu(n)$. However, this change also affects

$$\begin{aligned} P_{\mu v}(n-\hat{\mu}) &= U_\mu(n-\hat{\mu})U_\nu(n)U_\mu^\dagger(n-\hat{\mu}+\hat{\nu})U_\nu^\dagger(n-\hat{\mu}) \\ &\xrightarrow{1} U_\mu(n-\hat{\mu})U_\mu(n)U_\nu(n)U_\mu^\dagger(n-\hat{\mu}+\hat{\nu})U_\nu^\dagger(n-\hat{\mu}) \end{aligned} \tag{13}$$

So now change variables for the link

$$U_\mu(n-\hat{\mu}) \to U_\mu(n-\hat{\mu})U_\mu^\dagger(n) \quad 2 \tag{14}$$

So

$$P_{\mu v}(n-\hat{\mu}) \overset{2}{=} U_\mu(n-\hat{\mu})U_\nu(n)U_\mu^\dagger(n-\hat{\mu}+\hat{\nu})U_v^\dagger(n-\hat{\mu}) \tag{15}$$

independent of $U_\mu(n)$
Also,

$$P_{\mu\nu}(n-\hat{\nu}) = U_\mu(n-\hat{\nu})U_\nu(n-\hat{\nu}+\hat{\mu})U_\mu^\dagger(n)U_\nu^\dagger(n-\hat{\nu}) \tag{16}$$

under $\left(U_\nu^\dagger(n-\hat{\nu})\,(=U_{-\nu}(n)) \to U_\mu(n)U_\nu^\dagger(n-\hat{\nu})\right)$ 3 $\left[U_\nu(n-\hat{\nu}) \to U_\nu(n-\hat{\nu})U_\mu^\dagger(n)\right]$

$$P_{\mu\nu}(n-\hat{\nu}) \to U_\mu(n-\hat{\nu})U_\nu(n-\hat{\nu}+\hat{\mu})U_\nu^\dagger(n-\hat{\nu}) \tag{17}$$

indep of $U_\mu(n)$, and at the same time

$$\begin{aligned} P_{\mu\nu}(n-\hat{\nu}-\hat{\mu}) &= U_\mu(n-\hat{\nu}-\hat{\mu})U_\nu(n-\hat{\nu})U_\mu^\dagger(n-\hat{\mu})U_\nu^\dagger(n-\hat{\nu}-\hat{\mu}) \\ &\to U_\mu(n-\hat{\nu}-\hat{\mu})U_\nu(n-\hat{\nu})U_\mu^\dagger(n)U_\mu(n)U_\mu^\dagger(n-\hat{\mu})U_\nu^+(n-\hat{\nu}-\hat{\mu}) \\ &= U_\mu(n-\hat{\nu}-\hat{\mu})U_\nu(n-\hat{\nu})U_\mu^\dagger(n-\hat{\mu})U_\nu^\dagger(n-\hat{\nu}-\hat{\mu}) \quad \text{indep of } U_\mu(n) \end{aligned} \tag{18}$$

The same procedure proceeds in the other $D-2 \quad (\mu,\nu)$ planes and by changing variables on the $2D-1$ links from site n other than $U_\mu(n)$ the action has been made independent of $U_\mu(n)$!! Consequently

$$\begin{aligned} \langle U_\mu(n)\rangle &= \frac{1}{Z}\int \mathcal{D}\hat{U} e^{-\hat{S}[\hat{U}]}\Pi_{\nu=\pm1,\nu\neq\mu}^{\pm d}\mathrm{d}U_\nu(n)e^{-\tilde{S}[\hat{U},\{U_\nu(n)\}_{\nu\neq\mu}]}\int \mathrm{d}U_\mu(n)U_\mu(n) \\ &= 0. \end{aligned} \tag{19}$$