Computer Exercise 3.3.10

The following program will monitor the convergence of the ratio $\frac{e_{n+1}}{e_n e_{n-1}}$ when applying the secant method to $f(x) = arctan(x)$ where $e_n = r - x_n$ and r is a known root of $f(x)$. From properties of the indicated function we know that $r = 0$ is the only root of $arctan(x)$. From convergence properties of the secant method, we expect that the error ratio should converge to $-\frac{1}{2}\frac{f''(r)}{f'(r)}$ and from making the following evaluations:

$$f(x) = arctan(x) \Rightarrow f(x)|_{x=0} = 0$$

$$\frac{df}{dx} = \frac{1}{x^2+1} \Rightarrow \frac{df}{dx}|_{x=0} = 1$$

$$\frac{d^2f}{dx^2} = -\frac{2x}{(x^2+1)^2} \Rightarrow \frac{d^2f}{dx^2}|_{x=0} = 0$$

$$\Rightarrow \frac{e_{n+1}}{e_n e_{n-1}} \to -\frac{1}{2}\frac{f''(r)}{f'(r)} = -\frac{1}{2}\left(\frac{0}{1}\right) = 0 \text{ as } n \to \infty$$

Thus, we expect the error ratio to converge to zero.

```
r=0; %known root is used as input in the secant method function
f = @(x) atan(x);
x0 = -1.69; %initialize arbitrary points around x=0
x1 = 2.73;
N = 20; %max iterates
err = 0.5 * 10^(-20);

secant(f, x0, x1, N, err, r);
```

```
ratio =   0.073815824329
ratio =   0.580031172348
ratio =  -0.056781077976
ratio =  -0.171145861884
ratio =  -0.003155182463
ratio =   0.000320818071
ratio =  -0.000000015284
ratio =  -0.000000000000
```

Indeed, the error ratio does converge to zero with each iterate.

```
function secant(f, x0, x1, N, err, r) %No assigned output necessary
    %modified version of the earlier secant method I wrote that is
    %programmed to only output the error ratios
    xn = x1;
    xnm1 = x0;
    error = 1;
    n = 0;
    while (error > err) && (n < N)
        yn = f(xn);
        ynm1 = f(xnm1);
```

```
        xnp1 = xn - yn*((xn-xnm1)/(yn-ynm1));
        error = abs(xnp1 - xn);
        enm1 = r - xnm1; %error terms
        en = r - xn;
        enp1 = r - xnp1;
        q = enp1/(en*enm1); %error ratio
        fprintf('ratio = %16.12f\n', q)
        xnm1 = xn;
        xn = xnp1;
        n = n+1;
    end
end
```