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5.4.1 $\int_0^2 f(x)\,dx,\ f(x)=e^{-x^2}$

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} f\left[\tfrac{1}{2}(b-a)t+\tfrac{1}{2}(b+a)\right]dt \quad (1.1)$$

By eq. (1.1), $\int_0^2 e^{-x^2}dx = \int_{-1}^{1} e^{-(t+1)^2}dt$; by table 5.1:

nodes: $\{\pm\sqrt{\tfrac{1}{3}}\}$, weights: $\{1,1\} \Rightarrow \int_{-1}^{1} e^{-(t+1)^2}dt \approx f(-\sqrt{\tfrac{1}{3}}+1)+f(\sqrt{\tfrac{1}{3}}+1)$

$$\Rightarrow \boxed{\int_0^2 e^{-x^2}dx \approx e^{-(-\sqrt{\frac{1}{3}}+1)^2}+e^{-(\sqrt{\frac{1}{3}}+1)^2} \approx 0.9195}$$

5.4.2 (b) $n=3 \Rightarrow 2n+1=7$; $I \equiv \int_{-1}^{1} x^K dx = \dfrac{1+(-1)^K}{K+1}$ (2.1)

eq. (2.1) yields the following table:

K	0	1	2	3	4	5	6	7
I	2	0	2/3	0	2/5	0	2/7	0

Let $u=\sqrt{\tfrac{1}{7}\left(3-4\sqrt{\tfrac{3}{10}}\right)}$, $v=\sqrt{\tfrac{1}{7}\left(3+4\sqrt{\tfrac{3}{10}}\right)}$, $p=\tfrac{1}{2}+\tfrac{1}{12}\sqrt{\tfrac{10}{3}}$, $q=\tfrac{1}{2}-\tfrac{1}{12}\sqrt{\tfrac{10}{3}}$

Then, by table 5.1: Nodes $=\{-u,-v,u,v\}$, Weights $=\{p,q,p,q\}$

$\Rightarrow$ for $f(x)=x^K$: $\int_{-1}^{1} f(x)dx = p[f(-u)+f(u)]+q[f(-v)+f(v)]$

$\Rightarrow$ for odd K: $\int_{-1}^{1} f(x)dx = p[-f(u)+f(u)]+q[-f(v)+f(v)]=0$

$\Rightarrow$ for even K: $\int_{-1}^{1} f(x)dx = 2[pf(u)+qf(v)]$;

5.4.2(b)continued Thus, for even K where $f(x)=x^K$,

$$\int_{-1}^{1} f(x)\,dx \approx 2\left[\left(\frac{1}{2}+\frac{1}{12}\sqrt{\frac{10}{3}}\right)\left(\frac{3}{7}-\frac{4}{7}\sqrt{\frac{3}{10}}\right)^{\frac{K}{2}}+\left(\frac{1}{2}-\frac{1}{12}\sqrt{\frac{10}{3}}\right)\left(\frac{3}{7}+\frac{4}{7}\sqrt{\frac{3}{10}}\right)^{\frac{K}{2}}\right]$$

$$\Rightarrow K=2: \int_{-1}^{1} f(x)\,dx \approx 2\left[\left(\frac{1}{2}+\frac{1}{12}\sqrt{\frac{10}{3}}\right)\left(\frac{3}{7}-\frac{4}{7}\sqrt{\frac{3}{10}}\right)+\left(\frac{1}{2}-\frac{1}{12}\sqrt{\frac{10}{3}}\right)\left(\frac{3}{7}+\frac{4}{7}\sqrt{\frac{3}{10}}\right)\right]$$

$$=2\left[\frac{3}{14}-\frac{(\cancel{2\sqrt{30}}+20)}{420}+\frac{3}{14}+\frac{\cancel{2\sqrt{30}}-20}{420}\right]=2\left[\frac{6}{14}-\frac{40}{420}\right]=\frac{2}{3}$$

$$K=4: \int_{-1}^{1} f(x)\,dx \approx 2\left[\left(\frac{1}{2}+\frac{1}{12}\sqrt{\frac{10}{3}}\right)\left(\frac{3}{7}-\frac{4}{7}\sqrt{\frac{3}{10}}\right)^2+\left(\frac{1}{2}-\frac{1}{12}\sqrt{\frac{10}{3}}\right)\left(\frac{3}{7}+\frac{4}{7}\sqrt{\frac{3}{10}}\right)^2\right]$$

$$=2\left[\frac{69}{490}-\frac{(\cancel{49\sqrt{30}}+120)}{2940}+\frac{69}{490}+\frac{\cancel{49\sqrt{30}}-120}{2940}\right]=2\left[\frac{138}{490}-\frac{240}{2940}\right]=\frac{2}{5}$$

$$K=6: \int_{-1}^{1} f(x)\,dx \approx 2\left[\left(\frac{1}{2}+\frac{1}{12}\sqrt{\frac{10}{3}}\right)\left(\frac{3}{7}-\frac{4}{7}\sqrt{\frac{3}{10}}\right)^3+\left(\frac{1}{2}-\frac{1}{12}\sqrt{\frac{10}{3}}\right)\left(\frac{3}{7}+\frac{4}{7}\sqrt{\frac{3}{10}}\right)^3\right]$$

$$=2\left[\frac{351}{3430}-\frac{(\cancel{441\sqrt{30}}+1060)}{34300}+\frac{351}{3430}+\frac{(\cancel{441\sqrt{30}}-1060)}{34300}\right]=2\left[\frac{702}{3430}-\frac{2120}{34300}\right]=\frac{2}{7}$$

