LGTFA25-Unit3-day2

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1 Continuum chiral symmetry

Spontaneous chiral symmetry breaking is a very important aspect of QCD in the continuum theory, so we have plenty motivation to try to accurately represent this phenomena on the lattice. However, as we saw with Wilson fermions, to get rid of the doublers we needed to break Chiral symmetry explicitly in the Dirac operator. There is a deep reason for this; the Nielsen-Ninomiya theorem which is a no-go theorem relating to chiral fermions on a lattice. This leads us to generalize chiral symmetry on the lattice using the Ginsparg-Wilson equation. But first, we look at chiral symmetry in the continuum to get some idea of why it's so important to properly model on the lattice.

We first look at a theory with only a single flavor of fermion. The action for a massless fermion is

$$S_F[\psi,\overline{\psi},A]=\int d^4x\overline{\psi}\gamma_\mu(\partial_\mu+iA_\mu)\psi=\int d^4x\overline{\psi}D\psi \tag{1}$$

This Where D is our massless Dirac operator. We define the flavor singlet chiral transformation as

$$\psi\to\psi'=e^{i\alpha\gamma_5}\psi,\quad \overline{\psi}\to\overline{\psi}'=\overline{\psi}e^{i\alpha\gamma_5} \tag{2}$$

The Lagrangian density is invariant under this chiral transformation

$$\begin{aligned}\overline{\psi}'\gamma_\mu(\partial_\mu+iA_\mu)\psi'&=\overline{\psi}e^{i\alpha\gamma_5}\gamma_\mu(\partial_\mu+iA_\mu)e^{i\alpha\gamma_5}\psi\\&=\overline{\psi}e^{i\alpha\gamma_5}e^{-i\alpha\gamma_5}\gamma_\mu(\partial_\mu+iA_\mu)\psi=\overline{\psi}\gamma_\mu(\partial_\mu+iA_\mu)\psi\end{aligned} \tag{3}$$

Using the fact that $\{\gamma_\mu,\gamma_5\}=0$. Note that a mass term breaks this symmetry, and the sign in the exponential does not change,

$$m\overline{\psi}'\psi'=m\overline{\psi}e^{i2\alpha\gamma_5}\psi \tag{4}$$

Chiral symmetry allows us to treat left handed(lh) and right handed (rh) massless fermions separately, it decouples them. We can introduce the lh and rh projection operators,

$$P_R=\frac{\mathbb{1}+\gamma_5}{2},\quad P_L=\frac{\mathbb{1}-\gamma_5}{2} \tag{5}$$

These projection operators obey

$$\begin{gathered}P_R^2=P_R,\quad P_L^2=P_L,\quad P_RP_L=P_LP_R=0,\quad P_R+P_L=\mathbb{1}\\ \gamma_\mu P_L=P_R\gamma_\mu,\quad \gamma_\mu P_R=P_L\gamma_\mu\end{gathered} \tag{6}$$

These projectors allow us to define right and left handed fermion fields

$$\psi_R=P_R\psi,\quad \psi_L=P_L\psi,\quad \overline{\psi}_R=\overline{\psi}P_L,\quad \overline{\psi}_L=\overline{\psi}P_R \tag{7}$$

with these left and right handed fermion fields we can write the lagrangian as a sum of a left handed part and a right handed part

$$\mathcal{L}[\psi,\overline{\psi},A]=\overline{\psi}_LD\psi_L+\overline{\psi}_RD\psi_R \tag{8}$$

The left and right handed components are completely decoupled. A mass term, which breaks this flavor singlet chiral symmetry, also mixes the left and right handed components,

$$m\overline{\psi}\psi = m(\overline{\psi}_R\psi_L + \overline{\psi}_L\psi_R) \tag{9}$$

When we refer to the chiral limit, this is the limit of vanishing quark mass, since the action is only invariant under chiral symmetry with massless quarks. For a massless Dirac operator we can summarize this behavior with the anticommutator $\{\gamma_5, D\} = 0$.

Now we generalize to N_f quark flavors. The fields now carry a flavor index, but we suppress it by writing the action in matrix/vector notation

$$S_F[\psi, \overline{\psi}, A] = \sum_f^{N_f} \int d^4x \overline{\psi}^{(f)}(\gamma_\mu(\partial_\mu + iA_\mu) + M)\psi^{(f)} \tag{10}$$

Where M is a mass matrix, which is diagonal and whose eigenvalues in flavor space are the masses of each flavor of fermion.

Note this action is invariant under vector transformations

$$\begin{aligned} \psi' &= e^{i\alpha T_i}\psi, \quad \overline{\psi}' = \overline{\psi}e^{-i\alpha T_i} \\ \psi' &= e^{i\alpha \mathbb{1}}\psi, \quad \overline{\psi}' = \overline{\psi}e^{-i\alpha \mathbb{1}} \end{aligned} \tag{11}$$

Where the T_i are the generators of $\mathrm{SU}(N_f)$. The $M = 0$ action is clearly invariant under these vector transformations, but this symmetry also extends to the case where every flavor has the same mass. This is the familiar isospin symmetry for N_f flavors. The U(1) vector symmetry holds for arbitrary masses as well, and its corresponding conserved quantity is baryon number.

Now for chiral transformations,

$$\begin{aligned} \psi' &= e^{i\alpha\gamma_5 T_i}\psi, \quad \overline{\psi}' = \overline{\psi}e^{i\alpha\gamma_5 T_i} \\ \psi' &= e^{i\alpha\gamma_5 \mathbb{1}}\psi, \quad \overline{\psi}' = \overline{\psi}e^{i\alpha\gamma_5 \mathbb{1}} \end{aligned} \tag{12}$$

Where now the left handed and right handed components of the different flavors mix. Once again, like the flavor singlet case, we find the action is invariant under axial transformations when $M = 0$.

Altogether, the massless action has the flavor symmetry

$$\mathrm{U}(N_f)_V \otimes \mathrm{U}(N_f)_A = \mathrm{SU}(N_f)_L \otimes \mathrm{SU}(N_f)_R \otimes \mathrm{U}(1)_V \otimes \mathrm{U}(1)_A \tag{13}$$

Where we use a notation for the two factors of $\mathrm{SU}(N_f)$ to stress the fact that the left and right handed components are independently symmetric under $\mathrm{SU}(N_f)$ transformations.

