Lendel Deguia

I used a different letter ∵ I used f for another function without realizing

1.4.2 $x = 0.01$, $g(x) \overset{def}{=} e^x - x - 1$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^k}_{R_{n+1}}$$

$x - x_0 = h \Rightarrow$
$$f(x_0+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} h^k + R_{n+1} \quad (2.1)$$

$f(x) \overset{def}{=} e^x$; note that for some $x_0 \in \mathbb{R}$

$|f(x_0) - f_{approx.}| = |g(x_0) - g_{approx}|$ (2.2); let $x_0 = 0$ so that $h = x = 0.01$

Since $\frac{d^n}{dx^n}(e^x) = e^x \ \forall n \in \mathbb{N}$,

$$e^h = \sum_{k=0}^{n} \frac{h^k}{k!} + R_{n+1}; \quad R_{n+1} = \frac{e^{\xi}}{(n+1)!} h^{n+1}, \quad 0 < \xi < h$$

$\Rightarrow 1 < e^{\xi} < e^h$; $1 \leq (n+1)! \Rightarrow \frac{1}{(n+1)!} < 1 \Rightarrow \boxed{R_{n+1} < e^h h^{n+1}}$

$\exists n \in \mathbb{N}: e^h h^{n+1} \leq 10^{-5} \Rightarrow h^{n+1} \leq \frac{10^{-5}}{e^h} < 10^{-5} \Rightarrow h^{n+1} < 10^{-5}$

Now $\min(n)$ such that $h^{n+1} < 10^{-5}$ is $n = 2$ since

$n = 1 \Rightarrow h^{n+1} = 10^{-4}$ and $n = 2 \Rightarrow h^{n+1} = 10^{-6} < 10^{-5}$

$$\therefore e^{0.01} \approx 1 + \frac{0.01^1}{1!} + \frac{0.01^2}{2!} = 1 + 0.01 + 0.00005$$

$$\Rightarrow e^{0.01} \approx 1.01005 \Rightarrow \boxed{g(10^{-2}) = 0.00005} \quad (2.3)$$

Now supposing $e^{0.01} \approx 1.0101$ let $y = 1.0101 - 0.01 - 1 \Rightarrow y = 0.0001$

Contrasting this with (2.3): $\boxed{|y - g(10^{-2})| = 0.00005;\ g(10^{-2}) = \frac{1}{2}y}$

1.4.3) $f(x) = e^x - e = e(e^{x-1} - 1)$

We can use a Taylor series to reformulate $f(x)$ using

$$e^{x-1} = 1 + (x-1) + \frac{(x-1)^2}{2!} + \dots + \frac{(x-1)^n}{n!} + R_n(x) \quad (3.1).$$

limiting the bits lost to at most one bit requires

$$\frac{1}{2} \leq 1 - \frac{e}{e^x} \quad \text{for } x > 1$$

and

$$\frac{1}{2} \leq 1 - \frac{e^x}{e} \quad \text{for } x < 1$$

$\Rightarrow 1 + \ln(2) \leq x$, $x \leq 1 - \ln(2)$ for $x > 1$ and $x < 1$ respectively

Thus, for $x \in \mathbb{R} \setminus (1 - \ln(2), 1 + \ln(2))$, i.e, for x outside an epsilon neighborhood centered at one of radius $\ln(2)$, we will at most lose one bit by the operation given for $f(x)$.

Otherwise, let $u = x - 1$ so that $x = u + 1$ and $|u| < \ln(2)$ when $x \in (1 - \ln(2), 1 + \ln(2))$. Using this substitution, (3.1) becomes

$$e^u = 1 + u + \frac{u^2}{2!} + \dots + \frac{u^n}{n!} + R_n(u+1) \quad (3.2)$$

That is, for $|u| < \ln(2)$, ~~we evaluate (3.2) up to the appropriate number of terms for the precision desired~~ we use $f(u) = e(u + \frac{u^2}{2!} + \dots)$ up to the appropriate number of terms ~~desired~~ for the precision desired.

1.4.9 $f(x) = 1 - \cos(x)$

A loss of significance occurs within small neighborhoods of $x \in \{2\pi m \mid m \in \mathbb{Z}\}$. We can restrict this loss to at most one bit by considering $\frac{1}{2} \leq 1 - \cos(x) \Rightarrow \cos(x) \leq \frac{1}{2}$ which occurs for $x \in \left[\frac{\pi}{3} + 2\pi m, \frac{5\pi}{3} + 2\pi m\right]$ where $m \in \mathbb{Z}$. For points that ~~satisfy~~ are in this interval, evaluating $f(x)$ directly can only result in a loss of significance of at most one bit. Ultimately, doing the following rids of the subtraction entirely:

$$f(x) = 1 - \cos(x) \frac{1 + \cos(x)}{1 + \cos(x)} = \frac{1 - \cos^2(x)}{1 + \cos(x)}$$

$$\Rightarrow \quad f(x) = \frac{\sin^2(x)}{1 + \cos(x)}$$

Which would be suitable for evaluating $f(x)$ for $x \in \left(\frac{-\pi}{3} + 2\pi m, \frac{\pi}{3} + 2\pi m\right)$ where $m \in \mathbb{Z}$.

