Rendel Deguia

4.1.20

$$X \in \{\overset{x_0}{-2}, \overset{x_1}{-1}, \overset{x_2}{0}, \overset{x_3}{1}, \overset{x_4}{2}\}$$

$$Y \in \{2, 14, 4, 2, 2\}$$

$$P_4(x) = \sum_{i=0}^{4} l_i(x) f(x_i) \quad \text{where} \quad l_i(x) = \prod_{\substack{j \neq i \\ j=0}}^{n} \left(\frac{x - x_j}{x_i - x_j}\right) \quad 0 \le i \le n, \; n = 4$$

$$\Rightarrow l_0(x) = \left(\frac{x+1}{-2+1}\right)\left(\frac{x-0}{-2-0}\right)\left(\frac{x-1}{-2-1}\right)\left(\frac{x-2}{-2-2}\right)$$

$$\Rightarrow 2l_0(x) = \frac{1}{12}x^4 - \frac{1}{6}x^3 - \frac{1}{12}x^2 + \frac{1}{6}x \quad (20.1)$$

$$l_1(x) = \left(\frac{x+2}{-1+2}\right)\left(\frac{x-0}{-1-0}\right)\left(\frac{x-1}{-1-1}\right)\left(\frac{x-2}{-1-2}\right)$$

$$\Rightarrow 14l_1(x) = -\frac{7}{3}x^4 + \frac{7}{3}x^3 + \frac{28}{3}x^2 - \frac{28}{3}x \quad (20.2)$$

$$l_2(x) = \left(\frac{x+2}{0+2}\right)\left(\frac{x+1}{0+1}\right)\left(\frac{x-1}{0-1}\right)\left(\frac{x-2}{0-2}\right)$$

$$\Rightarrow 4l_2(x) = x^4 - 5x^2 + 4 \quad (20.3)$$

$$l_3(x) = \left(\frac{x+2}{1+2}\right)\left(\frac{x+1}{1+1}\right)\left(\frac{x-0}{1-0}\right)\left(\frac{x-2}{1-2}\right)$$

$$\Rightarrow 2l_3(x) = -\frac{1}{3}x^4 - \frac{1}{3}x^3 + \frac{4}{3}x^2 + \frac{4}{3}x \quad (20.4)$$

$$l_4(x) = \left(\frac{x+2}{2+2}\right)\left(\frac{x+1}{2+1}\right)\left(\frac{x-0}{2-0}\right)\left(\frac{x-1}{2-1}\right)$$

$$\Rightarrow 2l_4(x) = \frac{1}{12}x^4 + \frac{1}{6}x^3 - \frac{1}{12}x^2 - \frac{1}{6}x \quad (20.5)$$

Adding (20.1) to (20.5) gives $P(x)$

$$\Rightarrow P_4(x) = -\frac{3}{2}x^4 + 2x^3 + \frac{11}{2}x^2 - 8x + 4$$

4.1.21 We can use Newton's algorithm for $P_4(x)$ from 4.1.20 to interpolate the additional point $(x,y)=(3,10)$

$$\Rightarrow P_5(x)=P_4(x)+C(x+2)(x+1)(x-0)(x-1)(x-2)$$

$$\Rightarrow P_5(3)=10=P_4(3)+C\overset{5}{(3+2)}\overset{4}{(3+1)}\overset{3}{(3)}\overset{2}{(3-1)}\overset{1}{(3-2)}$$

$$\Rightarrow 10=-38+120C \Rightarrow C=\frac{48}{120}=\frac{2}{5}$$

$$\Rightarrow P_5(x)=P_4(x)+\frac{2}{5}x(x+2)(x+1)(x-1)(x-2)$$

$$\Rightarrow \boxed{P_5(x)=\frac{2}{5}x^5-\frac{3}{2}x^4+\frac{11}{2}x^2-\frac{32}{5}x+4}$$

4.1.23 For

X	1	2	3	1
Y	3	5	5	7

, we get the following divided dif table:

X				
1	3			
2	5	2	-1	undefined
3	5	0	1	
1	7	-1		

table 23.1

$$f[x_0,x_1]=\frac{5-3}{2-1}=2$$
$$f[x_1,x_2]=\frac{5-5}{3-2}=0$$
$$f[x_2,x_3]=\frac{7-5}{1-3}=-1$$
$$f[x_0,x_1,x_2]=\frac{0-2}{3-1}=-1$$
$$f[x_1,x_2,x_3]=\frac{-1-0}{1-2}=1$$
$$f[x_0,x_1,x_2,x_3]=\frac{1-(-1)}{1-1}=\text{undefined}$$

