Renormalization and Scale Setting

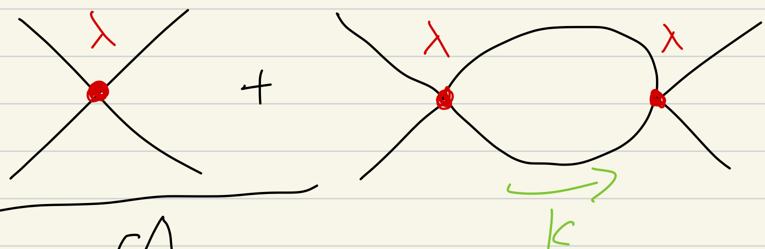
Renormalization/RG: bare parameters → physics!

N couplings → N experiments to fix theory.

"Scale setting": choice of which inputs to use.

Continuum ex: ϕ^4 theory, one-loop.

Scattering: $\phi\phi \to \phi\phi$

$iM =$ 

$$\sim \int^{\Lambda} d^4k \, \frac{1}{k^2} \cdot \frac{1}{(p+k)^2}$$

Log divergent: $$-M = \lambda + \frac{\lambda^2}{32\pi^2} \log \frac{s}{\Lambda^2}.$$

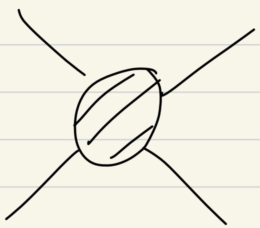
$\Lambda \to \infty$; ∞ amplitude! Is the theory "sick"?

(s: Mandelstam E_{CM}^2)

Theory is okay! Only cross-section differences.

$$\mathcal{M}(s_1) - \mathcal{M}(s_2) \sim \left(\log\frac{s_1}{\Lambda^2} - \log\frac{s_2}{\Lambda^2}\right) = \log\frac{s_1}{s_2}$$

Define renormalized coupling at ref. energy s_0.



$$\Bigg|_{s=s_0} = -i\lambda_R \Rightarrow \lambda_R = \lambda + \frac{\lambda^2}{32\pi^2}\log\frac{s_0}{\Lambda^2} + \dots$$

Scattering in terms of λ_R is finite:

$$\mathcal{M}(s) = -\lambda_R - \frac{\lambda_R^2}{32\pi^2}\log\frac{s}{s_0} + \dots$$

Definition of λ_R is our renormalization condition.

No divergences!*

*(except as $s_0 \to \infty$. No continuum limit on lattice!)

If we change $s_0 \to s_0'$, λ_R changes too.

calculate dependence $\to$ renormalization group.

$$\frac{d\lambda_R}{ds_0} = \frac{1}{s_0}\frac{\lambda^2}{32\pi^2}.$$

Define the beta function:

$$\beta(\lambda_R) \equiv s_0 \frac{d\lambda_R}{ds_0} = +(\ldots)\lambda_R^2$$

Arrows show UV flow (to $s_0 \to \infty$.)

Opposite is IR flow (to $s_0 \to 0$.)

($\lambda_R \to 0$.)

"Continuum limit" $s_0 \to \infty$ flows into $\lambda_R \to \infty$
(perturbative control lost.)

Alternatively hold $\lambda_R(s_0)$ fixed, look at $\lambda_R(s_1)$ w/ $s_1 \ll s_0$. then $\boxed{\lambda_R(s_1) \to 0}$ "triviality".

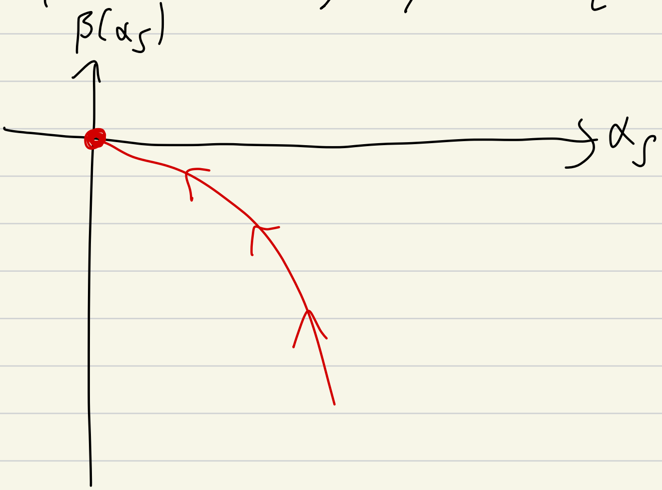
gauge coupling

For QCD first term in $\beta(\alpha_s)$ $(\alpha_s \equiv g_s^2/4\pi)$ has crucial minus sign.