1.4.14 $f(x)=\sqrt{x+2}-\sqrt{x}$

As $x\to\infty$, $f(x)\to 0$; this occurs because for very large x, $\sqrt{x+2}\approx\sqrt{x}$ which means loss of significance occurs for very large x. We can account for this simply by using the conjugate: $f(x)=\sqrt{x+2}-\sqrt{x}\left(\frac{\sqrt{x+2}+\sqrt{x}}{\sqrt{x+2}+\sqrt{x}}\right)$

$$\Rightarrow \quad f(x)=\frac{2}{\sqrt{x+2}+\sqrt{x}}$$

1.4.17 For very large x, $x\approx\sqrt{x^2-1}$ which may not only result in a loss of significance, but might result in an infinity error for

$$f(x)=\frac{\sin(x)}{x-\sqrt{x^2-1}}$$

We can circumvent both possible issues simply by evaluating $f(x)=\frac{\sin(x)}{x-\sqrt{x^2-1}}\,\frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$

$$\Rightarrow \quad f(x)=\sin(x)\left(x+\sqrt{x^2-1}\right)$$

We cannot have $|x|<1$; for $|x|>1$, we want $\frac{1}{2}\le 1-\frac{1}{x^2}$ so loss of precision is at most one bit. $\Rightarrow \quad |x|\ge\sqrt{2}$

Computer Exercise 1.4.6

This program will evaluate the following function

$f(x) = sin(x) - 1 + cos(x)$

$\Rightarrow f(x) = sin(x) - 2sin^2(x)$

The latter version of the function is what's used to perform the evaluations since it is a form such that loss of significance is minimized on the interval $[0, \frac{\pi}{4}]$. We wish to evaluate $f(x)$ to nearly full machine precision on various values of the aforementioned interval.

```
format long %display 15 decimal places

xvals0 = linspace(0, pi/4, 50);

yvals0 = p6(xvals0);

%use transpose arrays to display as a table
xvals = xvals0';
yvals = yvals0';

T = table(xvals, yvals);
disp(T)
```

xvals	yvals
0	0
0.0160285339468867	0.015899393430105
0.0320570678937734	0.031537793772343
0.0480856018406601	0.0469111833903016
0.0641141357875468	0.0620156127310493
0.0801426697344335	0.076847201339809
0.0961712036813202	0.0914021388568799
0.112199737628207	0.10567668599655
0.128228271575094	0.119667175507752
0.14425680552198	0.133370013116208
0.160285339468867	0.146781678447829
0.176313873415754	0.159898725933134
0.19234240736264	0.172717785692438
0.208370941309527	0.185235564401606
0.224399475256414	0.197448846138138
0.2404280092033	0.209354493207362
0.256456543150187	0.220949446948537
0.272485077097074	0.232230728520647
0.288513611043961	0.243195439667693
0.304542144990847	0.253840763463275
0.320570678937734	0.264163965034289
0.336599212884621	0.274162392263535
0.352627746831507	0.283833476471068
0.368656280778394	0.293174733074118
0.384684814725281	0.302183762225396
0.400713348672168	0.310858249429635
0.416741882619054	0.319195966138206
0.432770416565941	0.327194770321649
0.448798950512828	0.334852607019977
0.464827484459714	0.342167508870617

```
0.480856018406601     0.349137596613835
0.496884552353488     0.355761079575543
0.512913086300374     0.362036256127327
0.528941620247261     0.367961514123621
0.544970154194148     0.373535331315871
0.560998688141034     0.378756275743621
0.577027222087921     0.383623006102389
0.593055756034808     0.388134272088267
0.609084289981695     0.392288914719125
0.625112823928581     0.396085866632375
0.641141357875468     0.399524152359173
0.657169891822355     0.402602888575034
0.673198425769241     0.405321284326763
0.689226959716128     0.407678641235657
0.705255493663015     0.409674353676923
0.721284027609902     0.411307908935271
0.737312561556788     0.412578887336632
0.753341095503675      0.41348696235598
0.769369629450562     0.414031900701214
0.785398163397448     0.414213562373095
```

On the left column are the x values displayed in long format. On the right column are the y values displayed in long format. Fifty different values

evaluated on the interval $[0, \frac{\pi}{4}]$ have been displayed.

```
function y=p6(x)
   y = sin(x) - 2.*((sin(x./2)).^2);
end
```