We got "undefined" in the final column; this is expected behavior because the node values are not distinct which will result in a "divide by zero" for some $f[x_i,\dots,x_j,\dots]$

4.1.29 Divided differences are linear maps.

Proof

Base (n=0): $f[x_0]=f(x_0),\ g[x_0]=g(x_0)$

$$\therefore (\alpha f+\beta g)[x_0]=\alpha f[x_0]+\beta g[x_0]$$

Inductive Step (n=>n+1): Suppose $(\alpha f+\beta g)[x_i,\dots,x_n]=\alpha f[x_i,\dots,x_n]+\beta g[x_i,\dots,x_n]$ (29.1)

$$(\alpha f+\beta g)[x_i,\dots,x_n,x_{n+1}]=\Big[(\alpha f+\beta g)[x_{i+1},x_{i+2},\dots,x_{n+1}]-(\alpha f+\beta g)[x_i,x_{i+1},\dots,x_n]\Big]\Big/(x_{n+1}-x_i)$$

* Here, "i" and "n" are non-negative integers

By (29.1):

$$(\alpha f+\beta g)[x_i,\dots,x_{n+1}]=\left(\frac{\alpha f[x_{i+1},\dots,x_{n+1}]+\beta g[x_{i+1},\dots,x_{n+1}]}{x_{n+1}-x_i}\right)-\left(\frac{\alpha f[x_i,\dots,x_n]+\beta g[x_i,\dots,x_n]}{x_{n+1}-x_i}\right)$$

$$=\alpha\left(\frac{f[x_{i+1},\dots,x_{n+1}]-f[x_i,\dots,x_n]}{x_{n+1}-x_i}\right)+\beta\left(\frac{g[x_{i+1},\dots,x_{n+1}]-g[x_i,\dots,x_n]}{x_{n+1}-x_i}\right)$$

$$=\alpha f[x_i,\dots,x_{n+1}]+\beta g[x_i,\dots,x_{n+1}].$$

Since "i" can be any non-negative integer, let $i=0$ so that by (29.1),

$$(\alpha f+\beta g)[x_0,\dots,x_n]=\alpha f[x_0,\dots,x_n]+\beta g[x_0,\dots,x_n]$$

QED

$\boxed{4.2.7}$ (i) $f_a(x) = \sin(x)$; (ii) $f_b(x) = \cos(x)$; $(x_0, x_1) = (0.70, 0.71)$

(i) $P_1(x) = f_a(x_0)\left(\frac{x - x_1}{x_0 - x_1}\right) + f_a(x_1)\left(\frac{x - x_0}{x_1 - x_0}\right)$

$f_a(x_0) = 0.6442176872 \quad f_a(x_1) = 0.6518337710$

Using calculator:

$$\boxed{f_a(0.705) = 0.6480257291}$$

$\sin(0.705) = 0.6480338295$

$$\Rightarrow \boxed{\text{error} = 8.1 \times 10^{-6}}$$

(ii) $P_2(x) = f_b(x_0)\left(\frac{x - x_1}{x_0 - x_1}\right) + f_b(x_1)\left(\frac{x - x_0}{x_1 - x_0}\right)$

$f_b(x_0) = 0.7648421873 \quad f_b(x_1) = 0.7583618760$

$$\boxed{f_b(0.702) = 0.7635461250}$$

$\cos(0.702) = 0.7635522231$

$$\Rightarrow \boxed{\text{error} = 6.1 \times 10^{-6}}$$

4.2.9 $n=20$, $I=[0,2]$, $f(x)=e^{-x}$

Note that $f^{(n)}(x)=(-1)^n e^{-x}$ $\forall n\in\mathbb{N}\cup\{0\}$

$$\Rightarrow \max_{0\le x\le 2}\left[|f^{(n)}(x)|\right]=\max_{0\le x\le 2}\left[|(-1)^n e^{-x}|\right]=\max_{0\le x\le 2}\left[e^{-x}\right]=1$$

