# Functions

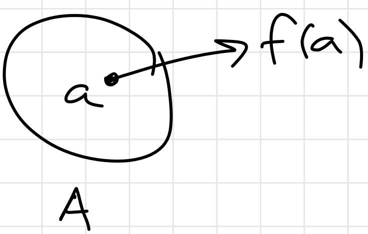
**Def** Let $A, B$ be sets.

$f: A \to B$ is a function if $f$ assigns to each $a \in A$ a single value $b \in B$, denoted $f(a)$.

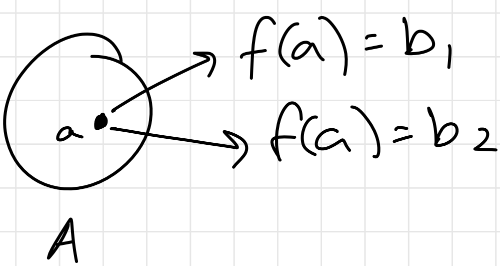
"$f$ from $A$ to $B$"

Equivalently, $f$ has 3 properties:

1) for each $a \in A$, $f(a)$ is defined.

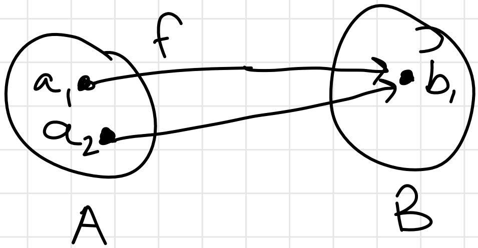



2) For each $a \in A$, $f(a)$ does not produce 2 different outputs.




only if $b_1 = b_2$

3) for each $a \in A$, $f(a) \in B$.




$f : A \to B$

A is the <u>domain</u> of f

B is the <u>codomain</u> of f

The <u>range</u> of f is $\{f(a) : a \in A\}$

range $\subseteq$ codomain

let $A = \{1, 2, 3\}$
let $B = \{x, y\}$

| $a \in A$ | $b \in B$ |
|---|---|
| 1 | $x = f(1)$ |
| 2 | $y = f(2)$ |
| 3 | $x = f(3)$ |

Props:

(1) $\forall a \in A$, $f(a)$ is defined ✓

(2) $\forall a \in A$, $f(a)$ does not produce 2 diff. outputs ✓

(3) $\forall a \in A$, $f(a) \in B$ ✓

- exactly 1 row for every element of A
- some elements of B can have zero rows, or elements of B can have multiple rows

ex $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.

domain: $\mathbb{R}$
codomain: $\mathbb{R}$
range: $\mathbb{R}^{\geq 0}$ (reals greater than or equal to 0)

Intuitive "proof" of 3 properties:

(1) $\forall x \in \mathbb{R}$, $f(x) = x^2$ ✓

(2) $\forall x \in \mathbb{R}$, $f(x) = x^2$, a single value

(3) $\forall x \in \mathbb{R}$, $f(x) \in \mathbb{R}$, because $x^2 \in \mathbb{R}$

ex $f: \mathbb{R} \to \mathbb{R}^{<0}$, $f(x) = x^2$

f is not a function.

Violates (3). Consider $2 \in \mathbb{R}$. $f(2) = 4 \notin \mathbb{R}^{<0}$

ex $s: \mathbb{Z} \to \mathbb{Z}$ defined by $s(x) = x+1$

"successor function"

domain, codomain: $\mathbb{Z}$

range: $\mathbb{Z}$

<u>claim</u> $s: \mathbb{Z} \to \mathbb{Z}$ is a function.

<u>Proof</u> We prove all 3 properties.

1) $\forall x \in \mathbb{Z}$, $s(x)$ is defined as $x+1$.

2) To show $\forall x \in \mathbb{Z}$, $s(x)$ does not produce 2 diff. outputs, we show that if $s(x) = a$ and $s(x) = b$, then $a = b$.

Assume $s(x) = a$ and $s(x) = b$.