For a theory with degenerate masses,

$$\mathrm{U}(N_f)_V = \mathrm{SU}(N_f)_V \otimes \mathrm{U}(1)_V \tag{14}$$

and for non-degenerate masses, the group breaks down into

$$U(1)_V \otimes U(1)_V \otimes ... \otimes U(1)_V \quad (N_f \text{ times}) \tag{15}$$

In the full quantized theory, the fermion determinant is not invariant under the chiral U(1) transformations, the symmetry is explicitly broken by the fermion integration measure, which leads to the axial anomaly, that is the divergence of the flavor singlet axial current is not zero. Introducing degenerate masses reduces the total symmetry from two independent left and right handed $\mathrm{SU}(N_f)$ factors to a subgroup of vector transformations $\mathrm{SU}(N_f)_V$. If we allow for non-degenerate masses, this symmetry further breaks down into a string of N_f U(1) vector transformations.

In terms of the measured quark masses, the up and down quark are nearly degenerate, giving us an approximate vector SU(2) symmetry in the QCD action. If we include the strange quark mass, we have a much more approximate vector SU(3) flavor symmetry. On the typical QCD scale of several hundred MeV, the approximate SU(2) symmetry is quite good.

If the u and d quarks were massless, we would have an exact $\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R \otimes \mathrm{U}(1)_V$ symmetry of the action. In this case we would expect the nucleon and its parity conjugate partner N^* to have the same mass. However, no such mass degeneracy is observed, the mass difference in nature is around 600 MeV, which is far too large a difference to be explained by explicit chiral symmetry breaking from the masses of the quarks, so there must be another mechanism at work. The action is invariant, so the strong chiral symmetry breaking effect must be from spontaneous breaking of chiral symmetry. The ground state is not invariant under chiral symmetry, even though the action is. An order parameter for chiral symmetry breaking is the so called chiral condensate,

$$\langle \overline{u}(x) u(x) \rangle \tag{16}$$

The chiral condensate transforms like a mass term, so it is not invariant under chiral transformations. Due to this, when the chiral condensate is non-zero, this implies that chiral symmetry is broken spontaneously.

Another result of spontaneously broken symmetry is the appearance of Nambu-Goldstone bosons by Goldstone's theorem. Since in nature the SU(2) symmetry is approximate, these bosons for spontaneous chiral SU(2) symmetry breaking are approximately massless, very light, and are referred to as pseudo Goldstone bosons. For this particular case, these are the three light pions, which have masses around 140MeV. If these were not pseudo-goldstone bosons, we would find the pion masses would be much heavier. This is evident in the case of the approximate $SU(3)$ flavor symmetry. Since there are 8 generators of $SU(3)$, there would be 8 Goldstone bosons from its spontaneous symmetry breaking - the pseudoscalar octet. It was theorized that $U(1)_A$ was also spontaneously broken, which would introduce a 9th Goldstone boson, the η', however there is a significant mass difference between the η and η', which is referred to as the $\eta - \eta'$ puzzle. $m_\eta = 548$ MeV, $m_{\eta'} = 958$ MeV. We know now that $U(1)_A$ is not spontaneously broken, it is explicitly broken by topological field configurations called instantons, thus the η' mass is not brought down because it is not a pseudo-goldstone boson.

2 Chiral symmetry on the lattice

2.1 Nielsen-Ninomiya theorem

We saw in the first lecture the appearance of doublers, several fermion species per fermion field in the lattice action. With the Wilson fermion formulation we saw that we could make these unwanted modes on the corners of the Brillouin zone infinitely heavy in the continuum limit. In a chiral theory, the left handed and right handed components of the fermion field is decoupled, how do the doublers effect this decoupling?

In the naive fermion action, we find doublers due to spectrum doubling symmetry. This is defined by the transformation

$$\psi'(x) = e^{-ix\cdot\pi_H} M_H \psi(x), \quad \overline{\psi}'(x) = \overline{\psi}(x) M^\dagger e^{ix\cdot\pi_H} \tag{17}$$

Where $\pi_H = \pi$ if $\mu \in H$, and zero otherwise, and the 4×4 matrices M_H for $H = \mu_1, ..., \mu_h$ are given by

$$M_H = M_{\mu_1} ... M_{\mu_h}, \quad M_\mu = i\gamma_5\gamma_\mu = M_\mu^\dagger \tag{18}$$

Using the relation $M_H^\dagger \gamma_\mu M_H = e^{\pm \pi_{H,\mu}} \gamma_\mu$ we can see the naive fermion action is invariant under such a transformation. This transformation exchanges the corners of the Brioullin zone,

$$\tilde{\psi}'_p = M_H \tilde{\psi}_{p+\pi_H}, \quad \tilde{\overline{\psi}}'_p = \tilde{\overline{\psi}}_{p+\pi_H} M_H^\dagger \tag{19}$$

Let's look at the case of a more general lattice fermion action - what can be said about the doubling under some general assumptions? We start with a general fermion action

$$S = \frac{1}{\Omega} \sum_p \tilde{\overline{\psi}}_p F(p) \tilde{\psi}_p \tag{20}$$

Where $F(p)$ is our inverse fermion propagator. Its zeros correspond to the poles of the propagator, which tells us about the particle content of the theory. We assume the following properties hold for $F(p)$.

- Reflection positivity
$$F(p) = \gamma_4 F^\dagger(\boldsymbol{p}, -p_4)\gamma_4 \tag{21}$$
- Invariance under the cubic group
$$F(p) = \gamma_\mu F^\dagger(R_\mu p)\gamma_\mu \tag{22}$$
- Chiral invariance - in the massless case, the flavor singlet chiral transformations leaves the action invariant.
$$F(p) = -\gamma_5 F(p)\gamma_5 \tag{23}$$
- Locality - The interaction vanishes sufficiently fast for large distances

Altogether this gives us the condition $F(p) = -F(-p)$. Note this assumption is not satisfied by the Wilson Dirac operator, as it explicitly breaks chiral symmetry. With periodic or antiperiodic BC's, $F(p)$ is a periodic function $p_\mu \to p_\mu + 2\pi$ in every direction, our condition implies that $F(p)$ has to vanish for every $p_\mu = 0, \pm\pi$. Thus under the above assumptions, there are always at least 16 poles in the fermion propagator at the corners of the Brioullin zone.