(i) By theorem 1, for interpolating polynomial $p(x)$ and for some $\xi\in(0,2)$:

$$|f(x)-p(x)|=\frac{1}{21!}f^{(21)}(\xi)\prod_{i=0}^{20}|x-x_i|\le\frac{1}{21!}\prod_{i=0}^{20}|x-x_i|$$

Now, note that: $\prod_{i=0}^{20}|x-x_i|=|x-x_0|\cdots|x-x_{20}|$

$$\le\underbrace{|2|\cdots|2|}_{21\text{ instances}}=2^{21}$$

$$\Rightarrow |f(x)-p(x)|\le\frac{1}{21!}\prod_{i=0}^{20}|x-x_i|\le\frac{2^{21}}{21!}\approx\boxed{4.105\times10^{-14}}$$

(ii) By theorem 2: $|f(x)-p(x)|\le\frac{1}{4(21)}M\left(\frac{1}{10}\right)^{21}$

Since $|f^{21}(x)|\le 1$ on $[0,2]$, let $M=1$.

$$\Rightarrow |f(x)-p(x)|\le\frac{1}{84}\left(\frac{1}{10}\right)^{21}\approx\boxed{1.1905\times10^{-23}}$$

Computer Exercise 4.4.1

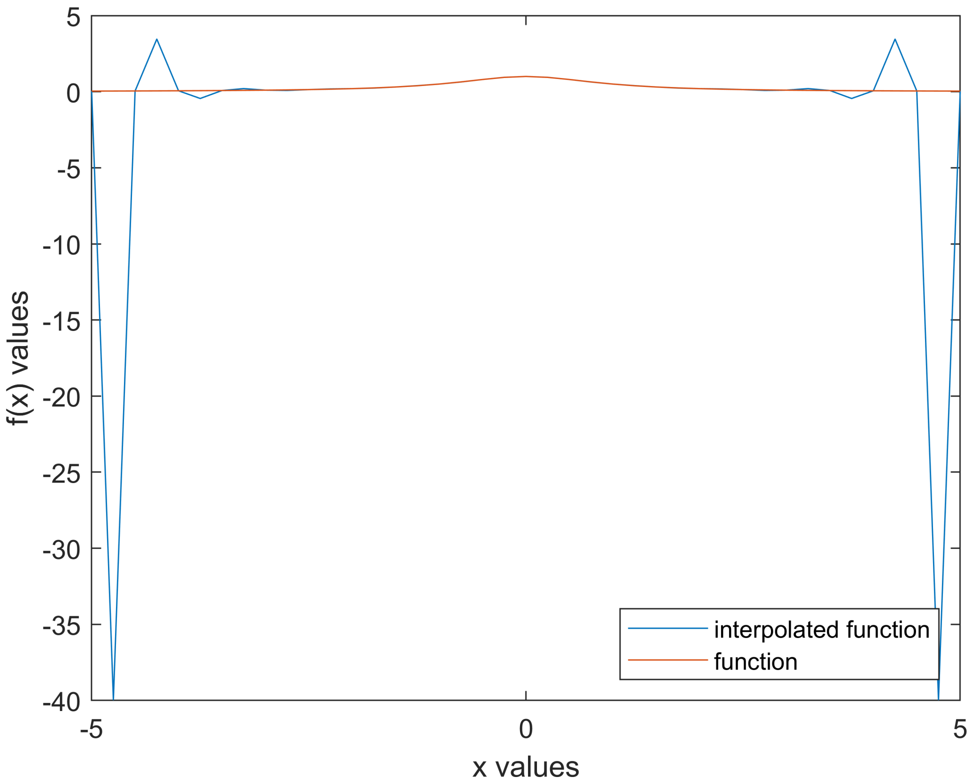
The following program will analyze the interpolation of the function $f(x) = (x^2 + 1)^{-1}$ by using 20 equally space nodes on the interval $[-5, 5]$. The interpolated values will then be tabulated and plotted to assist in analysis.

```
f = @(x) (x.^2 +1).^(-1); %initiate function
nodes = linspace(-5, 5, 21);
data = f(nodes);
xvals = linspace(-5, 5, 41); %x values to evaluate functions
yvals1 = interp_func(xvals, nodes, data); %p(x) evaluated at xvals
yvals2 = f(xvals); %f(x) evaluated at xvals
error = abs(yvals1 - yvals2);
table1 = table(xvals', yvals1', yvals2', error', ...
    'VariableNames', {'x', 'p(x)', 'f(x)', 'error'});
disp(table1)
```