$a = x+1$, $b = x+1$      def. of $s$

$a = b$      substitution

3) WTS (want to show) $\forall x \in \mathbb{Z}$, $s(x) \in \mathbb{Z}$.

$s(x) = x+1$, which is an integer because int + int = int.

Examples from last time:

1. $g : \mathbb{Z} \to \mathbb{Z}$ defined by $g(a) = 5$

Properties:

1) Defined $\forall x \in \mathbb{Z}$: yes. $g(x) = 5$.
2) $\forall x \in \mathbb{Z}$, $g(x)$ maps to only one output.

Proof:

let $x \in \mathbb{Z}$ and $g(x) = a$ and $g(x) = b$.

$a = 5$, $b = 5$ — def. of $g(x)$

$a = b$ — substitution. ∎

3) $\forall x \in \mathbb{Z}$, $g(x) \in \mathbb{Z}$. yes - $g(x) = 5$ $\forall x \in \mathbb{Z}$.

Domain: $\mathbb{Z}$
Codomain: $\mathbb{Z}$
Range: $\{5\}$

2. $E : \mathbb{Z} \to \{T, F\}$ defined by $E(x) = \begin{cases} T & x \text{ is even} \\ F & x \text{ is odd} \end{cases}$

Properties:

1) Defined $\forall x \in \mathbb{Z}$: yes.
2) $\forall x \in \mathbb{Z}$, $E(x)$ maps to only one output.

Proof:

let $x \in \mathbb{Z}$. WTS that if $E(x) = a$ and $E(x) = b$, then $a = b$.

we prove using cases.

Case 1: $x$ is even.

let $E(x) = a$ and $E(x) = b$. Since $x$ is even, $a = T$ and $b = T$, so $a = b$.

Case 2: $x$ is odd.

let $E(x) = a$ and $E(x) = b$. Since $x$ is odd, $a = F$ and $b = F$, so $a = b$.

Since the claim is true in all cases, the claim is true.

3) $\forall x \in \mathbb{Z}$, $E(x) \in \{T, F\}$. Yes.

Domain: $\mathbb{Z}$
Codomain: $\{T, F\}$
Range: $\{T, F\}$

3. $p : \mathbb{Z}^{\geq 0} \rightarrow \mathbb{Z}^{\geq 0}$ defined by $p(x) = x - 1$

Not a function! Fails property 3.

For $x = 0 \in \mathbb{Z}^{\geq 0}$, $p(x) = x - 1 = 0 - 1 = -1 \notin \mathbb{Z}^{\geq 0}$.

4. $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x)$ the number whose absolute value is $x$.

Not a function! Fails property 2.

For $x = 5 \in \mathbb{Z}$, the number whose abs. val. is 5 is both $-5$ and 5.

Violates property 1.

Consider $x = -5$. It's undefined which number has absolute value $-5$.

Another example:

$f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \frac{1}{x}$

1) $\forall x \in \mathbb{R}$, $f(x)$ is defined.

Consider $x = 0$. $f(x) = \frac{1}{x}$ is not defined.

<u>Def</u> A function $f: A \to B$ is

1. onto (surjective) if

$\forall b \in B \ \exists a \in A : f(a) = b$

$\equiv \forall b \in B$, something in $A$ maps to it

$\equiv \forall b \in B$, $b$ shows up in at least 1 row of the table

$\equiv$ codomain = range

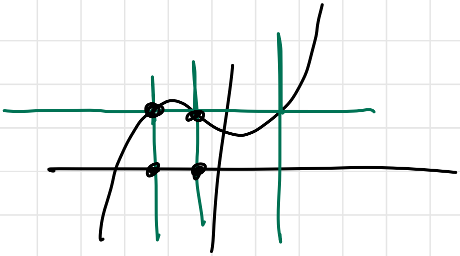
ex: $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = x$

2. one-to-one (injective) if
1:1

$\forall a_1, a_2 \in A, \quad a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$

$\equiv \forall b \in B$, at most 1 thing in $A$ maps to it

$\equiv \forall b \in B$, $b$ shows up in at most 1 row of the table!

