Renormalization and Scale Setting

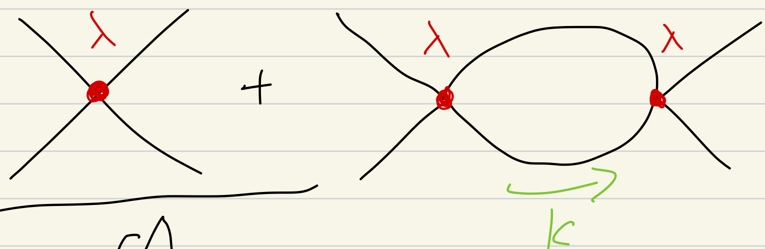
Renormalization/RG: bare parameters $\to$ physics!

N couplings $\to$ N experiments to fix theory.

"Scale setting": choice of which inputs to use.

Continuum ex: ϕ^4 theory, one-loop.

Scattering: $\phi\phi \to \phi\phi$

$i\mathcal{M} =$ 

$$\sim \int^{\Lambda} d^4k \, \frac{1}{k^2} \cdot \frac{1}{(p+k)^2}$$

Log divergent: $-\mathcal{M} = \lambda + \frac{\lambda^2}{32\pi^2} \log \frac{s}{\Lambda^2}$.

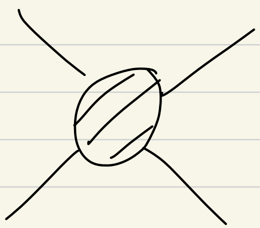
$\Lambda \to \infty$; ∞ amplitude! Is the theory "sick"?

(s: Mandelstam E_{CM}^2)

Theory is okay! Only cross-section differences.

$$\mathcal{M}(s_1) - \mathcal{M}(s_2) \sim \left(\log\frac{s_1}{\Lambda^2} - \log\frac{s_2}{\Lambda^2}\right) = \log\frac{s_1}{s_2}$$

Define renormalized coupling at ref. energy s_0.



$$\Big|_{s=s_0} = -i\lambda_R \Rightarrow \lambda_R = \lambda + \frac{\lambda^2}{32\pi^2}\log\frac{s_0}{\Lambda^2} + \dots$$

Scattering in terms of λ_R is finite:

$$\mathcal{M}(s) = -\lambda_R - \frac{\lambda_R^2}{32\pi^2}\log\frac{s}{s_0} + \dots$$

Definition of λ_R is our renormalization condition.

No divergences!*

*(except as $s_0 \to \infty$. No continuum limit on lattice!)

If we change $s_0 \to s_0'$, λ_R changes too.

calculate dependence $\to$ renormalization group.

$$\frac{d\lambda_R}{ds_0} = \frac{1}{s_0}\frac{\lambda^2}{32\pi^2}.$$

Define the beta function:

$$\beta(\lambda_R) \equiv s_0 \frac{d\lambda_R}{ds_0} = +(\dots)\lambda_R^2$$

Arrows show UV flow (to $s_0 \to \infty$.)

Opposite is IR flow (to $s_0 \to 0$.)
($\lambda_R \to 0$.)

"Continuum limit" $s_0 \to \infty$ flows into $\lambda_R \to \infty$
(perturbative control lost.)

Alternatively hold $\lambda_R(s_0)$ fixed, look at $\lambda_R(s_1)$ w/ $s_1 \ll s_0$. then $\boxed{\lambda_R(s_1) \to 0}$ "triviality".

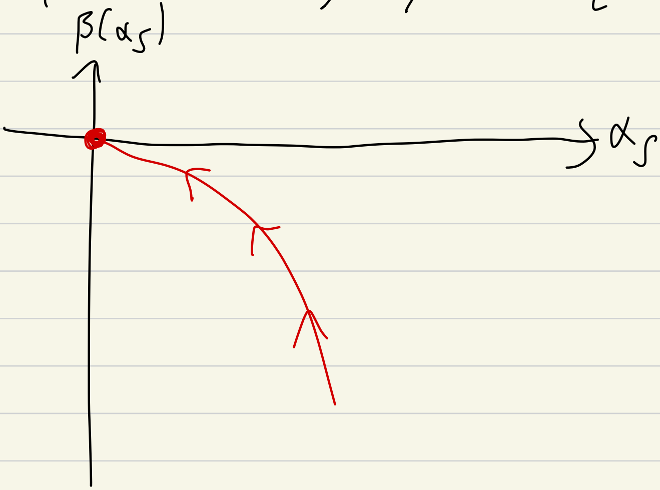
gauge coupling

For QCD first term in $\beta(\alpha_s)$ $(\alpha_s \equiv g_s^2/4\pi)$ has crucial minus sign.

