Lecture 2: Chiral symmetry

Continuum chiral symmetry

$$S_F[\psi,\bar\psi,A] = \int d^4x\, \bar\psi \gamma_\mu(\partial_\mu + iA_\mu)\psi = \int d^4x\, \bar\psi D \psi$$

massless Dirac operator

Chiral transformation

$$\psi \to \psi' = e^{i\alpha\gamma_5}\psi \quad ; \quad \bar\psi \to \bar\psi' = \bar\psi e^{i\alpha\gamma_5}$$

$$\bar\psi' \gamma_\mu(\partial_\mu + iA_\mu)\psi' = \bar\psi e^{i\alpha\gamma_5}\gamma_\mu(\partial_\mu + iA_\mu)\underline{e^{i\alpha\gamma_5}}\psi \qquad \{\gamma_\mu,\gamma_5\}=0$$

$$= \bar\psi \cancel{e^{i\alpha\gamma_5} e^{-i\alpha\gamma_5}}\gamma_\mu(\partial_\mu + iA_\mu)\psi \qquad \{\gamma_\mu, e^{\gamma_5}\} = 0$$

mass term

$$m\bar\psi'\psi = m\bar\psi e^{2i\alpha\gamma_5}\psi$$

Projection operator

$$P_{L/R} = \frac{\mathbb{1} \mp \gamma_5}{2} \; ; \quad P^2_{R/L} = P_{R/L}, \quad P_R P_L = P_L P_R = 0 \; , \quad P_R + P_L = \mathbb{1}$$

$$\gamma_\mu P_L = P_R \gamma_\mu$$

$$\psi_{R/L} = P_{R/L}\psi \; ; \quad \bar\psi_{R/L} = \bar\psi P_{L/R} \quad \to \quad \mathcal{L}[\psi,\bar\psi,A] = \bar\psi_L D\psi_L + \bar\psi_R D \psi_R$$

$$m\bar\psi\psi = m(\bar\psi_R\psi_L + \bar\psi_L\psi_R)$$

summarize $\boxed{\{\gamma_5, D\} = 0}$

Generalize to N_f flavors

$$S_F = \sum_f^{N_f}\int d^4x\, \bar\psi^f(\gamma_\mu(\partial_\mu + iA_\mu) + M)\psi^f$$

$N_f \times N_f$ matrix

Note: vector transformations

$$\psi' = e^{i\alpha T_i}\psi \; , \quad \bar\psi' = \bar\psi e^{-i\alpha T_i}$$

generators of $SU(N_f)_V$

$$\psi' = e^{i\alpha\mathbb{1}}\psi \; , \quad \bar\psi' = \bar\psi e^{-i\alpha\mathbb{1}} \quad - \; U(1)_V$$

$$U(N_f)_V = SU(N_f)_V \otimes U(1)_V$$

Axial transformation (chiral)

$\psi' = e^{i\alpha\gamma_5 T_i}\psi \; ; \; \bar\psi' = \bar\psi e^{i\alpha\gamma_5 T_i}$ $\quad SU(N_f)_A$

$\psi' = e^{i\alpha\gamma_5 \mathbb{1}}\psi \; ; \; \bar\psi' = \bar\psi e^{i\alpha\gamma_5}$ $\quad U(1)_A$

Full flavor symmetry $\quad U(N_f)_A \otimes U(N_f)_V \sim SU(N_f)_L \otimes SU(N_f)_R \otimes U(1)_V \otimes \cancel{U(1)_A}$ explicitly broken axial anomaly

for degen mass $\rightarrow$ $U(N_f)_V \sim SU(N_f)_V \otimes U(1)_V$

for any mass $\rightarrow$ $U(1)_V \otimes \ldots \otimes U(1)_V$ (N_f times)

$SU(N_f)_A$ $\rightarrow$ Spontaneously broken in QCD $\rightarrow$ order param $\langle \bar u(x) u(x) \rangle$

Goldstone bosons

2 flavor: $SU(2)_A$ flavour symm. $\rightarrow$ 3 Goldstone bosons $\rightarrow$ 3 pions $\pi^\pm, \pi^0$ ~ 140 MeV

3 flavor: $SU(3)_A$ flavor symm $\rightarrow$ 8 Goldstone bosons $\rightarrow$ pseudoscalar octet

$U(1)_A$ $\rightarrow$ 1 Goldstone boson $\rightarrow$ η'

$\eta \sim 548$ MeV, $\quad$ $U(1)_A \rightarrow$ explicitly broken by topological field configs $\rightarrow$ instantons.

