# Fermions on the lattice day1

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## 1 Introduction

In this unit we take a look at fermions on the lattice. We will start off looking at fermion doubling, and how this problem is fixed with Wilson fermions, followed by the construction of fermionic correlation functions.

## 2 Fermion doubling

### 2.1 Continuum QCD

In order to make the lattice discretization clear, we first look at the implementation of fermions in continuum QCD.

First, our fermion fields, denoted by $\psi$ and $\overline{\psi}$, are Dirac 4-spinors that carry a spin, color, and flavor index. Conventionally we separate the action into a fermionic part and gluonic part, where the fermionic part depends on the fermion fields and the gauge field, and the gluonic part is a pure gauge action.

We can write the free fermion action as

$$S_F^0[\psi, \overline{\psi}] = \int d^4x \overline{\psi}(x) \left(\gamma_\mu \partial_\mu + m\right) \psi(x) \tag{1}$$

This is not gauge invariant - this has nontrivial transformation properties under a local SU(3) transformation $\Omega(x)$. This is because the fields transform as

$$\psi(x) \to \psi'(x) = \Omega(x)\psi(x), \quad \overline{\psi}(x) \to \overline{\psi}'(x) = \overline{\psi}(x)\Omega^\dagger(x) \tag{2}$$

The action becomes

$$S_F^0[\psi', \overline{\psi}'] = \int d^4x \overline{\psi}\Omega(x)^\dagger(x) \left(\gamma_\mu \partial_\mu + m\right) \Omega(x)\psi(x) \tag{3}$$

This becomes gauge invariant if we introduce the gauge field with transformation properties

$$A_\mu(x) \to A'_\mu(x) = \Omega(x)A_\mu\Omega(x)^\dagger + i(\partial_\mu\Omega(x))\Omega(x)^\dagger \tag{4}$$

such that the extra term from the derivative with respect to the gauge transformation cancels. This introduces the covariant derivative

$$D_\mu = (\partial_\mu + iA_\mu) \tag{5}$$

$D_\mu\psi$ transforms like $\psi$ under gauge transformations, so a bilinear with the covariant derivative is invariant under gauge transformations.

Now we can write the full fermionic continuum QCD action,

$$S_f[\psi, \overline{\psi}, A] = \sum_{f=1}^{N_f} \int d^4x \overline{\psi}^{(f)}(x) \left(\gamma_\mu(\partial_\mu + iA_\mu(x)) + m^{(f)}\right) \psi^{(f)}(x) \tag{6}$$

Where we use vector or matrix notation for the flavor, Dirac, and color indices. We have written a gauge invariant action for fermions at the cost of introducing a term that couples the gauge field to the fermion fields.

## 2.2 Naive fermion discretization

Let's make an attempt at discretizing the free fermion action first. We introduce the lattice $\Lambda$ where a site $n \in \Lambda$ is a four component vector, $n = (n_1, n_2, n_3, n_4)$, with spatial extent $N$m and temporal extent $N_t$, and lattice spacing is $a$. We have toroidal boundary conditions $f(n+\hat{\mu}N_\mu) = e^{i2\pi\theta_\mu} f(n)$, for each direction. We place spinors at the lattice points, so we write the spacetime argument of our fermion fields as $\psi(n)$, where the physical spacetime point is $x = an$.

We want to discretize the free fermion action first. This requires two things, writing the integral as a sum, and writing the derivative as a finite difference.

$$\int d^4x \to a^4 \sum_{n\in\Lambda}, \quad \partial_\mu \psi(n) \to \frac{1}{2a}(\psi(n+\hat{\mu}) - \psi(n-\hat{\mu}))$$

$$S_F^0[\psi, \overline{\psi}] = a^4 \sum_{n\in\Lambda} \overline{\psi}(n) \left( \sum_{\mu=1}^{4} \frac{\psi(n+\hat{\mu}) - \psi(n-\hat{\mu})}{2a} + m\psi(n) \right) \tag{7}$$

Once again, we see this form of the action is not gauge invariant. In the same way as the continuum, we need to introduce the gauge links to make this object invariant under local SU(3) transformations.

