Prof Will Detmold | LGT4HEP | Sep 16, 2025

Unit 2: Scalar field theory → gauge theory

Last time: lightning review of QCD
brief intro to scalar field theory with a lattice regulator

1. Lattice geometry in 4D

$$\Lambda = \{ n \in (n_1, n_2, n_3, n_4) \mid n_1, n_2, n_3 \in \{0, 1, \ldots, N-1\}, n_4 \in \{0, 1, \ldots, N_T - 1\} \}$$

2. Euclidean action for free scalar

$$S_E = \int d^4x \left(\tfrac{1}{2} \phi (-\partial^2 + m_0^2) \phi \right)$$

$$\downarrow$$

$$S_E^{latt} = a^4 \sum_{n \in \Lambda} \left(\sum_{\nu=1}^{4} \frac{1}{2} \left(\frac{\phi(n+\hat{\nu}) - \phi(n)}{a} \right)^2 + \frac{m_0^2}{2} \phi(n)^2 \right) \xrightarrow{a \to 0} S_E$$

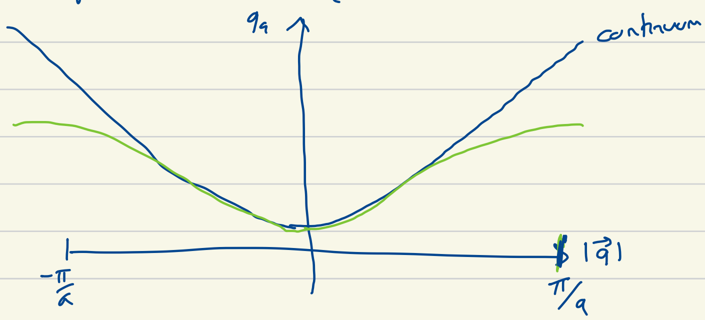
lattice spacing

3. Scalar field propagator: F.T. of $\langle 0 | \phi(x) \phi(0) | 0 \rangle$

$$G(q) \; \frac{1}{q^2 + m_0^2} \longrightarrow G_{latt}(q) = \left[m_0^2 + \sum_{\nu=1}^{4} \frac{4}{a^2} \sin^2\left(\frac{q_\mu a}{2}\right) \right]^{-1}$$

$a \to 0$

Dispersion relation ($m_0 = 0.2$)



Unit 2: More about scalar field theory: expansions & triviality
Intro to Lattice Gauge Theory

$O(n)$ models & interactions in scalar field theory

- Set of real scalar field $\varphi = (\varphi_0, \varphi_1, \dots \varphi_{n-1})^T$

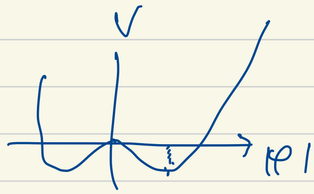
$$\mathcal{L} = \sum_i \partial_\mu \varphi_i \partial_\mu \varphi_i + V(\varphi^T\varphi)$$

Θ = parameters of transformation

invariant under global transformation : $\varphi(x) \to \varphi'(x) = \Omega(\Theta)\varphi(x)$
with $\Omega(\Theta) \in O(n)$ $[n \times n \text{ matrix}, \ \Omega^T\Omega = I]$

- Choose a quartic potential

$$V = \frac{m_0^2}{2}\varphi^T\varphi + \frac{\tilde{\lambda}_0}{4}(\varphi^T\varphi)^2 \qquad \tilde{\lambda}_0 \geq 0$$



- If $m_0^2 \geq 0$ and $\langle\varphi_i\rangle = 0$ preserves $O(n)$ symmetry
- If $m_0^2 \leq 0$ potential minimised when

$$\varphi_g^T\varphi_g = v^2 = -m_0^2/\lambda_0 > 0$$

vacuum state spontaneously breaks $O(n)$ symmetry (Goldstone's Thm)
$\Rightarrow$ massless scalar Goldstone boson

- After discretising

$$S_E^{latt} = a^4 \sum_{n\in\Lambda}\left(\frac{1}{2}\sum_{\nu=1}^{4}\left(\frac{\varphi^i_{n+\hat\nu} - \varphi^i_n}{a}\right)^2 + \frac{m_0^2}{2}\varphi_n^{i\,2} + \frac{\tilde{\lambda}_0}{4}(\varphi_n^{i\,2})^2\right) \qquad 1$$

$$= \frac{1}{2a^2}\sum_\nu\left(\varphi^{i\,2}_{n+\hat\nu} + \varphi_n^{i\,2} - 2\varphi^i_{n+\hat\nu}\varphi^i_n\right)$$

$$= 4\varphi_n^{i\,2} - \sum_\nu \varphi^i_{n+\hat\nu}\varphi^i_n$$

CHECK!!

