HW 9

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5.1.2 $\int_1^2 \frac{1}{x}dx \Rightarrow f(x)=\frac{1}{x}$; $P=\{1, \frac{3}{2}, 2\}$

$$\Rightarrow f(P)=\{1, \frac{2}{3}, \frac{1}{2}\} \Rightarrow \int_1^2 \frac{1}{x}dx \approx \frac{1}{2}\left[\left(\frac{3}{2}-1\right)\left(1+\frac{2}{3}\right)+\ldots\right.$$

$$\left.\ldots+\left(2-\frac{3}{2}\right)\left(\frac{1}{2}+\frac{2}{3}\right)\right] = \frac{1}{2}\left[\frac{5}{6}+\frac{7}{12}\right] \Rightarrow \boxed{\int_1^2 \frac{1}{x}dx \approx \frac{17}{24}}$$

5.1.9 $\int_0^1 \frac{1}{1+x^2}dx \Rightarrow f(x)=\frac{1}{1+x^2}$; $P=\{0, \frac{1}{2}, 1\}$

$$\Rightarrow f(P)=\{1, \frac{4}{5}, \frac{1}{2}\} \Rightarrow \int_0^1 \frac{1}{1+x^2}dx \approx \frac{1}{2}\left[\left(\frac{1}{2}-0\right)\left(1+\frac{4}{5}\right)+\ldots\right.$$

$$\left.\ldots+\left(1-\frac{1}{2}\right)\left(\frac{1}{2}+\frac{4}{5}\right)\right] = \frac{1}{2}\left[\frac{9}{10}+\frac{13}{20}\right] \Rightarrow \boxed{\int_0^1 \frac{1}{1+x^2}dx \approx \frac{31}{40} = 0.775}$$

actual value:

$$I=\int_0^1 \frac{1}{1+x^2}dx = \arctan(x)\Big|_0^1 = \frac{\pi}{4} \approx 0.785; \text{ let } T=0.775$$

$$\Rightarrow \boxed{I-T \approx 0.0104} \quad ; \quad f'(x)=\frac{-2x}{(1+x^2)^2}$$

$$f''(x)=\frac{6x^2-2}{(1+x^2)^3} \Rightarrow \max_{0\le x\le 1}\left(|f''(x)|\right) = |f''(0)| = 2$$

$$|I-T| = \left|\frac{b-a}{12}\right|\left|h^2 f''(\xi)\right| = \frac{1}{48}|f''(\xi)| \le \frac{2}{48} = \frac{1}{24}$$

upper bound

$$\Rightarrow \boxed{|I-T| \le \frac{1}{24} \approx 0.0417}$$

5.1.12 $\int_{-1}^{2} \sin(x)dx$, $a = -1$, $b = 2$ $\quad h = 0.01$

$f'(x) = \cos(x)$ $\quad f''(x) = -\sin(x) \Rightarrow \max\limits_{-1 \le x \le 2} |f''(x)| = 1$

$$\boxed{|I - T| = \left|\frac{b-a}{12} h^2 f''(\xi)\right| = \left|\frac{(0.01)^2}{4} f''(\xi)\right| \le 2.5 \times 10^{-5}}$$

5.1.24 $\int_a^{a+h} f(x)dx = hf(a) + \frac{h^2}{2} f'(a) + \frac{h^3}{3!} f''(a) + \dots$ (24.1)

(c) By (24.1): $\int_a^{a+h} f(x)\,dx = hf(a) + \frac{h^2}{2} f'(\xi)$ for some $\xi \in (a, a+h)$

$\Rightarrow$ $\boxed{\text{error term: } \frac{h^2}{2} f'(\xi) \text{ for some } \xi \in (a, a+h)}$

$\Rightarrow \left[\int_a^{x_1} f(x)dx - hf(a)\right] + \left[\int_{x_1}^{x_2} f(x)dx - hf(x_1)\right] + \dots + \left[\int_{x_{n-1}}^{b} f(x)dx - hf(x_{n-1})\right]$

$= \frac{h^2}{2} f'(\gamma_1) + \frac{h^2}{2} f'(\gamma_2) + \dots + \frac{h^2}{2} f'(\gamma_n) = \frac{h}{2}\left[hnf'(\xi)\right]$, $\xi \in (a,b)$

