Scalar Field Theory

Now we will make the transition from QM → QFT (scalar field theory). First, we start in the continuum

*Lattice Gauge Theories by Heinz Rothe Chapter 2

*Quantum Fields on the Lattice by Montvay & Munster

*Quantum Chromodynamics on the Lattice by Gattringer & Lang

$$\mathcal{L}[\phi,\partial_\mu\phi] = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \quad \text{with EOM} \quad (\Box + m^2)\phi = 0 \qquad \Box = \partial^2 = \partial_\mu\partial^\mu$$

The path integral

$$\int \mathcal{D}\phi\, e^{i\int d^4x\, \mathcal{L}[\phi,\partial_\mu\phi]} = \int \mathcal{D}\phi\, e^{iS[\phi,\partial_\mu\phi]} \qquad \mathcal{D}\phi = \prod_i d\phi_i$$

Perform an analytic continuation $t \to i\tau$

$$Z_E = \int \mathcal{D}\phi\, e^{-S_E[\phi,\partial_\mu\phi]} \quad \text{where} \quad S_E[\phi,\partial_\mu\phi] = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2\right)$$

We can rewrite via IBP

$$S_E = \int d^4x \left(\frac{1}{2}\phi(-\partial^2 + m^2)\phi\right)$$

As before, the partition function Z_E, is the generating functional of correlation functions

$$\langle\phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle = \frac{1}{Z}\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2)\ldots\phi(x_n)\, e^{-S_E(\phi,\partial_\mu\phi)}$$

$$Z[J] = \int \mathcal{D}\phi\, e^{iS[\phi,\partial_\mu\phi] + J\phi}$$

Since the path integral is a quadritic form in ϕ, we can compute path integral explicitly

$$Z_E[J] = \mathrm{Det}(-\partial^2 + m^2)^{1/2}\, e^{\frac{1}{2}\int d^4x \int d^4y\, J(x)G(x,y)J(y)}$$

*Formally $\langle\phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle = \dfrac{\delta Z[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_2)}$

The momentum space propagator is the Green function of the EOM

$$(-\partial^2 + m^2)G(x,y) = \delta(x,y) \qquad G(x,y) \text{ is the "inverse" of } -\partial^2 + m^2$$

Apply a fourier transformation

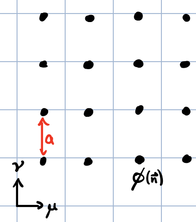
$$(p^2 + m^2)G(p) = 1$$

$$G(p) = \frac{1}{p^2 + m^2}$$

*Compare Minkowski, where are the poles

Spacetime discretization

Our quantum-fields will "live" in 4-d Euclidean spacetime



* Can discretize space but keep time continuous "Hamiltonian Lattice Gauge Theories"

$$\Lambda = \{ n = (n_1, n_2, n_3, n_4) \mid n_1, n_2, n_3 = 0, 1, \ldots, N-1; \; n_4 = 0, 1, \ldots, N_T - 1 \}$$

The vectors $n \in \Lambda$ lable points in spacetime separated by a lattice constant/spacing a. The lattice spacing plays the role of an ultraviolet cut-off in our theory $a = \frac{1}{\Lambda_{uv}}$
For the lattice to be a good approximation, a must be smaller than any other physical length scale of the theory.

We do the same translations from continuum to discrete (integrals → sums, derivatives → finite differences)

$$S_E = \sum_{n \in \Lambda} a^4 \left(\sum_{\mu=1}^{4} \frac{1}{2} \left(\frac{\phi(\vec{n}+\hat{\mu}) - \phi(\vec{n})}{a} \right)^2 + \frac{1}{2} m^2 \phi(\vec{n})^2 \right)$$

This is just one way! We can improve our action by using a different stencil.

What about propagator? For simplicity, consider $m=0$

$$D(n|m) = \sum_{\mu=1}^{4} \frac{-(\delta_{n+\hat{\mu},m} - 2\delta_{n,m} + \delta_{n-\hat{\mu},m})}{a}$$