On the lattice, we implement a gauge transformation by choosing $\Omega(n) \in \mathrm{SU}(3)$ for each lattice site, where the fields transform the same way as in the continuum. In particular though, we are interested in the transformation of the bilinear

$$\overline{\psi}(n)\psi(n+\hat{\mu}) \to \overline{\psi}'(n)\psi'(n+\hat{\mu}) = \overline{\psi}(n)\Omega(n)^\dagger\Omega(n+\hat{\mu})\psi(n+\hat{\mu}) \tag{8}$$

Clearly this isn't gauge invariant. However, if we introduce the gauge links, which transform like

$$U_\mu(n) \to U'_\mu(n) = \Omega(n)U_\mu(n)\Omega(n+\hat{\mu})^\dagger \tag{9}$$

in between the fermion fields in this bilinear form,

$$\overline{\psi}'(n)U'_\mu(n)\psi'(n+\hat{\mu}) = \overline{\psi}(n)\Omega(n)^\dagger U'_\mu(n)\Omega(n+\hat{\mu})\psi(n+\hat{\mu}) \tag{10}$$

we find that it is gauge invariant. Once again, in the discretized theory, we are required to introduce the gauge links to ensure gauge invariance of the fermion action. Similar to the covariant derivative in the continuum, these are called covariant shifts, and they are denoted by $C_\mu^\pm$. Explicitly they are defined as

$$C_\mu^+\psi(n) = U_\mu(n)\psi(n+\hat{\mu}), \quad C_\mu^-\psi(n) = U_\mu^\dagger(n-\hat{\mu})\psi(n-\hat{\mu}). \tag{11}$$

allowing us to write a lattice covariant derivative,

$$D_\mu\psi(n) = \frac{1}{2a}(C_\mu^+ - C_\mu^-)\psi(n) \tag{12}$$

Finally we can write the naive lattice fermion action as

$$S_F[\psi, \overline{\psi}, U] = a^4 \sum_{n\in\Lambda} \overline{\psi}(n) \left( \sum_{\mu=1}^{4} \gamma_\mu \frac{U_\mu(n)\psi(n+\hat{\mu}) - U_\mu^\dagger(n-\hat{\mu})\psi(n-\hat{\mu})}{2a} + m\psi(n) \right) \tag{13}$$

## 2.3 The appearance of doublers

Why do we call this discretization naive? it seems the standard integral to sum and derivative to finite difference substitution works perfectly fine, allowing us to write down some discretized fermion action, what is wrong with this? The problem arises when we look closer at the particle content of the theory we just wrote down. Generally speaking, the poles of the momentum space fermion propagator will tell us about the particle content of our theory, so our goal in this section is to first write down an explicit form for the naively discretized Dirac operator, Fourier transform it to momentum space, invert it, and look at the pole structure.

To write down the Dirac operator we can rewrite the action in a suggestive way,

$$S_F[\psi,\overline{\psi},U] = a^4 \sum_{n,m\in\Lambda} \sum_{a,b,\alpha,\beta} \overline{\psi}(n)_{\alpha,a} D(U)_{\alpha,a,n;\beta,b,m} \psi(m)_{\beta,b,j} \tag{14}$$

allowing us to write down the Dirac operator explicitly,

$$D(U)_{\alpha,a,n;\beta,b,m} = \sum_{\mu=1}^{4} (\gamma_\mu)_{\alpha\beta} \frac{U_\mu(n)_{ab}\delta_{n+\hat{\mu},m} - U_{-\mu}(n)_{ab}\delta_{n-\hat{\mu},m}}{2a} + m\delta_{\alpha\beta}\delta_{ab}\delta_{n,m}. \tag{15}$$

Next what we want to do is calculate the quark propagator for free lattice fermions. We set $U_\mu = \mathbb{1}$, and we use a discrete Fourier transform on the above form of the naive lattice Dirac operator. We implement the discrete Fourier transform as

$$\tilde{f}(p) = \frac{1}{\sqrt{|\Lambda|}} \sum_{n\in\Lambda} f(n)\exp(-ip\cdot na) \tag{16}$$

Where the lattice in momentum space is defined as

$$\tilde{\Lambda} = \left\{ p = (p_1,p_2,p_3,p_4) \mid p_\mu = \frac{2\pi}{aN_\mu}(k_\mu+\theta_\mu),\ k_\mu = -\frac{N_\mu}{2}+1,...,\frac{N_\mu}{2} \right\} \tag{17}$$

where $\theta_\mu$ is the boundary phase vector, generally for fermions we have periodic boundary conditions in space and antiperiodic boundary conditions in time. PBC corresponds to $\theta = 0$, and aPBC corresponds to $\theta = 1/2$. Thus the allowed lattice momenta are in the range $p_\mu \in (-\pi/a, \pi/a]$, with