(Gross, Wilczek '73; Politzer '73)

$$\Lambda \frac{d\alpha_s}{d\Lambda} \equiv \beta(\alpha_s) = -\frac{1}{2\pi}\left(11 - \frac{2}{3}N_f\right)\alpha_s^2 + \ldots$$

(N_f = # of light, active quarks.)



(UV flow: $\Lambda \to \infty$)

asymptotic freedom

($\Lambda \to \infty$ is $\alpha_s \to 0$.)

In QCD, α_s is inconvenient to renormalize the theory (we can't scatter quarks/gluons)

Instead, rely on dimensional transmutation

$$\boxed{\alpha_s(\Lambda_0) = \alpha_{ref} \longleftrightarrow m_H = c_{ref}\Lambda_0}$$

(roughly.)

(R. Sommer, 1401.3270)

Scale setting – focus on QCD.

Gauge coupling α_s + quark masses m_u, m_d, m_s, m_c, m_b.

(Top quark is too heavy/unstable; $\theta = 0$.)

Six experimental inputs (w/ dependence on parameters; proton mass $\not\to m_b$.)

can be any six processes ($\pi\pi \to \pi\pi$ scattering?)

We want highest precision → hadron masses
→ decay constants

Quark masses ⟷ direct meson ($\bar{q}q$) masses.

Heavy quarks: $\begin{cases} M_{D_s} \sim m_c & (\bar{c}s) \\ M_{B_s} \sim m_b & (\bar{b}s) \end{cases}$

Light quarks: chiral perturbation theory

(chiral symmetry restored at $m_q = 0$ ...)

Implies the Gell-Mann-Oakes-Renner relation:

(GMOR): $M^2_{q_1 q_2} = B(m_{q_1} + m_{q_2})$

(pseudoscalar)

Works well for light, strange. Pions, kaons!

$$\begin{cases} \pi^+ \sim \bar{d}u & K^+ \sim \bar{s}u \\ \pi^0 \sim (\bar{u}u + \bar{d}d) & K^0 \sim \bar{s}d \end{cases}$$

(Flavor Lattice Averaging Group)

FLAG 2024 ⟶ "Edinburgh consensus" (no QED)

$$\begin{cases} M_{\pi^+} = 135.0 \text{ MeV} \\ M_{K^+} = 491.6 \text{ MeV} \\ M_{K^0} = 497.6 \text{ MeV} \\ M_{D_s^+} = 1967 \text{ MeV} \\ M_{B_s^0} = 5367 \text{ MeV} \end{cases}$$

One more input to fix overall scale.

Simulation produces aM_{π^+}; need to set a to fix $M_{\pi^+} = 135$ MeV!

Three classes of scale setting:

A) Hadron masses;

B) Decay constants;

C) Theory scales.

Precision is essential! S to set scale.

(aM_H) from calculation, precision $\delta(aM_H)$.

Then: $$M_H = \frac{\cancel{a} M_H}{\cancel{a} S} \cdot S_{exp}$$ Frac error:

$$\frac{\delta M_H}{M_H} = \sqrt{\left(\frac{\delta(aM_H)}{aM_H}\right)^2 + \left(\frac{\delta(aS)}{aS}\right)^2 + \left(\frac{\delta S_{exp}}{S_{exp}}\right)^2}$$

lattice aM_H error — lattice aS error — experimental.

(https://pdg.lbl.gov)

$\delta S_{exp}/S_{exp}$ provides ultimate limit on error!

$\delta(aS)/aS$ also important.

A) Hadron masses - what are good options?

- Proton: $\dfrac{\delta m_p^{exp}}{m_p} \sim 3\times 10^{-8}$

But: exponential signal-to-noise degradation (see exercises)

- π, K: very precise, but we used them!

- η: also very precise, hard on lattice (disconnected diagrams...)

