Supplementary A: Implementation Details

This supplementary material provides the implementation details for the manuscript "Optimizing Capacitated Multi-Trip Vehicle Routing with Time Windows: Joint Utilization of Route-Based and Trip-Based Formulations".

EC.2. Valid Cuts and Separation Methods

EC.2.1. Families of Valid Cuts Applied to the CMTVRPTW

Subset-Row (SR) Cuts. Subset-row cuts, of its general form as shown below, are proposed by Jepsen et al. (2008) for the VRPTW.

$$\sum_{r\in\mathcal{R}}\left\lfloor\sum_{i\in\bar{\mathcal{V}}}\alpha_{ir}/k\right\rfloor x_r \le \left\lfloor|\bar{\mathcal{V}}|/k\right\rfloor, \qquad \forall\, \bar{\mathcal{V}}\subseteq\mathcal{V},$$

for any given integer $k\ge 2$.

Paradiso et al. (2020) apply a subset of the SR cuts, called SR3 cuts, to the CMTVRPTW by setting $|\bar{\mathcal{V}}|=3$ and $k=2$. Besides SR3 cuts, we also incorporate the SR52 cuts (i.e., $|\bar{\mathcal{V}}|=5$ and $k=2$) and the SR53 cuts (i.e., $|\bar{\mathcal{V}}|=5$ and $k=3$), as additional valid cuts to strengthen the LP relaxation. Given $r\in\mathcal{R}$, $\bar{\mathcal{V}}\subseteq\mathcal{V}$ and integer $k\ge 2$, let $\eta_{r\bar{\mathcal{V}}k}=\left\lfloor\sum_{i\in\bar{\mathcal{V}}}\alpha_{ir}/k\right\rfloor$. The SR3, SR52 and SR53 cuts are explicitly formulated as follows.

$$\text{(SR3)} \qquad \sum_{r\in\mathcal{R}}\eta_{r\bar{\mathcal{V}}2}x_r\le 1, \qquad \forall\,\bar{\mathcal{V}}\subseteq\mathcal{V}:|\bar{\mathcal{V}}|=3, \tag{EC.1a}$$

$$\text{(SR52)} \qquad \sum_{r\in\mathcal{R}}\eta_{r\bar{\mathcal{V}}2}x_r\le 2, \qquad \forall\,\bar{\mathcal{V}}\subseteq\mathcal{V}:|\bar{\mathcal{V}}|=5, \tag{EC.1b}$$

$$\text{(SR53)} \qquad \sum_{r\in\mathcal{R}}\eta_{r\bar{\mathcal{V}}3}x_r\le 1, \qquad \forall\,\bar{\mathcal{V}}\subseteq\mathcal{V}:|\bar{\mathcal{V}}|=5. \tag{EC.1c}$$

Elementary (EL) Cuts. The elementary cuts, introduced by Pecin et al. (2017), can be applied to the CMTVRPTW. Given $i\in\mathcal{V}$, $r\in\mathcal{R}$ and any nonempty set $\bar{\mathcal{V}}\subseteq\mathcal{V}\setminus\{i\}$, we use a binary parameter $\lambda_{ir\bar{\mathcal{V}}}$ which is determined by

$$\lambda_{ir\bar{\mathcal{V}}}=\left\lfloor\frac{|\bar{\mathcal{V}}|-1}{|\bar{\mathcal{V}}|}\alpha_{ir}+\sum_{j\in\bar{\mathcal{V}}}\frac{1}{|\bar{\mathcal{V}}|}\alpha_{jr}\right\rfloor.$$

Following Pecin et al. (2017), the EL cuts are formulated as follows.

$$\text{(EL)} \qquad \sum_{r\in\mathcal{R}}\lambda_{ir\bar{\mathcal{V}}}x_r\le 1, \qquad \forall\, i\in\mathcal{V},\bar{\mathcal{V}}\subseteq\mathcal{V}\setminus\{i\},\bar{\mathcal{V}}\neq\emptyset. \tag{EC.2}$$

For each customer $i\in\mathcal{V}$ and a nonempty customer subset $\bar{\mathcal{V}}\subseteq\mathcal{V}\setminus\{i\}$, we have such an EL cut to enforce that the number of routes that visit all the customers in $\bar{\mathcal{V}}$ or visit customer i together with at least one customer in $\bar{\mathcal{V}}$ cannot be greater than one.