x	p(x)	f(x)	error
-5	0.0384615384615385	0.0384615384615385	0
-4.75	-39.9524490330417	0.0424403183023873	39.9948893513441
-4.5	0.0470588235294118	0.0470588235294118	0
-4.25	3.45495779986408	0.0524590163934426	3.40249878347064
-4	0.0588235294117647	0.0588235294117647	0
-3.75	-0.447051960708839	0.0663900414937759	0.513442002202615
-3.5	0.0754716981132075	0.0754716981132075	0
-3.25	0.202422615704561	0.0864864864864865	0.115936129218074
-3	0.1	0.1	0
-2.75	0.0806599934216555	0.116788321167883	0.0361283277462277
-2.5	0.137931034482759	0.137931034482759	0
-2.25	0.179762629900598	0.164948453608247	0.0148141762923508
-2	0.2	0.2	0
-1.75	0.238445933738133	0.246153846153846	0.00770791241571311
-1.5	0.307692307692308	0.307692307692308	0
-1.25	0.395093053687779	0.390243902439024	0.0048491512487549
-1	0.5	0.5	0
-0.75	0.636755335916433	0.64	0.00324466408356672
-0.5	0.8	0.8	0
-0.25	0.942490379743985	0.941176470588235	0.00131390915574991
0	1	1	0
0.25	0.942490379743985	0.941176470588235	0.00131390915574958
0.5	0.8	0.8	0
0.75	0.636755335916433	0.64	0.00324466408356661
1	0.5	0.5	0
1.25	0.39509305368778	0.390243902439024	0.00484915124875535
1.5	0.307692307692308	0.307692307692308	0
1.75	0.238445933738133	0.246153846153846	0.00770791241571325
2	0.2	0.2	0
2.25	0.179762629900598	0.164948453608247	0.0148141762923507
2.5	0.137931034482759	0.137931034482759	0
2.75	0.0806599934216555	0.116788321167883	0.0361283277462277
3	0.1	0.1	0
3.25	0.202422615704561	0.0864864864864865	0.115936129218074
3.5	0.0754716981132075	0.0754716981132075	0
3.75	-0.447051960708834	0.0663900414937759	0.513442002202609
4	0.0588235294117647	0.0588235294117647	0
4.25	3.45495779986412	0.0524590163934426	3.40249878347068
4.5	0.0470588235294118	0.0470588235294118	0
4.75	-39.9524490330418	0.0424403183023873	39.9948893513442
5	0.0384615384615385	0.0384615384615385	0

We get alternating errors of zero in our table because the interpolated polynomial points at the nodes were based off of the function itself since we needed "data" points to perform interpolation.

```
plot(xvals, yvals1, xvals, yvals2)
xlabel('x values')
ylabel('f(x) values')
legend('interpolated function', 'function', "Location","southeast")
```



From the plot, we can see that the interpolation works great near zero, but as x approaches the end points, the errors become larger and we begin to see oscillation.

```
function sum = interp_func(x, nodes, data) %Lagrange Interpolation
    sum = 0;
    for i = 1:length(nodes)
        prod = 1;
        for j = 1:length(nodes)
            if j ~= i
                prod = prod .* ( (x - nodes(j))./(nodes(i) - nodes(j)) );
            end
        end
        sum = sum + data(i).*prod;
    end
end
```

Computer Exercise 4.4.2

The following program will analyze the interpolation of the function $f(x) = (x^2 + 1)^{-1}$ by using 20 equally space nodes on the interval $[-5, 5]$ first with nodes defined by $x_i = 5\cos(i\pi/20)$ (i.e. Chebyshev Nodes) and then with nodes defined by $x_i = 5\cos((2i+1)\pi/42)$ where $0 \le i \le 20$.

```
f = @(x) (x.^2 +1).^(-1); %initiate function

%initiate both sets of node points
nodes1 = zeros(1, 21);
nodes2 = zeros(1, 21);

for i = 1:21
    nodes1(i) = 5*cos(((i-1)*pi)/(20));
    nodes2(i) = vpa(5*cos(((2*i + 1)*pi)/(42)));
end

% final node points in the second set of nodes
% are equal so I will nudge the final node point
% so the program doesn't yield infinities for
% yvals2
nodes2(21) = nodes2(21) + (10^(-15));

data1 = f(nodes1);
data2 = f(nodes2);

xvals = linspace(-5, 5, 41); %x values to evaluate functions

yvals1 = interp_func(xvals, nodes1, data1); %p1(x) evaluated at xvals
yvals2 = interp_func(xvals, nodes2, data2); %p2(x) evaluated at xvals
yvals3 = f(xvals); %f(x) evaluated at xvals

error1 = abs(yvals1 - yvals3);
error2 = abs(yvals2 - yvals3);
```

