Rendel Deguia

5.4.1 $\int_0^2 f(x)\,dx,\ f(x)=e^{-x^2}$

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} f\left[\tfrac{1}{2}(b-a)t + \tfrac{1}{2}(b+a)\right]dt \quad (1.1)$$

By eq. (1.1), $\int_0^2 e^{-x^2}dx = \int_{-1}^{1} e^{-(t+1)^2}dt$; by table 5.1:

nodes: $\left\{\pm\sqrt{\tfrac{1}{3}}\right\}$, weights: $\{1, 1\}$ $\Rightarrow \int_{-1}^{1} e^{-(t+1)^2}dt \approx f\left(-\sqrt{\tfrac{1}{3}}+1\right) + f\left(\sqrt{\tfrac{1}{3}}+1\right)$

$$\Rightarrow \boxed{\int_0^2 e^{-x^2}dx \approx e^{-\left(-\sqrt{\frac{1}{3}}+1\right)^2} + e^{-\left(\sqrt{\frac{1}{3}}+1\right)^2} \approx 0.9195}$$

5.4.2 (b) $n=3 \Rightarrow 2n+1=7$; $I \equiv \int_{-1}^{1} x^k\,dx = \frac{1+(-1)^k}{k+1}$ (2.[illegible])

eq. (2.[illegible]) yields the following table:

K	0	1	2	3	4	5	6	7
I	2	0	2/3	0	2/5	0	2/7	0

Let $u = \sqrt{\frac{1}{7}\left(3-4\sqrt{\frac{3}{10}}\right)}$, $v = \sqrt{\frac{1}{7}\left(3+4\sqrt{\frac{3}{10}}\right)}$, $p = \frac{1}{2}+\frac{1}{12}\sqrt{\frac{10}{3}}$, $q = \frac{1}{2}-\frac{1}{12}\sqrt{\frac{10}{3}}$

Then, by table 5.1: Nodes $=\{-u,-v,u,v\}$, Weights $=\{p,q,p,q\}$

$\Rightarrow$ for $f(x)=x^k$: $\int_{-1}^{1} f(x)\,dx = p[f(-u)+f(u)] + q[f(-v)+f(v)]$

$\Rightarrow$ for odd K: $\int_{-1}^{1} f(x)\,dx = p[-f(u)+f(u)] + q[-f(v)+f(v)] = 0$

$\Rightarrow$ for even K: $\int_{-1}^{1} f(x)\,dx = 2[p f(u) + q f(v)]$;

$\boxed{\text{5.4.2(b) continued}}$ Thus, for even K where $f(x)=x^K$,

$$\int_{-1}^{1} f(x)\,dx \approx 2\left[\left(\frac{1}{2}+\frac{1}{12}\sqrt{\frac{10}{3}}\right)\left(\frac{3}{7}-\frac{4}{7}\sqrt{\frac{3}{10}}\right)^{\frac{K}{2}}+\left(\frac{1}{2}-\frac{1}{12}\sqrt{\frac{10}{3}}\right)\left(\frac{3}{7}+\frac{4}{7}\sqrt{\frac{3}{10}}\right)^{\frac{K}{2}}\right]$$

$$\Rightarrow K=2:\ \int_{-1}^{1} f(x)\,dx \approx 2\left[\left(\frac{1}{2}+\frac{1}{12}\sqrt{\frac{10}{3}}\right)\left(\frac{3}{7}-\frac{4}{7}\sqrt{\frac{3}{10}}\right)+\left(\frac{1}{2}-\frac{1}{12}\sqrt{\frac{10}{3}}\right)\left(\frac{3}{7}+\frac{4}{7}\sqrt{\frac{3}{10}}\right)\right]$$

$$=2\left[\frac{3}{14}-\frac{(2\sqrt{30}+20)}{420}+\frac{3}{14}+\frac{2\sqrt{30}-20}{420}\right]=2\left[\frac{6}{14}-\frac{40}{420}\right]=\frac{2}{3}$$

$$K=4:\ \int_{-1}^{1} f(x)\,dx \approx 2\left[\left(\frac{1}{2}+\frac{1}{12}\sqrt{\frac{10}{3}}\right)\left(\frac{3}{7}-\frac{4}{7}\sqrt{\frac{3}{10}}\right)^{2}+\left(\frac{1}{2}-\frac{1}{12}\sqrt{\frac{10}{3}}\right)\left(\frac{3}{7}+\frac{4}{7}\sqrt{\frac{3}{10}}\right)^{2}\right]$$

$$=2\left[\frac{69}{490}-\frac{(49\sqrt{30}+120)}{2940}+\frac{69}{490}+\frac{49\sqrt{30}-120}{2940}\right]=2\left[\frac{138}{490}-\frac{240}{2940}\right]=\frac{2}{5}$$

$$K=6:\ \int_{-1}^{1} f(x)\,dx \approx 2\left[\left(\frac{1}{2}+\frac{1}{12}\sqrt{\frac{10}{3}}\right)\left(\frac{3}{7}-\frac{4}{7}\sqrt{\frac{3}{10}}\right)^{3}+\left(\frac{1}{2}-\frac{1}{12}\sqrt{\frac{10}{3}}\right)\left(\frac{3}{7}+\frac{4}{7}\sqrt{\frac{3}{10}}\right)^{3}\right]$$

$$=2\left[\frac{351}{3430}-\frac{(441\sqrt{30}+1060)}{34300}+\frac{351}{3430}+\frac{(441\sqrt{30}-1060)}{34300}\right]=2\left[\frac{702}{3430}-\frac{2120}{34300}\right]=\frac{2}{7}$$