Strengthened Rounded Capacity (SRC) Cuts. The strengthened rounded capacity cuts, introduced by Baldacci et al. (2008) for the VRP, are applied to the CMTVRPTW. For route $r \in \mathcal{R}$ and set $\bar{\mathcal{V}} \subseteq \mathcal{V}$, let a binary parameter $\mu_{r\bar{\mathcal{V}}}$ indicate whether route r visits at least one customer in set $\bar{\mathcal{V}}$ (i.e., $\mu_{r\bar{\mathcal{V}}} = 1$ if $\mathcal{V}(r) \cap \bar{\mathcal{V}} \neq \emptyset$) or not (i.e., $\mu_{r\bar{\mathcal{V}}} = 0$ otherwise), where $\mathcal{V}(r)$ is the set of customers visited by route r. Following Baldacci et al. (2008), the SRC cuts can be formulated as follows.

$$\text{(SRC)} \qquad \sum_{r \in \mathcal{R}} \mu_{r\bar{\mathcal{V}}} x_r \geq \left\lceil \sum_{i \in \bar{\mathcal{V}}} q_i / Q \right\rceil, \qquad \forall\, \bar{\mathcal{V}} \subseteq \mathcal{V}. \tag{EC.3}$$

For each $\bar{\mathcal{V}} \subseteq \mathcal{V}$, there is an SRC cut to enforce that the number of routes that serve at least one customer in $\bar{\mathcal{V}}$ must be no less than $\lceil \sum_{i \in \bar{\mathcal{V}}} q_i / Q \rceil$, due to the capacity limit of each vehicle.

EC.2.2. Cut Separation for the Route-Based Formulation LF_1

For a solution $\boldsymbol{x}$ of LF_1, the valid cuts are identified in a cutting-plane fashion by the following separation methods.

Separation of the RSF Cuts. From the definition of β_{tr}, we know $\sum_{r \in \mathcal{R}} \beta_{tr} x_r = \sum_{r \in \mathcal{R}(\boldsymbol{x})} \beta_{tr} x_r$ is a step function of t, whose value may increase only at $t \in \{l_r : r \in \mathcal{R}(\boldsymbol{x})\}$ and may decrease only at $t \in \{e_r + d_r : r \in \mathcal{R}(\boldsymbol{x})\}$. Thus, the maximum value of $\sum_{r \in \mathcal{R}(\boldsymbol{x})} \beta_{tr} x_r$ over $t \in [a_0, b_0]$ must be achieved at some time point in $\{l_r : r \in \mathcal{R}(\boldsymbol{x})\}$. For the separation of the RSF cuts, it is sufficient to check the cuts of (6c) associated with $t \in \{l_r : r \in \mathcal{R}(\boldsymbol{x})\}$.

Separation of the RWT Cuts. To shorten the running time, we consider a subset of the RWT cuts to separate by enumeration. In particular, we only check cuts (6d) associated with some selected pairs of t_1 and t_2 where $t_1, t_2 \in \{e_r, l_r, e_r + d_r, l_r + d_r : r \in \mathcal{R}(\boldsymbol{x})\}$ and $t_1 < t_2$.

Separation of the SR3 Cuts. As the number of SR3 cuts (6e) is proportional to $|\mathcal{V}|^3$, the SR3 cuts can be separated by enumeration (see, e.g., Paradiso et al. 2020, Yang 2023).

Separation of the SR52 Cuts. The separation of SR52 cuts (6f) are performed sequentially by two phases. In the first phase, we consider a restricted subset of SR52 cuts to separate by enumeration. If the considered cuts are satisfied by $\boldsymbol{x}$, we solve the following IP formulation (IP-SR52) with respect to the given $\boldsymbol{x}$ in the second phase to further separate the SR52 cuts.

$$\text{(IP-SR52)} \qquad z_{\text{SR52}}(\boldsymbol{x}) = \max \sum_{r \in \mathcal{R}(\boldsymbol{x})} x_r u_r - 2, \tag{EC.4a}$$

$$\text{s.t.} \qquad u_r - \frac{1}{2} \sum_{i \in \mathcal{V}} \alpha_{ir} v_i \leq 0, \qquad \forall\, r \in \mathcal{R}(\boldsymbol{x}), \tag{EC.4b}$$

$$\sum_{i\in\mathcal{V}} v_i = 5, \tag{EC.4c}$$

$$u_r \in \mathbb{Z}_+, \qquad \forall\, r \in \mathcal{R}(\boldsymbol{x}), \tag{EC.4d}$$

$$v_i \in \{0,1\}, \qquad \forall\, i \in \mathcal{V}. \tag{EC.4e}$$

The objective function (EC.4a) is to find among cuts (6f) of LF_1 the maximum violation by the given solution $\boldsymbol{x}$. In (EC.4e), we introduce a binary variable v_i for $i\in\mathcal{V}$ to indicate whether customer i is selected. Constraint (EC.4c) enforces that exactly 5 customers are selected. Thus, a feasible solution of IP-SR52 corresponds to a set $\bar{\mathcal{V}}\subseteq\mathcal{V}$ with $|\bar{\mathcal{V}}|=5$. With the integer variables u_r for $r\in\mathcal{R}(\boldsymbol{x})$ introduced in (EC.4d), cuts (EC.4b) ensure that u_r is equal to the value of $\eta_{r\bar{\mathcal{V}}2}$ for the corresponding $\bar{\mathcal{V}}$ and $r\in\mathcal{R}(\boldsymbol{x})$. For IP-SR52, if there exists a feasible solution with objective value $z_{\text{SR52}}(\boldsymbol{x})>0$, the cut of (6f) with respect to $\bar{\mathcal{V}}=\{i\in\mathcal{V}: v_i=1\}$ in LF_1 is known to be violated.

