Londel Degenia

3.2.1 $f(x)=x^2-R \Rightarrow f'(x)=2x$

Newton's Method: $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)} \Rightarrow x_{n+1}=x_n-\frac{(x_n^2-R)}{2x_n}$

$\Rightarrow x_{n+1}=x_n-\frac{x_n}{2}+\frac{R}{2x_n}=\frac{1}{2}x_n+\frac{1}{2}\frac{R}{x_n}$

$\Rightarrow x_{n+1}=\frac{1}{2}\left(x_n+\frac{R}{x_n}\right)$

3.2.6 $f(x)=(x-1)^m,\ m\in\{8,12\}$

$\Rightarrow f'(x)=m(x-1)^{m-1}$; $x_0=1.1$

using a calculator for Newton's method yields up to $n=4$:

m=8

$n=0,\ x_n=1.1,\ f(x_n)=1.0000\times10^{-8}$

$n=1,\ x_n=1.0875,\ f(x_n)=3.4361\times10^{-9}$

$n=2,\ x_n=1.0766,\ f(x_n)=1.1807\times10^{-9}$

$n=3,\ x_n=1.0670,\ f(x_n)=4.0569\times10^{-9}$

$n=4,\ x_n=1.0586,\ f(x_n)=1.3940\times10^{-10}$

---

m=12

$n=0,\ x_n=1.1,\ f(x_n)=1.0000\times10^{-12}$

$n=1,\ x_n=1.0917,\ f(x_n)=3.5200\times10^{-13}$

$n=2,\ x_n=1.0840,\ f(x_n)=1.2390\times10^{-13}$

$n=3,\ x_n=1.0770,\ f(x_n)=4.3612\times10^{-14}$

$n=4,\ x_n=1.0706,\ f(x_n)=1.5351\times10^{-14}$

3.2.6 continued

The reason convergence is painfully slow ~~is~~ is because $f(x)$ has a root at $X=1$ of multiplicity 8 for $M=8$ and multiplicity 12 for $M=12$ — that is, $f(x)$ ~~has~~ shares a roots with all derivatives until the 7^{th} and 11^{th} derivatives for $M=8$ and $M=12$ respectively.

~~We can use instead, the modified Newton's method~~

The multiplicity of the roots causes the convergence of Newton's method to become linear instead of quadratic. However, we can, instead, use the modified Newton's method $\left(X_{n+1}=X_n-m\frac{f(X_n)}{f'(X_n)}\right)$ to recover the quadratic convergence. Indeed, using a calculator:

$M=8$ $\quad n=0,\ X_n=1.1,\ f(X_n)=1.000\times 10^{-8}$

$\quad n=1,\ X_n=1.0,\ f(X_n)=0$

$M=12$ $\quad n=0,\ X_n=1.1,\ f(X_n)=1.000\times 10^{-12}$

$\quad n=1,\ X_n=1.0,\ f(X_n)=0$

3.2.12

a) $x_{n+1} = \frac{1}{3}\left(2x_n - \frac{r}{x_n^2}\right) = \frac{1}{3}x_n - \frac{1}{3}x_n + \frac{2}{3}x_n - \frac{1}{3}\frac{r}{x_n^2}$

$$\Rightarrow x_n - \left(\frac{x_n^3 + r}{3x_n^2}\right) = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow f(x) = x^3 + r, \quad f'(x) = 3x^2$$

$\therefore$ For an appropriate choice of a nonzero initial point, the sequence $\boxed{\text{will converge}}$ to $\boxed{x = -(r)^{1/3}}$

---

b) $x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n} = x_n - \frac{1}{2}x_n + \frac{1}{x_n} = x_n - \frac{x_n^2}{2x_n} + \frac{2}{2x_n}$

$$\Rightarrow x_n - \left(\frac{x_n^2 - 2}{2x_n}\right) = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow f(x) = x^2 - 2, \quad f'(x) = 2x$$

$\therefore$ For an appropriate choice of a nonzero initial point, the sequence $\boxed{\text{will converge}}$ to either $\boxed{x = \sqrt{2}}$ or $\boxed{x = -\sqrt{2}}$

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$\boxed{3.3.2}$

$$X_{n+1} = X_n - f(X_n)\left[\frac{X_n - X_{n-1}}{f(X_n) - f(X_{n-1})}\right]$$

$f(x) = x^3 - 2x + 2$; $x_0 = 0$, $x_1 = 1$

$\Rightarrow f(x_0) = 2 \quad f(x_1) = 1 - 2 + 2 = 1$

$\Rightarrow X_2 = 1 - 1 \cdot \left[\frac{1-0}{1-2}\right] = 1 + 1 \Rightarrow \boxed{X_2 = 2}$

