HW 10

Lendel Degria

5.3.1 $\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$

Basic Simpson's rule

$f(x)=\frac{1}{1+x^2}$

$(x_0,x_1,x_2)=(0,\frac{1}{2},1) \Rightarrow (f(x_i))_{i=0}^{2}=(1,\frac{4}{5},\frac{1}{2})$

$\Rightarrow \int_0^1 \frac{1}{1+x^2}dx \approx \frac{1}{6}\left[1+4\left(\frac{4}{5}\right)+\frac{1}{2}\right]$

$\Rightarrow \int_0^1 \frac{1}{1+x^2}dx \approx \frac{47}{60} \approx 0.7833$

$\int_0^1 \frac{1}{1+x^2}dx = \frac{\pi}{4} \approx 0.7854 \Rightarrow \text{error} = \left|\frac{\pi}{4}-\frac{47}{60}\right|$

$\Rightarrow \text{error} = \left|\frac{15\pi-47}{60}\right| \approx 0.002065$

5.3.2 $\int_0^1 f(x)dx$, $f(x)=\sin\left(\frac{\pi x^2}{2}\right) \Rightarrow f'(x)=\pi x\cos\left(\frac{\pi x^2}{2}\right)$

$\Rightarrow f''(x)=\pi\cos\left(\frac{\pi x^2}{2}\right)-\pi^2x^2\sin\left(\frac{\pi x^2}{2}\right)$; $\varepsilon<10^{-3}$

a) $|f''(x)|=\left|\pi\cos\left(\frac{\pi x^2}{2}\right)-\pi^2x^2\sin\left(\frac{\pi x^2}{2}\right)\right| \leq \left|\pi\cos\left(\frac{\pi x^2}{2}\right)\right|+\left|\pi^2x^2\sin\left(\frac{\pi x^2}{2}\right)\right| \leq \pi+\pi^2$ $\forall x\in[0,1]$

for composite trapezoid rule: $\varepsilon=\left|\frac{1}{12}h^2f''(\xi)\right| \leq \frac{\pi+\pi^2}{12}h^2<10^{-3}$

$\Rightarrow h<\sqrt{\frac{12\times10^{-3}}{\pi+\pi^2}} \Rightarrow h<0.03037$

b) $|f^{(4)}(x)|=\left|(x^4\pi^4-3\pi^2)\sin\left(\frac{\pi x^2}{2}\right)-6x^2\pi^3\cos\left(\frac{\pi x^2}{2}\right)\right| \leq \left|(x^4\pi^4-3\pi^2)\sin\left(\frac{\pi x^2}{2}\right)\right|$
$+\left|6x^2\pi^3\cos\left(\frac{\pi x^2}{2}\right)\right| \leq |x^4\pi^4-3\pi^2|+6x^2\pi^3 \leq x^4\pi^4+3\pi^2+6x^2\pi^3 \leq \pi^4+3\pi^2+6\pi^3$

$\forall x\in[0,1]$ for composite Simpson's rule: $\varepsilon=\left|\frac{1}{180}h^4f^{(4)}(\xi)\right|$

$\Rightarrow \varepsilon \leq \frac{\pi^4+3\pi^2+6\pi^3}{180}h^4<10^{-3} \Rightarrow h<\left(\frac{180\times10^{-3}}{\pi^4+3\pi^2+6\pi^3}\right)^{\frac{1}{4}} \Rightarrow h<0.1548$

5.3.8 $\int_a^{a+3h} f(x)\,dx \approx \frac{3h}{8}\left[f(a)+3f(a+h)+3f(a+2h)+f(a+3h)\right]$

$$f(a+h) = f + hf' + \frac{1}{2}h^2 f'' + \frac{1}{6}h^3 f''' + \frac{1}{24}h^4 f^{(4)} + \dots \quad (8.1)$$

$$f(a+2h) = f + 2hf' + 2h^2 f'' + \frac{4}{3}h^3 f''' + \frac{2}{3}h^4 f^{(4)} + \dots \quad (8.2)$$

$$f(a+3h) = f + 3hf' + \frac{9}{2}h^2 f'' + \frac{9}{2}h^3 f''' + \frac{27}{8}h^4 f^{(4)} + \dots \quad (8.3)$$

$$\Rightarrow \left[f(a)+3f(a+h)+3f(a+2h)+f(a+3h)\right] = 8f + 12hf' + 12h^2 f'' + 9h^3 f''' + \frac{11}{2}h^4 f^{(4)} + \dots \quad (8.4)$$

multiplying (8.4) by $\frac{3h}{8}$ gives:

$$3hf + \frac{9}{2}h^2 f' + \frac{9}{2}h^3 f'' + \frac{27}{8}h^4 f''' + \frac{33}{16}h^5 f^{(4)} + \dots \quad (8.5)$$

$$F(x) = \int_a^x f(t)\,dt;\quad F(a+3h) = F + 3hF' + \frac{9}{2}h^2 F'' + \frac{9}{2}h^3 F''' + \frac{27}{8}h^4 F^{(4)} + \frac{81}{40}h^5 F^{(5)} + \dots$$