Now what we came here for, what about the chirality of such doublers? To start we include the assumption of invariance with respect to parity,

$$P\tilde{\psi}(\vec{p}, p_4) = \sigma_P \gamma_4 \tilde{\psi}(-\vec{p}, p_4), \quad \tilde{\overline{\psi}}(\vec{p}, p_4)P = \sigma_P^* \tilde{\overline{\psi}}(-\vec{p}, p_4)\gamma_4 \tag{24}$$

with $\sigma_P \sigma_P^* = 1$. Combined with the above assumptions, that lets us write the inverse propagator as

$$F(p) = \sum_\mu i f_\mu(p)\gamma_\mu = \sum_\mu i f_\mu(p)\gamma_\mu (P_L + P_R) \tag{25}$$

For example, the free naive propagator we saw in lecture 1 corresponds to $f_\mu(p) = \sin(p_\mu)$. From the locality assumption, it follows that $f_\mu(p)$ defines a continuous vector field on a four dimensional hypertorus, $T_4 = S_1 \otimes S_1 \otimes S_1 \otimes S_1$. A particle corresponds to a zero of this vector field. We are interested in the indices of this vector field, the behavior of the vector field near the zeros. Since we assumed chiral symmetry, the chiral parts of $F(p)$ are independent, since the propagator doesn't change chirality. For now, let's consider a fermion field with only one chirality, say, a left handed fermion. We look at the propagator in the vicinity of the zero,

$$F(p) = \sum_{\mu,\nu}^{4} i\gamma_\mu J_{\mu\nu}(\overline{p})(p_\nu - \overline{p}_\nu) + \mathcal{O}(p - \overline{p})^2 \tag{26}$$

Where $J_{\mu\nu} = \partial f_\mu / \partial p_\nu$. We now the continuum form of $F(p)$, and we know that the physical momentum in the continuum limit is the factor multiplying $i\gamma_\mu$. In lattice units, ak_μ. We need to transform from old momenta p_μ to the physical momenta k_μ. This transformation preserves chirality for $\det J > 0$, but changes chirality for $\det J < 0$. For the $\det J > 0$ case, the simplest example of a zero is

$$i\gamma_\mu (p_\mu - \overline{p}_\mu) P_L \tag{27}$$

Here, no coordinate rotation is needed at all, and the corresponding pole of the propagator describes a left handed fermion. A simple example of the other case is

$$i\gamma_\mu(-1)^{\delta_{\mu 1}}(p_\mu - \overline{p}_\mu)P_L \tag{28}$$

we transform the sign factor away using our spectrum doubling transformation M_1, but $M_1^\dagger i\gamma_\mu(-1)^{\delta_{\mu 1}} P_L M_1 = i\gamma_\mu P_R$, we switch chirality. This zero of the inverse propagator describes a right handed chiral fermion. According to the spectrum doubling transformations, for $h =$ odd, the chirality changes, and for even h the chirality is preserved.

The number of lh and rh fermions is determined by the number of zeros with index $+1$ (source or sink) ($\det J > 0$), and index -1 (saddle point) ($\det J < 0$). The Poincare-Hopf theorem implies that the sum of indices of the zeros of a continuous vector field on a compact manifold is equal to the Euler characteristic of the manifold, a topological invariant that tells us about the structure of the space. For the hypertorus T_4, the Euler-number is zero, thus under the above assumptions, there is always an equal number of right handed and left handed particles in the lattice fermion propagator. This is a simple proof for the Nielsen-Ninomiya theorem, which tells us that in the fermion propagator, there are an equal number of left handed and right handed particles for every set of quantum numbers.

We used rather general assumptions, but it is important to note that this theorem holds for fermion propagation on the lattice. This says nothing about the behavior of masses in the continuum limit or couplings, thus one way to escape its limitations is to decouple the spurious fermion states from the set of physical particles, we already saw this once, with Wilson fermions, where we gave the doublers a mass proportional to the cutoff. Again, this explicitly broke chiral symmetry.

2.2 Currents and Ward-Takahashi identites

We see that there are always an equal number of left handed and right handed fermions in a discretized theory. We now talk about the conserved currents from the flavor singlet chiral and vector transformations, looking specifically how the doublers effect said currents.

In the continuum, the axial charge, Q_5, is defined by the axial vector current

$$J_5(x)_\mu = \overline{\psi}\gamma_\mu\gamma_5\psi(x) \tag{29}$$

as the integral of the density

$$Q_5 = \int d^3x J_5(x,t)_4 = \int d^3x \overline{\psi}(x,t)\gamma_4\gamma_5\psi(x,t) = \int d^3x \psi^\dagger\gamma_5\psi(x,t) \tag{30}$$

We also define the vector current and charge

$$J(x)_\mu = \overline{\psi}\gamma_\mu\psi(x), \quad Q = \int d^3x J(x,t)_4 = \int d^3x \psi^\dagger(x,t)\psi(x,t) \tag{31}$$

According to this definition, Q_5 is the number of rh fermions minus the number of lh fermions minus the number of rh antifermions plus the number of lh antifermions. For massless fermions, Q_5 is conserved in the classical theory, even in an interacting theory. In the quantum theory, this chiral symmetry is broken by the chiral anomaly. For example, in massless QED,

$$\partial_\mu J_5(X)_\mu = \frac{g^2}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}F(x)_{\mu\nu}F(x)_{\rho\sigma} \tag{32}$$

This is proportional to the gauge configuration's winding number, a topological property of the gauge field we will talk about later. We can define an axial current that is conserved, but the consequence is that this conserved axial current is not gauge invariant.