- Ω baryon: $\dfrac{\delta m_\Omega^{exp}}{m_\Omega^{exp}} \sim 0.02\%$
(sss)

Also exp. S/N problem; better than proton.

Lattice calcs $\sim$ sub-percent aM_Ω.

B) Decay constants — more details next time.

$$\Gamma(\pi \to \mu\nu) \sim f_\pi^2 ; \quad \frac{\delta\Gamma^{exp}}{\Gamma^{exp}} \sim 10^{-3}$$

But disentangle EW physics. Systematics?

(f_π still commonly used.)

C) Theory scales: S_{exp} vs. aS precision;

(systematic effects, N/S, or expensive.)

Theory scale $1/T$ can be a proxy.

Easy to compute a/T precisely; relate to some physical S_{exp} once.

$$\lim_{a\to 0}\left(\frac{aS}{a/T}\right) = \boxed{TS} ; \quad S_{exp} \to T_{ref}.$$

Future calculations use TM_H vs. $\left(\frac{aM_H}{aS}\right)S_{exp}$.

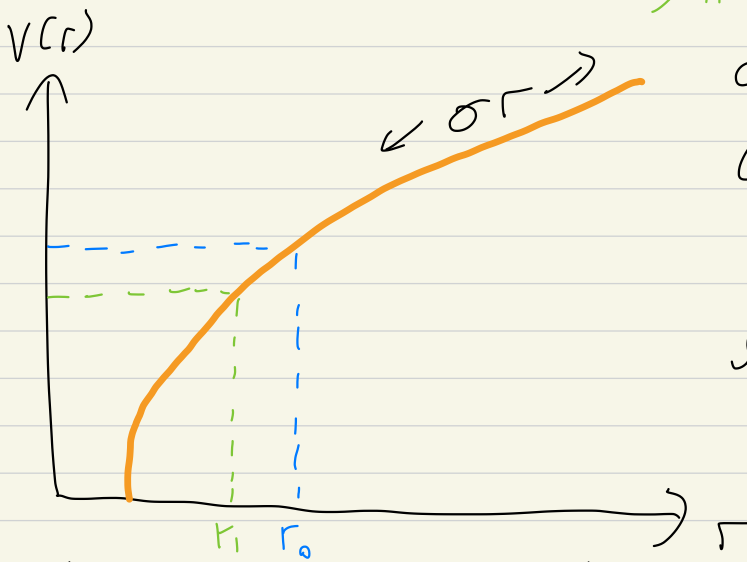
Static potential : r_0 , r_1

Gradient flow : $\sqrt{t_0}$, w_0

Static potential $\rightarrow$ r_0 , r_1 "Sommer parameters"

$$V(r) = A - \frac{\alpha}{r} + \sigma r$$

$\rightarrow$ linear - confinement!



σ : string tension

Can give overall scale, used in pure-gauge.

Subject to:

- Volume sensitivity
- String breaking

Avoid by going to "knee" region.

$$F(r) = -\frac{dV}{dr} = -\frac{\alpha}{r^2} - \sigma ,$$ then:

(FLAG'24, 2+1+1 = N_f) (1.6%)

$$r_0^2 \, |F(r_0)| = 1.65 \Rightarrow r_0 = 0.4580(73) \text{ fm}$$

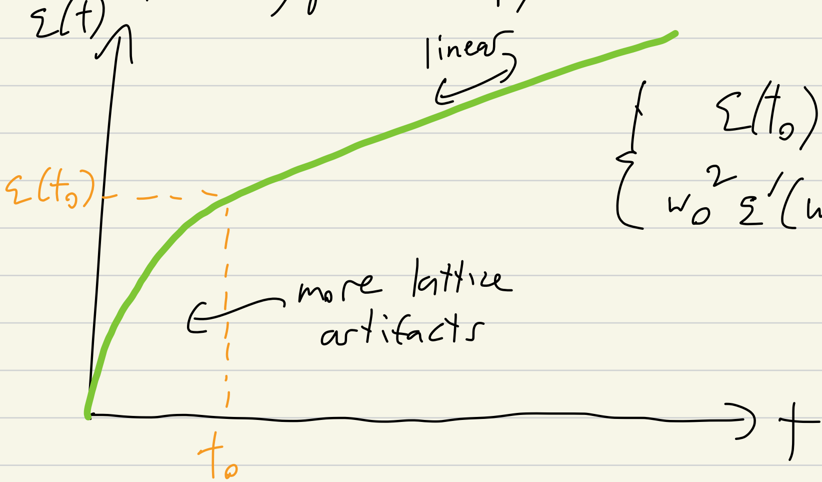
$r_1^2 |F(r_1)| = 1 \Rightarrow r_1 = 0.3068(37)$ fm
(1.2%)

