HW 10

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5.3.1 $\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$

Basic Simpson's rule

$f(x)=\frac{1}{1+x^2}$

$(x_0,x_1,x_2)=(0,\frac{1}{2},1) \Rightarrow (f(x_i))_{i=0}^{2}=(1,\frac{4}{5},\frac{1}{2})$

$\Rightarrow \int_0^1 \frac{1}{1+x^2}dx \approx \frac{1}{6}\left[1+4\left(\frac{4}{5}\right)+\frac{1}{2}\right]$

$\Rightarrow \int_0^1 \frac{1}{1+x^2}dx \approx \frac{47}{60} \approx 0.7833$

$\int_0^1 \frac{1}{1+x^2}dx = \frac{\pi}{4} \approx 0.7854 \Rightarrow \text{error} = \left|\frac{\pi}{4}-\frac{47}{60}\right|$

$\Rightarrow \text{error} = \left|\frac{15\pi-47}{60}\right| \approx 0.002065$

5.3.2 $\int_0^1 f(x)\,dx$, $f(x)=\sin\left(\frac{\pi x^2}{2}\right) \Rightarrow f'(x)=\pi x\cos\left(\frac{\pi x^2}{2}\right)$

$\Rightarrow f''(x)=\pi\cos\left(\frac{\pi x^2}{2}\right)-\pi^2x^2\sin\left(\frac{\pi x^2}{2}\right)$; $\varepsilon<10^{-3}$

a) $|f''(x)|=\left|\pi\cos\left(\frac{\pi x^2}{2}\right)-\pi^2x^2\sin\left(\frac{\pi x^2}{2}\right)\right| \le \left|\pi\cos\left(\frac{\pi x^2}{2}\right)\right|+\left|\pi^2x^2\sin\left(\frac{\pi x^2}{2}\right)\right| \le \pi+\pi^2 \quad \forall x\in[0,1]$

for composite trapezoid rule: $\varepsilon=\left|\frac{1}{12}h^2f''(\xi)\right| \le \frac{\pi+\pi^2}{12}h^2<10^{-3}$

$\Rightarrow h<\sqrt{\frac{12\times10^{-3}}{\pi+\pi^2}} \Rightarrow h<0.03037$

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b) $|f^{(4)}(x)|=\left|(x^4\pi^4-3\pi^2)\sin\left(\frac{\pi x^2}{2}\right)-6x^2\pi^3\cos\left(\frac{\pi x^2}{2}\right)\right| \le \left|(x^4\pi^4-3\pi^2)\sin\left(\frac{\pi x^2}{2}\right)\right|$
$+\left|6x^2\pi^3\cos\left(\frac{\pi x^2}{2}\right)\right| \le |x^4\pi^4-3\pi^2|+6x^2\pi^3 \le x^4\pi^4+3\pi^2+6x^2\pi^3 \le \pi^4+3\pi^2+6\pi^3$
$\forall x\in[0,1]$

for composite Simpson's rule: $\varepsilon=\left|\frac{1}{180}h^4f^{(4)}(\xi)\right|$

$\Rightarrow \varepsilon \le \frac{\pi^4+3\pi^2+6\pi^3}{180}h^4<10^{-3} \Rightarrow h<\left(\frac{180\times10^{-3}}{\pi^4+3\pi^2+6\pi^3}\right)^{\frac{1}{4}} \Rightarrow h<0.1548$

5.3.8 $\displaystyle\int_a^{a+3h} f(x)dx \simeq \frac{3h}{8}\left[f(a)+3f(a+h)+3f(a+2h)+f(a+3h)\right]$

$$f(a+h) = f + hf' + \frac{1}{2}h^2 f'' + \frac{1}{6}h^3 f''' + \frac{1}{24}h^4 f^{(4)} + \dots \quad (8.1)$$

$$f(a+2h) = f + 2hf' + 2h^2 f'' + \frac{4}{3}h^3 f''' + \frac{2}{3}h^4 f^{(4)} + \dots \quad (8.2)$$

$$f(a+3h) = f + 3hf' + \frac{9}{2}h^2 f'' + \frac{9}{2}h^3 f''' + \frac{27}{8}h^4 f^{(4)} + \dots \quad (8.3)$$

$$\Rightarrow \left[f(a)+3f(a+h)+3f(a+2h)+f(a+3h)\right] = 8f + 12hf' + 12h^2f'' + 9h^3 f''' + \frac{11}{2}h^4 f^{(4)} + \dots \quad (8.4)$$

multiplying (8.4) by $\frac{3h}{8}$ gives:

$$3hf + \frac{9}{2}h^2 f' + \frac{9}{2}h^3 f'' + \frac{27}{8}h^4 f''' + \frac{33}{16}h^5 f^{(4)} + \dots \quad (8.5)$$

$$F(x) = \int_a^x f(t)\,dt;\ F(a+3h) = F + 3hF' + \frac{9}{2}h^2 F'' + \frac{9}{2}h^3 F''' + \frac{27}{8}h^4 F^{(4)} + \frac{81}{40}h^5 F^{(5)} + \dots$$

$F = F(a) = 0$; also note $F^{(n)}(a) = f^{(n-1)}(a)$

$$\Rightarrow \int_a^{a+3h} f(x)dx = 3hf + \frac{9}{2}h^2 f' + \frac{9}{2}h^3 f'' + \frac{27}{8}h^4 f''' + \frac{81}{40}h^5 f^{(4)} + \dots \quad (8.6)$$

Subtracting (8.5) from (8.6) yields

$$\int_a^{a+3h} f(x)dx - \frac{3}{8}\left[f(a)+3f(a+h)+3f(a+2h)+f(a+3h)\right] = \left(\frac{81}{40}h^5 f^{(4)} - \frac{33}{16}h^5 f^{(4)}\right) + \dots$$

$$= \left(\frac{81}{40} - \frac{33}{16}\right)h^5 f^{(4)} + \dots = -\frac{3}{80}h^5 f^{(4)} + \dots$$; Denote error by "$\mathcal{E}$"

$$\Rightarrow \boxed{\mathcal{E} = -\frac{3}{80}h^5 f^{(4)}(\xi) \text{ for some } \xi \in (a, a+3h)}$$

The error term is $\mathcal{E} = O(h^5)$ for Simpson's 3/8 rule which is the same order as Simpson's 1/3 rule, yet an extra function evaluation is needed for Simpson's 3/8 rule so the cost of using the 3/8 rule outweighs the benefit.