We must Fourier transform the two spacetime arguments independently,

$$\begin{aligned}
\tilde{D}_{i,p;j,q} &= \frac{1}{|\Lambda|} \sum_{n,m\in\Lambda} e^{-ip\cdot na} D_{i,n;j,m} e^{iq\cdot ma} \\
&= \frac{1}{|\Lambda|} \sum_{n\in\Lambda} \left( \sum_{\mu=1}^{4} \gamma_\mu \frac{e^{+iq_\mu a} - e^{-iq_\mu a}}{2a} + m\mathbb{1} \right) \\
&= \delta(p-q)\tilde{D}(p)
\end{aligned} \tag{18}$$

Where

$$\tilde{D}(p) = m\mathbb{1} + \frac{i}{a} \sum_{\mu=1}^{4} \gamma_\mu \sin(p_\mu a) \tag{19}$$

We can see that the Dirac operator is diagonal in momentum space, thus to calculate the quark propagator, the inverse of the position space Dirac operator, we simply need to invert the $4\times 4$ matrix $\tilde{D}(p)$, and then do an inverse Fourier transform.

Using the formula for inverting a linear combination of gamma matrices,

$$\left( a\mathbb{1} + i\sum_{\mu=1}^{4} \gamma_\mu b_\mu \right)^{-1} = \frac{a\mathbb{1} - i\sum_{\mu=1}^{4}\gamma_\mu b_\mu}{a^2 + \sum_{\mu=1}^{4} b_\mu^2} \tag{20}$$

we can write

$$\tilde{D}(p)^{-1} = \frac{m\mathbb{1} - ia^{-1}\sum_\mu \gamma_\mu \text{sin}(p_\mu a)}{m^2 + a^{-2}\sum_\mu \sin(p_\mu a)^2} \tag{21}$$

Then we do an inverse Fourier transform

$$D^{-1}_{i,n;j,m} = \frac{1}{|\Lambda|}\sum_{p\in\tilde{\Lambda}} \tilde{D}(p)^{-1} e^{ip\cdot(n-m)a} \tag{22}$$

We eliminate the second momentum sum with the delta function. The sum is over all momenta sites in momenta space, where the lattice in momentum space is defined by sites $p_\mu = \frac{2\pi}{aN_\mu k_\mu}$.

The quark propagator is quite the important object, and it will come up in our study of fermionic correlation functions later. For free fermions we will analyze the quark propagator in momentum space, $\tilde{D}(p)^{-1}$. Let's look at the massless fermion case,

$$\tilde{D}(p)^{-1}\Big|_{m=0} = \frac{-ia^{-1}\sum_\mu \gamma_\mu \text{sin}(p_\mu a)}{a^{-2}\sum_\mu \sin(p_\mu a)^2} \to \frac{-i\sum_\mu \gamma_\mu p_\mu}{p^2} \tag{23}$$

Where the right arrow indicates the continuum limit.

The last expression is the continuum limit massless quark propagator, and it clearly has a pole at $p = (0,0,0,0)$. This corresponds to a single fermion described by the continuum Dirac operator. On the lattice, as we can see, the denominator of the massless free quark propagator has a sine squared, we have additional poles. Whenever all components are either $p_\mu = 0$ or $p_\mu = \pi/a$, we have a pole. Momentum space contains all momenta $p_\mu \in (-\pi/a, \pi/a]$ with the boundaries identified, so we cannot simply exclude these extra poles. Therefore, this naive propagator has $2^4 = 16$ poles, the one at $p = (0,0,0,0)$ and 15 other poles, $p = (\pi/a,0,0,0), (0,\pi/a,0,0), ..., (\pi/a,\pi/a,\pi/a,\pi/a)$. We have a quark propagator that describes the massless mode we are interested in, along with 15 other unwanted poles, which we call doublers. Each doubler is at a corner of the Brillouin zone. With free fermions, we might be able to tolerate an increase in the degrees of freedom, but in the interacting theory, these extra fermions can be pair-produced through interactions of the fermion field, so they affect the physics in a non-trivial way. Even if the external particles are the physical states at the $p = (0,0,0,0)$ corner of the Brillouin zone, the states at the other corners, the doublers, appear in virtual loops. How do we fix this issue?

## 2.4 Wilson fermions

The first fermion formulation we discuss that fixes these doubler solutions is Wilson fermions. In this framework we add a higher order term in our Dirac operator to remove only the unwanted poles. Let's see how that works.