and redefine

$$a\varphi_n^i \to \sqrt{2\kappa}\,\phi_n^i, \quad \boxed{a m_0^2 = \frac{1-2\lambda_0}{\kappa} - 8}, \quad \tilde{\lambda}_0 = \lambda_0/\kappa^2$$

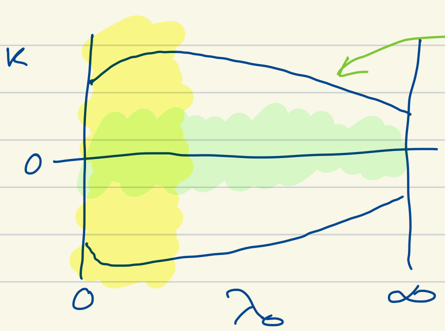
$(m_0, \tilde{\lambda}_0)$
$\downarrow$
(κ, λ_0)
- bare parameters
- $\kappa \to 0$ as $m_0 \to \infty$

$$\Rightarrow S_E^{latt} = \sum_{n\in\Lambda}\left(-2\kappa\sum_\nu \phi^i_{n+\hat\nu}\phi^i_n + \phi_n^{i\,2} + \lambda_0(\phi_n^{i\,2} - 1)^2 - \lambda_0\right)$$

$$= \sum_n s(\phi_n, \lambda) - 2\kappa \sum_{\langle n,m\rangle} \phi_n^i\phi_m^i \qquad 2$$

all neighbouring sites

Phase Structure



$\kappa = \kappa_{crit}(\lambda_0)$
corresponding to symm $\Leftrightarrow$ broken transition
- second order phase transition
- physical scales diverge in lattice units
$\Rightarrow$ continuum limit

A) Weak coupling ($\lambda_0 \to 0$) expansion

B) "Hopping expansion" in $\kappa \to 0$

A) Weak coupling: Feynman rules and loops similar to in QFT class but a bit more cumbersome

1) Vertices come from Lagrangian $\sim \lambda(\delta_{\alpha\beta}\delta_{\gamma\delta} + \dots)$

2) Edges come from free propagator

$$G_{latt}(q) = \left[m_0^2 + \sum_{\nu=1}^{4} \frac{4}{a^2} \sin^2\left(\frac{q_\mu a}{2}\right) \right]^{-1}$$

3) $\int d^4q \to \int_{-\pi/a}^{\pi/a} dq_1 \dots \int_{-\pi/a}^{\pi/a} dq_4$

Ex

$$\left(\text{———}\right)^{-1}_{1\,loop} = \left(\text{———}\right)^{-1}_{bare} + $$

Requires

$$J(am_0) \quad \int_{-\pi/a}^{\pi/a} dq_1 \dots \int_{-\pi/a}^{\pi/a} dq_4 \; \frac{1}{a^2(m_0^2 + \hat{q}^2)}$$

expand around $a \to 0$

$$J(am_0) = J(0) - a^2 m_0^2 \int_q \frac{1}{a^2\hat{q}^2(a^2m_0^2 + a^2\hat{q}^2)} + \dots$$

$J(0)$ convergent $\Rightarrow J(0) = r_0 = 0.1549\dots$

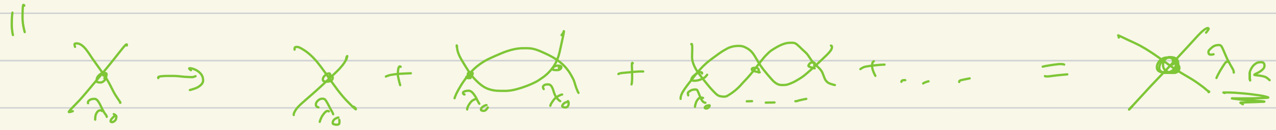
second term is IR divergent if $m_0 \to 0$

In continuum regulated version

$$\int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2(q^2+m^2)} = \int_0^\Lambda \frac{q^3 dq \int d\Omega}{(2\pi)^4} \frac{1}{q^2(q^2+m^2)} \overset{x=q^2}{=} \int_0^{\Lambda^2} \frac{1}{16\pi^2} \frac{x\,dx}{x(m^2+x)} = \frac{1}{16\pi^2} \ln\left(\frac{\Lambda^2+m^2}{m^2}\right)$$

lattice integral will have same IR divergence so

$$-\left[\int_q \frac{1}{a^4\hat{q}^2(m_0^2+\hat{q}^2)} + \frac{1}{16\pi^2}\ln(a^2m_0^2) \right] = r_1 = -0.030346\dots$$



