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2.1.1

$\alpha = 0 \Rightarrow \begin{cases} x_1 + 4x_2 = 6 \\ 2x_1 - x_2 = 3 \Rightarrow x_2 = 2x_1 - 3 \quad (1.1) \\ 3x_2 + x_3 = 5 \quad (1.2) \end{cases} \Rightarrow x_1 + 4(2x_1 - 3) = 6$

$\Rightarrow x_1 + 8x_1 - 12 = 6 \Rightarrow 9x_1 = 18 \Rightarrow x_1 = 2$

$(1.1) \Rightarrow x_2 = 2(2) - 3 \Rightarrow x_2 = 1 \quad (1.2) \Rightarrow x_3 = 2$

$\therefore \boxed{\begin{matrix} x_1 = 2 \\ x_2 = 1 \\ x_3 = 2 \end{matrix}}$

$\alpha = -1 \Rightarrow \begin{cases} x_1 + 4x_2 - x_3 = 6 \\ 2x_1 - x_2 - 2x_3 = 3 \\ -x_1 + 3x_2 + x_3 = 5 \end{cases} \xRightarrow{\text{augmented matrix}} A := \left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 2 & -1 & -2 & 3 \\ -1 & 3 & 1 & 5 \end{array}\right]$

$\Rightarrow A = \left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 0 & -9 & 0 & -9 \\ -1 & 3 & 1 & 5 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 0 & -9 & 0 & -9 \\ 0 & 7 & 0 & 11 \end{array}\right] \Rightarrow \begin{matrix} x_1 + 4x_2 - x_3 = 6 \\ -9x_2 = -9 \\ 7x_2 = 11 \end{matrix}$

$\Rightarrow A = \left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 0 & -9 & 0 & -9 \\ 0 & 0 & 0 & 4 \end{array}\right] \Rightarrow 0 = 4 \Rightarrow \boxed{\text{no solution}}$

$\boxed{\text{2.1.1 continued}}$ ("∞": infinitely many)

$\alpha=1 \Rightarrow \begin{cases} x_1+4x_2+x_3=6 \\ 2x_1-x_2+2x_3=3 \\ x_1+3x_2+x_3=5 \end{cases} \Rightarrow A := \left[\begin{array}{ccc|c} 1 & 4 & 1 & 6 \\ 2 & -1 & 2 & 3 \\ 1 & 3 & 1 & 5 \end{array}\right]$

$$\Rightarrow A = \left[\begin{array}{ccc|c} 1 & 4 & 1 & 6 \\ 0 & -9 & 0 & -9 \\ 1 & 3 & 1 & 5 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 4 & 1 & 6 \\ 0 & -9 & 0 & -9 \\ 0 & -1 & 0 & -1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 4 & 1 & 6 \\ 0 & -9 & 0 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_1+4x_2+x_3=6,\ x_2=1 \Rightarrow x_1=2-x_3$

where x_3 can be any real number

so $\boxed{\text{there are } \infty \text{ solutions}}$

For RHS replaced by 0's: $\begin{cases} x_1+4x_2+\alpha x_3=0 \\ 2x_1-x_2+2\alpha x_3=0 \\ \alpha x_1+3x_2+x_3=0 \end{cases}$

$\alpha=0 \Rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 0 & -9 & 0 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 0 & -9 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \boxed{\begin{array}{l} x_1=0 \\ x_2=0 \\ x_3=0 \end{array}}$

$\alpha=-1 \Rightarrow \left[\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 2 & -1 & -2 & 0 \\ -1 & 3 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 0 & -9 & 0 & 0 \\ -1 & 3 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 0 & -9 & 0 & 0 \\ 0 & 7 & 0 & 0 \end{array}\right]$

$\Rightarrow x_2=0,\ x_1=x_3 \in \mathbb{R} \Rightarrow \boxed{\infty \text{ solutions}}$

$\alpha=1 \Rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 1 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & 3 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 4 & 1 & 0 \\ 0 & -9 & 0 & 0 \\ 1 & 3 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 4 & 1 & 0 \\ 0 & -9 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$

$\Rightarrow x_2=0,\ x_1=-x_3 \in \mathbb{R} \Rightarrow \boxed{\infty \text{ solutions}}$