How does the axial anomaly appear on the lattice? We do get a contribution from the doublers. One may expect, that since the doublers appear with opposite chiralities, the anomaly cancels on the lattice. This is not true, and we will see how the axial anomaly is formulated on the lattice now. First we must define the vector and axial currents on the lattice. For simplicity, we consider a Wilson lattice fermion in an external gauge field,

$$U_{x\mu} = \exp\{igA_\mu\} \tag{33}$$

We can write our lattice action

$$S = \sum_x a^4 \left\{ \left(m + \frac{4r}{a} \right) (\overline{\psi}_x \psi_x) - \frac{1}{2a} \sum_{\mu=\pm 1}^{\pm 4} (\overline{\psi}_{x+\hat{\mu}} [r + \gamma_\mu] U_{x\mu} \psi_x) \right\} \tag{34}$$

This action is locally gauge invariant with respect to local U(1) transformations. The global U(1) symmetry acting on the fermion fields is

$$\psi'_x = e^{-i\alpha} \psi_x, \quad \overline{\psi}_x = \overline{\psi}_x e^{i\alpha} \tag{35}$$

In the continuum, from this global U(1) symmetry we could write down the corresponding Noether current that is conserved. In the Euclidean quantum field theory the global symmetry implies Ward-Takahashi identites which express the consequences of the symmetry in terms of Green's functions containing composite current operators.

We want to derive these lattice WT identites for the vector and axial currents. We start by rewriting the action

$$\begin{aligned} S = \sum_x a^4 \Bigg\{ & m(\overline{\psi}_x \psi_x) - \frac{1}{2a} \sum_{\mu=1}^{4} (\overline{\psi}_{x+\hat{\mu}} [r + \gamma_\mu] U_{x\mu} \psi_x) \\ & + (\overline{\psi}_x [r - \gamma_\mu] U^\dagger_{x\mu} \psi_{x+\hat{\mu}}) - 2r(\overline{\psi}_x \psi_x) \Big\} \end{aligned} \tag{36}$$

We consider the partition function

$$Z = \int \mathcal{D}[\psi, \overline{\psi}] e^{-S[\psi, \overline{\psi}, U]} \tag{37}$$

We perform a local infinitesimal transformation on the fermion fields

$$\psi_x = (1 + i\alpha_x) \psi'_x, \quad \overline{\psi}'_x = \overline{\psi}_x (1 - i\alpha_x) \tag{38}$$

From the rules of Grassmann integration, for a linear transformation, $\mathcal{D}[\psi, \overline{\psi}] = \det M \det \overline{M} \mathcal{D}[\psi, \overline{\psi}]$. Up to first order in α_x, the product of determinants is 1, and we introduce the change of variables into the partition function

$$Z = \int \mathcal{D}[\psi, \overline{\psi}] e^{-S[\psi, \overline{\psi}, U]} \left\{ 1 - i \sum_x \alpha_x \left[\frac{S \overleftarrow{\partial}}{\partial \psi'_x} \psi'_x - \overline{\psi}'_x \frac{\overrightarrow{\partial} S}{\partial \overline{\psi}'_x} \right] \right\} + \mathcal{O}(\alpha^2) \tag{39}$$

We subtract the original partition function, and implying a sum over μ, we get

$$\left\langle \Delta^b_\mu J_{x\mu} \right\rangle = 0 \tag{40}$$

where the vector current is defined as

$$J_{x\mu} = \frac{1}{2} \left\{ (\overline{\psi}_{x+\hat{\mu}} [r + \gamma_\mu] U_{x\mu} \psi_x) - (\overline{\psi}_x [r - \gamma_\mu] U^\dagger_{x\mu} \psi_{x+\hat{\mu}}) \right\} \tag{41}$$

Thus we found the lattice WT identity for the vector current, which is the consequence of the global vector symmetry in the quantized theory. In the continuum limit it implied the exact conservation of the vector current operator and its charge.

Now we look at the global $\mathrm{U}(1)_A$ chiral symmetry, an exact symmetry of the massless continuum action. The wilson term explicitly breaks this symmetry. We introduce infinitesimal chiral transformations

$$\psi_x = (1 + i\alpha_{5x}\gamma_5)\psi'_x, \quad \overline{\psi}'_x = \overline{\psi}_x(1 + i\alpha_{5x}\gamma_5) \tag{42}$$

Once again we rewrite the partition function

$$Z = \int \mathcal{D}[\psi, \overline{\psi}] e^{-S[\psi,\overline{\psi},U]} \left\{ 1 - i \sum_x \alpha_{5x} \left[\frac{S\overleftarrow{\partial}}{\partial \psi'_x} \gamma_5 \psi'_x - \overline{\psi}'_x \gamma_5 \frac{\overrightarrow{\partial} S}{\partial \overline{\psi}'_x} \right] \right\} + \mathcal{O}(\alpha^2) \tag{43}$$

then we define the axial current

$$J_{x\mu} = \frac{1}{2} \left\{ (\overline{\psi}_{x+\hat{\mu}} \gamma_\mu \gamma_5 U_{x\mu} \psi_x) + (\overline{\psi}_x \gamma_\mu \gamma_5 U^\dagger_{x\mu} \psi_{x+\hat{\mu}}) \right\} \tag{44}$$

Then from the transformed partition function we can write the anomalous WT identity for the axial current,

$$\begin{aligned} \Bigg\langle \Delta^b_\mu J^5_{x\mu} - 2am(\overline{\psi}_x \gamma_5 \psi_x) - r \sum_{\mu=1}^4 \Bigg\{ 2(\overline{\psi}_x \gamma_5 \psi_x) - \frac{1}{2} [(\overline{\psi}_{x+\hat{\mu}} \gamma_5 U_{x\mu} \psi_x) \\ + (\overline{\psi}_x \gamma_5 U^\dagger_{x\mu} \psi_{x+\hat{\mu}}) + (\overline{\psi}_x \gamma_5 U_{x-\hat{\mu},\mu} \psi_{x-\hat{\mu}}) + (\overline{\psi}_{x-\hat{\mu}} \gamma_5 U^\dagger_{x-\hat{\mu},\mu} \psi_x) \Bigg\} \Bigg\rangle = 0 \end{aligned} \tag{45}$$

Note we do not include the terms proportional to r here in the definition of the axial current. In the case of the vector current, we wanted to have an exact conservation for finite lattice spacings as well, so since the axial current is not conserved anyway, we elect to use a definition based on simplicity.

