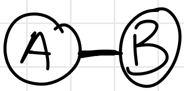
Intro to Graphs

<u>Def</u> An <u>undirected graph</u> $G = (V, E)$ is a non-empty set of vertices (nodes) V and a set $E = \{\{u, v\} : u, v \in V\}$ of edges joining pairs of nodes.

<u>ex</u>

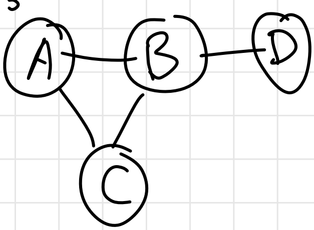
$V = \{A\}$
$E = \emptyset$



$V = \{A, B\}$

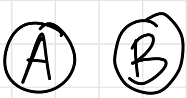
$E = \{\{A, B\}\}$

G_3:



$V = \{A, B, C, D\}$

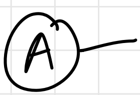
$E = \{\{A, B\}, \{B, D\}, \{B, C\}, \{A, C\}\}$



$V = \{A, B\}$

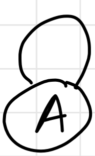
$E = \emptyset$

<u>non-ex</u>



all edges need 2 endpoints

<u>Q</u>



is this a graph? yes. $V = \{A\}$
$E = \{\{A, A\}\}$
$= \{\{A\}\}$

real-world examples

alice — bob
catherine

- Facebook friends

    nodes: people
    edge: 2 people are Facebook friends

- blood relationships

Q what property (or properties) would a mathematical relation need to have to be represented as an undirected graph?

ideas: symmetric $a \rightleftarrows b$ $\quad a - b$

reflexive a

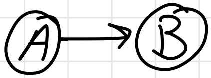
self loops are equivalent when directed or undirected

$c \quad c$

Def A directed graph $G = (V, E)$ has a set of vertices V and edges $E \subseteq V \times V = \{(u, v) : u, v \in V\}$ so that edges are directed from one vertex to another.

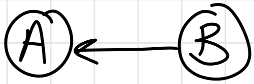
Note: relations on a single set and directed graphs are the same!

ex 



$V = \{A, B\}$
$E = \{(A, B)\}$

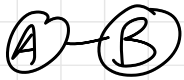
$\neq$



$V = \{A, B\}$
$E = \{(B, A)\}$

ordered pair
tuple
list
array

undirected:



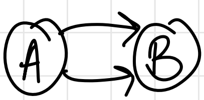
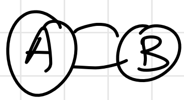
$E = \{\{A, B\}\}$

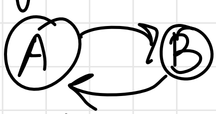
set

real-world example

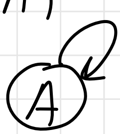
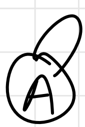
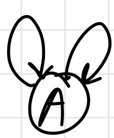
Twitter followers

Def A graph is simple if it contains no parallel edges or self-loops.

parallel edges:  

note that  has no parallel edges

$(A, B) \neq (B, A)$

self-loops:   

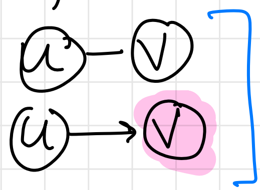
Example 11.3: Self-loops and parallel edges.

Suppose that we construct a graph to model each of the following phenomena. In which settings do self-loops or parallel edges make sense?

1. A social network: nodes correspond to people; (undirected) edges represent friendships.
2. The web: nodes correspond to web pages; (directed) edges represent links.
3. The flight network for a commercial airline: nodes correspond to airports; (directed) edges denote flights scheduled by the airline in the next month.
4. The email network at a college: nodes correspond to students; there is a (directed) edge $\langle u, v \rangle$ if u has sent at least one email to v within the last year.

	self-loops	parallel edges
social network	no	no
the web	yes	yes
flight network	no	yes
email network	yes	no

Def Let $e = \{u, v\}$ or (u, v)

