Discrete Structures (CSCI 246)
Homework 3

Purpose & Goals
The following problems provide practice relating to:

- direct proofs, proof by cases, and proofs by counter-example,
- mathematical definitions (rational, absolute value, divisibility, sets, etc.), and
- the problem solving process.

Submission Requirements

- **Type or clearly hand-write your solutions** into a pdf format so that they are legible and professional. Submit your pdf to Gradescope. **Illegible, non-pdf, or emailed solutions will not be graded.**
- Each problem should start on a new page of the document. When you submit to Gradescope, associate each page of your submission with the correct problem number. Please post in Discord if you are having any trouble using Gradescope.
- Try to model your formatting off of the proofs from lecture and/or the textbook.
- Submit to Gradescope early and often so that last-minute technical problems don't cause you any issues. Only the latest submission is kept. Per the syllabus, assignments submitted within 24 hours of the due date will receive a 25% penalty and assignments submitted within 48 hours will receive a 50% penalty. After that, no points are possible.

Academic Integrity

- You may work with your peers, but **you must construct your solutions in your own words on your own.**
- Do not search the web for solutions or hints, post the problem set, or otherwise violate the course collaboration policy or the MSU student code of conduct.
- Violations (regardless of intent) will be reported to the Dean of Students, per the MSU student code of conduct, and you will receive a 0 on the assignment.

Tips

- Answer each problem to the best of your ability. Partial credit is your friend!
- Read the hints for where to find relevant examples for each problem.
- Refer to the problem solving and homework tips guide.
- It is not a badge of honor to say that you spent 5 hours on a single problem or 15 hours on a single assignment. Use your time wisely and get help (see "How to Get Help" below).

How to Get Help
When you are stuck and need a little or big push, **please ask for help!**

- Timebox your effort for each problem so that you don't spend your life on the problem sets. (See the problem solving tips guide for how to do this effectively.)
- Post in Discord. If you're following the timebox guide, you can post the exact statement that you produced after spending 20 minutes being stuck.
- Come to office hours or visit the CS Student Success Center.

Problem 1 (14 points)

Consider the following claim.

Let n and m be integers. If nm is not divisible by 3, then neither n nor m is divisible by 3.

(a) (2 points) State the contrapositive of the claim.

Hint: Double check your work here! You need to get this right to be successful on the next part of the problem.

Grading Notes. This rubric is straightforward: correctly state the contrapositive.

(b) (8 points) Prove the claim by proving the contrapositive.

Grading Notes. While a detailed rubric cannot be provided in advance as it gives away the solution details, the following is a general idea of how points are distributed for this problem. We give partial credit where we can.

- (6) **Correctness.** If your proof is not correct, this is where you'll get docked.
  - (5) Regardless of how you formulate your proof, somewhere you'll need certain facts without which the proof wouldn't work. E.g., if it weren't true that the sum of two integers is an integer, would your proof fail? If so, then that is a fact I need to see stated somewhere.
  - (1) The order of these facts must make sense, so that you're not inferring something before you have all the facts to infer it. E.g., you cannot use the fact that the sum of two integers is integer if you don't already know that you have two integers to begin with.
- (2) **Communication.** We need to see a mix of notation and intuition, preferably in the "column" format with the statements in mathematical notation on the left and the reasons on the right. If you skip too many steps at once, or we cannot follow your proof, or if your proof is overly wordy or confusing, this is where you'll get docked.

(c) (2 points) Let $D(x)$ be the predicate that states that x is divisible by 3. Use this predicate to write a fully quantified universal expression that is equivalent to the **contrapositive** of the claim.

Hint: we learned about predicates, universal quantifiers, and fully quantified expressions in the lecture on predicates.

Grading Notes. This rubric is straightforward: you correctly formulated the expression.

(d) (2 points) Use the same predicate from (c) to write a fully quantified universal expression that is equivalent to the original claim.

Hint: we learned about predicates, universal quantifiers, and fully quantified expressions in the lecture on predicates.

Grading Notes. This rubric is straightforward: you correctly formulated the expression.

Problem 2 (16 points)

For each of the following two claims, either disprove with a counterexample or provide a proof.

(a) (8 points) $\forall x \in S : [P(x) \lor Q(x)] \Leftrightarrow [\forall x \in S : P(x)] \lor [\forall x \in S : Q(x)]$

(b) (8 points) $\exists x \in S : [P(x) \lor Q(x)] \Leftrightarrow [\exists x \in S : P(x)] \lor [\exists x \in S : Q(x)]$

Hint: Recall that if quantifiers are omitted, we assume the universal quantifier. Thus, the claims above contain an implicit $\forall S, \forall P, \forall Q$.

Hint: Double check that you have correctly determined whether the claim is correct or not. Partial credit will be sparse if you try to prove something true that is false or try to give a counterexample to a true statement.

Hint: To prove an if and only if ($\Leftrightarrow$) statement, a common strategy is to prove each "direction" of implication separately. That is, if the claim is $A \Leftrightarrow B$, provide one proof that $A \Rightarrow B$ and another that $B \Rightarrow A$.

Grading Notes. Each of the above is graded out of 8 points, broken down as described below.

(2) Correctly decide if the statement is true or false.

(6) Correctly prove your claim.

- A proof requires clearly stated facts and explanations in the column format with notation on the left and explanation on the right. Proving both directions of the implication separately is highly recommended.
- A disproof requires a clearly stated counterexample along with an explanation of why the counterexample is a counterexample.

Problem 3 (12 points)

Let $f(x) = \frac{3x}{2}$. For this problem, we use the notation $c\mathbb{Z}$ to denote the set of multiples of c. For example $2\mathbb{Z}$ denotes the even integers, and $3\mathbb{Z}$ denotes multiples of 3.

(a) (4 points) Prove that $f : \mathbb{Z} \to \mathbb{Z}$ is *not* a function (that is, it is *not* a function if both its domain and codomain are integers).

Hint: We discuss the definition of functions, domain, and codomain in the lecture on functions.

Grading Notes. While a detailed rubric cannot be provided in advance as it gives away the solution details, the following is a general idea of how points are distributed for this problem. We give partial credit where we can.

(4) **Correctness.** You must prove that f is not a function with a convincing proof modeled on the similar proofs from lectures.

(b) (8 points) Prove that $f : 2\mathbb{Z} \to \mathbb{Z}$ is a function (that is, it *is* a function if its domain is even integers and its codomain is all integers).

Note: f is a function from A to B if and only if:

- for each $a \in A$, $f(a)$ is defined,
- for each $a \in A$, $f(a)$ does not produce two different outputs, and,
- for each $a \in A$, $f(a) \in B$.

Grading Notes. While a detailed rubric cannot be provided in advance as it gives away the solution details, the following is a general idea of how points are distributed for this problem.

(6) **Correctness.** Regardless of how you formulate your proof, somewhere you'll need

(2) to show (not just state) that for each $a \in A$, $f(a)$ is defined.

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(2) **Communication.** We need to see a mix of notation and intuition, preferably in the "column" format with the statements in mathematical notation on the left and the reasons on the right. If you skip too many steps at once, or we cannot follow your proof, or if your proof is overly wordy or confusing, this is where you'll get docked.