HW 5

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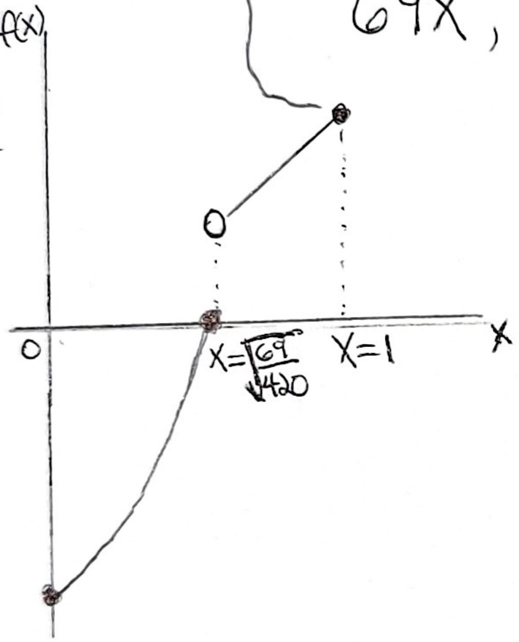
3.1.6

(i) Function that is discontinuous, yet bisection method converges to a root.

$$f(x)=\begin{cases} 420x^2-69, & x\in\left[0,\sqrt{\frac{69}{420}}\right] \\ 69x, & x\in\left(\sqrt{\frac{69}{420}},1\right] \end{cases}$$

*not drawn to scale

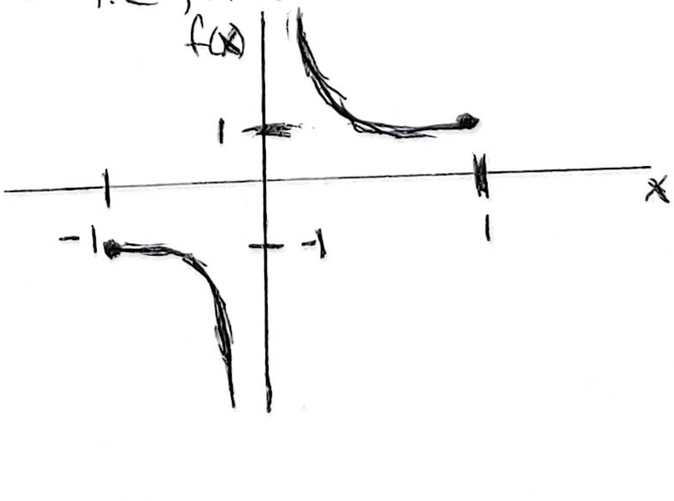


Bisection method applied on interval: $[0,1]$

(ii) Function that is discontinuous and bisection method is divergent – i.e., it does not converge to a root.

$f:[-1,1]\to\mathbb{R}$,

$f(x)=1/x$



Bisection method applied on interval: $[-1,1]$

$\boxed{3.1.8}$ By the Bisection Method Theorem (Pg. 119 Textbook), the bisection algorithm will yield an error of at most $\dfrac{(b-a)}{2^{n+1}}$ after n steps on interval $[a, b]$". For some error tolerance ε, we want $\dfrac{(b-a)}{2^{n+1}} < \varepsilon$, so

$$(b-a) < \varepsilon\left(2^{n+1}\right) \Rightarrow \log_B(b-a) < \log_B(2\varepsilon) + n\log_B(2)$$

$$\Rightarrow n > \frac{\log_B(b-a) - \log_B(2\varepsilon)}{\log_B(2)} \qquad (8.1)$$

where B is an arbitrary logarithm base

For $a = 0.1$, $b = 1$ and $\varepsilon = \frac{1}{2}\times 10^{-8}$ and using ~~the natural logarithm~~ base 10:

$$\frac{\log_{10}(9\times 10^{-1}) - \log_{10}(10^{-8})}{\log_{10}(2)} < n$$

$$\Rightarrow \frac{\log(9) + 7}{\log(2)} \approx 26.42 < n$$

$$\Rightarrow \boxed{n = 27}$$

3.1.11 The remark eliminates the problem of finding roots ONLY for invertible functions; functions are invertible if and only if they are bijective and hence, they must be injective (one to one). Thus, the problem is not eliminated for finding roots of equations in general.

(P1) Proposition | Let $C \in \mathbb{R}$ be a constant. Then, if $f(x)$ is not injective, $g(x) = f(x) + C$ is not injective.

Proof: Since $f(x)$ is not injective, $\exists a, b \in \text{Domain}(f)$ such that $f(a) = f(b)$ where $a \neq b$ which means that $f(a) + C = f(b) + C \Rightarrow g(a) = g(b)$. Since $g(x)$ and $f(x)$ only differ by a constant, they have the same domains. Thus, $\exists a, b \in \text{Domain}(g)$ such that $g(a) = g(b)$ where $a \neq b$, so $g(x)$ is not injective. $\square$

Define $f: \mathbb{R} \to [-1, 1]$ by $f(x) = \sin(x)$ and $g: \mathbb{R} \to [-1 - \frac{1}{\pi}, 1 - \frac{1}{\pi}]$ by $g(x) = \sin(x) - \frac{1}{\pi}$. $f(x)$ is not injective since, for example, $f(\frac{\pi}{2}) = f(\frac{5\pi}{2}) = 1$. Thus, by (P1), $g(x)$ is not injective, so it is not bijective and hence, $g(x)$ is not invertible, so the remark is not applicable* to the equation $\sin(x) = \frac{1}{\pi}$ when the domain of $\sin(x)$ is $\mathbb{R}$.