(Gross, Wilczek '73; Politzer '73)

$$\Lambda \frac{d\alpha_s}{d\Lambda} \equiv \beta(\alpha_s) = -\frac{1}{2\pi}\left(11 - \frac{2}{3}N_f\right)\alpha_s^2 + \ldots$$

(N_f = # of light, active quarks.)



(UV flow: $\Lambda \to \infty$)

asymptotic freedom

($\Lambda \to \infty$ is $\alpha_s \to 0$.)

In QCD, α_s is inconvenient to renormalize the theory (we can't scatter quarks/gluons)

Instead, rely on dimensional transmutation

$$\boxed{\alpha_s(\Lambda_0) = \alpha_{ref} \longleftrightarrow M_H = C_{ref}\Lambda_0}$$

(roughly.)

(R. Sommer, 1401.3270)

Scale setting — focus on QCD.

Gauge coupling α_s + quark masses m_u, m_d, m_s, m_c, m_b.

(Top quark is too heavy/unstable, $\theta = 0$.)

Six experimental inputs (w/ dependence on parameters; proton mass $\not\to m_b$.)

can be any six processes ($\pi\pi \to \pi\pi$ scattering?)

We want highest precision → hadron masses
→ decay constants

Quark masses ⟷ direct meson ($\bar{q}q$) masses.

Heavy quarks: $\begin{cases} M_{D_s} \sim m_c & (\bar{c}s) \\ M_{B_s} \sim m_b & (\bar{b}s) \end{cases}$

Light quarks: chiral perturbation theory

(chiral symmetry restored at $m_q = 0$ ...)

Implies the Gell-Mann-Oakes-Renner relation:

(GMOR): $M_{q_1 q_2}^2 = B(m_{q_1} + m_{q_2})$
(pseudoscalar)

works well for light, strange. Pions, kaons!

$\begin{cases} \pi^+ \sim \bar{d}u & K^+ \sim \bar{s}u \\ \pi^0 \sim (\bar{u}u + \bar{d}d) & K^0 \sim \bar{s}d \end{cases}$

(Flavor Lattice Averaging Group)
FLAG 2024 $\rightarrow$ "Edinburgh consensus" (no QED)

$$\begin{cases} M_{\pi^+} = 135.0 \text{ MeV} \\ M_{K^+} = 491.6 \text{ MeV} \\ M_{K^0} = 497.6 \text{ MeV} \\ M_{D_s^+} = 1967 \text{ MeV} \\ M_{B_s^0} = 5367 \text{ MeV} \end{cases}$$

One more input to fix overall scale.

Simulation produces aM_{π^+}; need to set a to fix $M_{\pi^+} = 135$ MeV!

Three classes of scale setting:

A) Hadron masses;

B) Decay constants;

C) Theory scales.

Precision is essential! S to set scale.

(aM_H) from calculation, precision $\delta(aM_H)$.

Then: $$M_H = \frac{\cancel{a} M_H}{\cancel{a} S} \cdot S_{exp}$$ Frac error:

$$\frac{\delta M_H}{M_H} = \sqrt{\left(\frac{\delta(aM_H)}{aM_H}\right)^2 + \left(\frac{\delta(aS)}{aS}\right)^2 + \left(\frac{\delta S_{exp}}{S_{exp}}\right)^2}$$

lattice aM_H error | lattice aS error | experimental.

(https://pdg.lbl.gov)

$\delta S_{exp}/S_{exp}$ provides ultimate limit on error!

$\delta(aS)/aS$ also important.

A) Hadron masses - what are good options?

- Proton: $\dfrac{\delta m_p^{exp}}{m_p} \sim 3\times10^{-8}$

But: exponential signal-to-noise degradation (see exercises)

- π, K: very precise, but we used them!

- η: also very precise, hard on lattice (disconnected diagrams...)

- Ω baryon: $\dfrac{\delta m_\Omega^{exp}}{m_\Omega^{exp}} \sim 0.02\%$

(sss)

Also exp. S/N problem, better than proton.

Lattice calcs $\sim$ sub-percent aM_Ω.

B) Decay constants – more details next time.

$$\Gamma(\pi \to \mu\nu) \sim f_\pi^2 ; \quad \frac{\delta\Gamma^{exp}}{\Gamma^{exp}} \sim 10^{-3}$$

But disentangle EW physics. Systematics?

(f_π still commonly used.)

C) Theory scales: S_{exp} vs. aS precision;

(systematic effects, N/S, or expensive.)

Theory scale $1/T$ can be a proxy.

Easy to compute a/T precisely; relate to some physical S_{exp} once.

$$\lim_{a\to 0} \left(\frac{aS}{a/T}\right) = \boxed{TS} ; \quad S_{exp} \to T_{ref}.$$

Future calculations use TM_H vs. $\left(\frac{aM_H}{aS}\right) S_{exp}$.

Static potential : Γ_0, Γ_1

Gradient flow : $\sqrt{t_0}$, w_0