$F = F(a) = 0$; also note $F^{(n)}(a) = f^{(n-1)}(a)$

$$\Rightarrow \int_a^{a+3h} f(x)\,dx = 3hf + \frac{9}{2}h^2 f' + \frac{9}{2}h^3 f'' + \frac{27}{8}h^4 f''' + \frac{81}{40}h^5 f^{(4)} + \dots \quad (8.6)$$

Subtracting (8.5) from (8.6) yields

$$\int_a^{a+3h} f(x)\,dx - \frac{3}{8}\left[f(a)+3f(a+h)+3f(a+2h)+f(a+3h)\right] = \left(\frac{81}{40}h^5 f^{(4)} - \frac{33}{16}h^5 f^{(4)}\right) + \dots$$

$$= \left(\frac{81}{40} - \frac{33}{16}\right)h^5 f^{(4)} + \dots = -\frac{3}{80}h^5 f^{(4)} + \dots$$; Denote error by "ε"

$$\Rightarrow \boxed{\varepsilon = -\frac{3}{80}h^5 f^{(4)}(\xi) \text{ for some } \xi \in (a, a+3h)}$$

The error term is $\varepsilon = O(h^5)$ for Simpson's 3/8 rule which is the same order as Simpson's 1/3 rule, yet an extra function evaluation is needed for Simpson's 3/8 rule so the cost of using the 3/8 rule outweighs the benefit.

Computer Exercise 5.3.1

The following Program will use an adaptive Simpson's scheme to approximate the integrals of $f(x) = \dfrac{4}{1+x^2}$ on $[0, 1]$ and $g(x) = 8(\sqrt{1-x^2} - x)$ on $\left[0, \dfrac{1}{\sqrt{2}}\right]$. The corresponding plots of $f(x)$ and $g(x)$ will also be displayed where the partitions used in the adaptive scheme will be indicated on the plots. In both cases, the following paramters will be used: $max\_level = 4$ and $\varepsilon = \dfrac{1}{2} \times 10^{-5}$.

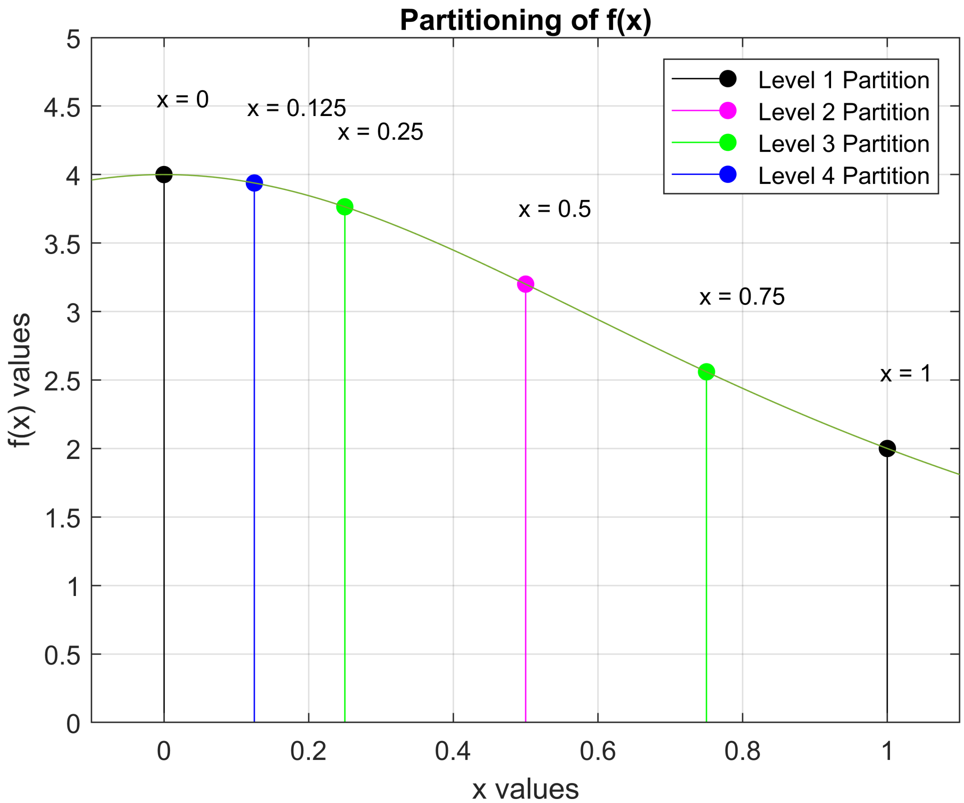
a) $4\int_0^1 \dfrac{1}{1+x^2}dx$

```
%initiate function
f = @(x) 4./((1 + (x.^2)));
%initiate input parameters
a = 0;
b = 1;
err = (0.5)*(10^(-5));
level = 0;
max_level = 4;
result1 = simpson_recur(f, a, b, err, level, max_level);
```

```
max level reached: level = 4, result = 0.4974, interval = (0.0000, 0.1250)

---------------------------------------------------------------------------

max level reached: level = 4, result = 0.4825, interval = (0.1250, 0.2500)

---------------------------------------------------------------------------

level = 3, result = 0.9799, interval = (0.0000, 0.2500)

---------------------------------------------------------------------------

error tolerance satisfied: level = 3, result = 0.8747, interval = (0.2500, 0.5000)

---------------------------------------------------------------------------

level = 2, result = 1.8546, interval = (0.0000, 0.5000)

---------------------------------------------------------------------------

error tolerance satisfied: level = 3, result = 0.7194, interval = (0.5000, 0.7500)

---------------------------------------------------------------------------

error tolerance satisfied: level = 3, result = 0.5676, interval = (0.7500, 1.0000)

---------------------------------------------------------------------------

level = 2, result = 1.2870, interval = (0.5000, 1.0000)

---------------------------------------------------------------------------

level = 1, result = 3.1416, interval = (0.0000, 1.0000)
```

