LGT1 Unit 3 day 14

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1 Introduction

In this lecture we focus on different fermion formulations on the lattice, in particular we talk about twisted mass fermions, staggered fermions, and domain wall fermions.

References

- Gattringer and Lang, chapter 10
- DeGrand and DeTar sections 6.2, 6.3
- Christoph Lehner's lecture notes chapter 12

Vadim Furman, Yigal Shamir, Axial symmetries in lattice QCD with Kaplan fermions, Nuclear Physics B, Volume 439, Issues 1–2, 1995, Pages 54-78, ISSN 0550-3213, https://doi.org/10.1016/0550-3213(95)00031-M.

2 Twisted mass fermions

First we look at fermion formulation for QCD with two quark flavors of Wilson fermions with degenerate mass, QCD with isospin symmetry. This isospin degree of freedom is used to introduce the so called "twisted mass" term, which has a nontrivial isospin structure. This twisted mass term provides an IR regulator, and can be utilized to obtain a $\mathcal{O}(a)$ improvement of the formulation.

We begin, as mentioned before, with two mass-degenerate Wilson quark flavors. Let $\chi, \overline{\chi}$ be quark fields which now also carry a flavor index, in this case $N_f = 2$. We write the fermion action for lattice twisted mass QCD (tmQCD) with Wilson fermions as

$$S_F^{\text{tw}}[\chi, \overline{\chi}, U] = a^4 \sum_{k,n\in\Lambda} \overline{\chi}(D_{i,k;j,n}\mathbb{1}_2) + m\mathbb{1}_2\delta_{k,n} + i\mu\gamma_5\tau^3\delta_{k,n})\chi(n) \tag{1}$$

Where the identity matrices displayed are only for flavor space, hence the 2 subscript. $D_{i,k;j,n}$ denotes the massless Wilson operator for a single flavor

$$D_{i,k;j,n} = \frac{4}{a}\delta_{k,n} - \frac{1}{2a}\sum_{\mu}(\mathbb{1} - \gamma_\mu)U_\mu(k)\delta_{k+\hat{\mu},n} \tag{2}$$

We have included a new term in the action, $i\mu\gamma_5\tau^3$. The real parameter μ is called the twisted mass. The actual mass term in the action is trivial in color, Dirac, and flavor indices, while this twisted mass term is only trivial in color indices, the γ_5 acts in Dirac space, and the third isospin generator, the Pauli matrix τ^3, acts in flavor space.

One can use this twisted mass term as an infrared regulator which removes exceptional configurations. We can see this from the following identity

$$\begin{aligned}
\det[D\mathbb{1}_2 + m\mathbb{1}_2 + i\mu\gamma_5\tau^3] &= \det[D + m + i\mu\gamma_5]\det[D + m - i\mu\gamma_5] \\
&= \det[D + m + i\mu\gamma_5]\det[\gamma_5(D + m + i\mu\gamma_5)\gamma_5] \\
&= \det[D + m + i\mu\gamma_5]\det[D^\dagger + m + i\mu\gamma_5] \\
&= \det[(D + m + i\mu\gamma_5)(D^\dagger + m + i\mu\gamma_5)] \\
&= \det[(D + m)(D + m)^\dagger + \mu^2] > 0 \quad \text{for} \quad \mu \neq 0
\end{aligned} \tag{3}$$

our twisted mass Dirac operator is diagonal in flavor space and the determinant for the two flavor operator was written as a product of two determinants for a single flavor. The inequality in the last line holds because the eigenvalues of the product $(D+m)(D+m)^\dagger$ are real and non-negative. Then the presence of a non-vanishing twisted mass term above ensures that the determinant of the twisted mass operator are strictly positive. Zero eigenvalues, which cause exceptional configurations, are excluded.

We can combine the twisted mass term with the usual mass term, since it is possible to work with both non-vanishing m and μ. We introduce the polar mass M and the twist angle α

$$M = \sqrt{m^2 + \mu^2}, \quad \alpha = \arctan(\mu/m) \tag{4}$$

We can then rewrite the mass term in the twisted mass action

$$m\mathbb{1} + i\mu\gamma_5\tau^3 = Me^{i\alpha\gamma_5\tau^3} \quad \text{with} \quad m = M\cos(\alpha), \quad \mu = M\sin(\alpha) \tag{5}$$

We call the $\alpha = \pi/2$ case 'full twist' and the $\alpha = 0$ 'zero twist'.