The proposed IP formulation is solved using a commercial MIP solver with an upper time limit of 0.5 seconds.

Separation of the SR53 Cuts. The separation of SR53 cuts (6g) is performed sequentially in two phases. In the first phase, we consider a restricted subset of SR53 cuts to separate by enumeration. If the considered cuts are satisfied by $\boldsymbol{x}$, we solve the following IP formulation (IP-SR53) with respect to the given $\boldsymbol{x}$ in the second phase to further separate the SR53 cuts.

$$\text{(IP-SR53)} \qquad z_{\text{SR53}}(\boldsymbol{x}) = \max \sum_{r\in\mathcal{R}(\boldsymbol{x})} x_r u_r - 1, \tag{EC.5a}$$

$$\text{s.t.} \qquad u_r - \frac{1}{3}\sum_{i\in\mathcal{V}} \alpha_{ir} v_i \le 0, \qquad \forall\, r\in\mathcal{R}(\boldsymbol{x}), \tag{EC.5b}$$

$$\sum_{i\in\mathcal{V}} v_i = 5, \tag{EC.5c}$$

$$u_r \in \mathbb{Z}_+, \qquad \forall\, r \in \mathcal{R}(\boldsymbol{x}), \tag{EC.5d}$$

$$v_i \in \{0,1\}, \qquad \forall\, i \in \mathcal{V}. \tag{EC.5e}$$

The objective function (EC.5a) is to find among cuts (6g) of LF_1 the maximum violation by the given solution $\boldsymbol{x}$. In (EC.5e), we introduce a binary variable v_i for $i\in\mathcal{V}$ to indicate whether customer i is selected. Constraint (EC.5c) enforces that exactly 5 customers are selected. Thus, a feasible solution of IP-SR53 corresponds to a set $\bar{\mathcal{V}}\subseteq\mathcal{V}$ with $|\bar{\mathcal{V}}|=5$. With the integer variables u_r for $r\in\mathcal{R}(\boldsymbol{x})$ introduced in (EC.5d), cuts (EC.5b) ensure that u_r is equal to the value of $\eta_{r\bar{\mathcal{V}}3}$ for the corresponding $\bar{\mathcal{V}}$ and $r\in\mathcal{R}(\boldsymbol{x})$. For IP-SR53, if there exists a feasible solution with objective value $z_{\text{SR53}}(\boldsymbol{x})>0$, the cut of (6g) with respect to $\bar{\mathcal{V}}=\{i\in\mathcal{V}: v_i=1\}$ in LF_1 is known to be violated.

The proposed IP formulation is solved using a commercial MIP solver with an upper time limit of 0.5 seconds.

Separation of the EL Cuts. To shorten the running time, we consider a subset of the EL cuts to separate by enumeration. In particular, we only check cuts (6h) associated with $\bar{\mathcal{V}} \in \{\mathcal{V}(r) : r \in \mathcal{R}(\boldsymbol{x})\}$.

Separation of the SRC Cuts. For the separation of SRC cuts, we solve the following IP formulation (IP-SRC) with respect to the given solution $\boldsymbol{x}$ of LF_1.

$$\text{(IP-SRC)} \qquad z_{\text{SRC}}(\boldsymbol{x}) = \min \sum_{r \in \mathcal{R}(\boldsymbol{x})} x_r u_r - w, \tag{EC.6a}$$

$$\text{s.t.} \qquad w - \sum_{i \in \mathcal{V}} q_i v_i / Q \geq 0, \tag{EC.6b}$$

$$w - \sum_{i \in \mathcal{V}} q_i v_i / Q \leq 1 - \epsilon, \tag{EC.6c}$$

$$u_r - \alpha_{ir} v_i \geq 0, \qquad \forall\, i \in \mathcal{V}, r \in \mathcal{R}(\boldsymbol{x}), \tag{EC.6d}$$

$$u_r \in \{0,1\}, \qquad \forall\, r \in \mathcal{R}(\boldsymbol{x}), \tag{EC.6e}$$

