Relations

CS application: relational databases

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Questions about data stored in relational databases can be posed precisely using the language of relations.

SQL (structured query language)

<u>Def</u> The <u>cartesian product</u> of two sets A, B is

$$A \times B = \{ (a, b) : a \in A \wedge b \in B \}$$

lists / tuples / arrays — order matters

ex

$\mathbb{R} \times \mathbb{R}$ = 2d plane, Cartesian plane

$\{red, blue\} \times \{1,2,3\} = \{(red,1), (red,2), (red,3), (blue,1), (blue,2), (blue,3)\}$

Q what is $|A \times B|$? $|A|, |B|$

$|A| \cdot |B|$

$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

$\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3$

Def A binary relation R on sets A, B is a subset $R \subseteq A \times B$.

We write $(x,y) \in R$ as xRy

$(x,y) \notin R$ as $x \not R y$

examples

1) R_1 "is (blood) related to" is a binary relation on people.

let P be the set of all people

"is blood related to" is

$\{(x,y) : x \in P \wedge y \in P \wedge x \text{ is related to } y\}$

(Serena Williams, Venus Williams) $\in R$,
(Lucy Williams, Serena Williams) $\notin R$,

② $<$ on $A = \{1, 2, 3, 4\}$

$< = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$

$1 < 2$ but $3 \not< 2$

③ Let $f : A \to B$ be a function

$\{(a, f(a)) : a \in A\} \subseteq A \times B$, so it is a relation

Q Is the converse true? Let R be a binary relation on A, B.

$\{(x,y) : x \in A \wedge y \in B \wedge xRy\}$
$\Rightarrow f : A \to B$ s.t. $f(x) = y$ is a function

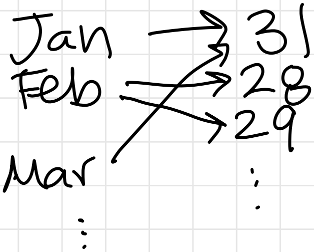
true or false

④ let A = months, B = number of days

Relation: month, its # days

$\{(\text{Jan}, 31), (\text{Feb}, 28), (\text{Feb}, 29), (\text{Mar}, 31) \dots \}$

Jan	31
Feb	28
Feb	29
Mar	31
⋮	⋮



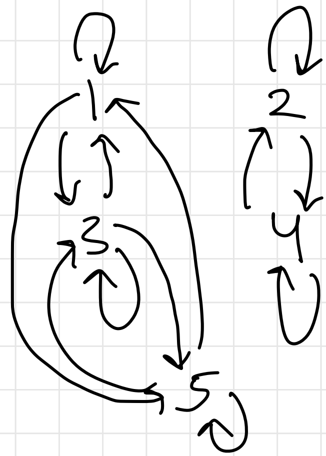
⑤ $A = \{1, 2, 3, 4, 5\}$

$(1,1) \in R_2$

$(2,4) \in R_2$

$(3,2) \notin R_2$

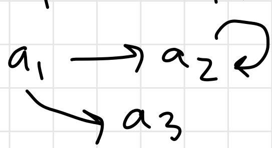
R_2:



Properties of relations

Let $R \subseteq A \times A$, so R is a relation on A

R:



R is <u>reflexive</u> if $\forall a \in A : aRa$

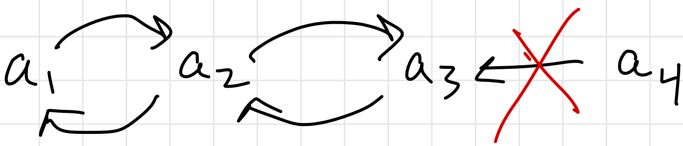
all nodes have self-loops

R is <u>irreflexive</u> if $\forall a \in A : a \not R a$

no nodes have self-loops

R is <u>symmetric</u> if

$$\forall a_1, a_2 \in A : \; a_1 R a_2 \Rightarrow a_2 R a_1$$

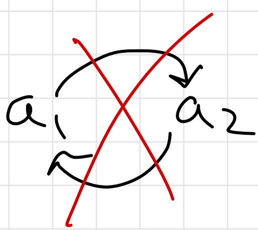


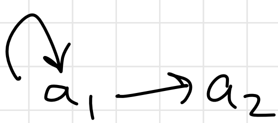
whenever we have a forward edge, we have the backward edge.

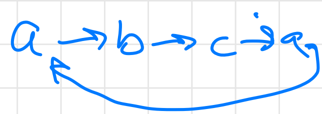
$\{(a,b), (b,c), (c,a)$

R is <u>anti-symmetric</u> if

$$\forall a_1, a_2 \in A : (a_1 R a_2 \wedge a_2 R a_1) \Rightarrow a_1 = a_2$$







never have backwards edges, but self-loops okay.