Let's perform a transformation on our two flavor fermion fields.

$$\psi = R(\alpha)\chi, \quad \overline{\psi} = \overline{\chi}R(\alpha), \quad R(\alpha) = e^{i\alpha\gamma_5\tau^3/2} \tag{6}$$

This allows us to rewrite the action as

$$S_F[\psi, \overline{\psi}, U] = a^4 \sum_{k,n\in\Lambda} \overline{\psi}(n)(D^{\text{tw}}_{k;n} + M\mathbb{1}_2\delta_{k,n})\psi(n) \tag{7}$$

The twisted mass term has disappeared and we find a conventional mass term with the polar mass as the mass parameter. The twisted mass Dirac operator is now a genuine two flavor operator. explicitly,

$$D^{\text{tw}}_{k;n} = \frac{4}{a}e^{-i\alpha\gamma_5\tau^3}\delta_{k,n} - \frac{1}{2a}\sum_\mu \left(e^{i\alpha\gamma_5\tau^3} - \gamma_\mu\right)U_\mu(k)\delta_{k+\hat{\mu},n} \tag{8}$$

Note the terms here that depend on γ_μ are not affected by the twist, these are the naive parts of the operator, while the Wilson parts are rotated. We have changed bases from the two flavor fields $\chi, \overline{\chi}$, referred to as the 'twisted basis', to the fields $\psi\overline{\psi}$, which is referred to as the 'physical basis'. In the physical basis, the mass term and kinetic term are in their conventional form, and only the doubler removing term, the Wilson term, is twisted.

sectionstaggered fermions We now move to the staggered fermion formulation. This formulation deals with fermion doublers by 'spin diagonalization', allowing us to write the action in terms of four identical spinor components, we throw away the extra three duplicate terms, and we reduce the 16-fold doubler degeneracy to a four-fold degeneracy.

We begin by introducing the transformations $\psi_n \to \psi'_n$, $\overline{\psi}_n \to \overline{\psi}\Omega^\dagger_n$, where

$$\Omega_n = \gamma_1^{n_1}\gamma_2^{n_2}\gamma_3^{n_3}\gamma_4^{n_4} \tag{9}$$

Noting that transforming fields at neighboring sites only differ by one power of γ, depending on the direction, since any of the gammas square to the identity. Using this transformation matrix, we can see how the γ_μ's transform,

$$\Omega_n^\dagger \gamma_\mu \Omega_{n+\hat{\mu}} = (-1)^{n_0+n_1+\ldots+n_{\mu-1}} = \alpha_\mu(n) \tag{10}$$

This is the 'staggered sign function'. Using this transformation we can rewrite the action

$$S = \frac{1}{2a}\sum_{n,\mu} \overline{\psi}'_n \alpha_\mu(n)[U_\mu(n)\psi'_{n+\hat{\mu}} - U_\mu(n-\hat{\mu})^\dagger \psi'_{n-\hat{\mu}}] + m\sum_n \overline{\psi}'_n \psi'_n \tag{11}$$

This action is diagonal in spinor space. The Fermi fields are four component spinors, but due to our spin diagonalization, we have rewritten the action in terms of four independent, identical spinor components. With this, we can reduce the multiplicity of naive fermions by a factor of four, simply by throwing away all but one spinor component. The resulting one-component field χ_n is the 'staggered fermion' field, with the corresponding one component action

$$S = \frac{1}{a}\overline{\chi}M(U)\chi = \frac{1}{2a}\sum_{n,\mu} \overline{\chi}_n \alpha_\mu(n)[U_\mu(n)\chi_{n+\hat{\mu}} - U_\mu(n-\hat{\mu})^\dagger \chi_{n-\hat{\mu}}] + m\sum_n \overline{\chi}_n \chi_n \tag{12}$$

Staggered fermions exhibit a remnant chiral symmetry under a modified U(1) transformation. The spin diagonalization acts on a pseudoscalar bilinear as,

$$\overline{\psi}'(n)\gamma_5\psi'(n) = \Gamma_5\overline{\psi}(n)\mathbb{1}\psi(n) \tag{13}$$

where we define $\Gamma_5 = (-1)^{n_1+n_2+n_3+n_4}$.