$$v_i \in \{0,1\}, \qquad \forall\, i \in \mathcal{V}, \tag{EC.6f}$$

$$w \in \mathbb{Z}_+. \tag{EC.6g}$$

The objective function (EC.6a) is to find among cuts (6i) of LF_1 the maximum violation by the given solution $\boldsymbol{x}$. In (EC.6f), we introduce a binary variable v_i for $i \in \mathcal{V}$ to indicate whether customer i is selected. Thus, a feasible solution of IP-SRC corresponds to a set $\bar{\mathcal{V}} = \{i \in \mathcal{V} : v_i = 1\}$. With the binary variables u_r for $r \in \mathcal{R}(\boldsymbol{x})$ introduced in (EC.6e), cuts (EC.6d) ensure that u_r is equal to the value of $\mu_{r\bar{\mathcal{V}}}$ for the corresponding $\bar{\mathcal{V}}$ and $r \in \mathcal{R}(\boldsymbol{x})$. By introducing an integer variable w in (EC.6g) and a very small positive number $\epsilon = 10^{-6}$, we develop constraints (EC.6b) and (EC.6c) to enforce that $w = \lceil \sum_{i \in \bar{\mathcal{V}}} q_i / Q \rceil$. For IP-SRC, if there exists a feasible solution with $z_{\text{SRC}}(\boldsymbol{x}) < 0$, the cut of (6i) with respect to $\bar{\mathcal{V}} = \{i \in \mathcal{V} : v_i = 1\}$ is known to be violated.

The proposed IP formulation is solved using a commercial MIP solver with an upper time limit of 0.5 seconds.

EC.2.3. Cut Separation for the Trip-Based Formulations $\tilde{F}_2$ and LF_2

For a trip-based solution $\boldsymbol{y}$, cuts (3c) of $\tilde{F}_2$ and (7c–7h) of LF_2 are identified in a cutting-plane fashion by the following separation methods.

Separation of Cuts (3c) and (7c). The relationship $\{\tau_s : s \in \tilde{\mathcal{S}}(\boldsymbol{y})\} \subseteq [a_0, b_0]$, together with Lemma EC.1.1, i.e., $\arg\max_{t \in [a_0,b_0]} \sum_{s \in \tilde{\mathcal{S}}} \gamma_{ts} y_s \subseteq \{\tau_s : s \in \tilde{\mathcal{S}}(\boldsymbol{y})\}$, implies that $\max_{t \in [a_0,b_0]} \sum_{s \in \tilde{\mathcal{S}}} \gamma_{ts} y_s = \max_{t \in \{\tau_s : s \in \tilde{\mathcal{S}}(\boldsymbol{y})\}} \sum_{s \in \tilde{\mathcal{S}}} \gamma_{ts} y_s$. Noting that $\{\tau_s : s \in \tilde{\mathcal{S}}(\boldsymbol{y})\} \subseteq \mathcal{H} \subseteq [a_0, b_0]$, we further have $\max_{t \in \mathcal{H}} \sum_{s \in \tilde{\mathcal{S}}} \gamma_{ts} y_s = \max_{t \in \{\tau_s : s \in \tilde{\mathcal{S}}(\boldsymbol{y})\}} \sum_{s \in \tilde{\mathcal{S}}} \gamma_{ts} y_s = \max_{t \in \{\tau_s : s \in \tilde{\mathcal{S}}(\boldsymbol{y})\}} \sum_{s \in \tilde{\mathcal{S}}(\boldsymbol{y})} \gamma_{ts} y_s$, where the second equality follows from the definition of $\tilde{\mathcal{S}}(\boldsymbol{y})$. Therefore, we only need to check cuts (3c) and (7c) associated with $t \in \{\tau_s : s \in \tilde{\mathcal{S}}(\boldsymbol{y})\}$, which can be done by enumeration.

Separation of the SR3, SR52, SR53, EL, and SRC Cuts. Because the SR3, SR52, SR53, EL, and SRC cuts are only related to the routes associated with trips selected in solution $\boldsymbol{y}$ of LF_2, and independent of the departure times of these trips, we can separate them by constructing a solution $\boldsymbol{x}$ of LF_1 based on $\boldsymbol{y}$, and identifying the cuts violated by $\boldsymbol{x}$ with the separation methods introduced in §EC.2.2. To achieve this, with denoting $\mathcal{R}(\boldsymbol{y}) = \{r_s : s \in \tilde{\mathcal{S}}(\boldsymbol{y})\}$ as the set of routes associated with these trips, we set $x_r = \sum_{s\in\tilde{\mathcal{S}}(\boldsymbol{y}):r_s=r} y_s, \forall\, r \in \mathcal{R}(\boldsymbol{y})$ and $x_r = 0, \forall\, r \in \mathcal{R} \setminus \mathcal{R}(\boldsymbol{y})$.