Also, for $K=0$: $\int_{-1}^{1} f(x)\,dx \approx 2\left[\frac{1}{2}+\cancel{\frac{1}{12}\sqrt{\frac{10}{3}}}+\frac{1}{2}-\cancel{\frac{1}{12}\sqrt{\frac{10}{3}}}\right]=2$

Thus, the Gaussian quadrature rule yields:

K	0	1	2	3	4	5	6	7
$\int_{-1}^{1} f(x)\,dx \approx$	2	0	2/3	0	2/5	0	2/7	0

Which is the same as the first table so the Gaussian quadrature rule is exact

5.4.3 (8) is given by:

$$\int_{-1}^{1} f(x)\,dx \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) := Y$$

From problem 5.4.2 where $f(x) = x^K$ and $I = \int_{-1}^{1} x^K dx$: $I = \frac{1+(-1)^K}{K+1}$

$\boxed{(8) \text{ is exact for degree } n \leq 5}$; Let $f(x) = x^K$

Note that

K	0	1	2	3	4	5	6
I	2	0	2/3	0	2/5	0	2/7

and $\forall K, f(0) = 0$

For odd K, (8) yields $\int_{-1}^{1} f(x)\,dx \approx \cancel{-\frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)} + \frac{8}{9} f(0) + \cancel{\frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)} = 0$

For even K, $K \neq 0$, (8) yields $\int_{-1}^{1} f(x)\,dx \approx \frac{10}{9} f\left(\sqrt{\frac{3}{5}}\right)$

For $K = 0$, (8) yields $\int_{-1}^{1} f(x)\,dx \approx \frac{5}{9} + \frac{8}{9} + \frac{5}{9} = 2$

Thus:

K	0	1	2	3	4	5
Y	2	0	2/3	0	2/5	0

However for $K = 6$: $Y = \frac{10}{9}\left(\sqrt{\frac{3}{5}}\right)^6 = \frac{10}{9}\left(\frac{3}{5}\right)^3 = \frac{270}{1125}$

$$\Rightarrow Y = \frac{6}{25} \neq \frac{2}{7}$$

$\therefore Y = \frac{1+(-1)^K}{1+K}$ only for $K \in \{0,1,2,3,4,5\}$

$\boxed{5.4.8}$ $\int_a^b f(x)dx \approx W_0 f(a) + W_1 f(b) + W_2 f'(a) + W_3 f'(b)$

4 unknowns $\Rightarrow$ (4 DOF)

First, let $a=0$, $b=1$; then, $\lambda : [0,1] \to [a,b] \Rightarrow \lambda(t) = (b-a)t + a$ (8.1)

$$f(x)=1 \Rightarrow \int_0^1 f(x)dx = 1 = W_0 + W_1$$

$$f(x)=x \Rightarrow \int_0^1 f(x)dx = \frac{1}{2} = W_1 + W_2 + W_3$$

$$f(x)=x^2 \Rightarrow \int_0^1 f(x)dx = \frac{1}{3} = W_1 + 2W_3$$

$$f(x)=x^3 \Rightarrow \int_0^1 f(x)dx = \frac{1}{4} = W_1 + 3W_3$$

We can cast this as a matrix equation:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \end{bmatrix} \Rightarrow \begin{bmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \end{bmatrix} \Rightarrow \begin{bmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/12 \\ -1/12 \end{bmatrix}$$

$$\therefore \int_0^1 f(x)dx = \frac{1}{2} f(0) + \frac{1}{2} f(1) + \frac{1}{12} f'(0) - \frac{1}{12} f'(1)$$

Using (8.1) and letting $x = \lambda(t)$ so $f'(x) = \lambda'(t) f'(\lambda(t))$:

$$\int_0^1 f(x)dx = \int_a^b f(t)dt = (b-a)\left[\frac{1}{2} f(a) + \frac{1}{2} f(b) + \frac{b-a}{12} f'(a) - \frac{b-a}{12} f'(b)\right] = \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b) + \frac{(b-a)^2}{12} f'(a) - \frac{(b-a)^2}{12} f'(b)$$

$t \mapsto x$

$$\boxed{\int_a^b f(x)dx \approx \frac{(b-a)}{2}\left[f(a) + f(b)\right] + \frac{(b-a)^2}{12}\left[f'(a) - f'(b)\right]}$$

which is ~~accurate~~ exact for polynomials of degree $n \le 3$

5.4.13 $\int_{-h}^{h} f(x)dx = Af(-h) + Bf(0) + Cf(h) - hDf'(h)$

4 unknowns $\Rightarrow$ 4 DOF

$f(x)=1 \Rightarrow \int_{-h}^{h} f(x)dx = 2h = A + B + C$

$f(x)=x \Rightarrow \int_{-h}^{h} f(x)dx = 0 = -hA + hC - hD \Rightarrow$ $A - C + D = 0$

$f(x)=x^2 \Rightarrow \int_{-h}^{h} f(x)dx = \frac{2h^3}{3} = h^2A + h^2C - 2h^2D$

$\Rightarrow A + C - D = \frac{2h}{3}$

$f(x)=x^3 \Rightarrow \int_{-h}^{h} f(x)dx = 0 = -h^3A + h^3C - 3h^3D$

$\Rightarrow A - C = -3D$

However, $A - C = -D$ so D must be zero

$\Rightarrow 2A + B = 2h \quad 2A = \frac{2h}{3} \Rightarrow A = \frac{h}{3} = C$

$\Rightarrow \frac{2h}{3} + B = 2h \Rightarrow B = \frac{6h}{3} - \frac{2h}{3} = \frac{4h}{3}$

$\therefore \int_{-h}^{h} f(x)dx = \frac{h}{3}f(-h) + \frac{4h}{3}f(0) + \frac{h}{3}f(h)$

which is accurate for polynomials of degree $n \leq 3$