looking back at the staggered action, we see that the kinetic term connects only even sites with odd sites. the mass term connects even with even and odd with odd. This means at zero mass, the staggered action is invariant under

$$\chi \to \exp(i\Gamma_5\theta)\chi, \quad \overline{\chi} \to \overline{\chi}\exp(i\Gamma_5\theta) \tag{14}$$

where Γ_5 is diagonal in the site and color index and at site n, $\Gamma_{5n} = 1$ for even n, and $\Gamma_{5n} = -1$ for odd n. At any mass, the one-component action satisfies $D(U)^\dagger = \Gamma_5 D(U)\Gamma_5$.

This remnant chiral symmetry is one of the reasons why the staggered action is interesting. At zero mass, D is antihermitian and has imaginary eigenvalues. The spectrum of $D^\dagger D$ is bounded from below by (am^2), allowing simulations using quark masses significantly smaller than the Wilson actions. Staggered fermions are preferred over Wilson fermions in situations where chiral properties of the fermions dominate the dynamics.

Due to the staggered sign function and the fact that this spin diagonal action mixes lattice and Dirac indices, a natural thing to do is group together the 16 sites of a hypercube, and interpret this hyper cube as 4 species of quarks with the familiar 4 component spinor structure. We will do a basis transformation to write the action in terms of this hypercubic structure, where the four species of quarks is referred to as tastes of staggered fermions. We will now transform to the spin-taste basis to investigate this further. we first label the 16 sites of the hypercube with four component vectors η with components $\eta_\mu = 0$ or 1. We define a four-taste Dirac field through a unitary change of basis,

$$\psi_y^{\alpha,a} = \frac{1}{8}\sum_\eta \Omega_\eta^{\alpha,a}\chi_{2y/(a+\eta)} \tag{15}$$

using the same Ω as before. The field ψ has four Dirac spinor components α, and four taste components a, and lives on a hypercube with origin at $2y$. We have now moved into the staggered fermion spin-taste basis. The inverse transformation is

$$\chi_{2y/(a+\eta)} = 2\mathrm{Tr}[\Omega_\eta^\dagger \psi_y] \tag{16}$$

We can express the action in this basis as

$$S = \sum_{y,\mu} b^4 \overline{\psi}_y \left[(\gamma_\mu \otimes \mathbb{1})\Delta_\mu + \frac{1}{2} b(\gamma_5 \otimes \gamma_\mu^* \gamma_5)\Box_\mu \right] \psi_y + mb^4 \sum_x \overline{\psi}_y \mathbb{1} \otimes \mathbb{1} \psi_y \tag{17}$$

where the first and second block derivatives are given by

$$\Delta_\mu \psi_y = \frac{1}{2b}[\psi_{y+b\hat{\mu}} - \psi_{y-b\hat{\mu}}], \quad \Box_\mu = \frac{\psi_{y+b\hat{\mu}} + \psi_{y-b\hat{\mu}} - 2\psi_y}{b^2} \tag{18}$$

where $b = 2a$, and the sum over y runs over all hypercubes of the blocked lattice. We explicitly write the tensor product notation as spin $\otimes$ taste. The taste symmetry is four-fold, so we use the gamma matrices as its generators. We also have the irrelevant dimension five term, $(\gamma_5 \otimes \gamma_\mu^* \gamma_5)$ breaks taste symmetry at non-zero lattice spacing.

We now move to the interacting theory. where the spin-taste basis is more complicated. Since the transformation from the one-component hypercube basis to the spin-taste basis collects fields from different lattice sites, to preserve gauge invariance in the interacting theory, the basis change needs to include the gauge connection

$$\psi_y^{\alpha,a} = \frac{1}{8} \sum_\eta \Omega_\eta^{\alpha,a} W(2y, 2y+\eta)\chi_{2y/(a+\eta)} \tag{19}$$

where $W(2y, 2y+\eta)$ is a product of gauge links connecting sites $2y$ and $2y+\eta$., or a suitable linear combination of such products. The inverse transformation is

$$\chi_{2y/(a+\eta)} = 2W^{-1}(2y, 2y+\eta)\mathrm{Tr}[\Omega_\eta^\dagger \psi_y] \tag{20}$$