EC.3. The Labeling Algorithm for Solving the Pricing Problem of Algorithm 2

The pricing problem considered in Algorithm 2 is to find an elementary route with the minimum reduced cost. We solve this pricing problem by a labeling algorithm similar to that of Yang (2023).

To simplify the representation, we define $T_{ij} = t_{ij} + \theta_i$ for each arc $(i,j) \in \mathcal{A}$. For any $(i,j),(j,k),(i,k) \in \mathcal{A}$, it can be seen that the triangle inequality $T_{ij} + T_{jk} \geq T_{ik}$ also holds, because $t_{ij} + t_{jk} \geq t_{ik}$ holds and the service time θ_i is nonnegative for each $i \in \mathcal{V}_0$.

Label Representation. Let $p = (i_0, i_1, \ldots, i_n)$ be a feasible elementary backward path where $i_0 = 0$ and $0 \notin \{i_1, \ldots, i_{n-1}\}$, such that a vehicle performing this backward path and starting serving node i_n at time a_{i_n} can serve nodes $i_n, i_{n-1}, \ldots, i_1$ sequentially and arrive at i_0 eventually, with time window constraints and capacity constraints satisfied. Similar to attributes e_r, l_r, d_r for a route $r \in \mathcal{R}$, the attributes e_p, l_p, d_p mean that for a vehicle performing the backward path p and starting serving node i_n at time t, the time window constraints will be satisfied if and only if $t \in [a_0, l_p]$, the vehicle will arrive at node i_0 at time $t + d_p$ if $t \in [e_p, l_p]$ and at time $e_p + d_p$ if $t \in [a_0, e_p)$.

Associated with such a feasible elementary backward path is a label $L = (p, i, q, e, l, d, \xi, \rho)$ where $i = i_n$ is the first visited node, $q = \sum_{w=1}^{n} q_{i_w}$ is the cumulative demand quantity, $e = e_p$, $l = l_p$, $d = d_p$, $\xi = \sum_{w=1}^{n} (c_{i_w i_{w-1}} - f_{i_w})$ is called the partial reduced cost and ρ is the full reduced cost calculated as equation (EC.7), where $\mathcal{V}(p) = \mathcal{V} \cap \{i_1, \ldots, i_n\}$ and parameters β_{tp}, $\bar{\beta}_{t_1 t_2 p}$, $\eta_{p\bar{\mathcal{V}}2}$, $\eta_{p\bar{\mathcal{V}}3}$, $\lambda_{jp\bar{\mathcal{V}}}$ are defined in the same way as parameters β_{tr}, $\bar{\beta}_{t_1 t_2 r}$, $\eta_{r\bar{\mathcal{V}}2}$, $\eta_{r\bar{\mathcal{V}}3}$, $\lambda_{jr\bar{\mathcal{V}}}$.

$$\begin{aligned} \rho = \xi - \sum_{t\in\mathcal{C}_{RSF}} g_t \beta_{tp} - \sum_{(t_1,t_2)\in\mathcal{C}_{RWT}} h_{t_1t_2} \bar{\beta}_{t_1t_2p} - \sum_{\bar{\mathcal{V}}\in\mathcal{C}_{SR3}} k_{\bar{\mathcal{V}}} \eta_{p\bar{\mathcal{V}}2} - \sum_{\bar{\mathcal{V}}\in\mathcal{C}_{SR52}} m_{\bar{\mathcal{V}}} \eta_{p\bar{\mathcal{V}}2} \\ - \sum_{\bar{\mathcal{V}}\in\mathcal{C}_{SR53}} o_{\bar{\mathcal{V}}} \eta_{p\bar{\mathcal{V}}3} - \sum_{(j,\bar{\mathcal{V}})\in\mathcal{C}_{EL}} u_{j\bar{\mathcal{V}}} \lambda_{jp\bar{\mathcal{V}}}. \end{aligned} \tag{EC.7}$$

The label $L = (p, i, q, e, l, d, \xi, \rho)$ is feasible if and only if $q \leq Q$ and $l \geq a_i$. We call L a complete label if $i = 0$ and $\mathcal{V}(p) \neq \emptyset$, for which p is a cyclic route. Otherwise, L is called an incomplete label. Since we can retrieve the forward route by reversing the node sequence of a backward route with the same reduced cost, the pricing problem is equivalent to finding the feasible and complete label with the minimum reduced cost.