Also, for $K=0$: $\int_{-1}^{1} f(x)\,dx \approx 2\left[\frac{1}{2}+\cancel{\frac{1}{12}\sqrt{\frac{10}{3}}}+\frac{1}{2}-\cancel{\frac{1}{12}\sqrt{\frac{10}{3}}}\right]=2$

Thus, the Gaussian quadrature rule yields:

K	0	1	2	3	4	5	6	7
$\int_{-1}^{1} f(x)\,dx \approx$	2	0	$2/3$	0	$2/5$	0	$2/7$	0

Which is the same as the first table so the Gaussian quadrature rule

$\boxed{\text{is exact}}$

5.4.3 (8) is given by:

$$\int_{-1}^{1} f(x)dx \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) := Y$$

From problem 5.4.2 where $f(x)=x^k$ and $I=\int_{-1}^{1} x^k dx$: $I = \frac{1+(-1)^k}{k+1}$

(8) is exact for degree $n \leq 5$; let $f(x)=x^k$

Note that

k	0	1	2	3	4	5	6
I	2	0	2/3	0	2/5	0	2/7

and $\forall k, f(0)=0$

For odd k, (8) yields $\int_{-1}^{1} f(x)dx \approx -\cancel{\frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)} + \frac{8}{9} f(0) + \cancel{\frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)} = 0$

For even k, $k \neq 0$, (8) yields $\int_{-1}^{1} f(x)dx \approx \frac{10}{9} f\left(\sqrt{\frac{3}{5}}\right)$

For $k=0$, (8) yields $\int_{-1}^{1} f(x)dx \approx \frac{5}{9} + \frac{8}{9} + \frac{5}{9} = 2$

Thus:

k	0	1	2	3	4	5
Y	2	0	2/3	0	2/5	0

However for $k=6$: $Y = \frac{10}{9}\left(\sqrt{\frac{3}{5}}\right)^6 = \frac{10}{9}\left(\frac{3}{5}\right)^3 = \frac{270}{1125}$

$$\Rightarrow Y = \frac{6}{25} \neq \frac{2}{7}$$

$$\therefore Y = \frac{1+(-1)^k}{1+k} \text{ only for } k \in \{0,1,2,3,4,5\}$$

$\boxed{5.4.8}$ $\int_a^b f(x)\,dx \approx W_0 f(a) + W_1 f(b) + W_2 f'(a) + W_3 f'(b)$

4 unknowns $\Rightarrow$ (4 DOF)

First, let $a=0$, $b=1$; then, $\lambda : [0,1] \to [a,b] \Rightarrow \lambda(t) = (b-a)t + a$ (8.1)

$$f(x)=1 \Rightarrow \int_0^1 f(x)\,dx = 1 = W_0 + W_1$$

$$f(x)=x \Rightarrow \int_0^1 f(x)\,dx = \frac{1}{2} = W_1 + W_2 + W_3$$

$$f(x)=x^2 \Rightarrow \int_0^1 f(x)\,dx = \frac{1}{3} = W_1 + 2W_3$$

$$f(x)=x^3 \Rightarrow \int_0^1 f(x)\,dx = \frac{1}{4} = W_1 + 3W_3$$

We can cast this as a matrix equation:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \end{bmatrix} \Rightarrow \begin{bmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \end{bmatrix} \Rightarrow \begin{bmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/12 \\ -1/12 \end{bmatrix}$$

$$\therefore \int_0^1 f(x)\,dx = \frac{1}{2} f(0) + \frac{1}{2} f(1) + \frac{1}{12} f'(0) - \frac{1}{12} f'(1)$$

Using (8.1) and letting $x = \lambda(t)$ so $f'(x) = \lambda'(t) f'(\lambda(t))$:

$$\int_0^1 f(x)\,dx = \int_a^b f(t)\,dt = (b-a)\left[\frac{1}{2} f(a) + \frac{1}{2} f(b) + \frac{b-a}{12} f'(a) - \frac{b-a}{12} f'(b)\right] = \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b) + \frac{(b-a)^2}{12} f'(a) - \frac{(b-a)^2}{12} f'(b)$$