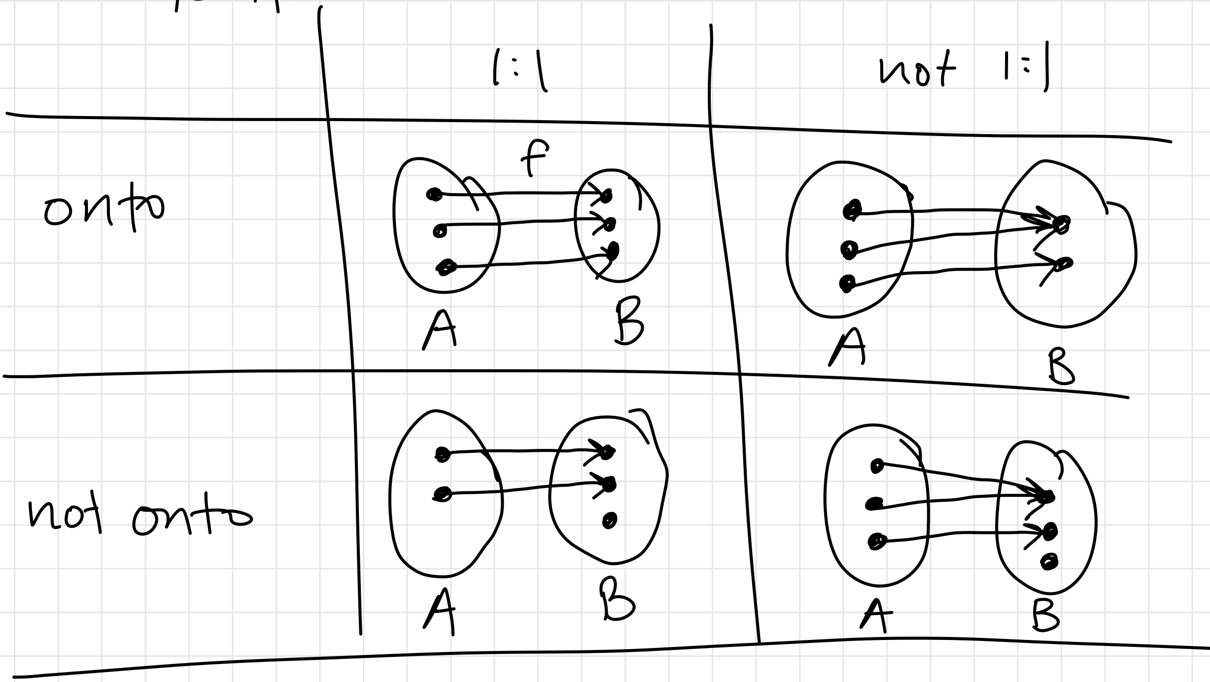


<u>ex</u> $f: \mathbb{R} \to \mathbb{R} : f(x) = x^2$

$f(-2) = 4$
$f(2) = 4$

3. a bijection if onto and 1:1

$\equiv \forall b \in B$, exactly 1 elt. of A maps to it.




How do we prove that f is onto or 1:1 ?

<u>onto</u>

WTS $\forall b \in B \; \exists a \in A : f(a) = b$.

$\equiv$ If $b \in B$, then $\exists a \in A : f(a) = b$.

Step 1: Assume $b \in B$.

Step 2: Construct a s.t. $f(a) = b$.

ex $s: \mathbb{Z} \to \mathbb{Z}$ defined by $s(x) = x+1$.

claim: $s$ is onto.

example: $b = 7$. What is $a \in \mathbb{Z}$ s.t. $f(a) = b = 7$?

$a = 6$, $f(a) = 6+1 = 7$.

Proof: Assume $b \in \mathbb{Z}$. Want to construct $a \in \mathbb{Z}$ s.t. $f(a) = b$.

Consider $a = b-1$. $a \in \mathbb{Z}$ since int − int = int