Gradient flow $\to \sqrt{t_0}$, w_0

(M. Lüscher, 1006.4518)

More abstract; gradient flow → "smeared out" gauge fields, radius $r_{smear} \approx \sqrt{8t}$ in "flow time" t.

Look at energy density, dimless: $\mathcal{E}(t) \equiv t^2 \langle E(t) \rangle$



$$\begin{cases} \mathcal{E}(t_0) \equiv 0.3 \\ w_0^2 \mathcal{E}'(w_0^2) \equiv 0.3 \end{cases}$$

(FLAG '24, 2+1+1:)

$$\begin{cases} \sqrt{t_0} = 0.14292(104) \text{ fm} & (0.7\%) \\ w_0 = 0.17256(103) \text{ fm} & (0.6\%) \end{cases}$$

Current "gold standard" theory scales!

perturbative renormalization

Why PT picture? Adding electroweak!

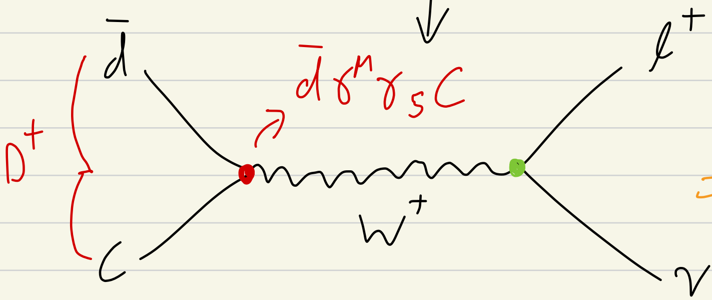
We don't usually include weak, QED, leptons... in our simulations. (Scale separation!)

Rely on $\boxed{\text{factorization!}}$ in strong + EW process, we can separate out QCD matrix elements.

ex: $D^+ \to \ell^+ \nu$. $(\ell = e, \mu)$ $\quad D^+ = (c\bar d)$

$$\boxed{\langle \ell^+ \nu | \mathcal{H}_{SM} | D^+ \rangle}$$

$$\approx \underline{\langle \ell^+ \nu | \mathcal{H}_{EW} | c\bar d \rangle} \cdot \underline{\langle c\bar d | \mathcal{H}_{QCD} | D^+ \rangle}$$



QCD vacuum

$$= \langle 0 | \bar d \gamma^\mu \gamma_5 c | D^+ \rangle$$

$$\Rightarrow \mathcal{M} \sim \frac{g^2}{M_W^2} \langle 0 | \bar d \gamma^\mu \gamma_5 c | D^+ \rangle \, .$$

(kinematics)

The QCD ME defines a decay constant:

$$\langle 0|\bar{d}\gamma^{\mu}\gamma_5 c|D^+(p)\rangle = ip^{\mu} f_D$$

Calculate on lattice: define currents

$$\begin{cases} A_{\mu} = \bar{d}\gamma_5\gamma^{\mu} c \\ P = \bar{d}\gamma_5 c \end{cases}$$

$\langle 0| A_{\mu} \quad P |0\rangle$

Compute:

$$C(t) = \sum_x \langle 0| A_{\mu}(x,t) P(0,0) |0\rangle$$

$$\xrightarrow[t\to\infty]{} \frac{e^{-m_D T}}{2m_D} \langle 0|A_{\mu}|D^+\rangle\langle D^+|P|0\rangle$$

$\sim f_D$

Add correlator $\langle PP\rangle$ to determine $\langle D^+|P|0\rangle$.

But, we need one more piece.

EW part of decay is perturbative.

Calculated w/ e.g. $\overline{MS}$ renormalization; depends on RG scale, μ.

