1 Homelessness dynamic matching formulation

Let there be I resources indexed by $i \in \{0, ..., I\}$ and J agents indexed by $j \in \{1, ..., J\}$. Each agent arrives for matching at time $t = j$ and remains available for d time steps. In total, there are T=J+d time steps in which a match can be made, indexed by $t \in \{1, ..., T\}$. $x_{i,j,t} \in \{0, 1\}$ represents whether resource i is allocated to agent j at time t, where $w_{i,j,t} \in \mathbb{R}$ denotes the quality of this match. We wish to find an optimal pairing between the set of agents and resources. Matches that are not possible due to arrival and departure timing are represented as $w_{i,j,t} = -1$, while self-matches are expressed as $w_{i,j,t} = 0$ for $i = 0$.

- Let k_i denote the utilization time of each resource i.
- Let c_i represent the capacity (number of copies) of resource i. Setting $c_0 = J$ allows for unlimited self-matches with resource 0.
- Let $J(t) = \{max(1, t - d) \leq j \leq min(J, t)\}$ represent the agents available at time t. Let $T(j) = \{j \leq t \leq j + d\}$ represent the times an agent j is available.

To maximize the total utility of matching the set of resources and agents we can solve the linear program (relaxing $x_{i,j,t} \in [0, 1]$):

$$max \sum_{i=0}^{I} \sum_{j=1}^{J} \sum_{t=1}^{T} x_{i,j,t} w_{i,j,t}$$

such that

$$\sum_{t'=0}^{k_i - 1} \sum_{j=0}^{J} x_{i,j,t+t'} \leq c_i, \forall i, t$$

$$\sum_{i=1}^{I} \sum_{t=0}^{T} x_{i,j,t} \leq 1, \forall j$$

$$x_{i,j,t} \geq 0, \forall i, j, t$$

The first constraint limits a resource to be allocated less than c_i times over its utilization interval k_i, while the second only permits an agent to match once. The dual of the above linear program can be expressed as:

$$min \sum_{i=0}^{I} \sum_{t=1}^{T} c_i \alpha_{i,t} + \sum_{j=1}^{J} \beta_j$$

such that

$$w_{i,j,t} - \sum_{t'=t}^{min(t+k_i-1, T)} \alpha_{i,t'} - \beta_j \leq 0, \forall i, j, t$$

$$\alpha_{i,t}, \beta_j \geq 0, \forall i, j, t$$

where $\alpha_{i,t}$ is the dual corresponding to the allocation of resource i at time t and the dual β_j corresponds to the constraint limiting each agent to only match once.