The above output is used to assist in constructing the following plot:

```
%plotting code
xvals1 = linspace(a,b, (2^(max_level-3))+1);
yvals1 = f(xvals1);
xvals2 = linspace(a,b, (2^(max_level-2))+1 );
yvals2 = f(xvals2);
xvals3 = linspace(a,b, (2^(max_level-1))+1 );
xvals3(xvals3 == 0.875) = [];
xvals3(xvals3 == 0.625) = [];
xvals3(xvals3 == 0.375) = [];
yvals3 = f(xvals3);
xvals4 = linspace(a,b, (2^(max_level-3))+1);
xvals4(xvals4 == a) = [];
xvals4(xvals4 == b) = [];
yvals4 = f(xvals4);

clf
stem4 = stem(xvals3, yvals3, 'filled', "Color", 'blue');
grid on
hold on
stem3 = stem(xvals2, yvals2, 'filled', "Color", 'green');
stem1 = stem(xvals1, yvals1, 'filled', "Color", 'black');
stem2 = stem(xvals4, yvals4, 'filled', "Color", 'magenta');
xfit = linspace(a-0.1,b+0.1);
yfit = f(xfit);
plot(xfit, yfit,  'DisplayName', 'Curve Fit')

legend([stem1, stem2, stem3, stem4], ...
    {'Level 1 Partition', 'Level 2 Partition', 'Level 3 Partition', 'Level 4 Partition'})

xlim([-0.1 1.1])
ylim([0 5])
lbl_dwny = 0.14*max(yvals3);
lbl_dwnx = 0.01*max(xvals3);
%label the individual points
for i = 1:length(xvals3)
     % Label the points with the corresponding 'x' value
     txt = ['x = ', num2str(round(xvals3(i), 3, 'significant'))];%, ',', ' ' , ...
         %num2str(round(yvals(i), 2, 'significant')), ')'];

     text(xvals3(i)-lbl_dwnx,yvals3(i)+lbl_dwny,txt, ...
         'FontSize',9);
end
title('Partitioning of f(x)')
xlabel('x values')
ylabel('f(x) values')
hold off
```



<u>Evaluated Partitions In the Recursive Simpson Scheme for</u> $f(x)$:

Level 1: $\text{Partition} = \{[0, 1]\}$

Level 2: $\text{Partition} = \{[0, 0.5], [0.5, 1]\}$

Level 3: $\text{Partition} = \{[0, 0.25], [0.25, 0.5], [0.5, 0.75], [0.75, 1]\}$

Level 4: $\text{Partition} = \{[0, 0.125], [0.125, 0.25]\}$

We see that only the first interval of level 3 is paritioned into level 4. From the above output of the recusive Simpson algorithm, we can indeed see that this is the case since three out of four of the level 3 intervals indicate that the error tolerance has been satisfied.

```
fprintf('Approximate value for the integral: %5.4f', result1)
```

```
Approximate value for the integral: 3.1416
```