* Symmetric 2^{nd} derivative finite stencil

Apply a discrete fourier transform

$$\tilde{D}(p|q) = \sum_{n \in \Lambda} e^{-ip \cdot na} \sum_{\mu=1}^{4} \frac{-(\delta_{m+\hat{\mu},m} - 2\delta_{n,m} + \delta_{n-\hat{\mu},m})}{a^2} e^{iq\, ma}$$

$$= \sum_{n \in \Lambda} e^{-i(p-q)na} \sum_{\mu=1}^{4} \frac{-(e^{iq_\mu a} + e^{-iq_\mu a} - 2)}{a^2}$$

$$= \delta(p-q) \sum_{\mu=1}^{4} \frac{1}{a^2} (2 - 2\cos(q_\mu a))$$

$$= \delta(p-q) \underbrace{\sum_{\mu=1}^{4} \frac{4}{a^2} \sin^2\left(\frac{q_\mu a}{2}\right)}_{\tilde{D}(q)}$$

$$G(q) = \frac{1}{\sum_{\mu=1}^{4} \frac{4}{a^2} \sin^2\left(\frac{q_\mu a}{2}\right)}$$

Taking the naive limit of $a \to 0$ returns the continuum propagator. The momentum takes values $p_\mu \in [-\pi/a, \pi/a]$ with $-\pi/a$ and π/a identified. This defines ou
Brillouin zone. We only have one pole corresponding to $p_\mu = 0$ which is our physical pole.

As we will see later in the course, this is not the case with fermions. There will be "doublers"

QCD Lightning fast review

$$\mathcal{L} = \sum_{i,j=1}^{N} \sum_{f} \bar{\psi}_i^f \Big(\underbrace{\delta_{ij}\, i\partial\!\!\!/ + g_s A\!\!\!/^a T^a_{ij}}_{\text{Gauge covariant derivative } D} - \delta_{ij} m \Big) \psi_j^f - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a}$$

Gauge covariant derivative D $\quad = -\frac{1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_s f^{abc} A_\mu^b A_\nu^c\right)^2$; A_μ^a is gluon field f^{abc} are structure constants

g_s is strong coupling $\alpha_s = \frac{g_s^2}{4\pi}$

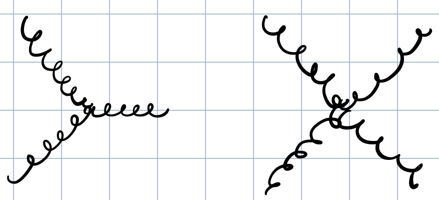
Can also include θ term: $\theta \frac{g^2}{32\pi^2} G^{\mu\nu a} \tilde{G}^a_{\mu\nu}$. This induces a non-zero neutron electric dipole moment. Hard to include on the lattice because sign problems

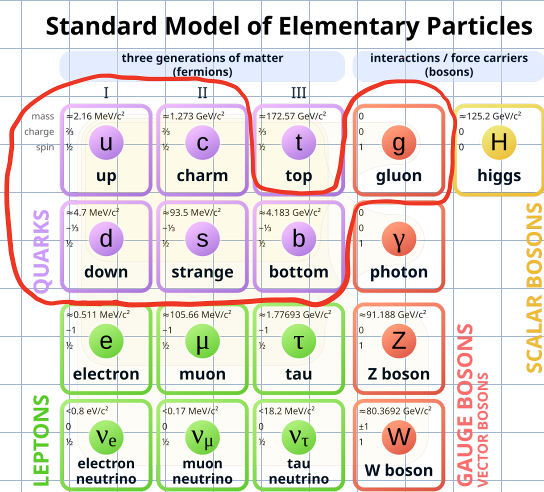
$\theta \leq 10^{-10}$, strong CP problem

This Lagrangian is invariant under SU(3) transformations $\psi \to e^{i\alpha^a(x) T^a} \psi$

Also have global U(1) $\to$ conservation of Baryon number

The crucial difference between QED and QCD is gluon self interaction (Non-abelian)





The path integral can be written as

$$\int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \mathcal{D}A\, e^{i[S_F[\bar{\psi},\psi,A] + S_G[A]]}$$

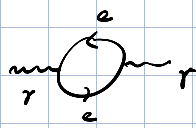
Integrating out fermionic degrees of freedom

$$\int \mathcal{D}A\, \underbrace{\det[i D\!\!\!/ - m]}\, e^{iS[A]}$$

Contains sea quarks, $\det[D\!\!\!/ - m] = 1$ is called the quenched approximation

At low energies, $\alpha_s \approx 1$ so perturbation theory breaks down. Two options
- Lattice QCD, path integral numerically
- Use symmetry arguments to find a change of variables to describe some subset of particles (Chiral perturbation theory)