Perturbation theory relate bare & renormalised couplings $(\lambda_0, m_0) \to (\lambda_R, m_R)$

$$m_R^2 = m_0^2 + \lambda_0(n+2)\left(\frac{r_0}{a^2} + r_1 m_0^2 + \frac{m_0^2}{16\pi^2}\ln(a^2m_0^2)\right) + \mathcal{O}(\lambda_0^2)$$

$$\lambda_R = \lambda_0 + \lambda_0^2(\dots) + \mathcal{O}(\lambda_0^3)$$

B) Hopping expansion (n=1 here)

$$\langle 0|\phi_{n_1}\phi_{n_2}\cdots\phi_{n_k}|0\rangle = \frac{1}{Z}\int \mathcal{D}\phi\, e^{-S_E^{latt}[\phi]}\,\phi_{n_1}\cdots\phi_{n_k}$$

$$= \frac{1}{Z}\int\mathcal{D}\phi \prod_n e^{-s(\phi_n,\lambda)}\, \underbrace{\exp\Big[-2\kappa\sum_{\langle n,m\rangle}\phi_n\phi_m\Big]}_{\text{expand}}\, \phi_{n_1}\cdots\phi_{n_k}$$

$$= \frac{1}{Z}\sum_j \frac{(-2\kappa)^j}{j!}\int\mathcal{D}\phi\prod_n e^{-s(\phi_n,\lambda_0)}\Big(\sum_{\langle n,m\rangle}\phi_n\phi_m\Big)^j \phi_{n_1}\cdots\phi_{n_k}$$

$$Z = \sum_j \int\mathcal{D}\phi \prod_n e^{-s(\phi_n,\lambda)}\Big(\sum\phi_n\phi_m\Big)^j \frac{(-2\kappa)^j}{j!}$$

$$Z = Z_0+Z_1+Z_2+\cdots \qquad Z_j \propto \kappa^j$$

\# sites

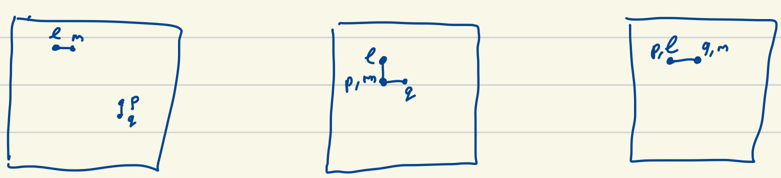
$$Z_0 = \int\mathcal{D}\phi\prod_n e^{-s(\phi_n,\lambda_0)} = \left(\int_{-\infty}^{\infty}d\phi\, e^{-s(\phi,\lambda_0)}\right)^{\Omega} = (z_0)^\Omega$$

$$Z_1 = \int\mathcal{D}\phi\prod_n e^{-s(\phi_n,\lambda_0)}\sum_{\langle \ell,m\rangle}\phi_\ell\phi_m$$

$$= (z_0)^{\Omega-2}\sum_{\langle\ell,m\rangle}\left(\int_{-\infty}^\infty d\phi_\ell\, e^{-s(\phi_\ell)}\phi_\ell\right)\left(\underbrace{\int_{-\infty}^{\infty} d\phi_m\, e^{-s(\phi_m,\lambda_0)}\phi_m}_{=0}\right)$$

$$=0$$

$$Z_2 = \int\mathcal{D}\phi\prod_n e^{-s(\phi_n,\lambda_0)}\,\frac{4\kappa^2}{2}\Big(\sum_{\ell,m}\phi_\ell\phi_m\Big)\Big(\sum_{p,q}\phi_{p,q}\Big)$$



a) b) c)

for c) $z_0^{\Omega-2}\left(\int d\phi_\ell\, e^{-s(\phi_\ell,\lambda)}\phi_\ell\phi_p\delta_{p,\ell}\right)\left(\int d\phi_m\, e^{-s(\phi_m,\lambda)}\phi_m\phi_q\delta_{m,q}\right)$

$$\Rightarrow Z_2 = 2\kappa^2 z_0^{\Omega-2}\,\gamma_2^2(\lambda) \qquad \boxed{\gamma_n(\lambda) = \int_{-\infty}^{\infty}d\phi\, e^{-s(\phi,\lambda)}\phi^n}$$

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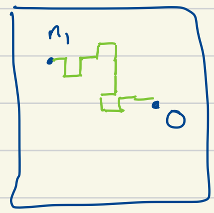
$$\langle 0 | \phi_{n_1} \phi_0 | 0 \rangle \longrightarrow$$  $\sim (-2\kappa)^{\# \text{links}}$

$\boxed{\boxed{\text{Lüscher \& Weisz } 1987, 1989}}$

Computable !