$\boxed{3.3.11}$ (i) $\lim_{n\to\infty} X_n = r \ni \left[\left(X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}\right) \wedge \left(f'(r) \neq 0\right)\right], n \in \mathbb{N}_0$

$\Rightarrow f(r) = 0$ where $f(x) \in C^1$. (Newton method proof)

$\boxed{\text{proof}}$ Suppose $\lim_{n\to\infty} X_n = r$; then $\lim_{n\to\infty} X_{n+1} = r$, so:

$\lim_{n\to\infty} X_n = \lim_{n\to\infty} X_{n+1} \Rightarrow \lim_{n\to\infty} X_n = \lim_{n\to\infty}\left[X_n - \frac{f(X_n)}{f'(X_n)}\right]$

$\Rightarrow r = r - \frac{f(r)}{f'(r)} \Rightarrow -\frac{f(r)}{f'(r)} = 0 \Rightarrow f(r) = 0$ $\boxed{\text{QED}}$

(ii) $\lim_{n\to\infty} X_n = r \ni \left[\left(X_{n+1} = X_n - f(X_n)\left[\frac{f(X_n) - f(X_{n-1})}{X_n - X_{n-1}}\right]^{-1}\right) \wedge \left(f'(r) \neq 0\right)\right], n \in \mathbb{N}$

$\Rightarrow f(r) = 0$ where $f(X) \in C^1$; suppose WLOG, $X_{n-1} \leq X_n$. (secant method proof)

$\boxed{\text{proof}}$ Suppose $\lim_{n\to\infty} X_n = r$; then $\lim_{n\to\infty} X_{n+1} = r$. By the Mean Value Theorem since $f(X) \in C^1$, $\exists c_n \in (X_{n-1}, X_n) : f'(c_n) = \frac{f(X_n) - f(X_{n-1})}{X_n - X_{n-1}}$.

Thus, the secant method sequence becomes $X_{n+1} = X_n - \frac{f(X_n)}{f'(c_n)}$ and since $\forall n \in \mathbb{N}$, $X_{n-1} \leq c_n \leq X_n$, $\lim_{n\to\infty} c_n = r$. Therefore:

$r = \lim_{n\to\infty}\left(X_n - \frac{f(X_n)}{f'(c_n)}\right) = r - \frac{f(r)}{f'(r)} \Rightarrow -\frac{f(r)}{f'(r)} = 0 \Rightarrow f(r) = 0$

$\boxed{\text{QED}}$

$\boxed{3.3.13}$ b) For $x_n = \frac{1}{2^n}$, $\lim_{n\to\infty} x_n = 0$

Define error as: $|\varepsilon| = |x_n - a|$ where $\lim_{n\to\infty} x_n = a$.

Note that $\forall n \in \mathbb{N}$, $\frac{x_{n+1}}{x_n} = \frac{2^n}{2^{n+1}} = \frac{1}{2}$

so $|x_{n+1} - 0| = \frac{1}{2}|x_n - 0| < \frac{3}{4}|x_n - 0|$

$\therefore \exists C \in [0,1) : |x_{n+1} - 0| \le C|x_n - 0|$

So $\{x_n\}$ is $\boxed{\text{linearly convergent}}$

---

d) $a_0 = a_1 = 1$; $a_2 = 2$, $a_3 = 3$, $a_4 = 5$, $a_5 = 8, \dots$

$\Rightarrow a_n \in \{1, 1, 2, 3, 5, 8, \dots\} \Rightarrow$ a_n is a Fibonacci sequence

$a_{n+1} = a_n + a_{n-1} \Rightarrow \forall n \in \mathbb{N}$, $a_{n+1} > a_n$

$x_n = 2^{-a_n} \Rightarrow \lim_{n\to\infty} x_n = 0$ since a_n is an increasing sequence.

$$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n\to\infty} \frac{2^{-a_n - a_{n-1}}}{2^{-a_n}} = \lim_{n\to\infty} 2^{-a_{n-1}} = 0$$

also note that $\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|^\alpha} = \lim_{n\to\infty} 2^{(\alpha-1)a_n - a_{n-1}}$

so if $\alpha \le 1$, $\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|^\alpha} = 0$ and if $\alpha \ge 2$, the limit diverges. There must be some α such that $\frac{|x_{n+1}|}{|x_n|^\alpha}$ converges to a nonzero value, so we can conclude that $1 < \alpha < 2 \Rightarrow$ $\boxed{\{x_n\} \text{ is superlinearly convergent}}$