We want to investigate this expression in the continuum limit. To do so, we rewrite it in a form where the left hand side tends to the expectation value of the divergence of the axial current;

$$\langle a^{-1} \Delta^b_\mu J^5_{x\mu} \rangle = 2m \langle \overline{\psi}_x \gamma_5 \psi_x \rangle + \langle X_x \rangle \tag{46}$$

Where X_x is everything in the sum in EQ. 29. This form makes it particularly clear that the Wilson term, proportional to r explicitly breaks chiral symmetry even for zero quark mass, as it does not vanish in the chiral limit. We can express the rhs of this rewritten WT identity in terms of the matrix elements of the full fermion propagator in the background gauge field. in the continuum limit, one can show that if the mass and typical momentum q carried by the external gauge field satisfies $r/a >> m, |q|$. Then for $r > 0$, the limit of the triangle graph contributions to the last term in our WT identity, which arise from doubler regions, reproduces the correct axial anomaly

$$\lim_{a \to 0} \frac{g^2}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} F(x)_{\mu\nu} F(x)_{\rho\sigma} \tag{47}$$

In the chiral limit, we wish to see the rhs of the WT identity go to zero, indicating the axial current has a vanishing divergence, it is conserved. However, while the term proportional to the mass goes to zero in the chiral limit, the term which came from the doublers in the lattice theory, X_x, does not vanish in the chiral limit or the continuum limit, in fact this term that came from the doubler regions, reduces to the $(F, \tilde{F})$ term, the rhs of the continuum axial anomaly, in the continuum limit.

3 Ginsparg-Wilson equation

3.1 A generalization of chiral symmetry

The Ginsparg-Wilson equation reformulatesthe essential equation for chiral symmetry using renormalization group transformations - this replaces $\{\gamma_5, D\} = 0$ with

$$D\gamma_5 + \gamma_5 D = aD\gamma_5 D \tag{48}$$

Where the factor of the lattice spacing on the right hand side enforces the correct behavior in the naive continuum limit, where $a \to 0$.

We start looking at the consequences of this equation by investigating the quark propagator D^{-1}. Assuming no zero modes and invertibility, we can multiply the GW equation with D^{-1} on both sides

$$\gamma_5 D^{-1}_{n,m} + D^{-1}_{n,m}\gamma_5 = a\gamma_5 \delta(n-m) \tag{49}$$

We see from this version of the GW equation, the anticommutator of γ_5 and the quark propagator is only modified for $n = m$, we've added a contact term to the continuum anticommutator.

With this new essential equation for chiral symmetry on the lattice, we can reformulate the lattice version of chiral symmetry in analogy to the continuum case. Let D be an operator obeying the GW equation. We define chiral transformations that in the continuum limit reduces to the familiar continuum chiral transformations,

$$\psi' = \exp\left(i\alpha\gamma_5\left(\mathbb{1} - \frac{a}{2}D\right)\right)\psi, \quad \overline{\psi}' = \overline{\psi}\exp\left(i\alpha\left(\mathbb{1} - \frac{a}{2}D\right)\gamma_5\right) \tag{50}$$

Where the Dirac operator in this transformation acts on the Fermi fields Dirac, color, and spacetime indices, as we expect.

The action and Lagrange density for massless fermions for such a Dirac operator is invariant under our lattice chiral transformation

$$\begin{aligned}\mathcal{L}[\psi', \overline{\psi}'] &= \overline{\psi}' D\psi' \\ &= \overline{\psi}\exp\left(i\alpha\gamma_5\left(\mathbb{1} - \frac{a}{2}D\right)\right) D \,\exp\left(i\alpha\left(\mathbb{1} - \frac{a}{2}D\right)\gamma_5\right)\psi \\ &= \overline{\psi}\exp\left(i\alpha\gamma_5\left(\mathbb{1} - \frac{a}{2}D\right)\right)\exp\left(-i\alpha\left(\mathbb{1} - \frac{a}{2}D\right)\gamma_5\right) D\psi \\ &= \overline{\psi}D\psi = \mathcal{L}[\psi, \overline{\psi}]\end{aligned} \tag{51}$$

Where we used a rewritten GW equation,

$$D\gamma_5\left(\mathbb{1} - \frac{a}{2}D\right) + \left(\mathbb{1} - \frac{a}{2}D\right)\gamma_5 D = 0 \tag{52}$$

In the lattice formulation of chiral symmetry we can do the decomposition into the left handed and right handed components, defining new projectors

$$\hat{P}_R = \frac{\mathbb{1} + \hat{\gamma}_5}{2}, \quad \hat{P}_L = \frac{\mathbb{1} - \hat{\gamma}_5}{2}, \quad \hat{\gamma}_5 = \gamma_5(\mathbb{1} - aD) \tag{53}$$

Due to the GW euation, we find $\hat{\gamma}_5^2 = \mathbb{1}$, implying the same projection operator relations for the hatted projectors as we saw in the continuum for our usual lh and rh projectors. Once again invoking the GW equation,

$$D\hat{P}_R = P_L D, \quad D\hat{P}_L = P_R D \tag{54}$$

We can then write left handed and right handed field components

$$\psi_R = \hat{P}_R\psi, \quad \psi_L = \hat{P}_L\psi, \quad \overline{\psi}P_L, \quad \overline{\psi}_L = \overline{\psi}P_R \tag{55}$$

These definitions imply that the Dirac operator does not mix lattice chirality, and the lattice action can be decomposed similarly to the continuum

$$\overline{\psi}D\psi = \overline{\psi}_L D\psi_L + \overline{\psi}_R D\psi_R \tag{56}$$

We can write the symmetry breaking mass term

$$m(\overline{\psi}_R\psi_L + \overline{\psi}_L\psi_R) = m\overline{\psi}\left(\mathbb{1} - \frac{a}{2}D\right)\psi \tag{57}$$

Thus we can describe massive GW fermions with the operator

$$D_M = D + m\left(\mathbb{1} - \frac{a}{2}D\right) = \omega D + m\mathbb{1} \tag{58}$$

where $\omega = 1 - am/2$. Thus we have implemented the continuum structures on the lattice. We can generalize to multiple flavors in a similar way as in the continuum.

