Rendel Deguia

4.1.20

$$X \in \{-2, -1, 0, 1, 2\} \quad (x_0, x_1, x_2, x_3, x_4)$$
$$Y \in \{2, 14, 4, 2, 2\}$$

$$P_4(x) = \sum_{i=0}^{4} l_i(x) f(x_i) \quad \text{where} \quad l_i(x) = \prod_{\substack{j \neq i \\ j=0}}^{n} \left(\frac{x - x_j}{x_i - x_j}\right) \quad 0 \le i \le n, \ n=4$$

$$\Rightarrow l_0(x) = \left(\frac{x+1}{-2+1}\right)\left(\frac{x-0}{-2-0}\right)\left(\frac{x-1}{-2-1}\right)\left(\frac{x-2}{-2-2}\right)$$

$$\Rightarrow 2l_0(x) = \frac{1}{12}x^4 - \frac{1}{6}x^3 - \frac{1}{12}x^2 + \frac{1}{6}x \quad (20.1)$$

$$l_1(x) = \left(\frac{x+2}{-1+2}\right)\left(\frac{x-0}{-1-0}\right)\left(\frac{x-1}{-1-1}\right)\left(\frac{x-2}{-1-2}\right)$$

$$\Rightarrow 14l_1(x) = -\frac{7}{3}x^4 + \frac{7}{3}x^3 + \frac{28}{3}x^2 - \frac{28}{3}x \quad (20.2)$$

$$l_2(x) = \left(\frac{x+2}{0+2}\right)\left(\frac{x+1}{0+1}\right)\left(\frac{x-1}{0-1}\right)\left(\frac{x-2}{0-2}\right)$$

$$\Rightarrow 4l_2(x) = x^4 - 5x^2 + 4 \quad (20.3)$$

$$l_3(x) = \left(\frac{x+2}{1+2}\right)\left(\frac{x+1}{1+1}\right)\left(\frac{x-0}{1-0}\right)\left(\frac{x-2}{1-2}\right)$$

$$\Rightarrow 2l_3(x) = -\frac{1}{3}x^4 - \frac{1}{3}x^3 + \frac{4}{3}x^2 + \frac{4}{3}x \quad (20.4)$$

$$l_4(x) = \left(\frac{x+2}{2+2}\right)\left(\frac{x+1}{2+1}\right)\left(\frac{x-0}{2-0}\right)\left(\frac{x-1}{2-1}\right)$$

$$\Rightarrow 2l_4(x) = \frac{1}{12}x^4 + \frac{1}{6}x^3 - \frac{1}{12}x^2 - \frac{1}{6}x \quad (20.5)$$

Adding (20.1) to (20.5) gives $P(x)$

$$\Rightarrow P_4(x) = -\frac{3}{2}x^4 + 2x^3 + \frac{11}{2}x^2 - 8x + 4$$

4.1.21 We can use Newton's algorithm for $P_4(x)$ from 4.1.20 to interpolate the additional point $(x_i, y_i) = (3, 10)$

$$\Rightarrow P_5(x) = P_4(x) + C(x+2)(x+1)(x-0)(x-1)(x-2)$$

$$\Rightarrow P_5(3) = 10 = P_4(3) + C(3+2)(3+1)(3)(3-1)(3-2)$$

$$\Rightarrow 10 = -38 + 120C \Rightarrow C = \frac{48}{120} = \frac{2}{5}$$

$$\Rightarrow P_5(x) = P_4(x) + \frac{2}{5}x(x+2)(x+1)(x-1)(x-2)$$

$$\Rightarrow \boxed{P_5(x) = \frac{2}{5}x^5 - \frac{3}{2}x^4 + \frac{11}{2}x^2 - \frac{32}{5}x + 4}$$

4.1.23 For

x	1	2	3	1
y	3	5	5	7

, we get the following divided dif table:

x				
1	3	2	−1	undefined
2	5	0	1	
3	5	−1		
1	7			

table 23.1

$$f[x_0, x_1] = \frac{5-3}{2-1} = 2$$

$$f[x_1, x_2] = \frac{5-5}{3-2} = 0$$

$$f[x_2, x_3] = \frac{7-5}{1-3} = -1$$

$$f[x_0, x_1, x_2] = \frac{0-2}{3-1} = -1$$

$$f[x_1, x_2, x_3] = \frac{-1-0}{1-2} = 1$$

$$f[x_0, x_1, x_2, x_3] = \frac{1-(-1)}{1-1} = \text{undefined}$$

We got "undefined" in the final column; this is expected behavior because the node values are not distinct which will result in a "divide by zero" for some $f[x_i, \ldots, x_j, \ldots]$