<u>Results for the node points given by</u> $x_i = 5\cos(i\pi/20)$:

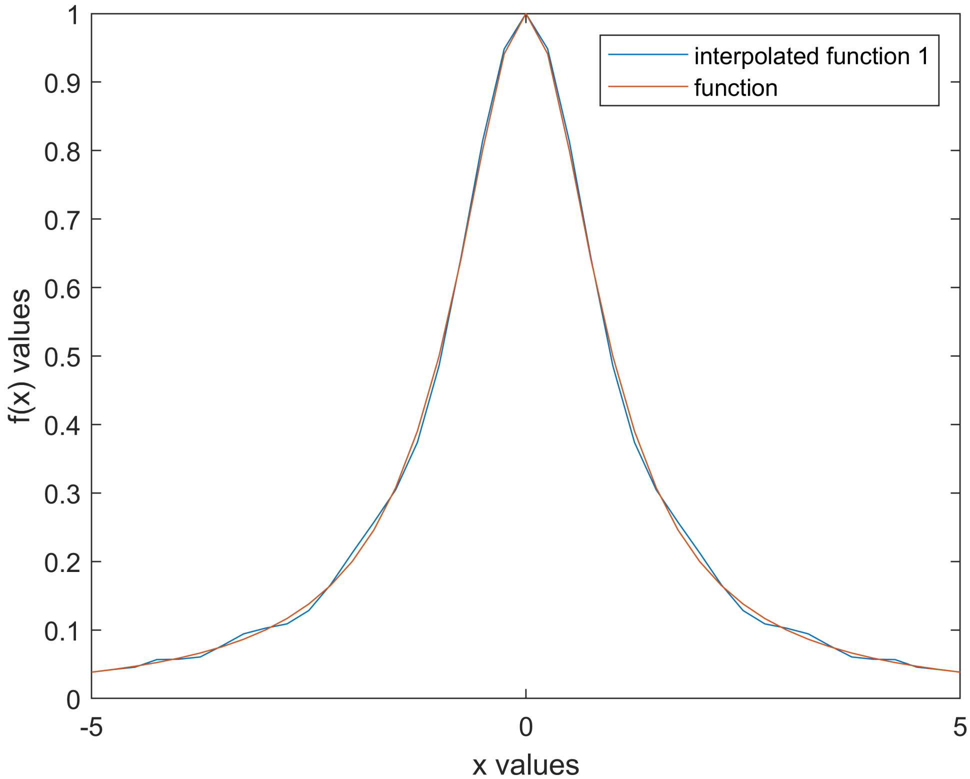
```
table1 = table(xvals', yvals1', yvals3', error1', ...
    'VariableNames', {'x', 'p1(x)', ...
    'f(x)', 'error1'});
disp(table1)
```