Computed to $O(\kappa^{14})$

- Zero momentum 2 pt function

$$\chi_2(\lambda_0, \kappa) = \sum_n \underbrace{\langle 0 | \phi_n \phi_0 | 0 \rangle_{conn}}_{e^{ip\cdot n}} = \gamma_2 + 16\kappa\gamma_2^2 + (2\kappa)^2(52\gamma_2^3 + 4\gamma_2\gamma_4) + \ldots$$

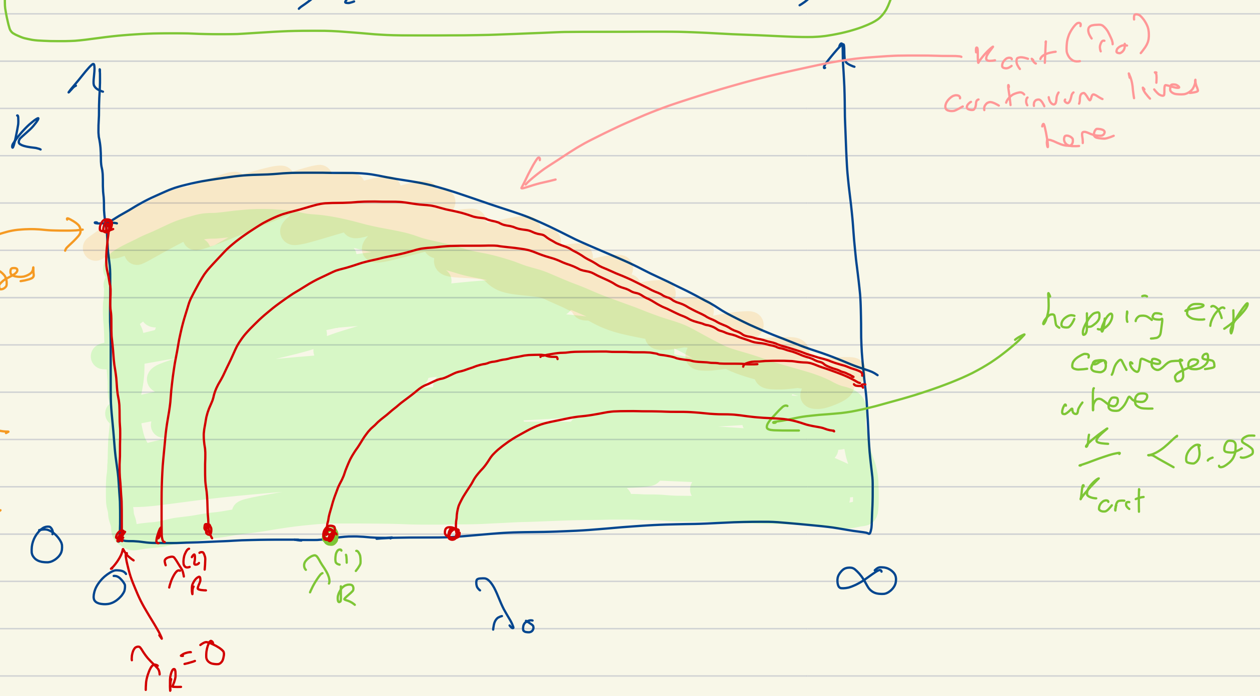
$$\chi_4(\lambda_0, \kappa) = \sum_{n,m,\ell} \langle 0 | \phi_n \phi_m \phi_\ell \phi_0 | 0 \rangle_{con} = (\gamma_4 - 3\gamma_2^2) + 64\kappa\gamma_2(\gamma_4 - 3\gamma_2^2) + \ldots$$

$$\mu_2(\lambda_0, \kappa) = \sum_n n^2 \langle 0 | \phi_n \phi_0 | 0 \rangle_{con} = 2\kappa\gamma_2^2 + 512\kappa^2\gamma_2^3 + \ldots$$

CHECK THE LEADING TERM

in terms of these 2-pt, 4pt functions

$$\boxed{m_R^2(\lambda_0, \kappa) = 8\frac{\chi_2}{\mu_2}, \quad \lambda_R(\lambda_0, \kappa) = -64\frac{\chi_4}{\mu_2}}$$



$\frac{\lambda_0}{16\pi^2}$	$\frac{\lambda_R}{16\pi^2}$	$a m_R$
0	0	0.65
∞	0.3	0.49

Except for $\lambda_R = 0$ curves never reach $\kappa_{crit}(\lambda)$ so there is no continuum limit

