Hamiltonian Simulation of the Lattice Schwinger Model Using Exact Diagonalization

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The goal of this problem is to obtain, numerically, the energy spectrum of the Schwinger model with open boundary conditions (OBC) for a given set of parameters, and to evaluate both the continuous and the Trotterized evolution of the Schwinger model. Consider the Schwinger Hamiltonian in the purely fermionic formulation:

$$H = \mathrm{x}\sum_{n=0}^{N-2}\left[\psi^\dagger(n)\psi(n+1) + \text{h.c.}\right] + \mu\sum_{n=0}^{N-1}(-1)^n\psi^\dagger(n)\psi(n) + \sum_{n=0}^{N-2}\left\{\epsilon_0 + \sum_{m=0}^{n}\left[-\psi^\dagger(m)\psi(m) + \frac{1-(-1)^m}{2}\right]\right\}^2. \tag{1}$$

Part (a) Show that after a Jordan-Wigner transformation, one can map the fermionic Hamiltonian to a qubit Hamiltonian of the form:

$$H = \mathrm{x}\sum_{n=0}^{N-2}\left(\sigma_n^+\sigma_{n+1}^- + \text{h.c.}\right) + \frac{\mu}{2}\sum_{n=0}^{N-1}(-1)^{n+1}\sigma_n^{\mathrm{z}} + \sum_{n=0}^{N-2}\left\{\epsilon_0 + \frac{1}{2}\sum_{m=0}^{n}\left[\sigma_m^{\mathrm{z}} - (-1)^m\right]\right\}^2 \tag{2}$$

$$:= \sum_{n=0}^{N-2} H_{n,n+1}^{(\mathrm{XX})} + \sum_{n=0}^{N-2} H_{n,n+1}^{(\mathrm{YY})} + H^{(\mathrm{ZZ})} + H^{(\mathrm{Z})}. \tag{3}$$

Make sure to identify $H_{n,n+1}^{(\mathrm{XX})}$, $H_{n,n+1}^{(\mathrm{YY})}$, $H^{(\mathrm{ZZ})}$, and $H^{(\mathrm{Z})}$, where $H_{n,n+1}^{(\mathrm{XX/YY})}$ are terms proportional to the product of two Pauli-X/Y matrices on qubits n and $n+1$, $H^{(\mathrm{ZZ})}$ are all terms proportional to the product of two Pauli-Z matrices on two distinct qubits, while $H^{(\mathrm{Z})}$ are all terms proportional to a Pauli-Z matrix on a single qubit.

Part (b) Consider a system of $N = 4$ staggered sites. Construct the Hamiltonian matrix associated with the Hamiltonian in Part (a). This means that you have to find all the matrix elements in terms of x, μ, and ϵ_0. Now try to diagonalize the Hamiltonian to find the energy eigenvalues for the following model parameters: $\epsilon_0 = 0$, $\mathrm{x} = 0.6$, and $\mu = 0.1$.

Part (c) Consider the strong-coupling vacuum (i.e., the eigenstate of the Hamiltonian in the limit $\mathrm{x} = 0$) and call it $|\psi(0)\rangle$. Apply the time-evolution operator e^{-itH} onto this state for the total evolution time $t = 5$. For all other parameters, use the values given in

Part (b). This procedure gives you $|\psi(t)\rangle = e^{-itH}|\psi(0)\rangle$. Evaluate and plot, as a function of time, the Loschmidt echo, i.e., the survival probability the initial state, defined as

$$\mathcal{P}(t) := |\langle\psi(0)|\psi(t)\rangle|^2 \,, \tag{4}$$

What do you learn from this quantity?

Part (c) For the same procedure and parameters as in the previous part, evaluate and plot, as a function of time, the staggered fermion density defined as

$$\nu(t) := \frac{1}{N}\sum_{n=0}^{N-1}\nu_n(t), \tag{5}$$

where

$$\nu_n(t) := \langle\psi(t)|\frac{(-1)^{n+1}\sigma^{\mathrm{z}}_n + 1}{2}|\psi(t)\rangle \,. \tag{6}$$

How many electron-positron pairs do you have in the initial state? (Recall the mapping we talked about in the lecture.) Does the dynamics generate electron-positron pairs? The phenomenon of pair production out of "vacuum" is one of the hallmarks of a relativistic quantum field theory, which can be seen in nonequilibrium dynamics as simple as the one you just studied! Compare your results to the lower panel in fig.6 of 2112.14262.

Part (d) Now divide the time evolution for duration $t = 5$ into $N_T = 10$ Trotter steps. Hence, the evolution time for each step is $\delta t = 0.5$. Apply the first-order Trotter-Suzuki approximation

$$V(t) = \prod_{i=1}^{N_T}\left(e^{-i\delta t H^{(Z)}}e^{-i\delta t H^{(ZZ)}}\prod_{n=0}^{N-2}e^{-i\delta t H^{(XX)}_{n,n+1}}\prod_{n=0}^{N-2}e^{-i\delta t H^{(YY)}_{n,n+1}}\right) \tag{7}$$

to evolve the strong-coupling vacuum state $|\psi(0)\rangle$ as in Part (c) for time t. Notice that there are more than one way to do the term decomposition, but we are going to adopt the form above. Plot both the Loschmidt echo and the staggered fermion density defined in the previous parts as a function of Trotterized time. Overlay your plots with the continuous time evolution you saw previously to clearly see any deviation from exact result. If you are up for it, change the Trotter step size to explore how Trotter error responds to this change.

Part (e) [**Bonus**] First show that the total charge operator

$$Q := \sum_{n=0}^{N-1}\frac{\sigma^{\mathrm{z}}_n + (-1)^{n+1}}{2} \tag{8}$$

commutes with the Hamiltonian. What are the eigenvalues of this operator? Since $[H, Q] = 0$, the eigenvalues of Q are conserved quantities: if you start the evolution in a given Q-eigenvalue sector, Hamiltonian time dynamics should not take you out of this sector. Does the Trotter-Suzuki expansion we picked in the previous part conserve the total charge Q? If yes, argue why. If no, can you come up with a decomposition that conserves the total charge? This example shows that approximate algorithms can break the symmetries of the model!