R is <u>transitive</u> if

$$\forall\, a,b,c \in A : (\overset{T}{aRb \wedge bRc}) \Rightarrow (\overset{F}{aRc})$$

$$\quad a_1 R a_2 \wedge a_2 R a_1 \qquad\qquad a_1 \not R a_1$$

$$\begin{tikzcd} a \arrow[r] \arrow[rr, bend right] & b \arrow[r] & c \end{tikzcd}$$

shortcut edges always exist

Q Is $\begin{tikzcd} a_1 \arrow[r, bend left] & a_2 \arrow[l, bend left] \end{tikzcd}$ transitive? $a_1 \neq a_2$

let $a = a_1$

$b = a_2$

$c = a_1$

$$\begin{tikzcd} a_1 \arrow[r] & a_2 \arrow[l, bend left] \end{tikzcd}$$

Q Is $\begin{tikzcd} a_1 \arrow[loop above] \end{tikzcd}$ transitive?

$$(\overset{T}{a_1 R a_1 \wedge a_1 R a_1}) \Rightarrow (\overset{T}{a_1 R_1 1})$$

Relations review

$(a,b) \in R$

Let A, B be sets. aRb

$R \subseteq A \times B$ is a binary relation

often, we are concerned with relations over a single set:

$R \subseteq S \times S$ "R is a relation on S"

Properties of relations on single sets:

- reflexive: $\forall a \in A: aRa$

$$\begin{tikzcd} a \arrow[loop above] \end{tikzcd}$$

- irreflexive: $\forall a \in A: a \not R a$

- symmetric: $\forall a_1, a_2 \in A: a_1 R a_2 \Rightarrow a_2 R a_1$

$$\begin{tikzcd} a_1 \arrow[r, bend left] & a_2 \arrow[l, bend left] \end{tikzcd}$$

- anti-symmetric: $\forall a_1, a_2 \in A:$

$$\begin{tikzcd} a_1 \arrow[loop above] & a_2 \end{tikzcd}$$

$$\underbrace{(a_1 R a_2 \wedge a_2 R a_1)} \Rightarrow a_1 = a_2$$

- transitive:
$\forall a_1, a_2, a_3 \in A: (a_1 R a_2 \wedge a_2 R a_3) \Rightarrow a_1 R a_3$

a b

symmetric ✓

anti-symmetric ✓

$A = \{a, b\}$

$R \subseteq A \times A$

$\emptyset$

$B \subseteq \text{People} \times \text{People}$

(Lucy, Britney Spears) $\in B$

Lucy B Britney Spears

Lucy $\not B$ Braeden

$> \subseteq \mathbb{R} \times \mathbb{R}$

$2 > 1.5$

$2 \not> 2$

ex relation $<$ on $\mathbb{Z}$:

- reflexive?

no - disproof by counterexample.

$1 \in \mathbb{Z}$. $1 \not< 1$

- irreflexive? yes.

$\forall a \in \mathbb{Z}: a \not R a$.

let $a \in \mathbb{Z}$. $a \not< a$ because no integer is less than itself. $\square$

- symmetric?

disproof by counterexample:

$1 < 2$ but $2 \not< 1$.

- antisymmetric? $\forall a_1, a_2 \in A$:

$(a_1 R a_2 \wedge a_2 R a_1) \Rightarrow a_1 = a_2$

Let $a_1, a_2 \in \mathbb{Z}$. Assume $a_1 < a_2$ and $a_2 < a_1$.

Since no a_1, a_2 satisfy $a_1 < a_2$ and $a_2 < a_1$,

$(a_1 < a_2 \wedge a_2 < a_1) \Rightarrow a_1 = a_2$ is

vacuously true.

- tranisitive?

$\forall a_1, a_2, a_3 \in A: (a_1 R a_2 \wedge a_2 R a_3) \Rightarrow a_1 R a_3$

<u>Proof</u> Assume $a_1, a_2, a_3 \in \mathbb{Z}$ and $a_1 < a_2$ and $a_2 < a_3$. By the def. of $<$, $a_1 < a_3$

	$< \subseteq \mathbb{R} \times \mathbb{R}$	Same Bday $\subseteq P \times P$	Subset $\subseteq \mathcal{P}(S) \times \mathcal{P}(S)$
reflexive			
irreflexive			
symmetric			
anti-symmetric			
transitive			
equivalence relation			
partial order			
strict partial order			
total order			
strict total order			