The approximation of the integral for $f(x)$ turns out to be an approximation of π. Of course, this is to be expected since analytically: $4\int_0^1 \frac{1}{1+x^2}dx = 4[arctan(1) - arctan(0)] = 4\left(\frac{\pi}{4}\right) = \pi$.

b) $8\int_0^{1/\sqrt{2}}(\sqrt{1-x^2} - x)dx$

```
%initiate function
```

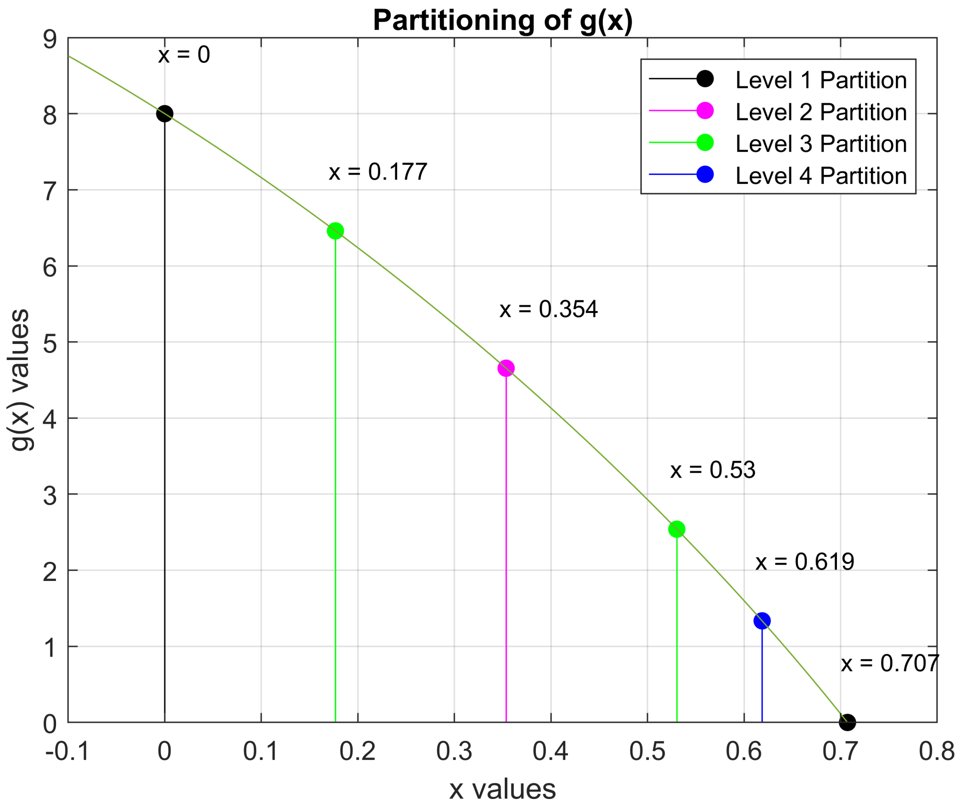
```
g = @(x) 8.*(sqrt(1 - x.^2) - x);
%initiate input parameters
a = 0;
b = 1/(sqrt(2));
err = (0.5)*(10^(-5));
level = 0;
max_level = 4;
result2 = simpson_recur(g, a, b, err, level, max_level);
```

```
error tolerance satisfied: level = 3, result = 1.2818, interval = (0.0000, 0.1768)

---------------------------------------------------------------------------

error tolerance satisfied: level = 3, result = 0.9865, interval = (0.1768, 0.3536)

---------------------------------------------------------------------------

level = 2, result = 2.2683, interval = (0.0000, 0.3536)

---------------------------------------------------------------------------

error tolerance satisfied: level = 3, result = 0.6411, interval = (0.3536, 0.5303)

---------------------------------------------------------------------------

max level reached: level = 4, result = 0.1721, interval = (0.5303, 0.6187)

---------------------------------------------------------------------------

max level reached: level = 4, result = 0.0601, interval = (0.6187, 0.7071)

---------------------------------------------------------------------------

level = 3, result = 0.2322, interval = (0.5303, 0.7071)

---------------------------------------------------------------------------

level = 2, result = 0.8732, interval = (0.3536, 0.7071)

---------------------------------------------------------------------------

level = 1, result = 3.1416, interval = (0.0000, 0.7071)

---------------------------------------------------------------------------
```