State Transition Equations. The labeling algorithm is a label extension procedure that starts from an initial label $((0), 0, 0, a_0, b_0, 0, 0, 0)$. By extending a feasible and incomplete label $L = (p, i, q, e, l, d, \xi, \rho)$ toward a node $j \in \mathcal{V}_0 \setminus \mathcal{V}(p)$, the newly generated label $L' = (p', j, q', e', l', d', \xi', \rho')$ is determined by the following state transition equations, which are the same as that of Yang (2023) except for the computation of ρ':

$$p' = (i_0, i_1, \ldots, i_n = i, i_{n+1} = j), \tag{EC.8a}$$

$$q' = q + q_j, \tag{EC.8b}$$

$$e' = \max\{a_j, \min\{b_j, e - T_{ji}\}\}, \tag{EC.8c}$$

$$l' = \min\{l - T_{ji}, b_j\}, \tag{EC.8d}$$

$$d' = d + \max\{T_{ji}, e - b_j\}, \tag{EC.8e}$$

$$\xi' = \xi + c_{ji} - f_j, \tag{EC.8f}$$

$$\rho' \quad \text{is updated by equation (EC.7)} \tag{EC.8g}$$

Pruning Techniques. In order to accelerate the label extension process and reduce the number of labels generated, we apply the dominance rule, proposed by Yang (2023) for a different pricing problem, and show in §EC.3.1 that this rule is also valid for our pricing problem. By applying the dominance rule, we can eliminate labels that are dominated. To further reduce the number of labels generated, we employ two extra pruning techniques: rollback pruning (see, e.g., Lozano et al. 2016) and completion bound pruning (see, e.g., Baldacci et al. 2012, Lozano et al. 2016). The implementations of rollback pruning and completion bound pruning are explained in §EC.3.2 and §EC.3.3, respectively.

Heuristic. In the earlier stages of the column generation process, we adopt a heuristic labeling algorithm instead of seeking exact solutions for the pricing problems. Specifically, we select only a subset of incomplete labels with smaller reduced costs for extension. This heuristic strategy allows us to quickly identify complete labels with negative reduced costs, thereby expediting the column generation iterations. When the heuristic labeling algorithm fails to find routes with a negative reduced cost, we disable the heuristic strategy and use the exact labeling algorithm to find the optimal solution for the pricing problem, so as to prove the optimality.

EC.3.1. Dominance Rule

The dominance rule is stated in Proposition EC.1.

PROPOSITION EC.1. *Let $L_1 = (p_1, i_1, q_1, e_1, l_1, d_1, \xi_1, \rho_1)$ and $L_2 = (p_2, i_2, q_2, e_2, l_2, d_2, \xi_2, \rho_2)$ be two incomplete labels. L_1 dominates L_2 if the following conditions are satisfied simultaneously:* (i) $i_1 = i_2$, (ii) $\mathcal{V}(p_1) \subseteq \mathcal{V}(p_2)$, (iii) $l_1 \geq l_2$, (iv) $e_1 + d_1 \leq e_2 + d_2$, (v) $d_1 \leq d_2$, *and* (vi) $\xi_1 \leq \xi_2$.

Proof. Given the two labels L_1 and L_2, we first show that $\rho_1 \le \rho_2$ due to (ii–vi). According to the definitions of β_{tp}, $\bar{\beta}_{t_1t_2p}$, $\eta_{p\bar{\mathcal{V}}2}$, $\eta_{p\bar{\mathcal{V}}3}$ $\lambda_{jp\bar{\mathcal{V}}}$, we have $\eta_{p_1\bar{\mathcal{V}}2} \le \eta_{p_2\bar{\mathcal{V}}2}$, $\eta_{p_1\bar{\mathcal{V}}3} \le \eta_{p_2\bar{\mathcal{V}}3}$, $\lambda_{jp_1\bar{\mathcal{V}}} \le \lambda_{jp_2\bar{\mathcal{V}}}$ due to (ii). We have $\beta_{tp_1} \le \beta_{tp_2}$ and $\bar{\beta}_{t_1t_2p_1} \le \bar{\beta}_{t_1t_2p_2}$ due to (iii–v), which, together with (vi), (EC.7), and $\boldsymbol{g} \le \mathbf{0}, \boldsymbol{h} \le \mathbf{0}, \boldsymbol{k} \le \mathbf{0}, \boldsymbol{m} \le \mathbf{0}, \boldsymbol{o} \le \mathbf{0}, \boldsymbol{u} \le \mathbf{0}$, verify that $\rho_1 \le \rho_2$.

Following the same argument of Yang (2023), we show that if (i–vi) hold for L_1 and L_2, then for any $j_1 \in \mathcal{V}_0 \setminus \mathcal{V}(p_2)$ such that L_2 can be extended toward j_1 to form a feasible label $L_2' = (p_2', i_2' = j_1, q_2', e_2', l_2', d_2', \xi_2', \rho_2')$, L_1 can be extended toward j_1 to form a feasible label $L_1' = (p_1', i_1' = j_1, q_1', e_1', l_1', d_1', \xi_1', \rho_1')$, and (i–vi) still hold for L_1' and L_2'.