*However, if domain[$\sin(x)$] is $[\frac{-\pi}{2}, \frac{\pi}{2}]$, then the remark is applicable.

§3.1.14

(a) • Starting with $[a_0, b_0]$, let $c_1 = \frac{a_0+b_0}{2}$. Then $a_0 \le c_1 \le b_0$. After this iteration, we will have either $(a_1 = c_1 \text{ and } b_1 = b_0)$ or $(a_1 = a_0 \text{ and } b_1 = c_1)$. In either case, $a_0 \le a_1$ and $b_0 \ge b_1$.

• For $[a_{k-1}, b_{k-1}]$, let $c_k = \frac{a_{k-1}+b_{k-1}}{2}$; then $a_{k-1} \le c_k \le b_{k-1}$. After this iteration, we will have $(a_k = c_k \text{ and } b_k = b_{k-1})$ or $(a_k = a_{k-1} \text{ and } b_k = c_k)$ so $a_{k-1} \le a_k$ and $b_{k-1} \ge b_k$.

• For $[a_k, b_k]$, let $c_{k+1} = \frac{a_k+b_k}{2}$; then, $a_k \le c_{k+1} \le b_k$. After this iteration, we will have $(a_{k+1} = c_{k+1} \text{ and } b_{k+1} = b_k)$ or $(a_{k+1} = a_k \text{ and } b_{k+1} = c_{k+1})$, so $a_k \le a_{k+1}$ and $b_k \ge b_{k+1}$. From the previous bullet point, we have

$$a_{k-1} \le a_k \le a_{k+1} \text{ and } b_{k-1} \ge b_k \ge b_{k+1}.$$

Thus, by induction, we can conclude that

$$a_0 \le a_1 \le \ldots a_{k-1} \le a_k \le a_{k+1} \le \ldots \text{ and}$$

$$b_0 \ge b_1 \ge \ldots b_{k-1} \ge b_k \ge b_{k+1} \ge \ldots$$

where $k \in \mathbb{N}$

3.1.14 continued

(b) Since $a_1 = \frac{b_0 + a_0}{2}$ and $b_1 = b_0$ or $a_1 = a_0$ and $b_1 = \frac{b_0 + a_0}{2}$, we have that $b_1 - a_1 = \frac{b_0 - a_0}{2}$ in either case. Similarly, for successive iterations:

$$b_2 - a_2 = \frac{b_1 - a_1}{2} = \frac{b_0 - a_0}{2^2}$$

$$b_3 - a_3 = \frac{b_2 - a_2}{2} = \frac{b_0 - a_0}{2^3}$$

$$\vdots$$

$$b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2} = \frac{1}{2}\left(\frac{b_0 - a_0}{2^{n-1}}\right) \Rightarrow \boxed{b_n - a_n = \frac{b_0 - a_0}{2^n}}$$

(c) $$a_n b_n + b_{n-1} a_{n-1} = a_{n-1} b_n + a_n b_{n-1} \quad (14.c.1)$$

$$\Rightarrow a_n(b_n - b_{n-1}) = a_{n-1}(b_n - b_{n-1})$$

$$\Rightarrow a_n(b_n - b_{n-1}) - a_{n-1}(b_n - b_{n-1}) = 0$$

$$\Rightarrow (a_n - a_{n-1})(b_n - b_{n-1}) = 0 \quad (14.c.2)$$

Now either $a_n = \frac{1}{2}(a_{n-1} + b_{n-1})$ and $b_n = b_{n-1}$ or

$a_n = a_{n-1}$ and $b_n = \frac{1}{2}(a_{n-1} + b_{n-1})$.

In both cases, (14.c.2) is true; thus (14.c.1) must also be true.

3.1.18

(a) $\boxed{\text{False}}$ consider $f(x) = x^3 - 4$ on $[a_0, b_0] = [1, 2]$ where $r = 4^{\frac{1}{3}}$; applying the bisection algorithm yields the following table.

n	(a_n, b_n)	$\lvert r - a_n\rvert$	$2\lvert r - b_n\rvert$	$\lvert r - a_n\rvert \le 2\lvert r - b_n\rvert$
0	$(1, 2)$	0.587	0.825	True
1	$(1.5, 2)$	0.0874	0.825	True
2	$(1.5, 1.75)$	0.0874	0.325	True
3	$(1.5, 1.625)$	0.0874	0.0752	False

We see that for $n=3$, the inequality is not true.

(c) $\boxed{\text{False}}$ consider $f(x) = x^2 - 2$ on $[a_0, b_0] = [1, 2]$ where $r = \sqrt{2}$; applying bisection yields:

n	(a_n, b_n)	$\lvert r - \frac{1}{2}(a_n + b_n)\rvert$	$2^{-n-2}(b_0 - a_0)$	$\lvert r - \frac{1}{2}(a_n + b_n)\rvert \le \frac{b_0 - a_0}{2^{n+2}}$
0	$(1, 2)$	0.0858	0.25	True
1	$(1, 1.5)$	0.164	0.125	False

We see that for $n=1$, the inequality is not true

3.1.8 continued

(e) Consider the same function from (c):

n	(a,b)	$\lvert r-b_n\rvert$	$2^{-n-1}(b_0-a_0)$	$\lvert r-b_n\rvert \leq 2^{-n-1}(b_0-a_0)$
0	(1,2)	0.586	0.5	False

Already, without even evaluating $f\left(\frac{b_0-a_0}{2}\right)$, we can determine that the inequality given by $\lvert r-b_n\rvert \leq 2^{-n-1}(b_0-a_0)$ is $\boxed{\text{False}}$