Non-perturbatively $\lambda\phi^4$ $O(n)$ models in 4D are <u>trivial</u>

<u>Gauge Fields:</u>

Yang Mills $\quad \mathcal{L}_{YM} = \frac{1}{2} tr[F_{\mu\nu} F_{\mu\nu}] \quad \left(+ \Theta\, tr[F_{\mu\nu} F_{\rho\sigma}] \varepsilon_{\mu\nu\rho\sigma} \right)$

ignore this

$F_{\mu\nu} = T_a F^a_{\mu\nu}$ $\qquad T^a$ = generators of gauge group G

$tr[T^a] = 0 \quad , \quad [T_a, T_b] = i f_{abc} T_c$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} [A^b_\mu, A^c_\nu]$$

Invariant under local transformations: $\Omega(x) = e^{-\frac{i}{2}\Theta_a(x) T_a} \in G^\Omega$

$$A_\mu(x) \xrightarrow[G.T.]{} A'_\mu(x) = \Omega(x) A_\mu(x) \Omega^{-1}(x) - i (\partial_\mu \Omega) \Omega^{-1}(x)$$

$$F_{\mu\nu}(x) \rightarrow F'_{\mu\nu}(x) = \Omega(x) F_{\mu\nu} \Omega^{-1}(x)$$

$$\mathcal{L}_{YM} \rightarrow \frac{1}{2} tr\left[\cancel{\Omega(x)} F_{\mu\nu}(x) \cancel{\Omega^{-1}(x) \Omega(x)} F_{\mu\nu}(x) \cancel{\Omega^{-1}(x)}\right]$$

$$= \mathcal{L}_{YM}$$

<u>global symm</u>

Consider $O(n)$ model $\quad \varphi(x) = \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_n(x) \end{pmatrix}$ $\qquad \varphi \rightarrow \Omega(\Theta)\varphi = \varphi'$

$\varphi^\dagger \rightarrow \varphi^\dagger \Omega^{-1}(\Theta)$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi^\dagger(x) \partial_\mu \varphi(x) + V(\varphi^\dagger \varphi)$$

Try a local symmetry transformation $\Omega(\Theta(x))$

$$\partial_\mu \varphi(x) \rightarrow \partial_\mu \varphi'(x) = (\partial_\mu \Omega(x)) \varphi(x) + \Omega(x) \partial_\mu \varphi(x)$$

$\mathcal{L}$ is not invariant!

covariant derivative

Fix this by $\partial_\mu \to D_\mu = \partial_\mu - iA_\mu$

$$D_\mu\varphi \to D'_\mu\varphi' = (\partial_\mu - iA'_\mu)\varphi'$$
$$= (\partial_\mu\Omega)\varphi + \Omega\partial_\mu\varphi - i\Omega A_\mu\Omega^{-1}\varphi - (\partial_\mu\Omega)\Omega^{+}\varphi$$
$$= \Omega(\partial_\mu - iA_\mu)\varphi$$
$$= \Omega D_\mu\varphi$$

$$D_\mu\varphi^+ \to D_\mu\varphi^+\Omega^+$$

$$\mathcal{L} = \frac{1}{2}D_\mu\varphi^+ D_\mu\varphi \to \mathcal{L}$$ gauge invariant

With lattice discretisation

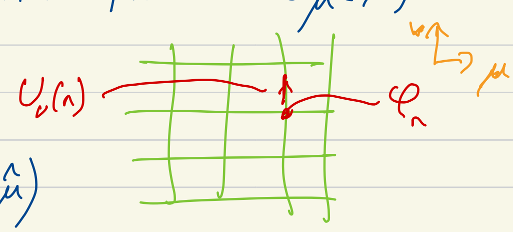
$$\partial_\mu\varphi^+\partial_\mu\varphi \to \frac{1}{a^2}\sum_\nu\left(\varphi^+_{n+\hat\nu} - \varphi^+_n\right)\left(\varphi_{n+\hat\nu} - \varphi_n\right)$$

$$\varphi_n \to \varphi'_n = \Omega_n\varphi_n$$

so

$$\varphi^+_{n+\hat\nu}\varphi_n \xrightarrow{G.T.} \varphi^+_{n+\hat\nu}\underbrace{\Omega^{-1}_{n+\hat\nu}\Omega_n}_{\neq 1}\varphi_n$$ not gauge invariant

Solve this by introducing a new "link field" $U_\mu(x)$ that is a link between two neighboring sites



demand

$$U_\mu(n) \to \Omega(n)U_\mu(n)\Omega^{-1}(n+\hat\mu)$$

$$\mathcal{L}_{latt} = \frac{1}{a^2}\sum_\nu\Big(\varphi^+_{n+\hat\nu}\varphi_{n+\hat\nu} - \varphi^+_{n+\hat\nu}U_{-\nu}(n+\hat\nu)\varphi_n - \varphi^+_n U_\nu(n)\varphi(n+\hat\nu) + \varphi^+_n\varphi_n$$

$$U_{-\nu}(n) \equiv U_\nu^{-1}(n-\hat\nu)$$

$$\xrightarrow{GT} \mathcal{L}_{latt}$$ invariant

As in continuum, constructed gauge invariant theory by introducing a new field

What are link fields & how are they related to continuum gauge field?