We cannot express the action simply in the spin-taste basis in the interacting theory, we can write the first few terms in the expansion in the lattice spacing

$$S = \sum_{y,\mu} b^4 \overline{\psi}[(\gamma_\mu \otimes \mathbb{1})D_\mu \psi_y + aS_{tb,1} + \mathcal{O}(a^2)] + mb^4 \sum_x \overline{\psi}_y \mathbb{1} \otimes \mathbb{1} \psi_y \tag{21}$$

The zeroth order in a term is the continuum action with a four-fold taste degeneracy. We write the irrelevant first order taste breaking contribution as $S_{tb,1}$, which contains dimension five fermion bilinears. Even in the interacting theory only irrelevant operators break the taste symmetry, in the continuum limit this taste breaking is suppressed, and we get four degenerate tastes.

At nonzero lattice spacing, most staggered fermion taste and spin rotations are replaces by shifts and rotations in the hypercube. Continuous symmetries of QCD are broken to discrete symmetries. In particular, continuum taste symmetry is broken and taste mulitplets are split. At zero mass, we have one transformation on the lattice that survives in continuum form; a $\mathrm{U}(1)_A$ symmetry;

$$\psi_y \to \exp(i\theta\gamma_5 \otimes \gamma_5)\psi_y, \quad \overline{\psi}_y \to \overline{\psi}_y \exp(i\theta\gamma_5 \otimes \gamma_5) \tag{22}$$

We see this remnant chiral transformation mixes taste and spin.

A conventional isovector chiral transformation would be generated by the taste singlet operator $\tau_i\gamma_5\gamma_\mu \otimes \mathbb{1}$, and an isosinglet chiral transformation by $\gamma_5\gamma_\mu \otimes \mathbb{1}$. In the one-component basis, a current from these transformations would connect the one component fields at pairs of lattice sites separated by three links in the hypercube. They are not conserved at nonzero lattice spacing. The axial anomaly originates in the gluon sector, which has no taste structure, so it must be a taste singlet and couple to the pseudoscalar density generated by $\gamma_5 \otimes \mathbb{1}$. This leads to the consequence that there is no connection between the zero modes of the staggered fermion Dirac operator and the winding number of the gauge fields. These modes only emerge in the continuum limit, and their chirality is not exactly ± 1.

3 Domain wall fermions

For the final topic of this unit, we will discuss domain wall fermions. This fermion formulation introduce infinitely many heavy regulator fields, and realize light ordinary fermions as zero modes bound on four dimensional slices in a theory of five dimensional Dirac fermions. This constructs a clear measure of chiral symmetry breaking, as this theory is exactly chirally symmetric in the limit of an infinite fifth dimensional extent. In practice we cannot simulate a theory with an infinite extent in the fifth dimension, but we do have a handle on how much explicit chiral symmetry breaking there is due to the finite fifth dimension. Chiral right handed and left handed components of the physical quark arise as surface states on opposite boundaries of a five dimensional slab with free boundary conditions. For every physical quark we need one five dimensional fermion field.

When this fifth dimension is finite, there is overlap of the lh and rh components of the five dimensional quark wave function. At tree level, the tail of the quark wave function goes like $(1-Ma)^s$ where s is the coordinate in the fifth dimension. The perturbative overlap of the left and right handed components vanishes exponentially with increasing L_s, the extent of the fifth dimension.

Using DWF, one can define axial currents whose divergences in the massless case are completely localized in the two middle layers of the fifth dimension, at $s=N$ and $s=N+1$. In this case the anomalous term in Ward identities of the non-singlet axial symmetries is governed only by the small tail of the quarks wave functions at the center of the fifth dimension. We do get the divergence of the singlet axial current to couple to two gluons, giving rise to the expected axial anomaly in the $L_s \to \infty$ limit. Let's explicitly define the formulation.

For a four dimensional target theory, the fermion fields and regulator fields are five dimensional, and the gauge field is four dimensional. We let ordinary four coordinates, x_μ range from 1 to L, while the extra coordinate ranges from $s=1$ to $s=2N$ while the Pauli-Villars(PV) regulator fields are only half as big, ranging from 1 to N. We realize this by requiring the link variables in the fermion and PV action obey $U_{x,s,5}=1$ and $U_{x,s,\mu}=U_{x,\mu}$, independently of s. We take the topology of the fifth dimension to be a circle, and the couplings which reside on the links connecting $s=1$ and $s=2N$ are proportional to parameter $-m_i$. $m_i=1$ corresponds to aPBC, where the model supports no light fermionic state, and $m_i=0$ corresponds to open BC's, where we get the physics of QCD with massless quarks by taking the chiral limit then the continuum limit.