We want to identify spurious poles and remove them. Wilson's solution was to add a term to the action such that in the Dirac operator we get

$$\tilde{D}(p) = m\mathbb{1} + \frac{1}{a}\sum_\mu \gamma_\mu \text{sin}(p_\mu a) + \mathbb{1}\frac{1}{a}\sum_\mu (1-\cos(p_\mu a)) \tag{24}$$

This cosine, the Wilson term, removes the doublers. for $p_\mu = 0$, the entire term vanishes. For each component with $p_\mu = \pi/a$, we get an extra contribution of $2/a$. This acts like an additional mass term, and the total mass of the doublers is $m + \frac{2l}{a}$ where $l$ is the number of momentum components with $\pi/a$. In the continuum limit, these doubler modes become very heavy and they decouple from the theory.

We can use the same trick to write the Wilson quark propagator, where we see the doubler poles are gone and we only have the physical pole we are interested in remains (exercise?)

We can inverse Fourier transform the Wilson term in the momentum space Wilson Dirac operator to write it in position space,

$$-a \sum_{\mu=1}^{4} \frac{U_\mu(n)_{ab}\delta_{n+\hat{\mu},m} - 2\delta_{ab}\delta_{n,m} + U_{-\mu}(n)_{ab}\delta_{n-\hat{\mu},m}}{2a^2} \tag{25}$$

This is a discretization of a second derivative, $-(a/2)\partial_\mu\partial_\mu$. We combine this with the naive discretized Dirac operator to get the complete Wilson Dirac operator.

$$D^{(f)}(U)_{\alpha,a,n;\beta,b,m} = \left(m^{(f)} + \frac{4}{a}\right)\delta_{\alpha\beta}\delta_{ab}\delta_{n,m} - \frac{1}{2a}\sum_{\mu=\pm1}^{\pm4}(\mathbb{1} - \gamma_\mu)_{\alpha\beta}U_\mu(n)_{ab}\delta_{n+\hat{mu},m} \tag{26}$$

We can write this in a more compact notation using the covariant shift operators,

$$D(U) = \sum_\mu \gamma_\mu D_\mu + m - \frac{1}{2}\sum_\mu (C_\mu^+ + C_\mu^- - 2) \tag{27}$$

And we have fixed the doubler issue by adding the Wilson term to our naive discretization. The main drawback of this fermion formulation is how the Wilson term explicitly breaks chiral symmetry, which, in the interacting theory, will then allow certain explicit chiral symmetry breaking terms, such as mass terms and 5D operators, which must be fine tuned away via the bare quark mass.

# 3 Fermion correlators

## 3.1 Grassmann Numbers

In order to express the fermionic degrees of freedom in the path integral, we look briefly at fully anticommuting numbers called Grassmann numbers, which will let us describe the fermionic fields in the path integral language.

First, let $\eta_i$ be a Grassman number in a larger set labelled by $i$. The fundamental property of these variables is their anticommutaion, $\{\eta_i, \eta_j\} = 0 \quad \forall\ i, j$. This means that $\eta_i^2 = 0$, so a power series in these numbers truncates after the linear term. Integration and differentiation turn out be the same operation on these variables,

$$\partial_{\eta_i}1 = 0, \quad \partial_{\eta_i}\eta_i = 1, \quad \{\partial_{\eta_i}, \partial_{\eta_j}\} = 0, \quad \{\partial_{\eta_i}, \eta_j\} = 0 \tag{28}$$

$$\int d\eta_i 1 = 0, \quad \int d\eta_i\eta_i = 1, \quad \{d\eta_i, d\eta_j\} = 0. \tag{29}$$

This can be extended to integrals over a set of Grassmann variables, where we have the normalization condition,

$$\int d^N\eta\eta_1\eta_2...\eta_N = 1 \tag{30}$$

From this condition it is possible to show that under a linear transformation

$$\eta_i' = \sum_{j=1}^{N} M_{ij}\eta_j \tag{31}$$

the integral measure transforms as

$$d^N\eta = \det[M]d^N\eta'. \tag{32}$$

Note the key difference from the bosonic case here, we have the determinant in the numerator, while in the bosonic case the determinant is in the denominator.