$t \mapsto x$

$$\boxed{\int_a^b f(x)\,dx \approx \frac{(b-a)}{2}\left[f(a) + f(b)\right] + \frac{(b-a)^2}{12}\left[f'(a) - f'(b)\right]}$$

which is exact for polynomials of degree $n \leq 3$

5.4.13 $\displaystyle\int_{-h}^{h} f(x)\,dx = Af(-h) + Bf(0) + Cf(h) - hDf'(h)$

4 unknowns $\Rightarrow$ 4 DOF

$f(x)=1 \Rightarrow \displaystyle\int_{-h}^{h} f(x)\,dx = 2h = A + B + C$

$A - C + D = 0$

$f(x)=x \Rightarrow \displaystyle\int_{-h}^{h} f(x)\,dx = 0 = -hA + hC - hD \Rightarrow$

$f(x)=x^2 \Rightarrow \displaystyle\int_{-h}^{h} f(x)\,dx = \frac{2h^3}{3} = h^2A + h^2C - 2h^2D$

$\Rightarrow A + C - D = \dfrac{2h}{3}$

$f(x)=x^3 \Rightarrow \displaystyle\int_{-h}^{h} f(x)\,dx = 0 = -h^3A + h^3C - 3h^3D$

$\Rightarrow A - C = -3D$

However, $A - C = -D$ so D must be zero

$\Rightarrow 2A + B = 2h \quad 2A = \dfrac{2h}{3} \Rightarrow A = \dfrac{h}{3} = C$

$\Rightarrow \dfrac{2h}{3} + B = 2h \Rightarrow B = \dfrac{6h}{3} - \dfrac{2h}{3} = \dfrac{4h}{3}$

$\therefore \displaystyle\int_{-h}^{h} f(x)\,dx = \frac{h}{3}f(-h) + \frac{4h}{3}f(0) + \frac{h}{3}f(h)$

which is accurate for polynomials of degree $n \leq 3$

Computer Exercise 5.4.1

This program contains an algorithm written based on formula (8) (on page 243 in the textbook**); the program also uses tests the algorithm on the indicated test function $f(x) = \frac{1}{e^{x^2}}$ in the textbook using the same test parameters: $a = 0,\ b = 1$. The objective is to match the answer indicated in the textbook: $\int_0^1 f(x)dx \approx 0.746814584$.

**Reference: Cheney, E.W. and Kincaid, D.R. Numerical Mathematics and Computing 7th edition

Formula (8) evaluates $f(x)$ on the interval $[-1, 1]$ and is given by

$$\int_{-1}^{1} f(x)dx \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

When modified to be evaluated on the interval $[a, b]$, formula (8) becomes:

$$\int_{a}^{b} f(x)dx \approx \frac{b-a}{2}\left(\frac{5}{9} f\left(-\frac{b-a}{2}\sqrt{\frac{3}{5}} + \frac{b+a}{2}\right) + \frac{8}{9} f\left(\frac{b+a}{2}\right) + \frac{5}{9} f\left(\frac{b-a}{2}\sqrt{\frac{3}{5}} + \frac{b+a}{2}\right)\right)$$

```
%initiate function
f = @(x) exp(-(x^2));
%initiate parameters
a=0;
b=1;
%execute algorithm
sum1 = formula8(f, a, b);
%display approximation
fprintf('integral approximation = %9.9f', sum1)
```

```
integral approximation = 0.746814584
```

We see that the approximation acquired here is exactly the same as the one acquired in the textbook.

```
function sum = formula8(f, a, b)
    x1 = (-(b-a)/2)*(sqrt(3/5)) + (b+a)/2;
    x2 = (b+a)/2;
    x3 = ((b-a)/2)*(sqrt(3/5)) + (b+a)/2;
    sum = ((5/9)*f(x1) + (8/9)*f(x2) + (5/9)*f(x3))*((1/2)*(b-a));
end
```

Computer Exercise 5.4.3

The following program will use the modified version of formula (8) from computer exercise 5.4.1 to approximate $\int_0^1 \frac{\sin(x)}{x} dx$

```
%initiate function
f = @(x) sin(x)./x;
%initiate parameters
a=0;
b=1;
%execute algorithm
sum1 = formula8(f, a, b);
%display approximation
fprintf('integral approximation = %9.9f', sum1)
```

```
integral approximation = 0.946083134
```

We can compare this to Matlab's 'integral' function to check for consistency:

```
int1 = integral(f, a, b);
fprintf('Matlab integral = %9.9f', int1)
```

```
Matlab integral = 0.946083070
```

We see that the approximation made here is mostly consistent with the value given by Matlab's integral function.

```
function sum = formula8(f, a, b)
    x1 = (-(b-a)/2)*(sqrt(3/5)) + (b+a)/2;
    x2 = (b+a)/2;
    x3 = ((b-a)/2)*(sqrt(3/5)) + (b+a)/2;
    sum = ((5/9)*f(x1) + (8/9)*f(x2) + (5/9)*f(x3))*((1/2)*(b-a));
end
```