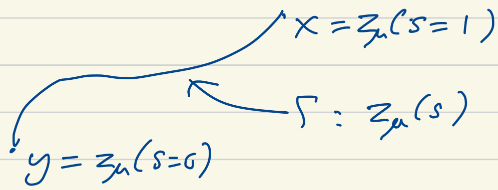
Wilson lines "parallel transporters" along a path Γ

$$E_\Gamma(x,y) = P\exp\left(-i\int_\Gamma A\cdot ds\right)$$

path ordered exponential

$$= \lim_{N\to\infty} \exp\left(-i\int_{z_N}^{x} dz_\mu A_\mu(z)\right) \cdots \exp\left(-i\int_{z_j}^{z_{j+1}} dz_\mu A_\mu(z)\right) \cdots \exp\left(-i\int_{y}^{z_1} dz_\mu A_\mu(z)\right)$$

solve the diff eq



$$D_s E_\Gamma(x,y) = 0$$

$$D_s = \frac{dz_\mu(s)}{ds} D_\mu = \frac{dz_\mu(s)}{ds}\left(\partial_\mu - iA_\mu(z)\right)$$

$$= \frac{d}{ds} - i\frac{dz_\mu}{ds} A_\mu$$

$\Rightarrow$

$$\frac{d}{ds} E_\Gamma(x,y) = i\frac{dz_\mu}{ds} A_\mu E_\Gamma(x,y)$$

CHECK E_Γ SATISFIES THIS

Under a gauge transformation

$$E_\Gamma(x,y) \to \Omega(x) E_\Gamma(x,y) \Omega^{-1}(y)$$

CHECK THIS

$$\Rightarrow \quad \mathrm{tr}\, E_\Gamma(x,x) \to \mathrm{tr}\left[\Omega(x) E_\Gamma(x,x) \Omega^{-1}(x)\right] = \underbrace{\mathrm{tr}\left[E_\Gamma(x,x)\right]}_{\text{gauge invariant}}$$

Now lets take Γ to be straight line path from $n \to n+\hat{\nu}$

$$E_{\text{straight}}(n+\hat{\nu}, n) \equiv U_\nu(n) = P\exp\left[-i\int_n^{n+\hat{\nu}} dz_\mu A^\mu(z)\right]$$

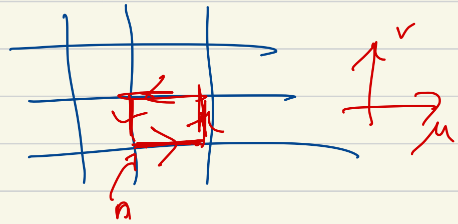
lets take $a \to 0$, $A^\mu(z) = A^\mu(n)$ constant on the path $\int_n^{n+\hat{\nu}} dz_\rho = a\delta_{\nu\rho}$

$$U_\mu(n) = \exp\left[-ia A_\mu(n)\right]$$

What about $\mathrm{tr}(F_{\mu\nu}F_{\mu\nu})$: pure gauge?

Wilson 1974 : Wilson action built in terms of closed paths
– simplest closed path

Plaquette $P_{\mu\nu} = U_\mu(n)U_\nu(n+\hat{\mu})U^\dagger_\mu(n+\hat{\nu})U^\dagger_\nu(n)$



$$S_{Wilson} = \frac{\beta}{N}\sum_n \sum_{\mu<\nu=1}^{4} \mathrm{Re\,Tr}[1 - P_{\mu\nu}(n)]$$

SU(N)

$a \to 0$

$$\frac{a^4\beta}{N}\sum_n \sum_{\mu,\nu} \mathrm{tr}[F_{\mu\nu}^2] + \mathcal{O}(a^6)$$

HW : Show this! You should do this once

① use BCH : $P_{\mu\nu} = \exp(\ldots)\exp(\ldots)\exp(\ldots)\exp(\ldots)$
$= \exp(\ldots\ldots)$

② Taylor expand $A_\mu(n+\hat{\nu})$ around $A_\mu(n)$

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Last time Plaquette $P_{\mu\nu} = U_\mu(n) U_\nu(n+\hat\mu) U^\dagger_\mu(n+\hat\nu) U^\dagger_\nu(n)$

$$S_{Wilson} = \frac{\beta}{N} \sum_n \sum_{\mu<\nu=1}^{4} \mathrm{Re}\,\mathrm{Tr}\left[1 - P_{\mu\nu}(n)\right] \qquad \boxed{\beta = \frac{2N}{g^2}}$$

Partition function.