The key takeaway here is that in the continuum, chiral transformations was a strictly local operation independent of the gauge field. We only use spinors at a given spacetime point. On the lattice both the chiral transformation and the projections require the application of the Dirac operator. The chiral transformation and decomposition in to lh and rh components involce neighboring sites and the gauge field, so the chirality of a lattice fermion depends on neighboring sites and the gauge field as well.

3.2 Spectrum of the Dirac operator

The first thing we will look at with our new formulation of chiral symmetry on the lattice has to do with the spectrum of a Dirac operator on a finite lattice. We write the eigenvalue equation as

$$Dv_\lambda = \lambda v_\lambda \tag{59}$$

We won't invoke the Ginsparg-Wilson equation just yet, but we will require the Dirac operator to be γ_5-Hermitian. This requirement imposes a condition on the characteristic polynomial, $P(\lambda)$

$$\begin{aligned} P(\lambda) &= \det[D - \lambda\mathbb{1}] = \det[\gamma_5^2(D - \lambda\mathbb{1})] = \det[\gamma_5(D - \lambda\mathbb{1})\gamma_5] \\ &= \det[(D^\dagger - \lambda\mathbb{1})] = \det[D - \lambda^*\mathbb{1}]^* = P(\lambda^*)^* \end{aligned} \tag{60}$$

Where we insert the identity as $\gamma_5^2 = \mathbb{1}$ in the second step, and used the cyclic invariance of the determinant in the third. The eigenvalues are zeros of the characteristic polynomial, and this condition, which came solely from γ_5-Hermiticity, implies that if λ is a zero, so is its complex conjugate λ^*. This means that either the eigenvalues are purely real, or they come in complex conjugate pairs.

This isn't all when it comes to γ_5-Hermiticity, if we write the inner product of two vectors as $u^\dagger v = (u, v)$, we find

$$\lambda(v_\lambda, \gamma_5 v_\lambda) = (v_\lambda, \gamma_5 D v_\lambda) = (v_\lambda, D^\dagger \gamma_5 v_\lambda) = (Dv_\lambda, \gamma_5 v_\lambda) = \lambda^*(v_\lambda, \gamma_5 v_\lambda) \tag{61}$$

Since the first expression and last expression are equal, the real parts cancel, and $(\mathrm{Im}\lambda)(v_\lambda, \gamma_5 v_\lambda) = 0$, which means $(v_\lambda, \gamma_5 v_\lambda) = 0$ unless $\lambda \in R$. Therefore, only eigenvectors v_r with real eigenvalues r can have nonvanishing chirality, meaning $v_r^\dagger \gamma_5 v_r \neq 0$.

With this in mind we can now require that our Dirac operator satisfies the Ginsparg-Wilson (GW) equation. If we multiply the GW equation with γ_5 on either the left or right and using $\gamma_5 D \gamma_5 = D^\dagger$, we find

$$D^\dagger + D = aD^\dagger D, \quad D + D^\dagger = aDD^\dagger \tag{62}$$

a γ_5-Hermitian GW Dirac operator is normal - D and $D^\dagger$ commute. This implies that the eigenvectors form a orthogonal basis. Now we multiply the first of EQ. 62 on the left with a normalized eigenvector v_λ and on the right with $v_\lambda^\dagger$,

$$\lambda^* + \lambda = a\lambda^*\lambda. \tag{63}$$

We can write these eigenvalues as $\lambda = x + iy$, and we get

$$\left(x - \frac{1}{a}\right)^2 + y^2 = \frac{1}{a^2} \tag{64}$$

This shows that the eigenvalues of a γ_5-Hermitian GW Dirac operator are restricted to a circle in the complex plane. This is called the Ginsparg-Wilson circle, which has its center at $1/a$ on the real axis and has a radius of $1/a$. We recall that the doubler modes that showed up in the naive discretization end up near $2/a$ in the complex plane and decouple in the continuum limit.

An alternate parameterization is one where we write the eigenvalues as

$$\lambda = \frac{1}{a}(1 - e^{i\phi}), \quad \psi \in (-\pi, \pi] \tag{65}$$

With this parameterization we see the eigenvalues of the quark propagator $1/\lambda$ fall on a line parallel to the imaginary axis,

$$\frac{1}{\lambda} = \frac{a}{2} + i\frac{a}{2}\frac{\sin(\phi)}{1 - \cos(\phi)} \tag{66}$$

We can see that the GW circle does touch the origin of the complex plane, our Dirac operator can have exact zero-modes. If we write a zero mode as v_0,

$$Dv_0 = 0, \quad \gamma_5 D v_0 = 0, \quad D\gamma_5 v_0 = 0 \tag{67}$$

In the subspace of the zero modes, γ_5 commutes with D due to the GW equation, $[\gamma_5, D] = 0$ so the zero modes can be chosen as eigenstates of γ_5 itself. Because $\gamma_5^2 = \mathbb{1}$, the eigenvalues can only be ± 1, and we conclude $\gamma_5 v_0 = \pm v_0$. This implies that the zero modes are chiral - a zero mode with positive chirality is referred to as right handed, and a zero mode with negative chirality is referred to as left-handed. The exact same argument can be made for the eigenmodes with no imaginary part at $2/a$ on the real axis, which turn out to be chiral as well. These are the doubler partners of the zero modes, (remember the Nielsen-Ninomiya theorem, we have an equal amount of lh and rh particles in a discretized theory) which decouple in the continuum limit.

3.3 Topological charge and the axial anomaly

In the continuum, the Atiyah-Singer index theorem relates the number of left handed and right handed zero modes of the massless Dirac operator so the so called winding number, ν, which tells us how many times the gauge group wraps around the physical space. This winding number is a topological property of the gauge fields, and in the lattice we can define an analogous quantity, the topological charge Q_{top}. This is defined by

$$\begin{aligned} Q_{\text{top}} &= \frac{a}{2}\text{Tr}[\gamma_5 D] = -\frac{1}{2}\text{Tr}[\gamma_5(2 - aD)] = -\frac{1}{2}\sum_{\lambda}(v_\lambda, \gamma_5(2 - aD)v_\lambda) \\ &= -\frac{1}{2}\sum_{\lambda}(2 - a\lambda)(v_\lambda, \gamma_5 v_\lambda) = n_- - n_+ \end{aligned} \tag{68}$$

Where n_- and n_+ are the number of left handed and right handed zero modes. In the second step we used the fact that γ_5 is traceless, and we expressed the trace as a sum over the eigenvectors of D, which we are allowed to do because of the normality condition we showed earlier. We also used the condition we derived that only real eigenvalues have nonvanishing chirality, and the $(2 - aD)$ factor cancels the contributions from the doubler modes.