$\Rightarrow$ $\boxed{\text{general rule: } \frac{h}{2}(b-a)f'(\xi) \text{ for some } \xi \in (a,b)}$

(d) By (24.1): $\int_a^{a+h} f(x)dx = hf(a) + \frac{h^2}{2} f'(a) + \frac{h^3}{6} f''(\xi)$, $\xi \in (a, a+h)$

$\Rightarrow$ $\boxed{\text{error term: } \frac{h^3}{6} f''(\xi) \text{ for some } \xi \in (a, a+h)}$

$\Rightarrow \sum_{k=1}^{n} \left[\int_{x_{k-1}}^{x_k} f(x)dx - hf(x_{k-1}) - \frac{h^2}{2} f'(x_{k-1})\right] = \frac{h^3}{6} \sum_{k=1}^{n} f''(\gamma_k)$

$= \frac{h^2}{6}\left[nhf''(\xi)\right]$, $\xi \in (a,b)$ $\quad nh = (b-a)$

$\Rightarrow$ $\boxed{\text{general rule: } \frac{h^2}{6}(b-a)f''(\xi) \text{ for some } \xi \in (a,b)}$

Computer Exercise 5.1.1

This program contains an algorithm written based on the pseudocode composite trapezoid rule with uniform spacing (on page 204 in the textbook**); the program also uses tests the algorithm on the indicated test function $f(x) = \frac{1}{e^{x^2}}$ in the textbook using the same test parameters: $a = 0,\ b = 1,\ n = 60$. The objective is to match the answer indicated in the textbook: $\int_0^1 f(x)dx \approx 0.74681$.

**Reference: Cheney, E.W. and Kincaid, D.R. Numerical Mathematics and Computing 7th edition

```
%initiate function
f = @(x) 1/exp(x^2);

%initiate parameters
n = 60;
a = 0;
b = 1;

%execute algorithm
I1 = Trapezoid_Uniform(f, a, b, n);

%display approximation
fprintf('integral approximation = %5.5f', I1)
```

```
integral approximation = 0.74681
```

We see that the approximation acquired here is exactly the same as the one acquired in the textbook.

```
function I = Trapezoid_Uniform(f, a, b, n)
    h = ((b-a)/n);
    sum = (1/2)*(f(a) + f(b));
    for k = 1:(n-1)
        xk = a + k*h;
        sum = sum + f(xk);
    end
    I = h*sum;
end
```

Computer Exercise 5.1.2

The following program will use the same integral approximation algorithm from exercise 5.1.2 for two functions and compare the approximations (for 100 iterations) with known actual values. The corresponding approximations, actual values, and errors will be displayed.

a) $\int_0^{\pi} \sin(x)dx = 2$

```
%initiate function
f = @(x) sin(x);

%initiate parameters
n = 100;
a = 0;
b = pi;

%known actual value
actual = 2;

%execute algorithm
I1 = Trapezoid_Uniform(f, a, b, n);

%display approximation
fprintf('integral approximation = %12.12f, acutal = %12.12f, error = %8.8e', I1, ...
    actual, abs(actual - I1))
```

```
integral approximation = 1.999835503887, acutal = 2.000000000000, error = 1.64496113e-04
```

b) $\int_0^{\pi} e^x dx = e - 1$

```
%initiate function
f = @(x) exp(x);

%initiate parameters
n = 100;
a = 0;
b = 1;

%known actual value
actual = exp(1)-1;

%execute algorithm
I2 = Trapezoid_Uniform(f, a, b, n);

%display approximation
fprintf('integral approximation = %12.12f, acutal = %12.12f, error = %8.8e', I2, ...
    actual, abs(actual - I2))
```

```
integral approximation = 1.718296147450, acutal = 1.718281828459, error = 1.43189914e-05
```

The approximations are pretty decent for a relatively simple integral approximation method.

```
function I = Trapezoid_Uniform(f, a, b, n)
    h = ((b-a)/n);
    sum = (1/2)*(f(a) + f(b));
    for k = 1:(n-1)
        xk = a + k*h;
        sum = sum + f(xk);
    end
    I = h*sum;
end
```