$\eta' - 958$ MeV

Chiral symmetry on the lattice

Nielsen-Ninomiya theorem

- Reflection pos. $\quad$ Dirac operator
- cubic group inv. $\Rightarrow$ $D(p) = -D(-p)$
- chiral inv.
- Locality

$$D(p) = \sum_\mu i f_\mu(p)\gamma_\mu = \sum_\mu i f_\mu(p)\gamma_\mu(P_L + P_R)$$

Naive Dirac $\rightarrow$ $f_\mu(p) = \sin(p_\mu)$

$T_4 = S_1 \otimes S_1 \otimes S_1 \otimes S_1$ $\quad$ particle $\rightarrow$ zero of vector field

+1 index $\rightarrow$ source or sink ; $\quad$ -1 index $\rightarrow$ saddle point

Poincare - Hopf theorem - Sum of indices of the zeros of a continuous vector field on a compact manifold is equal to the Euler characteristic of that manifold

$\Rightarrow$ There are an equal number of lh and r.h particles for every set of q.n.

Ginsparg - Wilson equation

in continuum: $\{\gamma_5, D\} = 0$

on lattice $\{\gamma_5, D\} = aD\gamma_5 D$ continuum limit $a \to 0$

$$D\gamma_5(\mathbb{1} - \tfrac{a}{2}D) + (\mathbb{1} - \tfrac{a}{2}D)\gamma_5 D = 0$$

$$D\gamma_5 + \gamma_5 D = aD\gamma_5 D \rightarrow \gamma_5 D^{-1}_{n,m} + D^{-1}_{n,m}\gamma_5 = a\gamma_5\,\delta(n-m)$$

D^{-1} D^{-1}

field transformation

$$\psi' = \exp\left(i\alpha\gamma_5(\mathbb{1} - \tfrac{a}{2}D)\right)\psi \; ; \; \bar{\Psi}' = \bar{\Psi}\exp\left(i\alpha\gamma_5(\mathbb{1} - \tfrac{a}{2}D)\right)$$

$$\mathcal{L} = \bar{\Psi}' D\psi' = \bar{\Psi}\exp\left(i\alpha\gamma_5(\mathbb{1} - \tfrac{a}{2}D)\right) D \exp\left(i\alpha\gamma_5(\mathbb{1} - \tfrac{a}{2}D)\right)\psi$$

$$= \bar{\Psi}\exp(i\alpha\gamma_5(\ldots))\exp(-i\alpha\gamma_5(\ldots))D\psi = \bar{\Psi}D\psi$$

$$\hat{\gamma}_5 = \gamma_5(\mathbb{1} - aD) \rightarrow \hat{P}_{L/R} = \frac{\mathbb{1} \mp \hat{\gamma}_5}{2}$$

$$\mathcal{L} = \bar{\Psi}D\psi = \bar{\Psi}_L D\psi_L + \bar{\Psi}_R D\psi_R, \quad m(\bar{\Psi}_R\psi_L + \bar{\Psi}_L\psi_R) = m\bar{\Psi}(\mathbb{1} - \tfrac{a}{2}D)\psi$$

Spectra of GW Dirac operator

$$Dv_\lambda = \lambda v_\lambda \qquad \gamma_5^2 = \mathbb{1}$$

$$P(\lambda) = \det[D - \lambda\mathbb{1}] = \det[\gamma_5^2(D - \lambda\mathbb{1})] = \det[\gamma_5(D - \lambda\mathbb{1})\gamma_5]$$

$$= \det[(D^\dagger - \lambda\mathbb{1})] = \det[D - \lambda^*]^* = P(\lambda^*)^*$$

if λ is a zero of $P(\lambda)$, so is λ^* either λ is purely real, or they come in c.c. pairs.

$u^\dagger v = (u,v)$: $\lambda(v_\lambda, \gamma_5 v_\lambda) = (v_\lambda, \gamma_5 D v_\lambda) = (v_\lambda, D^\dagger \gamma_5 v_\lambda) = (D v_\lambda, \gamma_5 v_\lambda) = \lambda^*(v_\lambda, \gamma_5 v_\lambda)$

$(v_\lambda, \gamma_5 v_\lambda) = 0$ unless $\lambda \in \mathbb{R}$ $\qquad v^\dagger \gamma_5 v \neq 0$

GW operator

$$D\gamma_5 + \gamma_5 D = a D \gamma_5 D$$

$$D^\dagger + D = a D^\dagger D \;;\; D + D^\dagger = a D D^\dagger \rightarrow [D, D^\dagger] = 0$$

$$\lambda^* + \lambda = a \lambda^* \lambda \qquad \lambda = x + iy$$

$$\Rightarrow \left(x - \frac{1}{a}\right)^2 + y^2 = \frac{1}{a^2} \Rightarrow \text{GW circle}$$

doubler mass term

$\frac{2l}{a}$