4.1.29 Divided differences are linear maps.

Proof

Base $(n=0)$: $f[x_0]=f(x_0)$, $g[x_0]=g(x_0)$

$$\therefore (\alpha f+\beta g)[x_0]=\alpha f[x_0]+\beta g[x_0]$$

Inductive Step $(n \Rightarrow n+1)$: Suppose $(\alpha f+\beta g)[x_i,\dots,x_n]=\alpha f[x_i,\dots,x_n]+\beta g[x_i,\dots,x_n]$ (29.1)

$$(\alpha f+\beta g)[x_i,\dots,x_n,x_{n+1}]=\Big[(\alpha f+\beta g)[x_{i+1},x_{i+2},\dots,x_{n+1}]-(\alpha f+\beta g)[x_i,x_{i+1},\dots,x_n]\Big]/(x_{n+1}-x_i)$$

* Here, "i" and "n" are non-negative integers

By (29.1):

$$(\alpha f+\beta g)[x_i,\dots,x_{n+1}]=\left(\frac{\alpha f[x_{i+1},\dots,x_{n+1}]+\beta g[x_{i+1},\dots,x_{n+1}]}{x_{n+1}-x_i}\right)-\left(\frac{\alpha f[x_i,\dots,x_n]+\beta g[x_i,\dots,x_n]}{x_{n+1}-x_i}\right)$$

$$=\alpha\left(\frac{f[x_{i+1},\dots,x_{n+1}]-f[x_i,\dots,x_n]}{x_{n+1}-x_i}\right)+\beta\left(\frac{g[x_{i+1},\dots,x_{n+1}]-g[x_i,\dots,x_n]}{x_{n+1}-x_i}\right)$$

$$=\alpha f[x_i,\dots,x_{n+1}]+\beta g[x_i,\dots,x_{n+1}].$$

Since "i" can be any non-negative integer, let $i=0$ so that by (29.1),

$$(\alpha f+\beta g)[x_0,\dots,x_n]=\alpha f[x_0,\dots,x_n]+\beta g[x_0,\dots,x_n]$$

QED

4.2.7 (i) $f_a(x) = \sin(x)$; (ii) $f_b(x) = \cos(x)$; $(x_0, x_1) = (0.70, 0.71)$

(i) $P_1(x) = f_a(x_0)\left(\frac{x - x_1}{x_0 - x_1}\right) + f_a(x_1)\left(\frac{x - x_0}{x_1 - x_0}\right)$

$f_a(x_0) = 0.6442176872 \quad f_a(x_1) = 0.6518337710$

Using calculator:

$\boxed{f_a(0.705) = 0.6480257291}$

$\sin(0.705) = 0.6480338295$

$\Rightarrow \boxed{\text{error} = 8.1 \times 10^{-6}}$

(ii) $P_2(x) = f_b(x_0)\left(\frac{x - x_1}{x_0 - x_1}\right) + f_b(x_1)\left(\frac{x - x_0}{x_1 - x_0}\right)$

$f_b(x_0) = 0.7648421873 \quad f_b(x_1) = 0.7583618760$

$\boxed{f_b(0.702) = 0.7635461250}$

$\cos(0.702) = 0.7635522231$

$\Rightarrow \boxed{\text{error} = 6.1 \times 10^{-6}}$

4.2.9 $n=20$, $I=[0,2]$, $f(x)=e^{-x}$

Note that $f^{(n)}(x)=(-1)^n e^{-x}$ $\forall n\in \mathbb{N}\cup\{0\}$

$$\Rightarrow \max_{0\le x\le 2}\left[\left|f^{(n)}(x)\right|\right]=\max_{0\le x\le 2}\left[\left|(-1)^n e^{-x}\right|\right]=\max_{0\le x\le 2}\left[e^{-x}\right]=1$$

(i) By theorem 1, for interpolating polynomial $p(x)$ and for some $\xi\in(0,2)$:

$$|f(x)-p(x)|=\frac{1}{21!}f^{(21)}(\xi)\prod_{i=0}^{20}|x-x_i|\le\frac{1}{21!}\prod_{i=0}^{20}|x-x_i|$$

Now, note that $\prod_{i=0}^{20}|x-x_i|=|x-x_0|\cdots|x-x_{20}|$

$$\le\underbrace{|2|\cdots|2|}_{21\text{ instances}}=2^{21}$$

$$\Rightarrow |f(x)-p(x)|\le\frac{1}{21!}\prod_{i=0}^{20}|x-x_i|\le\frac{2^{21}}{21!}\approx\boxed{4.105\times10^{-14}}$$

(ii) By theorem 2: $|f(x)-p(x)|\le\frac{1}{4(21)}M\left(\frac{1}{10}\right)^{21}$

Since $|f^{21}(x)|\le 1$ on $[0,2]$, let $M=1$

$$\Rightarrow |f(x)-p(x)|\le\frac{1}{84}\left(\frac{1}{10}\right)^{21}\approx\boxed{1.1905\times10^{-23}}$$