continuum

$$Z = \int \mathcal{D}A_\mu \, e^{-S[A]} \longrightarrow Z = \int \mathcal{D}U \, e^{-S[U]}$$

- need to gauge fixing
- Faddeev-Popov ghosts

$U_\mu(n) = \exp[iaA_\mu(n)] \in$ gauge group G ← compact Lie group

No need to gauge fix or add ghosts

- Integration measure

$$\mathcal{D}U = \prod_{n\in\Lambda} \prod_{\mu=1}^{4} \underbrace{dU_\mu(n)}_{\in G}$$

- Individual link integrations of G ⟹ "Haar measure" define for $V, U \in G$

$$\underbrace{dU = d(VU) = d(UV)}_{\text{left-right invariant}}, \quad \underbrace{\int dU = 1}_{\text{normalized}}, \quad \underbrace{\int dU(\alpha f(U) + \beta g(U)) = \alpha \int f\, dU + \beta \int g\, dU}_{\text{linear}}$$

- concrete realisation : integrate over the parameters of some parametrisation of group elements in G. For $G = SO(3)$ this is just Euler angles

$$dU_\mu(n) \longrightarrow d\alpha_1 \ldots d\alpha_k \quad \text{for some angles } k$$

[Broznan, Phys Rev D38 (1988) 1944]

- Haar measure : only singlets under G (invariants) integrate to something $\neq 0$

$$\int dU\, U = \int d(VU)\, U \overset{U' = VU}{=} \int dU' V^\dagger U' = \int d(U'W^\dagger) V^\dagger U' \overset{U'' = U'W^\dagger}{=} \int dU'' V^\dagger U'' W$$

$$= V^\dagger \left[\int dU'' U''\right] W$$

true for arbitrary $V, W \in G$ ⟹ $\int dU\, U = 0$

Correlation function

$$\langle 0 | U_\mu(n) U_\nu(p) \ldots | 0 \rangle \equiv \frac{1}{Z} \int \mathcal{D}U \, e^{-S[U]} U_\mu(n) U_\nu(p) \ldots$$

- Physical quantities do not depend on any individual link (physics is in the gauge invariant combination of links)

some particular link

all other links

$$S[U] = \hat{S}[\hat{U}] + \tilde{S}[\hat{U}, \{U_\mu(n)\}]$$

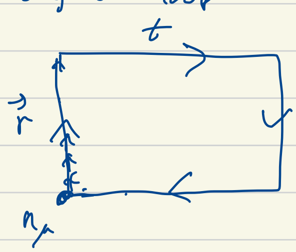
$$\langle U_\mu(n) \rangle = \frac{1}{Z} \int \mathcal{D}U \, e^{-S[U]} U_\mu(n)$$

$$= \frac{1}{Z} \int \mathcal{D}\hat{U} \, e^{-\hat{S}[\hat{U}]} \prod_{\nu=1}^{4} dU_\nu(n) \, e^{-\tilde{S}[\hat{U}, U_\nu(n)]} U_\mu(n)$$

$$= 0$$

Wilson loop expectation values

Rectangular loop



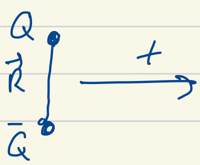
$$W_C = \mathrm{tr}\left[S(\vec{n}, \vec{r}, 0) \, T(\vec{n}+\vec{r}, t) \, S^\dagger(\vec{n}, \vec{r}, t) \, T^\dagger(\vec{n}, t) \right]$$

$$S(\vec{n}, r\hat{j}, t) = \prod_{\vec{m}=\vec{n}}^{\vec{n}+r\hat{j}} U_j(\vec{m}, t)$$

$$T(\vec{n}, t) = \prod_{j=1}^{t} U_4(\vec{n}, j)$$

$$\langle W_C \rangle = \frac{1}{Z} \int dU \, e^{-S[U]} W_C[U]$$

- Can use the freedom of removing the dependence on particular gauge links to set $T = T^\dagger = I$



$\sum_n |n\rangle\langle n|$ energy eigenstates

$$\langle W_C \rangle = \langle 0 | Q\bar{Q}(\vec{R}, 0) \, \bar{Q}Q(\vec{R}, t) | 0 \rangle$$

$$e^{-\hat{H}t} \bar{Q}Q(\vec{R}, 0) e^{\hat{H}t}$$

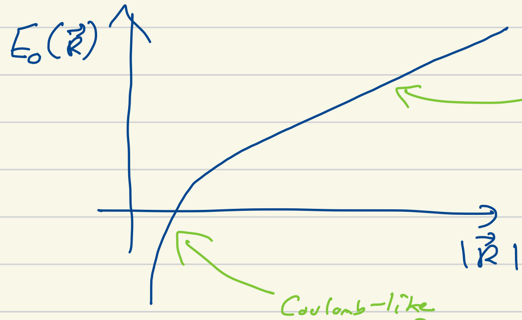
$$= \sum_n \langle 0 | Q\bar{Q}(\vec{R}, 0) | n \rangle \langle n | \bar{Q}Q(\vec{R}, t) | 0 \rangle$$