Following the above argument repeatedly, for any node sequence $j_1, j_2, \ldots, j_n \in \mathcal{V}_0 \setminus \mathcal{V}(p_2)$ such that $j_n = 0$, $j_1, j_2, \ldots, j_n$ are different from each other and L_2 can be extended toward $j_1, j_2, \ldots, j_n$ sequentially to form a feasible label $L_2^n = (p_2^n, i_2^n = j_n, q_2^n, e_2^n, l_2^n, d_2^n, \xi_2^n, \rho_2^n)$, L_1 can be extended toward $j_1, j_2, \ldots, j_n$ sequentially to form a feasible label $L_1^n = (p_1^n, i_1^n = j_n, q_1^n, e_1^n, l_1^n, d_1^n, \xi_1^n, \rho_1^n)$, and (i–vi) still hold for L_1^n and L_2^n.

Since (ii–vi) hold for L_1^n and L_2^n, we have $\rho_1^n \le \rho_2^n$ according to the earlier procedure for showing $\rho_1 \le \rho_2$. Since $j_n = 0$, both L_1^n and L_2^n are complete labels. Due to the arbitrariness of $j_1, j_2, \ldots, j_n$, we conclude that for every complete label L_2^n extended from L_2, there exists a complete label L_1^n extended from L_1 that has an equal or smaller reduced cost. Therefore, L_1 dominates L_2. □

EC.3.2. Rollback Pruning

The rollback pruning stated in Proposition EC.2 is a sufficient condition to quickly decide whether a label is dominated, through efficient arithmetic operations, instead of by directly comparing the label with a large number of other labels possibly dominating it.

Proposition EC.2. *For an incomplete label $L_2 = (p_2, i = i_n, q_2, e_2, l_2, d_2, \xi_2, \rho_2)$ where $p_2 = (i_0, i_1, \ldots, i_{n-1}, i_n)$, if there exists an integer $\bar{w} \in \{0, 1, \ldots, n-2\}$, such that $c_{i_n i_{\bar{w}}} - f_{i_n} \le \sum_{w=\bar{w}+1}^{n} (c_{i_w i_{w-1}} - f_{i_w})$, then label L_2 can be pruned since it is dominated by another incomplete label $L_1 = (p_1, i = i_n, q_1, e_1, l_1, d_1, \xi_1, \rho_1)$ where $p_1 = (i_0, i_1, \ldots, i_{\bar{w}-1}, i_{\bar{w}}, i_n)$.*

Proof. We prove this proposition by showing that L_1 dominates L_2 according to the dominance rule stated in §EC.3.1 (see Proposition EC.1). In particular, we show (i–vi) of Proposition EC.1 are all satisfied with respect to L_1 and L_2.

According to the definitions of L_1 and L_2, (i) is naturally satisfied. Since p_1 can be derived by removing $i_{\bar{w}+1}, i_{\bar{w}+2}, \ldots, i_{n-1}$ from p_2, we have $\mathcal{V}(p_1) \subset \mathcal{V}(p_2)$, and (ii) is hence satisfied.

Recall that the extension from $(p, i, q, e, l, d, \xi, \rho)$ to $(p', j, q', e', l', d', \xi', \rho')$ incurs the label updates following (EC.8a–EC.8g).

Consider a label $L = (p, i_{\bar{w}}, q, e, l, d, \xi, \rho)$ where $p = (i_0, i_1, \ldots, i_{\bar{w}-1}, i_{\bar{w}})$. Note that L can be further extended to form L_1 and L_2, due to $T_{i_n i_{\bar{w}}} \le \sum_{w=\bar{w}+1}^{n} T_{i_w i_{w-1}}$ according to the triangle inequality.

Thus, from (EC.8d), we have $l_1 = \min\{l - T_{i_n i_{\bar{w}}}, b_{i_n}\}$, $l_2 \le b_{i_n}$, and $l_2 \le l - \sum_{w=\bar{w}+1}^{n} T_{i_w i_{w-1}} \le l - T_{i_n i_{\bar{w}}}$. Note that $l_1 \ge l_2$, so (iii) is satisfied.

According to (EC.8c) and (EC.8e), we have $e' + d' = \max\{e + d, a_j + T_{ji} + d\}$. Thus, for the labels L_1 and L_2 extended from L, we have $e_1 + d_1 = \max\{e + d, a_{i_n} + T_{i_n i_{\bar{w}}} + d\}$, $e_2 + d_2 \ge e + d$, and $e_2 + d_2 \ge a_{i_n} + \sum_{w=\bar{w}+1}^{n} T_{i_w i_{w-1}} + d \ge a_{i_n} + T_{i_n i_{\bar{w}}} + d$. Note that $e_1 + d_1 \le e_2 + d_2$, so (iv) is satisfied.