2.1.5

$$A = \begin{bmatrix} 0.780 & 0.563 \\ 0.913 & 0.659 \end{bmatrix} \quad b = \begin{bmatrix} 0.217 \\ 0.254 \end{bmatrix}$$

$$\tilde{r} = A\tilde{x} - b = \begin{bmatrix} 0.2157 \\ 0.2524 \end{bmatrix} - \begin{bmatrix} 0.217 \\ 0.254 \end{bmatrix} = \begin{bmatrix} -0.001343 \\ -0.001572 \end{bmatrix}$$

$$\hat{r} = A\hat{x} - b = \begin{bmatrix} 0.216999 \\ 0.254 \end{bmatrix} - \begin{bmatrix} 0.217 \\ 0.254 \end{bmatrix} = \begin{bmatrix} (216{,}999 - 217{,}000) \times 10^{-6} \\ 0 \end{bmatrix}$$

exact solution:

$$x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \hat{r} = \begin{bmatrix} -1 \times 10^{-6} \\ 0 \end{bmatrix} \Rightarrow$$ residual vectors indicate $\hat{x}$ is the better solution

$$\tilde{e} = \tilde{x} - x = \begin{bmatrix} 0.999 \\ -1.001 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -0.001 \\ -0.001 \end{bmatrix}$$

$$\hat{e} = \hat{x} - x = \begin{bmatrix} 0.341 \\ -0.087 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -0.659 \\ 0.913 \end{bmatrix}$$

$\Rightarrow$ error vectors indicate $\tilde{x}$ is the better solution

$\Rightarrow$ Basing the better solution on the smaller residual vector is not always a reliable method

2.1.6 (a) $b_1=1,\ b_2=2 \Rightarrow \begin{cases} 10^{-4}x_1+x_2=1 \\ x_1+x_2=2 \end{cases}$

$$\Rightarrow \left[\begin{array}{cc|c} 10^{-4} & 1 & 1 \\ 1 & 1 & 2 \end{array}\right] = \left[\begin{array}{cc|c} 10^{-4} & 1 & 1 \\ 0 & -9999 & -9998 \end{array}\right]$$

$$\Rightarrow x_2=\frac{9998}{9999},\quad 10^{-4}x_1+\frac{9998}{9999}=1$$

relative error: $100\frac{|x_{approx}-x_{exact}|}{|x_{exact}|}$

$$\Rightarrow x_2\approx 1.00 \Rightarrow 10^{-4}x_1+1\approx 1 \Rightarrow x_1\approx 0.00$$

$$\Rightarrow \boxed{x_1\approx 0.00,\ x_2\approx 1.00}$$

exact

$x_1=\frac{10{,}000}{9999}$

$x_2=\frac{9998}{9999}$ $\Rightarrow$ relative error for x_1 is %100; relative error for x_2 is ~ %0.01

(b) interchange $\Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 10^{-4} & 1 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0.9999 & 0.9998 \end{array}\right]$

$$\Rightarrow x_2=\frac{9998}{9999}\approx 1.00 \Rightarrow x_1+1.00=2 \Rightarrow x_1=1.00$$

$\Rightarrow \boxed{x_1\approx 1.00,\ x_2\approx 1.00}$ relative error for both x_1 and x_2 is ~%0.01

(c) Let $b_1=1,\ b_2=1$, then $\left[\begin{array}{cc|c} 10^{-4} & 1 & 1 \\ 1 & 1 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 10^{-4} & 1 & 1 \\ 0 & -9999 & -9999 \end{array}\right]$

$$\Rightarrow x_2=\frac{-9999}{-9999}=1.00 \Rightarrow 10^{-4}x_1+1.00=1.00 \Rightarrow x_1=0.00$$

$\Rightarrow$ Answer is exact; $\boxed{b_1=1,\ b_2=1}$

2.3.1 ~~[illegible]~~ A banded system $A\vec{x} = \vec{b}$ subjected to Gaussian elimination with partial pivoting will make A become an upper-triangular matrix such that it is able to solve the system via backwards substitution provided A is not close to being singular.

2.3.5

(a) Starting from $i=4$, the first $i-3$ entries are zero, that is, for $4 \le i \le n$, $1 \le j \le i-3$, $a_{ij}=0$

$$A = \begin{bmatrix} a_{11} & a_{12} & & \cdots & a_{1n} \\ a_{21} & a_{22} & & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ 0 & a_{42} & a_{43} & \cdots & a_{4n} \\ 0 & 0 & a_{53} & \cdots & a_{5n} \\ 0 & 0 & 0 & a_{64} \cdots & a_{6n} \\ & & \cdots & & \end{bmatrix}$$

forms triangle of zeros

(b) For each row "i", there are $n-(i+1)$ zeros starting from column $j=i+2$; i.e. $a_{ij}=0$ where $1 \le i \le n$, $i+2 \le j \le n$ until $i+2>n$.

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 & 0 & \cdots & 0 \\ & & & \vdots & & & & \\ & & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \\ & & & \cdots & & & & \vdots \end{bmatrix}$$

also forms triangle of zeros