x	p1(x)	f(x)	error1
-5	0.0384615384615385	0.0384615384615385	0
-4.75	0.0422824977197068	0.0424403183023873	0.000157820582680421
-4.5	0.0457196254657967	0.0470588235294118	0.00133919806361509
-4.25	0.056769278871316	0.0524590163934426	0.004310262477873 4
-4	0.0572664536391745	0.0588235294117647	0.00155707577259024
-3.75	0.0606194623090939	0.0663900414937759	0.00577057918468203
-3.5	0.0768541918171254	0.0754716981132075	0.00138249370391784

```
-3.25    0.0943647104785681    0.0864864864864865     0.00787822399208159
   -3     0.102647028813403                   0.1     0.00264702881340348
-2.75     0.108922142789554     0.116788321167883      0.0078661783783289
 -2.5      0.12839144305709     0.137931034482759     0.00953959142566849
-2.25     0.166040814650425     0.164948453608247      0.0010923610421778
   -2     0.212576330192443                   0.2      0.0125763301924428
-1.75     0.257513572129776     0.246153846153846      0.0113597259759296
 -1.5     0.304635825507642     0.307692307692308      0.0030564821846662
-1.25     0.373826298022184     0.390243902439024      0.0164176044168409
   -1     0.486008433799535                   0.5      0.0139915662004648
-0.75     0.642273210162905                  0.64     0.00227321016290527
 -0.5     0.813328509497763                   0.8      0.0133285094977625
-0.25      0.94847254819309     0.941176470588235     0.00729607760485518
    0                     1                     1    1.11022302462516e-16
 0.25      0.94847254819309     0.941176470588235     0.00729607760485496
  0.5     0.813328509497762                   0.8      0.0133285094977623
 0.75     0.642273210162906                  0.64     0.00227321016290549
    1     0.486008433799535                   0.5      0.0139915662004647
 1.25     0.373826298022184     0.390243902439024      0.0164176044168409
  1.5     0.304635825507641     0.307692307692308     0.00305648218466631
 1.75     0.257513572129776     0.246153846153846      0.0113597259759299
    2     0.212576330192443                   0.2      0.0125763301924429
 2.25     0.166040814650425     0.164948453608247     0.00109236104217797
  2.5      0.12839144305709     0.137931034482759     0.00953959142566849
 2.75     0.108922142789554     0.116788321167883     0.00786617837832897
    3     0.102647028813403                   0.1     0.00264702881340342
 3.25     0.094364710478568    0.0864864864864865     0.00787822399208153
  3.5    0.0768541918171254    0.0754716981132075     0.00138249370391788
 3.75    0.0606194623090939    0.0663900414937759     0.00577057918468205
    4    0.0572664536391744    0.0588235294117647     0.00155707577259029
 4.25     0.056769278871316    0.0524590163934426     0.00431026247787337
  4.5    0.0457196254657967    0.0470588235294118     0.00133919806361509
 4.75    0.0422824977197069    0.0424403183023873    0.000157820582680386
    5    0.0384615384615385    0.0384615384615385                       0
```

...

```
h = plot(xvals, yvals1, xvals, yvals3);
xlabel('x values')
ylabel('f(x) values')
legend('interpolated function 1', 'function')%, "Location","southeast")
```



From the plot, we can see that Chebyshev node points work really well. However, at the tails, we can observe a slight amount of oscillation.

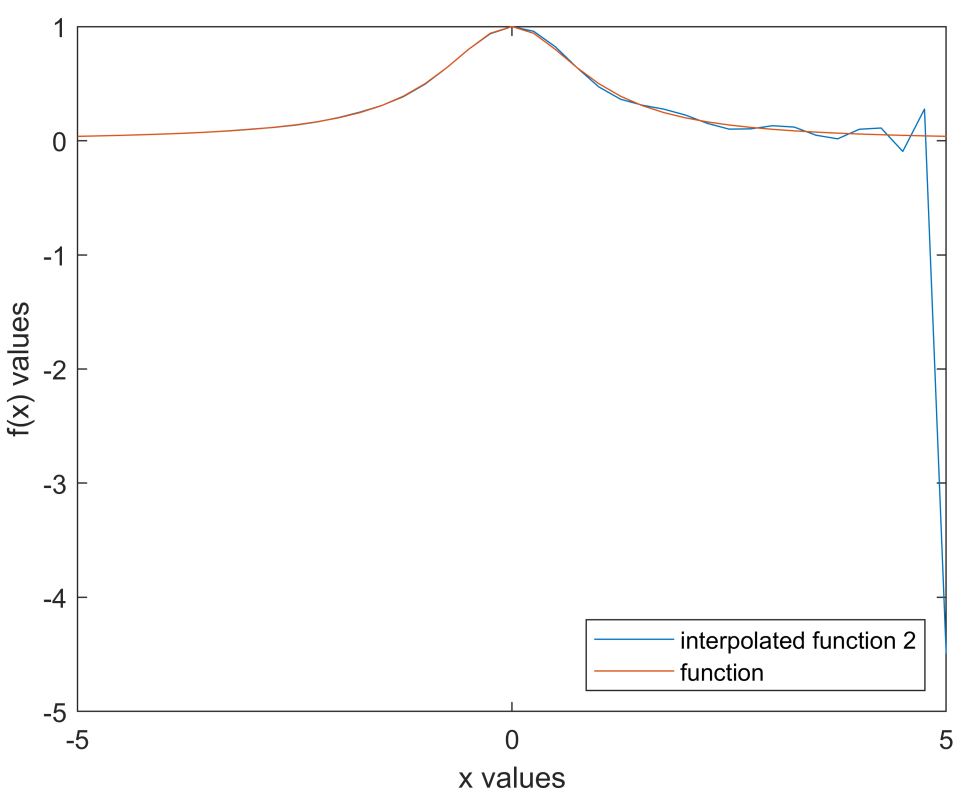
<u>Results for the node points given by</u> $x_i = 5\cos((2i+1)\pi/42)$

```
table2 = table(xvals', yvals2', yvals3', error2', ...
    'VariableNames', {'x', 'p2(x)', ...
    'f(x)', 'error2'});
disp(table2)
```