Must account for RG dependence (μ) in f_D calculation $\longrightarrow$ matching.

Consider bare operator $\mathcal{O}_0$.

Continuum: $\quad \mathcal{O}^{\overline{MS}}(\mu) = Z_C(\mu) \cdot \mathcal{O}_0$

Lattice: $\quad \mathcal{O}^{lat}(1/a) = Z_L(1/a) \cdot \mathcal{O}_0$

Either regulator handles UV divergences. But details are different. Consider matrix elements:

$$\langle \alpha | \mathcal{O}_0 | \beta \rangle = \langle \alpha | \mathcal{O}_0 | \beta \rangle$$

$$\Rightarrow \boxed{\langle \alpha | \mathcal{O}^{\overline{MS}}(\mu) | \beta \rangle = \frac{Z_C(\mu)}{Z_L(1/a)} \cdot \langle \alpha | \mathcal{O}^{lat}(1/a) | \beta \rangle}$$

"matching factor"

μ-dependent – always partly perturbative. ($Z_C(\mu)$)

We can get $Z_L(1/a)$ using PT or not.

PT matching factors are widely available in literature.

Focus on methods, tricks, phenomena.

How to get Z-factors:

Back to example: $D^+ \to \ell^+ \nu$.

$$\langle 0 | \bar{d} \gamma_5 \gamma^\mu c | D^+(p) \rangle = i p^\mu f_D$$

want $\overline{MS}$ renormalized ME, Lattice current A^μ:

$\overline{MS}$ current.

$$\langle 0 | \mathcal{A}^\mu | D^+ \rangle = \langle 0 | Z_A A^\mu | D^+ \rangle$$

matching factor.

Axial current satisfies partial conservation:

"PCAC"

$$\boxed{\langle \partial_\mu A^\mu \mathcal{O} \rangle = 2m \langle P \mathcal{O} \rangle + \mathcal{O}(a)}$$

Renormalize: $Z_A \langle \partial_\mu A^\mu \mathcal{O} \rangle = 2m Z_m(\mu) Z_P(\mu) \langle P \mathcal{O} \rangle$

Scale-dependence cancels on RHS:

$$Z_m(\mu) Z_P(\mu) = 1$$

$\langle 0 | A^\mu P | 0 \rangle$

Choose zero component:

$$\langle 0 | Z_A A^0 | D^+(p) \rangle = M_{D^+} f_{D^+}$$

Use PCAC:

$$2m \langle 0 | P | D^+(p) \rangle = (p_\mu p^\mu) f_{D^+} = M_{D^+}^2 f_{D^+}$$

$$Z_A \langle 0 | A^0 | D^+ \rangle = \frac{2m}{M_{D^+}} \langle 0 | P | D^+ \rangle$$

$$\Rightarrow \boxed{Z_A = \frac{2m \langle 0 | P | D^+ \rangle}{\langle 0 | A^0 | D^+ \rangle}}$$

"ratio method"

Other "ratio tricks" can be useful, e.g., for Wilson fermions,

Two vector currents. Simple, local current: (two-flavor)

$$V_\mu^a(x) = \bar{\psi}(x) \frac{\tau^a}{2} \gamma_\mu \psi(x)$$

Conserved isovector current:

$$V^a_{\mu,c}(x) = \frac{1}{2}\Big[\bar\psi(x)\frac{\tau^a}{2}(\gamma_\mu - 1)U(x,\mu)\psi(x+a\hat\mu) + \bar\psi(x+a\hat\mu)\frac{\tau^a}{2}(\gamma_\mu+1)U^\dagger(x,\mu)\psi(x)\Big]$$

"Point-split", but <u>doesn't renormalize</u>: $\boxed{Z_{V,c} = 1.}$

Simple current <u>does</u> renormalize:

$$Z_V = 1 + \mathcal{O}(a).$$

<u>Ratio trick</u>: $\boxed{Z_V = \dfrac{\langle H | V_\mu^c | H\rangle}{\langle H | V_\mu | H \rangle}}$

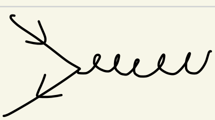
compute once, work w/ simpler local current.

<u>Another approach</u>: <u>lattice perturbation theory</u>.

Calculations are available for most things; just show interesting features.