We can write the generating functional;

$$\begin{aligned} Z &= Z(g_0, L, N, m_i) \\ &= \prod_x \left(\prod_\mu \int dU_{x,\mu} \prod_{s=1}^{2N} d\overline{\psi}_{x,s} d\psi_{x,s} \prod_{s'=1}^{N} \int d\phi^\dagger_{x,s'} d\phi_{x,s'} \right) e^{-S} \end{aligned}$$

With the action given by

$$S = S_G(U) + S_F(\overline{\psi}, \psi, U) + S_{PV}(\phi^\dagger, \phi, U) \tag{23}$$

The fermion and PV actions contain a sum over flavors, and the only difference between the flavors in these actions is the parameter m_i. Below the one flavor action is written. The fermionic part has the following form

$$S_F(\overline{\psi}, \psi, U) = -\sum_{x,y,s,s'} \overline{\psi}_{x,s} (D_F)_{x,s;y,s'} \psi_{y,s'} \tag{24}$$

The fermionic matrix is defined by

$$(D_F)_{x,s;y,s'} = \delta s, s' D^{||}_{x,y} + \delta_{x,y} D^{\perp}_{s,s'} \tag{25}$$

$$D^{||}_{x,y}(U) = \frac{1}{2} \sum_\mu ((1+\gamma_\mu) U_\mu(x) \delta_{x+\hat{\mu},y} + (1-\gamma_\mu) U^\dagger_\mu(x-\hat{\mu}) \delta_{x-\hat{\mu},y} + (M-4)\delta_{x,y} \tag{26}$$

$$D^{\perp}_{s,s'}(U) = \begin{cases} P_R\delta_{2,s'} - mP_L\delta_{2N,s'} - \delta 1, s', & s = 1 \\ P_R\delta_{s+1,s'} + P_L\delta_{s-1,s'} - \delta_{s,s'}, & 1 < s < 2N \\ -mP_R\delta_{1,s'} + P_L\delta_{2N-1,s'} - \delta_{2N,s'}, & s = 2N \end{cases} \tag{27}$$

Notice $D^{\perp}_{s,s'}$ is independent of the gauge field, and apart from the unconventional sign of the mass term, $D^{||}_{x,y}$ is the usual four dimensional gauge covariant Dirac operator for massive Wilson fermions.

With our open BC's in the fifth dimension, the spectrum of states on the surfaces of the fifth dimension contains one right handed Weyl fermion near the boundary $s = 1$, and one left handed Weyl fermion near the other boundary for every five dimensional field. These Weyl fermions have the same coupling to the gauge field, so they describe N_f quarks. Aside from the exponentially vanishing overlap between these quark states at the midpoint, these states describe massless quarks.

As for the regulator fields, they are necessary to cancel the contribution of heavy fermion modes to the effective action. This contribution is proportional to N. Every five dimensional fermion field describes one light quark field and $2N - 1$ four dimensional fields whose mass if of the order of the cutoff, if we do not subtract the contribution of the massive fields by hand, they will dominate the effective action in the chiral limit.

The PV fields live on a five dimensional lattice with N sites in the s direction. We denote the dependence of the Dirac operator on m_i and number of sites in the s direction, $D_F = D_F(2N, m_i)$.

$$S_{PV}(\phi^\dagger, \phi, U) = \sum_{x,y,z,s,s',s''} \phi^\dagger_{x,s} D^\dagger_F(N,1)_{x,s;z,s''} D_F(N,1)_{z,s'';y,s'} \phi_{y,s'} \tag{28}$$

The differential operator in this action is simply the square of the Dirac operator on a smaller lattice. To keep $S_{eff}(U)$ finite in the chiral limit, we choose $m_i = 1$ to prevent the appearance of light scalar modes on the boundaries.