The final thing we must establish is how to do a Gaussian integral over Grassmann variables. We consider a Grassmann algebra with $2N$ generators, $\eta_i, \overline{\eta}_i, i = 1, 2, ..., N$. All of these generators commute with all of the other generators. The first important formula is the Matthews-Salam formula

$$Z_F = \int d\eta_N d\overline{\eta}_N ... d\eta_1 \overline{\eta}_1 \exp\left(\sum_{i,j=1}^{N} \overline{\eta}_i M_{ij} \eta_j\right) = \det[M] \tag{33}$$

This can be proven by the same transformation we defined in the previous section, and by the transformation of the measure we pick up a determinant of the transformation matrix $M$.

$$\begin{aligned} Z_F = \det[M] \int \prod_{i=1} d\eta_i' d\overline{\eta}_i \exp\left(\sum_{j=1}^{N} \overline{\eta}_j \eta_j'\right) &= \det[M] \prod_{i=1}^{N} \int d\eta_i' d\overline{\eta}_i \exp\left(\overline{\eta}_j \eta_j'\right) \\ &= \det[M] \prod_{i=1}^{N} \int d\eta_i' d\overline{\eta}_i (1 + \overline{\eta}_i \eta_i') = \det[M] \end{aligned} \tag{34}$$

expanding the exponential in a power series, which terminates after the first two terms because these are Grassmann variables.

The Fermi fields behave like Grassmann variables in the path integral, so we can generalize our discussion of integrals over Grassmann variables very simply by setting the $M$ in the Gaussian form to the negative Dirac operator, and we can write the fermionic partition function as a determinant,

$$Z_F[U] = \int \mathcal{D}[\psi, \overline{\psi}] e^{-S_E} = \int \mathcal{D}[\psi, \overline{\psi}] e^{\sum \overline{\psi}(x)_i D_{i,x;j,y}(U)\psi(y)_j} = \det(D(U)) \tag{35}$$

$$S_E = \sum_{x,y,i,j} \overline{\psi}(x)_i D_{i,x;j,y}(U) \psi(y)_j \tag{36}$$

From here we can define correlation functions in the path integral language, and see how these give rise to expressions of correlators in terms of quark propagators.

### 3.2 The generating functional

With the path integral defined, we can now develop the method of evaluating fermionic correlation functions. We start by defining the fermionic generating functional, including Grassmannian sources.

We generalize to a $4N$-dimensional algebra generated by $\eta_i, \overline{\eta}_i, \theta_i, \overline{\theta}_i, i = 1, 2, ..., N$. All $4N$ Grassmann variables anticommute with every other one. Let's write an integral over the $\eta, \overline{\eta}$ and use the other generators as source terms.

$$W[\theta, \overline{\theta}] = \int \prod_{i=1}^{N} d\eta_i d\overline{\eta}_i \exp\left(\sum_{k,l=1}^{N} \overline{\eta_k} M_{kl} \eta_l + \sum_{k=1}^{N} \overline{\theta}_k \eta_k + \sum_{k=1}^{N} \overline{\eta}_k \theta_k\right) \tag{37}$$

To evaluate this, we must complete the square in the exponential, and then shift our variables. First we rewrite the exponential as

$$\left(\overline{\eta}_i + \overline{\theta}_j (M^{-1})_{ji}\right) M_{ik} \left(\eta_k + (M^{-1})_{kl} \theta_l\right) - \overline{\theta}_n (M^{-1})_{nm} \theta_m \tag{38}$$

Grassmann integrals are invariant to shifts, $\int d\psi f(\psi) = \int d\psi f(\psi + \alpha)$, so we shift our integration variables

$$\eta_k' = \eta_k + (M^{-1})_{kl} \theta_l, \quad \overline{\eta}_i' = \overline{\eta}_i + \overline{\theta}_j (M^{-1})_{ji} \tag{39}$$

Putting this into $W$,

$$\begin{aligned} W[\theta,\overline{\theta}] &= \exp\left(-\sum_{n,m=1}^{N}\overline{\theta}_n(M^{-1})_{nm}\theta_m\right)\int\prod_{i=1}^{N}d\eta_i d\overline{\eta}_i \exp\left(\sum_{k,l=1}^{N}\overline{\eta_k}M_{kl}\eta_l\right) \\ &= \det[M]\exp\left(-\sum_{n,m=1}^{N}\overline{\theta}_n(M^{-1})_{nm}\theta_m\right) \end{aligned} \tag{40}$$