<u>Quark-gluon vertex</u>: in continuum QCD,

$$\mathcal{L} \supset \bar\psi i D^\mu \psi \sim g_s \bar\psi A^\mu_a T^a \psi$$

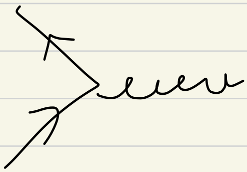
 $\sim g_s$.

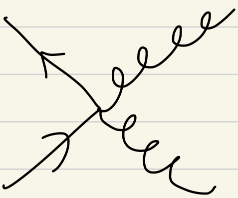
$\exp(igA_\mu^a T^a)$

On the lattice:

$$\mathcal{L} \supset \bar{\psi}(x)\,\Gamma \ldots U_\mu(x+y) \ldots \psi(x+y)$$

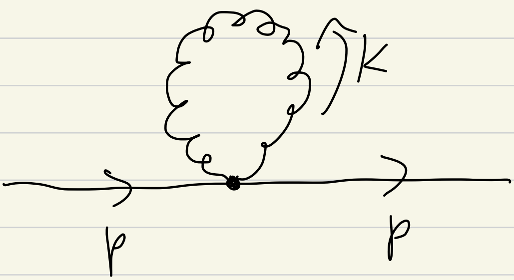
$$\Rightarrow \bar{\psi} A_\mu \psi + \bar{\psi} A_\mu A_\nu \psi + \ldots$$

 $\sim g$ ← continuum vertex, $a \to 0$.

 $\sim g^2 a$ ← lattice artifact vertex.

Gives a lattice-only contribution to quark prop:

"tadpole diagram"



$$\sim (g^2 a) \int^{\pi/a} d^4k \cdot \frac{1}{k^2}$$

$$\sim (g^2 a) \int^{\pi/a} k\, dk \sim \frac{1}{a}.$$

UV divergence!   Not a disaster (we have cutoff)

$$\delta m_0 \sim \frac{1}{a} \Rightarrow \delta(am_0) \sim a^0.$$

$\Rightarrow$ additive renormalization of quark mass.

($am_0 = 0 \not\Rightarrow m_R = 0.$)

Need to deal with this...

- Chiral symmetry - prevents this contribution (for mass.) Staggered, overlay, DWF.
- Calculate m_R, shift bare mass am_0 as needed. (Wilson.)

For Wilson, axial ward identity (AWI) helps:

$$\langle \partial_\mu A^\mu \mathcal{O} \rangle = 2m \langle P \mathcal{O} \rangle$$

$$\Rightarrow \boxed{m_{AWI} \equiv \frac{\langle 0 | \partial_\mu A^\mu | \pi \rangle}{2 \langle 0 | P | \pi \rangle}}$$

Caveat: need Z_A, Z_P; currents must be renormalized.

But, these are multiplicative. Guaranteed

$$\boxed{m_R \propto m_{AWI}.} \qquad m_{AWI} = 0 \to m_R = 0.$$

For other quantities in lattice PT, "tadpoles" () lead to strong cutoff dependence.

Don't spoil $a \to 0$, but "uncomfortably large"

Problematic modes are gluon high modes (UV)

Idea: (Lepage + Mackenzie) $\Rightarrow$ split apart.

$$U_\mu(x) = e^{iagA_\mu(x)} = 1 + iagA_\mu(x) - a^2 g^2 A_\mu A_\nu + \ldots$$

$$\boxed{(\text{integrate})} \longrightarrow \; = u_0 \left(1 + iagA_\mu(x) + \ldots\right)$$

Rescale out "mean-field value":

$$\tilde{U}_\mu(x) = U_\mu(x) / u_0 . \qquad (0 < u_0 < 1)$$

Compute on lattice:

$$u_0 = \left\langle \frac{1}{N_c} \operatorname{Re} \operatorname{Tr} U_\mu(x) \right\rangle$$

$$u_0 \simeq u_P = \left\langle \frac{1}{N_c} \mathrm{Re}\,\mathrm{Tr}\, P_{\mu\nu}(x) \right\rangle^{1/4}$$

Calculate and rescale coupling for PT:

$$\boxed{\tilde{g}_0^2 = \frac{g_0^2}{u_0^4}}$$

Better convergent PT!