$$= \sum_n e^{-E_n(\vec{R}) t} |\langle 0 | Q\bar{Q}(\vec{R}, 0) | n \rangle|^2$$

$$\xrightarrow[t \to \infty]{} \# \, e^{-E_0(\vec{R}) t}$$

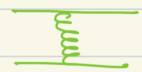
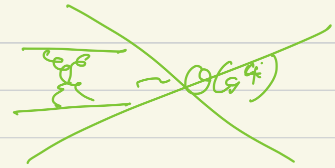
$\langle W_C \rangle \xrightarrow[t \to \infty]{}$ tells us about $E_0(\vec{R})$

• Numerical calculations of $\langle W_{R\times T}\rangle$



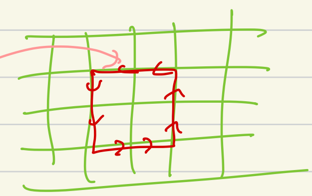
linear rising energy
$\Rightarrow$ constant force
$\Rightarrow$ infinite energy to separate Q & $\overline{Q}$

Coulomb-like contribution from one gluon exchange

 $\sim \mathcal{O}(g^2)$  $\sim \cancel{\mathcal{O}(g^4)}$

<u>Strong Coupling Expansion</u> $\beta = \frac{2N}{g^2}$, expand in β = strong coupling $g\to\infty$

$$\langle W_C\rangle = \frac{1}{Z}\int \mathcal{D}U\, e^{-\frac{\beta}{N}\sum_P(\mathrm{tr}(P)+\mathrm{tr}(P^\dagger))}\ \mathrm{tr}\Big[\prod_{\ell\in C} U_\ell\Big]$$



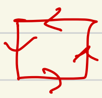
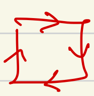
$\int dU_\mu(n)\, U_\mu(n) = 0$

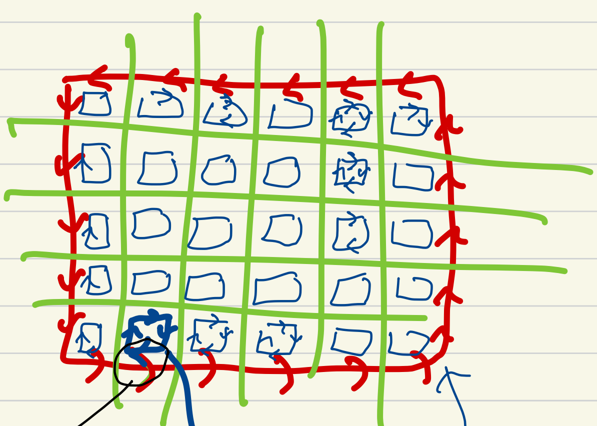
What is $\langle W_C\rangle$ for $\beta = 0$?

$\langle W_C\rangle_{\beta=0} = 0$

But $\boxed{\int dU\ U_{ab} U^\dagger_{cd} = \frac{1}{N}\delta_{ad}\delta_{bc}}$ $SU(N)$ $\qquad \int dU\, U_{ab} U_{cd} = 0$

So need a $U_\mu(n)U_\mu(n)^\dagger$ in integration to get non-zero

$P_{\mu\nu} =$  $\qquad P^\dagger_{\mu\nu} =$ 



$\int dU\, U_{ab} U^\dagger_{cd} = \frac{1}{N}\delta_{ad}\delta_{bc}$

$P^\dagger_{\mu\nu}$ from action

N_A plaquettes
area $A = a^2 N_A$

$R\times T$

$$\langle W_C\rangle = \frac{1}{Z}\int \mathcal{D}U\, \frac{1}{N_A!}\left(\frac{\beta}{N}\right)^{N_A}\left(\prod_{P\in A_C}\mathrm{tr}(P^\dagger)\right)\mathrm{tr}\Big[\prod_{\ell\in C} U_\ell\Big]$$

$$= \mathrm{tr}[I]\left(\frac{\beta}{N}\right)^{N_A}\left(\frac{1}{N}\right)^{N_A}$$

$$\sim \exp\left(N_A \log\frac{\beta}{N^2}\right)$$

$$\sim \exp(-\sigma A)$$

string tension

$\Rightarrow E_0(\vec{R}) = \sigma|\vec{R}|$

$\Rightarrow$ Confinement in YM theory