Asymptotic Freedom

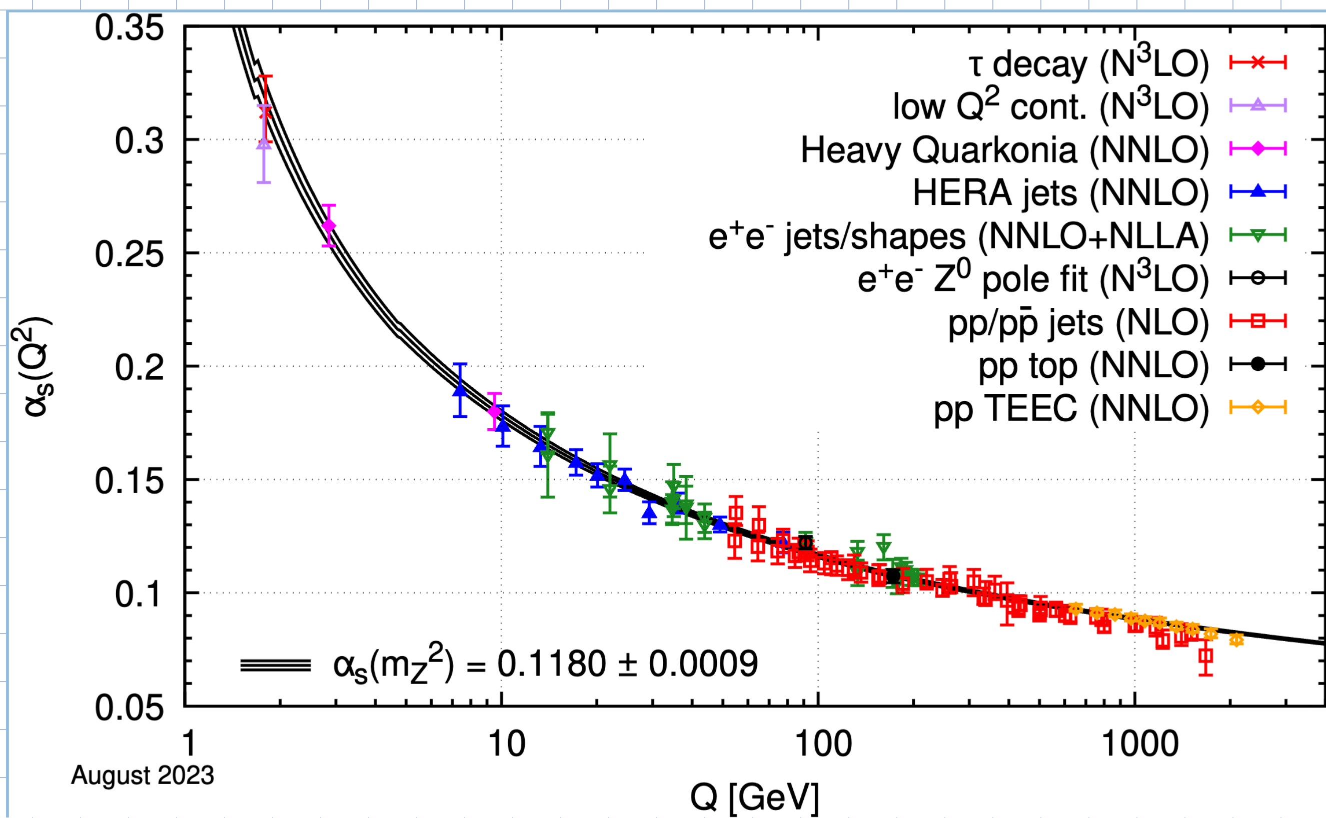
In QED, 1-loop vacuum polarization diagrams  lead to vacuum polarization. e^+e^- pairs screen electric charge at long distances

$$\mu \frac{de}{d\mu} = \frac{e^3}{12\pi^2}$$ some energy scale μ. (β-function)

QCD the opposite happens. Diagrams like  lead to anti screening.

$$\mu \frac{dg_s}{d\mu} = \frac{-g_s^3}{16\pi^2}\left[11C_A - \frac{4}{3}n_f T_F\right]$$

This is <0 so g_s decreases as μ increases. This is known as asymptotic freedom



Can also be determined by Lattice QCD.

Exercises:

1.) Review: What is the spectral decomposition of the two point function $\langle \phi(t)\phi(0)\rangle$
What about the three-point function $\langle \phi(t)\hat{J}(\tau)\varphi(0)\rangle$ where $\hat{J}$ is some operator

2.) What is the energy dispersion relation based on the scalar propagator

3.) Compute the lattice fermion propagator Where are the poles?

4.) What is the conserved current associated with the SU(3) symmetry. Is it physical?