We can write the topological charge Q_{top} as a spacetime sub of the topological charge density,

$$Q_{\text{top}} = a^4 \sum_{n\in\Lambda} q(n), \quad q(n) = \frac{1}{2a^3}\text{Tr}_{CD}[\gamma_5 D], \quad Q_{\text{top}} = n_- - n_+ \tag{69}$$

This is a really interesting result - The topological charge, a topological property of the gauge field alone, can be expressed in terms of the zero modes of the Dirac operator. Another interesting property is the topological charge is an integer, this comes from arguments from homotopy theory, hence the name topological charge.

Practically, we never find zero modes of both chiralities on the lattice, n_- and n_+ are never both nonzero simultaneously. One could argue that a configuration where both were zero simultaneously has a zero measure in the path integral.

In the continuum, the index theorem takes the form $\nu = n_- - n_+$, with

$$\nu = \frac{1}{32\pi^2} \int d^4x \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}_C[F_{\mu\nu} F_{\rho\sigma}] \tag{70}$$

In the continuum, topological gauge configurations with nonzero winding number corresponds to instantons - local minima of the Euclidean gauge action, which give a contribution from a semiclassical or saddle point approximation of the path integral.

On the lattice, Q_{top} is an integer through its definition from the number of zero modes of a chiral Dirac operator. The question that remains is if the topological charge approaches the winding number in the continuum limit, and it can be shown that under certain smoothness conditions for the gauge fields, $q(n) = q(n)^{\mathrm{cont}} + \mathcal{O}(a)$.

The QCD path integral is symmetric with respect to configurations of positive and negative topological charges and thus $\langle Q_{\mathrm{top}} \rangle$ vanishes. However, we can consider the topological susceptibility

$$\chi_{\mathrm{top}} = \frac{1}{V} \langle Q_{\mathrm{top}}^2 \rangle = \frac{1}{a^4|\Lambda|} a^8 \sum_{m,n} \langle q(m) q(n) \rangle = a^4 \sum \langle q(0) q(n) \rangle \tag{71}$$

Using translational invariance of the correlator in the last step to get rid of one of the summations. Notice this quantity is volume dependent. We can consider the topological susceptibility in the infinite volume limit to gain information about the distribution of topological charge as a function of N_f and m. We will come back to this in a bit.

3.4 The Banks-Casher relation

The chiral condensate, $\langle \overline{u}(x) u(x) \rangle$, transforms like a mass term, thus it is not invariant under any of the chiral rotations we introduced. This means that the chiral condensate can serve as an order parameter for spontaneous chiral symmetry breaking. A non-vanishing condensate indicates that the flavor non-singlet chiral symmetry is being spontaneously broken. Earlier we identified the correct mass term on the lattice - according to the Ginsparg-Wilson formalism the bilinear that maximally mixes left and right handed components of the field. This allows us to write a form of the chiral condensate that transforms like a lattice mass term

$$\Sigma_{\mathrm{lat}}(a, m, |\Lambda|) = - \left\langle \overline{u}(n) \left(\mathbb{1} - \frac{a}{2} D \right) u(n) \right\rangle \tag{72}$$

We consider N_f flavors or fermion, all with mass m, and massive Dirac operator given by $D_m = D + m\left(\mathbb{1} - \frac{a}{2}D\right)$. The term coming from the GW equation cancels the real part of the eigenvalues of the massless propagator, shifting the spectrum onto the imaginary axis. Our formulation is translationally invariant, so it is independent of n, and we can average over all lattice sites. Doing so and performing the Grassmann integration, we find

$$\Sigma_{\mathrm{lat}}(a, m, |\Lambda|) = \frac{1}{a^4|\Lambda|} \left\langle \mathrm{Tr} \left[\left(\mathbb{1} - \frac{a}{2} D \right) D_m^{-1} \right] \right\rangle_G \tag{73}$$

Where we are tracing over spacetime, color, and Dirac indices, and

$$\langle X \rangle_G = \frac{1}{Z} \int \mathcal{D}[U] e^{-S_G[U]} \det[D_m]^{N_f} X \tag{74}$$

We can rewrite the propagator like $D_m^{-1} = (\omega D + m\mathbb{1})^{-1}$ with $\omega = 1 - am/2$, to get for our full lattice chiral condensate expression

$$\Sigma_{\mathrm{lat}}(a, m, |\Lambda|) = \frac{1}{\omega a^4|\Lambda|} \left\langle \mathrm{Tr} \left[(\omega D + m\mathbb{1})^{-1} - \frac{a}{2}\mathbb{1} \right] \right\rangle_G \tag{75}$$

There are two limits we are interested in here. Spontaneous symmetry breaking requires an infinite system, so we need to take the infinite volume limit at some point. We also need to remove explicit breaking through the mass term, so the massless limit also needs to be taken. These limits cannot be interchanged, and we get the bare condensate after these limits are taken. The physical condensate is regularlization dependent.

Numerically, these proper sequence of limits is hard to establish. We can define the chiral condensate $\Sigma(m,V)$ in a 4D box of finite volume V at finite mass, where this particular chiral condensate contains the physical condensate

$$\Sigma = \lim_{m\to 0}\lim_{V\to\infty}\Sigma(m,V) \tag{76}$$

Once the form of $\Sigma(m,V)$ is known, one can use lattice simulations to compute the lattice condensate $\Sigma_{\mathrm{lat}}(a,m,|\Lambda|)$ at finite volume and mass, and fit the data to $\Sigma(m,V)$ to obtain the true bare condensate as a fit parameter. This is usually done in the quenched approximation. Within partially quenched perturbation theory, one can derive the functional form of the chiral condensate for different topological sectors as a function of the volume and mass. We talk about the topological sector ν when we only consider gauge configurations in the path integral with definite winding number ν. We can implement this on the lattice by sorting gauge configurations with respect to the topological charge defined through the index theorem. The quenched result for the condensate in the topological sector ν is given by

$$\Sigma(m,V)_\nu = \Sigma z(I_{|\nu|}(z)K_{|\nu|}(z) + I_{|\nu|+1}(z)K_{|\nu|-1}(z)) + \frac{|\nu|}{mV} \tag{77}$$

where $z = mV\Sigma$ is a dimensionless scaling variable and Σ is the bare condensate. The lattice condensate is then computed for each topological sector separately and fit to the functional form above.