Computer Exercise 5.1.5

The following program will apply the composite trapezoid algorithm with uniform spacing to $f(x) = \frac{\sin(x)}{x}$ from $x = 0$ to $x = \infty$. Since $f(x)$ is undefined at $x = 0$ and (of course) $x = \infty$, we will have to approximate these 'endpoints' with a small and big value respectively. An approximation will then be made to the same function by using the following change of variables $x = \frac{1}{t}$ to the same function. The corresponding approximations, actual values, and errors will be displayed.

```
%initiate function
f = @(x) sin(x)/x;

%%%initiate parameters
exponent = 6;
n = 10^(exponent);

%small number to approximate zero
a = 10^(-exponent);
%big number to approximate 'infinity'
b = n;

%known actual value
actual = pi/2;

%execute algorithm
I1 = Trapezoid_Uniform(f, a, b, n);

%display approximation
fprintf('integral approximation = %12.12f, acutal = %12.12f, error = %8.8e', ...
    I1, actual, abs(actual - I1))
```

```
integral approximation = 1.570794497499, acutal = 1.570796326795, error = 1.82929585e-06
```

This approximation is decent compared to the actual value: $\pi/2$.

Now by making the change of variable, we get:

$$x = \frac{1}{t} \;\Rightarrow\; \int_0^\infty \frac{\sin(x)}{x}\,dx = \int_{x=0}^{x=\infty} \frac{\sin(1/t)}{(1/t)}\left(\frac{-1}{t^2}\right)dt = \int_\infty^0 \frac{-\sin(1/t)}{t}\,dt = \int_0^\infty \frac{\sin(1/t)}{t}\,dt$$

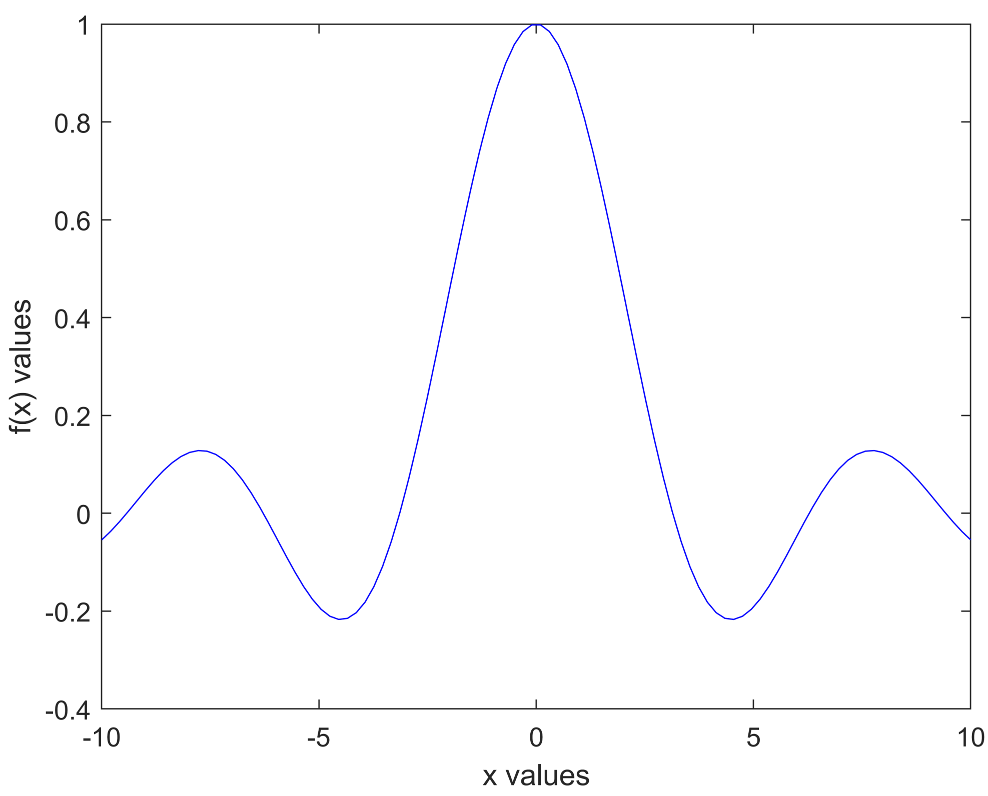
```
%initiate function
f = @(t) sin((1/t))/t;

%%%initiate parameters
exponent = 6;
n = 10^(exponent);

%small number to approximate zero
```

```
a = 10^(-exponent);
%big number to approximate 'infinity'
b = n;

%known actual value
actual = pi/2;

%execute algorithm
I2 = Trapezoid_Uniform(f, a, b, n);

%display approximation
fprintf('integral approximation = %12.12f, acutal = %12.12f, error = %8.8e', ...
    I2, actual, abs(actual - I2))
```

```
integral approximation = -174995.278262264415, acutal = 1.570796326795, error = 1.74996849e+05
```