Furthermore, from (EC.8e) and $e' + d' = \max\{e + d, a_j + T_{ji} + d\}$, we have $d_1 = \max\{d + T_{i_n i_{\bar{w}}}, d + e - b_{i_n}\}$, $d_2 \ge d + \sum_{w=\bar{w}+1}^{n} T_{i_w i_{w-1}} \ge d + T_{i_n i_{\bar{w}}}$, and $d_2 \ge d + e - b_{i_n}$. Thus, we have $d_1 \le d_2$, and (v) is also satisfied.

From (EC.8f), we have $\xi_1 = \xi + c_{i_n i_{\bar{w}}} - f_{i_n}$ and $\xi_2 = \xi + \sum_{w=\bar{w}+1}^{n} (c_{i_w i_{w-1}} - f_{i_w})$. Since $c_{i_n i_{\bar{w}}} - f_{i_n} \le \sum_{w=\bar{w}+1}^{n} (c_{i_w i_{w-1}} - f_{i_w})$, we obtain that $\xi_1 \le \xi_2$, and (vi) is satisfied.

Since (i–vi) of Proposition EC.1 are satisfied, we know L_2 is dominated by L_1 and can be pruned through the rollback pruning. $\square$

EC.3.3. Completion Bound Pruning

For any incomplete label, we compute a completion bound that provides a lower bound on the minimum increment of reduced cost for extending it to a complete label. That is, for an incomplete label $L = (p, i, q, e, l, d, \xi, \rho)$, if the completion bound plus the reduced cost ρ exceeds any upper bound on the optimal solution value of the pricing problem, there is no need to further extend the incomplete label because it cannot lead to any optimal complete label for the pricing problem.

Different from the completion bound computation in Yang (2023), we take into account both the demand and time information to compute the completion bound of an incomplete label. Specifically, consider a triplet (i, q, l) for customer $i \in \mathcal{V}$, demand quantity $q \in [q_i, Q]$ and time point $l \in [a_i, b_i]$. Let $\xi^*(i, q, l)$ be the minimum increment of the reduced cost for extending an incomplete label L with $i(L) = i$, $q(L) = q$ and $l(L) = l$ to form a complete label. Thus, any value $\tilde{\xi}(i, q, l) \le \xi^*(i, q, l)$ is a completion bound associated with triplet (i, q, l).

Let c'_{ub} be an upper bound on the optimal solution value of the pricing problem, which is initialized as zero and updated when a better solution to the pricing problem is found. The completion bound pruning is stated as follows. For any incomplete label $L = (p, i, q, e, l, d, \xi, \rho)$ and completion bound $\tilde{\xi}(i, q', l')$ associated with triplet (i, q', l') where $q_i \le q' \le q$ and $l \le l' \le b_i$, if $\rho + \tilde{\xi}(i, q', l') > c'_{ub}$, then label L can be pruned, since no complete labels with reduced cost less than or equal to c'_{ub} can be derived by extending L.

EC.3.4. Rollback Pruning for the DRP

In the labeling algorithm for the DRP, state transition equations, and dominance rules introduced by Yang (2023) are still valid. However, rollback pruning is not applied by them. Moreover, Proposition EC.2 is not valid for the DRP because the objective function includes not only travel costs but also energy costs. Instead, we apply rollback pruning with the following corollary for the DRP, where ce_{ijq} is the energy cost for a drone to travel along arc (i,j) with load quantity q.

COROLLARY EC.1. *For a feasible label* $L_2 = (p_2, i_n, q_2, e_2, l_2, d_2, \xi_2, \rho_2)$ *where* $p_2 = (i_0, i_1, \ldots, i_n)$, *denote* $q(\tilde{w}) = \sum_{w=1}^{\tilde{w}} q_{i_w}$. *If there exists an integer* $0 \leq \bar{w} \leq n-2$, *such that* $c_{i_n i_{\bar{w}}} + ce_{i_n i_{\bar{w}} q(\bar{w})} - f_{i_n} \leq \sum_{w=\bar{w}+1}^{n} (c_{i_w i_{w-1}} + ce_{i_w i_{w-1} q(w-1)} - f_{i_w})$, *then label* L_2 *can be pruned since it is dominated by another label* $L_1 = (p_1, i_n, q_1, e_1, l_1, d_1, \xi_1, \rho_1)$ *where* $p_1 = (i_0, i_1, \ldots, i_{\bar{w}}, i_n)$.

We omit the proof for Corollary EC.1 since it can be derived in the same way as Proposition EC.2.

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