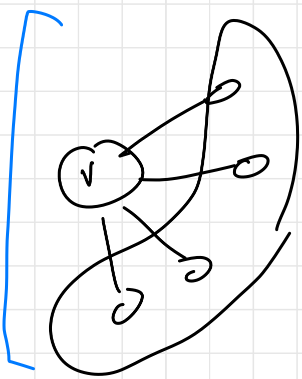


- nodes u, v are <u>adjacent</u> or <u>neighbors</u>
- in a directed graph, v is an <u>out-neighbor</u> of u and u is an <u>in-neighbor</u> of v
- u, v are <u>endpoints</u> of e
- u, v are <u>incident</u> to e

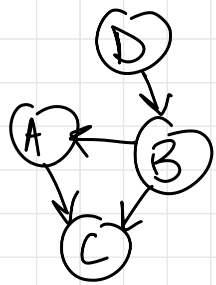
Let v be a node in a simple undirected graph.

degree(v) = deg(v) = $d(v)$ = # of neighbors of v

$$= \left| \{ u \in V : \underbrace{\{v, u\}}_{\text{or } \{u,v\}} \in E \} \right|$$



deg$(v) = 4$



for directed graphs, indeg(v) = # of in-neighbs of v

outdeg(v) = # of out-neighbors of v

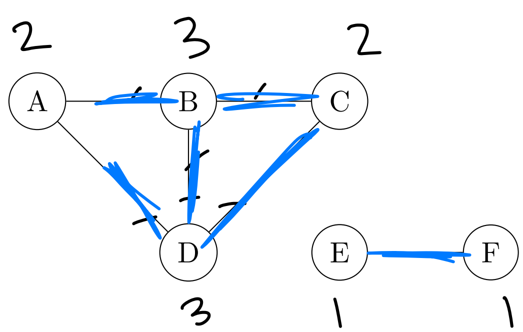
Proofs about graphs

Discrete Structures (CSCI 246)

in-class activity

Names: ____

1. For each of the two graphs, label each node v with $deg(v)$, and give $\sum_{v \in V} deg(v)$, the total degree of the graph, and $|E|$, the number of edges in the graph.

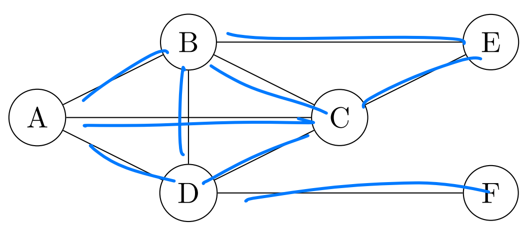


$$\sum_{v \in V} deg(v) = 2+3+2+3+1+1 = 12$$

$$|E| = 6$$

$$|E| = \frac{1}{2} \sum_{v \in V} deg(v)$$

$$2|E| = \sum_{v \in V} deg(v)$$



9 edges

total degree: 18

2. Can you give a conjecture about the relationship between $\sum_{v \in V} deg(v)$ and and $|E|$?

Theorem 11.8 "Handshaking Lemma"

Let $G = (V, E)$ be a simple undirected graph. Then

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Proof Let $G = (V, E)$ be an undirected graph. Notice that every edge is connected to exactly 2 nodes, meaning that it contributes 1 to the degree of 2 nodes.

So $\sum_{v \in V} \deg(v) = 2|E|$.

Corollary → fact that follows simply from a previous theorem/lemma

Let n_{odd} denote the number of nodes whose degree is odd. Then n_{odd} is even.

Proof Aiming for a contradiction, suppose n_{odd} is odd.

Note that

$$\underbrace{\sum_{v \in V} \deg(v)}_{\text{this is } 2|E| \text{, which is even}} = \underbrace{\sum_{\substack{v \in V: \\ \deg(v) \text{ is odd}}} \deg(v)}_{\text{this must be odd, because sum of odd \# of odds is odd}} + \underbrace{\sum_{\substack{v \in V: \\ \deg(v) \text{ is even}}} \deg(v)}_{\text{this must be even, because sum evens is even}}$$

even = odd + even, $\uparrow$

a contradiction!

So n_{odd} must be even. $\square$

3. Go back to the previous page and give n_{odd}, the number of nodes with odd degree, for each graph.

4. Give a conjecture relating to n_{odd}. If you are having trouble, try drawing more graphs! Ask for a hint if you still don't have a conjecture after 5 minutes.

5. Prove your conjecture. Hint: use the handshaking lemma!