Path Integrals in the Continuum

Consider a general quantum mechanical system in one-dimension with Hamiltonian

$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(x)$ (I will drop operator notation from here)

The quantity of interest is the partition function

$$Z = \mathrm{Tr}\left[e^{-iHt}\right] = \int dx \langle x| e^{-iHt} |x\rangle$$

which will give us the generating functional for correlation functions. First, lets compute the free matrix element $\langle x| e^{-iH_0 t} |y\rangle$

$$\begin{aligned}
\langle x| e^{-iH_0 t} |y\rangle &= \langle x| e^{-ip^2 t/2m} |y\rangle \\
&= \int dp \langle x| e^{-ip^2 t/2m} |p\rangle\langle p|y\rangle \\
&= \int dp \langle x|p\rangle\langle p|y\rangle e^{-ip^2 t/2m} \\
&= \int \frac{dp}{2\pi} e^{-ip^2 t/2m + ip(x-y)} \\
&= \sqrt{\frac{m}{2\pi i t}}\, e^{(x-y)^2 m/2t}
\end{aligned}$$

*Resolution of identity $1 = \int dp |p\rangle\langle p|$

*real space representation of momentum eigenstates $\langle x|p\rangle = \frac{1}{\sqrt{2\pi}} e^{ipx}$

Unfortunately, momentum eigenstates are not eigenstates of the full Hamiltonian. However, we can use the Trotter formula to write the potential symmetrically

$$e^{-iH\delta t} = e^{-iV(x)\delta t/2} e^{-iH_0\delta t} e^{-iV(x)\delta t/2} + \mathcal{O}(\delta t^2)$$

$$= \int dx dy\, e^{-iV(x)\delta t/2} |x\rangle\langle x| e^{-iH_0 \delta t} |y\rangle\langle y| e^{-iV(x)\delta t/2} + \mathcal{O}(\delta t^2)$$

We can now reconstruct our original amplitude via

$$e^{-iHt} = \underbrace{e^{-iHt/N} e^{-iHt/N} \dots e^{-iHt/N}}_{N \text{ times}}$$

where $\delta t = t/N$.

$$e^{-iHt} = \int dx_1 dy_1 \dots dx_N dy_N\, e^{-iV(x)\delta t/2} |x_1\rangle\langle x_1| e^{-iH_0 t/2} |y_1\rangle\langle y_1| e^{-iV(x)\delta t/2} |x_2\rangle \dots \langle x_N| e^{iH_0\delta t} |y_N\rangle\langle y_N| e^{-iV(x)\delta t/2} + \mathcal{O}(\delta t^2)$$

Continue the derivation to show that

$$Z = \left(\frac{m}{2\pi i \delta t}\right)^{N/2} \int dx_1 \dots dx_{N-1}\, e^{iS(x \dots x_{N-1})}$$

where $S(x_1 \dots x_{N-1}) = \frac{1}{2} m \frac{1}{\delta t} \sum_{j=1}^{N} (x_{j+1} - x_j)^2 - \delta t \sum_{j=1}^{N} \frac{V(x_j) + V(x_{j+1})}{2}$

We can see that $\lim_{\to 0} \lim_{N\to\infty} S(x_1 \dots x_{N-1}) = \int_0^t dt' \left(\frac{1}{2} m \dot{x}(t')^2 - V(x(t'))\right)$ which is the continuum action!

Analytic Continuation

Since we have a finite number of integration measures, we can compute the path integral numerically. However, our integrand is an oscillating phase which poses a computational problem. We can remedy this by performing an analytic continuation into imaginary time $t = -i\tau$.

$$Z = \int dx_1 \dots dx_{N-1} e^{-S_E(x_1 \dots x_{N-1})}$$

Where $S_E = \int_0^\tau d\tau' \left(\frac{1}{2} m \dot{x}(\tau)^2 + V(x(\tau))\right)$ is the Euclidean action. Now we can compute the path integral numerically.

Coding exercise: Use the provided jupyter notebook to perform the path integral numerically for Harmonic Oscillator.

This is very similar to a Boltzmann like distribution in statistical mechanics. Thus we can interpret the Euclidean path integral with periodic boundary conditions as the canonical partition function of the corresponding thermal system.

Correlation Functions

Correlation functions are of great importance as they can give us information about the mass spectrum of a particle, hadronic contributions to μ g-2, weak decays, etc. Correlation functions follow an operator expectation value. Lets first consider a Euclidean two-point correlator

$$
\begin{aligned}
\langle x(t)x(0)\rangle &= \frac{1}{Z}\int dx_1 \dots dx_N \, x(t_f) x(t_i) e^{-S(x_1 \dots x_{N-1})} \\
&= \frac{1}{Z} \mathrm{Tr}\left[x e^{-Ht} x e^{-H(T-t)}\right] \\
&= \frac{1}{Z}\sum_{m,n} \langle n|e^{-Ht} x|m\rangle\langle m|e^{-H(T-t)}|n\rangle \\
&= \frac{1}{Z}\sum_{m,n} \langle n|x|m\rangle\langle m|x|n\rangle e^{-t\Delta E_n} e^{-(T-t)E_m} \\
&= \frac{\sum_{m,n} \langle n|x|m\rangle\langle m|x|n\rangle e^{-t\Delta E_n} e^{-(T-t)\Delta E_m}}{1 + e^{-T\Delta E_1} + e^{-T\Delta E_2} + \dots}
\end{aligned}
$$

T is a maximal formal distance, which will eventually be taken to ∞

In the limit of large T

$$\boxed{\langle x(t)x(0)\rangle = \sum_{n,m} \langle n|x|m\rangle\langle m|x|n\rangle e^{-t\Delta E_n}}$$

Thus we can extract the mass/energy spectrum from a two-point function.