### 3.3 Correlation functions

Once again we readily generalize to Fermi fields, setting $M = -D$. The generating functional is key in evaluating correlation functions. For example, a general correlation function of operator $O$ can be written as

$$\langle O(U,\psi(x)_i\overline{\psi}(y)_j\rangle = \frac{1}{Z}\int\mathcal{D}[U,\psi,\overline{\psi}]O(U,\psi(x)_i\overline{\psi}(y)_j)e^{-S_F(U)-S_g(U)} \tag{41}$$

Writing the full partition function as

$$Z = \int\mathcal{D}[U,\psi,\overline{\psi}]e^{-S_F(U)-S_G(U)} = \int\mathcal{D}[U]\det(D(U))e^{-S_G(U)} \tag{42}$$

Writing our correlation function with additional source terms $\theta,\overline{\theta}$,

$$\begin{aligned} \langle O(U,\psi(x)_i\overline{\psi}(y)_j\rangle &= \frac{1}{Z_F}\int\mathcal{D}[U,\psi,\overline{\psi}]O(U,\psi(x)_i\overline{\psi}(y)_j)e^{-S_F(U)-S_g(U)-\overline{\theta}\psi-\overline{\psi}\theta} \\ &= \frac{1}{Z}\int\mathcal{D}[U,\psi,\overline{\psi}]O(U,\psi(x)_i\partial_{\theta(y)_j})e^{-S_F(U)-S_g(U)-\overline{\theta}\psi-\overline{\psi}\theta} \\ &= \frac{1}{Z}\int\mathcal{D}[U,\psi,\overline{\psi}]O(U,-\partial_{\theta(y)_j}\psi(x)_i)e^{-S_F(U)-S_g(U)-\overline{\theta}\psi-\overline{\psi}\theta} \\ &= \frac{1}{Z}\int\mathcal{D}[U,\psi,\overline{\psi}]O(U,\partial_{\theta(y)_j}\partial_{\overline{\theta}(x)_i})e^{-S_F(U)-S_g(U)-\overline{\theta}\psi-\overline{\psi}\theta} \\ &= \frac{1}{Z}\int\mathcal{D}[U]O(U,-\partial_{\overline{\theta}(x)_i}\partial_{\theta(y)_j})e^{-S_g(U)}\int\mathcal{D}[U,\psi,\overline{\psi}]e^{-S_F(U)-\overline{\theta}\psi-\overline{\psi}\theta} \\ &= \frac{1}{Z}\int\mathcal{D}[U]O(U,-\partial_{\overline{\theta}(x)_i}\partial_{\theta(y)_j})e^{-S_g(U)}\times W[\theta,\overline{\theta}] \\ &= \frac{1}{Z}\int\mathcal{D}[U]\det(D(U))e^{-S_g(U)}\left(O(U,-\partial_{\overline{\theta}(x)_i}\partial_{\theta(y)_j})e^{\overline{\theta}D^{-1}(U)\theta}\right) \end{aligned} \tag{43}$$

As a quick example, consider the two point function,

$$\langle\psi(x)_i\overline{\psi}(y)_j\rangle = \langle -\partial_{\overline{\theta}(x)_i}\partial_{\theta(y)_j})\times W[\theta,\overline{\theta}]\rangle = \langle D^{-1}(U)_{i,x;j,y}\rangle \tag{44}$$

When we want to define an n-point function, or any type of Green's function higher than a two point function, we need to be careful about permutations of fields - the calculation is exactly the same as the two point case, but we also pick up a sum over permutations $P$

$$\langle\psi(x_1)_{i_1}\overline{\psi}(y_1)_{j_1}...\psi(x_n)_{i_n}\overline{\psi}(y_n)_{j_n}\rangle = \sum_P(-1)^P\langle D^{-1}(U)_{i_1,x_1;j_{P_1},y_{P_1}}...D^{-1}(U)_{i_n,x_n;j_{P_n},y_{P_n}}\rangle \tag{45}$$

Note we can rewrite this using the generating functional,

$$\langle\psi(x_1)_{i_1}\overline{\psi}(y_1)_{j_1}...\psi(x_n)_{i_n}\overline{\psi}(y_n)_{j_n}\rangle = \frac{1}{Z}\int D[U]e^{-S_g(U)}\left(\frac{\partial}{\partial\theta_{j_1}}\frac{\partial}{\partial\overline{\theta}_{i_1}}...\frac{\partial}{\partial\theta_{j_n}}\frac{\partial}{\partial\overline{\theta}_{i_n}}\times W[\theta,\overline{\theta}]\right) \tag{46}$$