Let R denote a reflection relative to the midpoint, $s = N + 1/2$. The DWF Dirac operator satisfies an analog of γ_5-Hermiticity

$$\gamma_5 R D_F \gamma_5 R = D^\dagger_F \tag{29}$$

This implies that the operator $\gamma_5 R D_F$ is Hermitian, and since $\det(\gamma_5 R) = 1$, so $\det(D_f) = \det(\gamma_5 \mathcal{R} D_F)$, and the fermionic determinant is real. We can use the Hermitian Dirac operator in the definition of the fermionic action instead of what we used above, this is facilitated by

$$\begin{aligned} \psi &\rightarrow \psi' = \psi \\ \overline{\psi} &\rightarrow \overline{\psi}' = \overline{\psi}\gamma_5\mathcal{R} \end{aligned}$$

I will now take a second to show the relation between this formulation and the overlap fermion scheme which was mentioned in the lecture 2 exercise. The overlap operator acts as a projection of a non-chiral, doubler free Dirac operator onto a chiral one, and DWF works in a similar way, aside from the unconventional sign of the mass term, the first term in the domain wall operator uses the Wilson fermion operator as a kernel. Through a series of transformations one can show

$$\det[D^{\rm dw}] = \det[D^{\rm ov}_{L_s}]\det[D^{\rm PV}] \tag{30}$$

we can apply this to the generating functional, and do the fermion and regulator field integration,

$$\int \mathcal{D}[\psi, \overline{\psi}, \phi, \phi^\dagger] e^{-S_F[\psi,\overline{\psi},U] - S_{PV}[\phi^\dagger,\phi,U]} = \frac{\det[D^{\rm dw}]}{\det[D^{\rm PV}]} = \det[D^{{\rm ov}_{L_s}}] \tag{31}$$

because the PV fields are bosonic, their determinant appears in the denominator. We refer to the operator $D^{{\rm ov}_{L_s}}$ as the truncated overlap operator, given by

$$D^{{\rm ov}_{N_5}} = \mathbb{1} + \gamma_5 \tanh\left(\frac{L_s}{2}\tilde{H}\right) \xrightarrow{L_s \to \infty} \mathbb{1} + \gamma_5 {\rm sign}[\tilde{H}] \tag{32}$$

Where $\tilde{H}$ is a nonlocal variant of the kernel operator H we used when constructing the overlap operator. We see in the limit of infinite 5th dimension this truncated overlap operator reduces to the standard overlap operator we have seen previously.

Now we talk a bit about the chiral properties of this DWF formulation. The five dimensional action is invariant under a global $\mathrm{U}(N_f)$ symmetry, and we can write the conserved five-dimensional current, For $\mu = 1, ..., 4$

$$j^a_\mu(x,s) = \frac{1}{2}\left(\overline{\psi}_{x,s}(1+\gamma_\mu)U_{x,\mu}\lambda^a\psi_{x+\hat{\mu},s} - \overline{\psi}_{x+\hat{\mu},s}(1-\gamma_\mu)U^\dagger_{x,\mu}\lambda^a\psi_{x,s}\right) \tag{33}$$

for the fifth component, we define

$$j^a_5(x,s) = \begin{cases} \overline{\psi}_{x,s}P_R\lambda^a\psi_{x,s+1} - \overline{\psi}_{x,s+1}P_L\lambda^a\psi_{x,s}, & 1 \le s < 2N \\ \overline{\psi}_{x,2N}P_R\lambda^a\psi_{x,1} - \overline{\psi}_{x,1}P_L\lambda^a\psi_{x,2N}, & s = 2N \end{cases} \tag{34}$$

This five dimensional current satisfies the continuity equation

$$\sum_\mu \Delta_\mu j^a_\mu = \begin{cases} -j^a_5(x,1) - mj^a_5(x,2N), & s = 1 \\ -\Delta_5 j^a_5(x,s), & 1 < s < 2N-1 \\ j^a_5(x,2N-1) + mj^a_5(x,2N), & s = 2N \end{cases} \tag{35}$$

Where λ^a is a flavor symmetry generator. Here

$$\begin{aligned} \Delta_\mu f(x,s) &= f(x,s) - f(x-\hat{\mu},s) \\ \Delta_5 f(x,s) &= f(x,s) - f(x,s-1). \end{aligned}$$

In the end though, we want to know how this connects to a physical, four dimensional theory. We can define the four dimensional vector current

$$V^a_\mu(x) = \sum_{s=1}^{2N} j^a_\mu(x,s) \tag{36}$$

This is conserved, and this can be checked by using the five dimensional continuity equation.