x	p2(x)	f(x)	error2
-5	0.0380859375	0.0384615384615385	0.000375600961538464
-4.75	0.04248046875	0.0424403183023873	4.01504476127343e-05
-4.5	0.0469970703125	0.0470588235294118	6.17532169117641e-05
-4.25	0.0526123046875	0.0524590163934426	0.000153288294057377
-4	0.0589599609375	0.0588235294117647	0.000136431525735295
-3.75	0.0660400390625	0.0663900414937759	0.000350002431275934
-3.5	0.075439453125	0.0754716981132075	3.22449882075443e-05
-3.25	0.087158203125	0.0864864864864865	0.000671716638513509
-3	0.101318359375	0.1	0.00131835937499999
-2.75	0.1160888671875	0.116788321167883	0.000699453980383208
-2.5	0.1357421875	0.137931034482759	0.00218884698275862
-2.25	0.163818359375	0.164948453608247	0.00113009423324742
-2	0.2021484375	0.2	0.00214843749999999
-1.75	0.2503662109375	0.246153846153846	0.00421236478365383
-1.5	0.308212280273438	0.307692307692308	0.000519972581129791

-1.25	0.385986328125	0.390243902439024	0.0042575743140244
-1	0.49560546875	0.5	0.00439453125
-0.75	0.639919281005859	0.64	8.07189941406383e-05
-0.5	0.800537109375	0.8	0.000537109374999956
-0.25	0.93701171875	0.941176470588235	0.00416475183823528
0	1	1	1.11022302462516e-16
0.25	0.958740234375	0.941176470588235	0.0175637637867647
0.5	0.822998046875	0.8	0.022998046875
0.75	0.639389038085938	0.64	0.000610961914062513
1	0.4716796875	0.5	0.0283203125
1.25	0.36376953125	0.390243902439024	0.0264743711890244
1.5	0.3114013671875	0.307692307692308	0.00370905949519229
1.75	0.2763671875	0.246153846153846	0.0302133413461538
2	0.224365234375	0.2	0.024365234375
2.25	0.152099609375	0.164948453608247	0.0128488442332474
2.5	0.1015625	0.137931034482759	0.0363685344827586
2.75	0.10400390625	0.116788321167883	0.0127844149178832
3	0.13134765625	0.1	0.03134765625
3.25	0.11962890625	0.0864864864864865	0.0331424197635135
3.5	0.04833984375	0.0754716981132075	0.0271318543632075
3.75	0.0166015625	0.0663900414937759	0.0497884789937759
4	0.1005859375	0.0588235294117647	0.0417624080882353
4.25	0.111328125	0.0524590163934426	0.0588691086065574
4.5	-0.09375	0.0470588235294118	0.140808823529412
4.75	0.27734375	0.0424403183023873	0.234903431697613
5	-4.5	0.0384615384615385	4.53846153846154

```
h = plot(xvals, yvals2, xvals, yvals3);
xlabel('x values')
ylabel('f(x) values')
legend('interpolated function 2', 'function', "Location","southeast")
```



The second set of node points appear to have the effect of confining the oscillation effects only to the right tail of the plot. Nevertheless, the estimations appear to be good for points far enough to the left of $x = 5$.

```
function sum = interp_func(x, nodes, data) %Lagrange Interpolation
    sum = 0;
    for i = 1:length(nodes)
        prod = 1;
        for j = 1:length(nodes)
            if j ~= i
                prod = prod .* ( (x - nodes(j))./(nodes(i) - nodes(j)) );
                %fprintf('node difference = %8.8f \n', (nodes(i) - nodes(j)))
            end
        end
        sum = sum + data(i).*prod;
    end
end
```