Discrete Structures (CSCI 246)
Homework 6

Purpose & Goals
The following problems provide practice relating to:

- relations,
- properties of relations,
- equivalence relations, partial orders,
- direct proof and proof by example, and
- the problem solving process.

Submission Requirements

- **Type or clearly hand-write your solutions** into a pdf format so that they are legible and professional. Submit your pdf to Gradescope. **Illegible, non-pdf, or emailed solutions will not be graded.**
- Each problem should start on a new page of the document. When you submit to Gradescope, associate each page of your submission with the correct problem number. Please post in Discord if you are having any trouble using Gradescope.
- Try to model your formatting off of the proofs from lecture and/or the textbook.
- Submit to Gradescope early and often so that last-minute technical problems don't cause you any issues. Only the latest submission is kept. Per the syllabus, assignments submitted within 24 hours of the due date will receive a 25% penalty and assignments submitted within 48 hours will receive a 50% penalty. After that, no points are possible.

Academic Integrity

- You may work with your peers, but **you must construct your solutions in your own words on your own.**
- Do not search the web for solutions or hints, post the problem set, or otherwise violate the course collaboration policy or the MSU student code of conduct.
- Violations (regardless of intent) will be reported to the Dean of Students, per the MSU student code of conduct, and you will receive a 0 on the assignment.

Tips

- Answer each problem to the best of your ability. Partial credit is your friend!
- Read the hints for where to find relevant examples for each problem.
- Refer to the problem solving and homework tips guide.
- It is not a badge of honor to say that you spent 5 hours on a single problem or 15 hours on a single assignment. Use your time wisely and get help (see "How to Get Help" below).

How to Get Help
When you are stuck and need a little or big push, **please ask for help!**

- Timebox your effort for each problem so that you don't spend your life on the problem sets. (See the problem solving tips guide for how to do this effectively.)
- Post in Discord. If you're following the timebox guide, you can post the exact statement that you produced after spending 20 minutes being stuck.
- Come to office hours or visit the CS Student Success Center.

Problem 1 (36 points)

For each of the following relations on $S = \mathcal{P}(\{0, 1, 2, 3\})$, determine whether

- it is reflexive, irreflexive, or neither,
- it is symmetric, anti-symmetric, both, or neither,
- it is transitive or not, and
- all pairs of elements are comparable (that is, $\forall a \neq b \in S : (aRb \vee bRa)$).

Use these facts to establish whether the relation is any type of order (partial order, strict partial order, total order, strict total order) and/or an equivalence relation. If it is an equivalence relation, list the equivalence classes under the relation.

Grading Notes. For each relation below, of the four bullet points above, you get one point for correctly identifying whether the property applies or not and one point for giving a valid proof. Recall that a proof of a "for all" statement should be a direct proof showing why the property holds in general. A disproof of a "for all" statement just requires one counterexample. You also get one point for correctly identifying whether the relation is one of the types of orders and one for giving a valid proof, and one correctly identifying whether it is an equivalence relation (and if it is, identifying the equivalence classes) and giving a valid proof.

(a) (12 points) $(A, B) \in R_1$ if (i) A and B are nonempty and the largest element in A equals the largest element in B, or (ii) if $A = B = \emptyset$.

(b) (12 points) $(A, B) \in R_2$ if the sum of elements in A is equal to the sum of elements in B. (Formally, $AR_2B \Leftrightarrow \sum_{x\in A} x = \sum_{y\in B} y$.)

(c) (12 points) $(A, B) \in R_4$ if $A \cap B \neq \emptyset$.

Problem 2 (6 points)

In this problem, you will learn two new definition and give some examples. Note that we can define the *size* of a relation to be the number of elements in the corresponding set. For example, consider the set $A = \{1, 2, 3, 4\}$ and the relation $R = \{\langle 2, 4\rangle, \langle 4, 3\rangle, \langle 4, 4\rangle\}$. We say that $|R| = 3$.

We define the *closure* of a relation R with respect to some property as the smallest relation that includes everything from R but also has the property. To be precise:

- The *reflexive closure* of a relation R is the smallest possible $R' \supseteq R$ such that R' is reflexive.
- The *symmetric closure* of a relation R is the smallest possible $R' \supseteq R$ such that R' is symmetric.
- The *transitive closure* of a relation R is the smallest possible $R' \supseteq R$ such that R' is transitive.

If R already has the property, then the closure is just R! For example, the reflexive closure of the $=$ relation is just the $=$ relation, since $=$ is already reflexive.

For the set A and relation R given above, give the following:

(a) The reflexive closure of R.

(b) The symmetric closure of R.

(c) The transitive closure of R.

Grading Notes. You get two points for giving the correct closures for each of (a), (b), and (c).

Problem 3 (10 points)

(a) (2 points) List all partial orders on $\{0, 1\}$. Hint: there are four of them.

(b) (8 points) List all partial orders on $\{0, 1, 2\}$. Be careful!