One can formally perform the two limits discussed above, and parameterize the unknown non-perturbative information in terms of the eigenvalue density of the Dirac operator near the origin. This gives rise to the Banks-Casher relation.

We start with the trace being expressed as a sum over all eigenvalues λ_i of the normal operator D, whose eigenvalues define a complete basis,

$$\begin{aligned}
&\mathrm{Tr}\left[(\omega D + m\mathbb{1})^{-1} - \frac{a}{2}\mathbb{1}\right] = \sum_{\lambda_i}\left(\frac{1}{\omega\lambda_i + m} - \frac{a}{2}\right)\\
&= (n_+ + n_-)\left(\frac{1}{m} - \frac{a}{2}\right) + (n'_+ + n'_-)\left(\frac{1}{2\omega/a + m} - \frac{a}{2}\right)\\
&+ \sum_{\lambda_i\neq 0,2/a}\left(\frac{1}{\omega\lambda_i + m} - \frac{a}{2}\right)
\end{aligned} \tag{78}$$

Where we split the sum over eigenvalues into the contributions from the zero modes, n_+ and n_-, the contributions from the zero mode chiral partners at $2/a$, n'_- and n'_+, and a remaining sum over the complex eigenvalues. From the definition of ω, we find that the contribution of the zero modes reduces to ω/m and the real doubler contribution vanishes. We once again express the eigenvalues through the phases ϕ_i, and combining the complex conjugate pairs of eigenvalues,

$$\frac{\omega}{m}(n_+ + n_-) + \frac{\omega}{2}\sum_{\phi_i\neq 0,\pi}\frac{(1+\cos(\phi_i))m}{(2-2\cos(\phi_i))(am+\omega)\omega/a^2 + m^2} \tag{79}$$

As mentioned earlier, practically we only have zero modes of one chirality, meaning we can replace

$(n_+ + n_-) \to |Q_{\rm top}|$. Inserting this for the trace in the definition of the lattice condensate,

$$\begin{aligned}\lim_{|\Lambda|\to\infty} \Sigma^{\rm lat}(a,m,|\Lambda|) = & \lim_{|\Lambda|\to\infty} \frac{1}{a^4|\Lambda|}\frac{\langle |Q^2_{\rm top}\rangle\rangle_G}{m} \\ & + \frac{1}{2a^4}\int_{-\pi}^{\pi} d\phi \rho_A(\phi) \frac{(1+\cos(\phi))m}{(2-2\cos(\phi))(1-a^2m^2/4)/a^2+m^2}\end{aligned} \tag{80}$$

Where we defined angular density $\rho_A(\phi)$ for the angles of the eigenvalues on the Ginsparg-Wilson circle,

$$\rho_A(\phi) = \lim_{|\Lambda|\to\infty}\left\langle \frac{1}{|\Lambda|}\sum_{\phi_i\neq 0,\pi}\delta(\phi-\phi_i)\right\rangle_G \tag{81}$$

In the definition of this angular density we perform the infinite volume limit necessary for spontaneous symmetry breaking. In this limit the eigenvalues become dense on the Ginsparg-Wilson circle and $\rho_A(\phi)$ becomes a density.

The first term in the lattice condensate vanishes in the thermodynamic limit, since the expectation value of the topological charge does not grow faster than $1/\sqrt{V}$. For the second term, we can use

$$\delta_m(X) = \frac{1}{\pi}\frac{m}{X^2(1+\mathcal{O}(m^2))+m^2} \tag{82}$$

which approaches a proper delta function in the chiral limit. Thus for the second term of the lattice condensate we get

$$\begin{aligned}&\frac{\pi}{2a^4}\int_{-\pi}^{\pi} d\phi(1+\cos(\phi))\delta\left(a^{-1}\sqrt{2-2\cos(\phi)}\right)\rho_A(\phi) \\ &\qquad = \frac{\pi}{2a^3}\int_{-\pi}^{\pi} d\phi(1+\cos(\phi))\delta(\phi)\rho_A(\phi) = \frac{\pi}{a^3}\rho_A(0)\end{aligned} \tag{83}$$

This shows that after we take the infinite volume limit, the chiral condensate is proportional to the angular density $\rho_A(0)$. Usually one uses the density ρ_λ instead, the density of eigenvalues on the imaginary axis, $\rho_\lambda = \Delta n/\Delta y$. For small angles, $\Delta y = \Delta\phi.a$, implying that $\rho_\lambda = a\rho_A$. Thus

$$\Sigma^{\rm lat}(a) = \lim_{m\to 0}\lim_{|\Lambda|\to\infty}\Sigma^{\rm lat}(a,m,|\Lambda|) = \frac{\pi}{a^3}\rho_A(0) = \pi/a^4\rho_\lambda(0) = \pi\rho(0) \tag{84}$$

The Banks-Casher relation has given us some important insight into the mechanism for chiral symmetry breaking. One idea is that the chiral condensate is formed through a fluid if weakly interacting topological field configurations, such as instantons. In the case of instantons, the topological configurations are localized, so called topological lumps. According to the index theorem, if we have some lump structure in our gauge configuration, we have zero eigenvalues. For a mixture of topological lumps with different charges, one cannot assume that the zero modes are left unperturbed. One expects the eigenvalues to move in the imaginary direction. The instanton liquid model gives rise to an accumulation of eigenvalues on the imaginary axis, thus building up the eigenvalue density $\rho(0)$ near the origin. This in turn gives rise to a non-vanishing chiral condensate via the Banks-Casher relation.