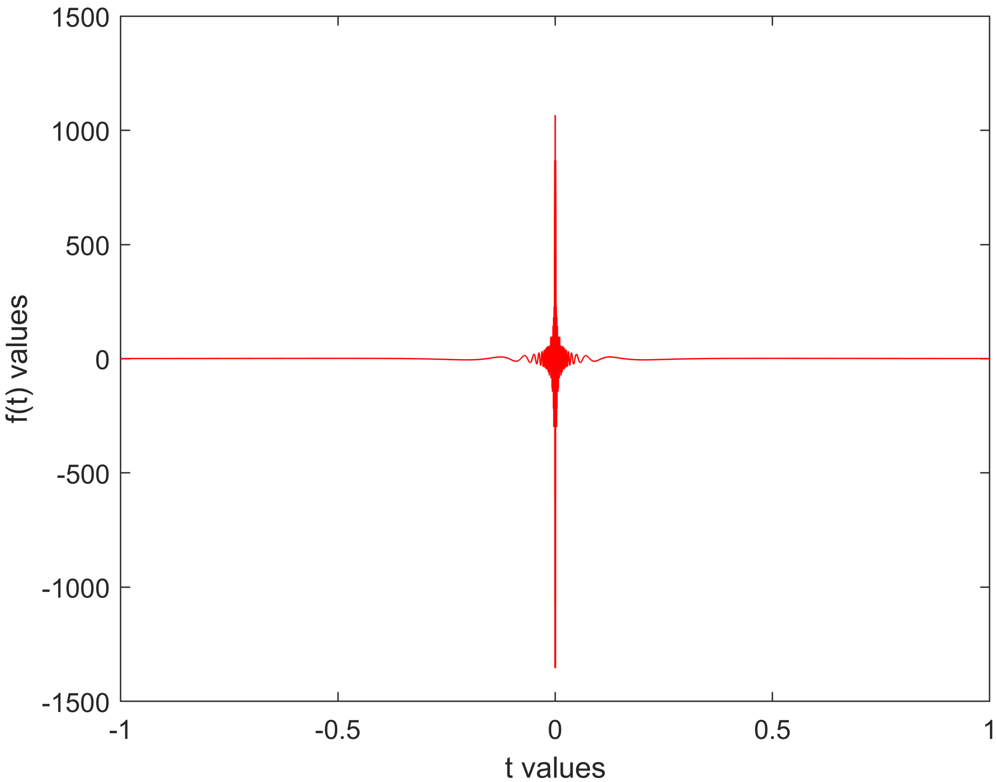
That is a terrible approximation. What is going on? Perhaps plotting out the functions will help out.

```
tvals = linspace(-1,1, 4500);
xvals = linspace(-10, 10);
yvals1 = sin(xvals)./xvals;
yvals2 = sin(1./tvals)./tvals;

figure(1)
plot(xvals, yvals1, "Color", 'b')
xlabel('x values')
ylabel('f(x) values')
```



Here is the function plotted out before the change of variables from $x = -10$ to $x = 10$. It is symmetric and well behaved near zero despite being undefined there.

```
figure(2)
plot(tvals, yvals2, "Color",'r')
xlabel('t values')
ylabel('f(t) values')
```



However, here is the plot for $f(t) = \frac{\sin(1/t)}{t}$ plotted out over 4500 points from $t = -1$ to $t = 1$. It is clear that the function is terribly behaved near zero (increasing the amount of points increases the range substantially) which explains our lack of convergence after the change of variables.

```
function I = Trapezoid_Uniform(f, a, b, n)
    h = ((b-a)/n);
    sum = (1/2)*(f(a) + f(b));
    for k = 1:(n-1)
        xk = a + k*h;
        sum = sum + f(xk);
    end
    I = h*sum;
end
```