The above output is used to assist in constructing the following plot:

```
%plotting code
xvals5 = linspace(a,b, (2^(max_level-1))+1 );
xvals5(2) = [];
xvals5(3) = [];
xvals5(4) = [];
yvals5 = g(xvals5);
xvals6 = linspace(a,b, (2^(max_level-2))+1 );
yvals6 = g(xvals6);
xvals7 = linspace(a,b, (2^(max_level-3))+1 );
yvals7 = g(xvals7);
xvals8 = linspace(a,b, (2^(max_level-3))+1 );
xvals8(1) = [];
```

```
xvals8(2) = [];
yvals8 = g(xvals8);

clf
stem5 = stem(xvals5, yvals5, 'filled', "Color", 'blue');
hold on
grid on
xlim([-0.1 0.8])
ylim([0 9])

stem6 = stem(xvals6, yvals6, 'filled', "Color", 'green');
stem7 = stem(xvals7, yvals7, 'filled', "Color", 'black');
stem8 = stem(xvals8, yvals8, 'filled', "Color", 'magenta');

xfit = linspace(a-0.1,b+0.1);
yfit = g(xfit);
plot(xfit, yfit,  'DisplayName', 'Curve Fit')
lbl_dwny = 0.1*max(yvals5);
lbl_dwnx = 0.01*max(xvals5);
%label the individual points
for i = 1:length(xvals5)
     % Label the points with the corresponding 'x' value
     txt = ['x = ', num2str(round(xvals5(i), 3, 'significant'))];%, ',', ' ' , ...
         %num2str(round(yvals(i), 2, 'significant')), ')'];

     text(xvals5(i)-lbl_dwnx,yvals5(i)+lbl_dwny,txt, ...
         'FontSize',9);
end

legend([stem7, stem8, stem6, stem5], ...
    {'Level 1 Partition', 'Level 2 Partition', 'Level 3 Partition', 'Level 4 Partition'})
title('Partitioning of g(x)')
xlabel('x values')
ylabel('g(x) values')
hold off
```



<u>Evaluated Partitions In the Recursive Simpson Scheme for</u> $g(x)$:

Level 1: $\text{Partition} = \{[0, 0.707]\}$

Level 2: $\text{Partition} = \{[0, 0.354], [0.354, 0.707]\}$

Level 3: $\text{Partition} = \{[0, 0.177], [0.177, 0.354], [0.354, 0.53], [0.53, 0.707]\}$

Level 4: $\text{Partition} = \{[0.53, 0.619], [0.619, 0.707]\}$

We see that only the final interval of level 3 is partitioned into level 4 intervals which means error condidtions were satisfied for the other intervals of level 3. Indeed, this can be seen in the output of the recursive simpson algorithm.

```
fprintf('Approximate value for the integral: %5.4f', result2)
```

```
Approximate value for the integral: 3.1416
```

We see that the integral for $g(x)$ is also an approximation of π and is hence, equal to that of $f(x)$.

```
function result = simpson_recur(f, a, b, err, level, max_level)
    level = level + 1; %increment level upon recursion
    h = b - a;
    c = (a + b)/2;
    simpson_1 = (h/6)*(f(a) + 4*f(c) + f(b));
    d = (a + c)/2;
    e = (c + b)/2;
```

```
    simpson_2 = (h/12)*(f(a)+ 4*f(d) + 2*f(c) + 4*f(e) + f(b));
    %recursion and output
    if level >= max_level
        result = simpson_2;
        fprintf(['max level reached: level = %d, result = %4.4f, ' ...
            'interval = (%4.4f, %4.4f) \n\n'], level, result, a, b)
        fprintf(['---------------------------------------' ...
            '-------------------------------------- \n\n'])
    else
        if abs(simpson_1 - simpson_2) < 15*err
            result = simpson_2 + (simpson_2 - simpson_1)/15;
            fprintf(['error tolerance satisfied: level = %d, result = %4.4f, ' ...
                'interval = (%4.4f, %4.4f) \n\n'], level, result, a, b)
            fprintf(['---------------------------------------' ...
                '----------------------------------------- \n\n'])
        else
            %left recursion
            left_simpson = simpson_recur(f, a, c, err/2, level, max_level);
            %right recursion
            right_simpson = simpson_recur(f, c, b, err/2, level, max_level);
            result = left_simpson + right_simpson;
            fprintf('level = %d, result = %4.4f, interval = (%4.4f, %4.4f) \n\n', ...
                level, result, a, b)
            fprintf(['----------------------------------' ...
                '------------------------------------------- \n\n'])
        end
    end
end
```