As for the axial current, there is arbitrariness when defining chiral transformations in this model. In particular, any transformation which assigns opposite charges to left handed and right handed chiral modes with reduce to the proper chiral transformation in the continuum limit. For our model, since the lh and rh modes are globally separated in the fifth dimension, we define the chiral transformation to act vectorially on a given four dimensional layer, while assigning different charges to fermions in the two half spaces

$$\begin{aligned} \delta^a_A\psi_{x,s} &= +iq(s)\lambda^a\psi_{x,s} \\ \delta^a_A\overline{\psi}_{x,s} &= -iq(s)\overline{\psi}_{x,s}\lambda^a \end{aligned}$$

$$q(s) = \begin{cases} 1, & 1 \le s \le N \\ -1, & N < s \le 2N \end{cases} \tag{37}$$

The corresponding axial currents are

$$A^a_\mu(x) = -\sum_{s=1}^{2N} \mathrm{sgn}(N - s + \frac{1}{2}) j^a_\mu(x,s) \tag{38}$$

which satisfies the following divergence relation

$$\Delta_\mu A^a_\mu(x) = 2mJ^a_5(x) + 2J^a_{5q}(x) \tag{39}$$

Where

$$J_5^a(x) = j_5^a(x, 2N)$$
$$J_{5q}^a(x) = j_5^a(x, N)$$

we can also define four dimensional quark operators

$$q_x = P_R \psi_{x,1} + P_L \psi_{x,2N}$$
$$\overline{q}_x = \overline{\psi}_{x,2N} P_R + \overline{\psi}_{x,1} P_L$$

Where these projections from the 5D fields onto the 4D fields can be used to define observables, such as the pseudoscalar density,

$$\overline{q}_x q_x = \overline{\psi}_{x,2N} P_R \psi_{x,1} + \overline{\psi}_{x,1} P_L \psi_{x,2N} \tag{40}$$

We can also write J_5^a in terms of these surface quark states

$$J_5^a(x) = \overline{\psi}_x \gamma_5 \lambda^a \psi_x \tag{41}$$

We note that based on this definition of J_5^a, we see the first term in the divergence of the axial current is the expected contribution from a classical mass term, while the midpoint current term, proportional to J_{5q}^a is a term that measures the residual chiral symmetry breaking for finite L_s. This corresponds to the so called residual mass of domain wall fermions.

We would expect that with massless quarks, the flavor non-singlet axial current it exaclty conserved in the continuum limit, and it was shown by Furman and Shamir that this is indeed the case. explicitly, they found that the midpoint current term in the non-singlet axial Ward identity vanishes in the limit of an infinite fifth dimension. The non-singlet axial Ward identity take the form

$$\begin{aligned}\Delta_\mu \left\langle A_\mu^a O(y_1, y_2, ...) \right\rangle &= 2m \left\langle J_5^a(x) O(y_1, y_2, ...) \right\rangle \\ &+ 2 \left\langle J_{5q}^a(x) O(y_1, y_2, ...) \right\rangle \\ &+ i \left\langle \delta_A^a O(y_1, y_2, ...) \right\rangle\end{aligned}$$

Where the first term on the right vanishes for massless quarks, but we still an additional term proportional to the midpoint current. We know that in the continuum limit, this Ward identity must agree with the corresponding identity in the continuum, in which the axial current is conserved. Thus the sum of the two terms on the right hand side of the Ward identity must be equivalent to an effective quark mass, $m_{\text{eff}} = m_f + m_{\text{res}}$, times the pseudoscalar density J_5^a. at low energy, $J_{5q}^a = m_{\text{res}} J_5^a$.

Furman and Shamir showed that if the operator of interest was made up of only quark operators, then the anomalous term,

$$\left\langle J_{5q}^a(x) O(y_1, y_2, ...) \right\rangle \tag{42}$$

vanishes in the limit $L_s \to \infty$. For the flavor singlet case, the axial anomaly is also reproduced when taking such a chiral limit, then taking the continuum limit.

Thus we see that the effective chiral limit in this scheme is the one $L_s \to \infty$, and we can get a measure of the residual chiral symmetry breaking due to the finite extent of the fifth dimension from the residual mass term, the anomalous term in the non-singlet axial current Ward identity.