Discrete Structures (CSCI 246)

Homework 1

Purpose & Goals

The following problems provide practice relating to:

- direct proofs, proof by cases, and proofs by counter-example,
- mathematical definitions (rational, absolute value, divisibility, sets, etc.), and
- the problem solving process.

Submission Requirements

- **Type or clearly hand-write your solutions** into a pdf format so that they are legible and professional. Submit your pdf to Gradescope. **Illegible, non-pdf, or emailed solutions will not be graded.**
- Each problem should start on a new page of the document. When you submit to Gradescope, associate each page of your submission with the correct problem number. Please post in Discord if you are having any trouble using Gradescope.
- Try to model your formatting off of the proofs from lecture and/or the textbook.
- Submit to Gradescope early and often so that last-minute technical problems don't cause you any issues. Only the latest submission is kept. Per the syllabus, assignments submitted within 24 hours of the due date will receive a 25% penalty and assignments submitted within 48 hours will receive a 50% penalty. After that, no points are possible.

Academic Integrity

- You may work with your peers, but **you must construct your solutions in your own words on your own.**
- Do not search the web for solutions or hints, post the problem set, or otherwise violate the course collaboration policy or the MSU student code of conduct.
- Violations (regardless of intent) will be reported to the Dean of Students, per the MSU student code of conduct, and you will receive a 0 on the assignment.

Tips

- Answer each problem to the best of your ability. Partial credit is your friend!
- Read the hints for where to find relevant examples for each problem.
- Refer to the problem solving and homework tips guide.
- It is not a badge of honor to say that you spent 5 hours on a single problem or 15 hours on a single assignment. Use your time wisely and get help (see "How to Get Help" below).

How to Get Help

When you are stuck and need a little or big push, **please ask for help!**

- Timebox your effor for each problem so that you don't spend your life on the problem sets. (See the problem solving tips guide for how to do this effectively.)
- Post in Discord. If you're following the timebox guide, you can post the exact statement that you produced after spending 20 minutes being stuck.
- Come to office hours or visit the CS Student Success Center.

1. Problem 1 (18 points)

   (a) (8 points) Use a direct proof to prove that if v and z are rational numbers and $v \neq 0$ then z/v is a rational number.

   Hint: We saw a direct proof involving rational numbers in the lecture on direct proofs and disproof by example. You should first try to understand that proof before attempting this one. Don't forget to apply the problem solving process by trying the claim out on examples first to get a sense for the result.

   Grading Notes. While a detailed rubric cannot be provided in advance as it gives away the solution details, the following is a general idea of how points are distributed for this problem. If you can get at least partway, we give partial credit where we can.

   (6) **Correctness**. If your proof is not correct, this is where you'll get docked.

   (5) Regardless of how you formulate your proof, somewhere you'll need certain facts without which your proof wouldn't work. For example, if it weren't true that the sum of two integers is an integer, would your proof fail? If so, then that is a fact I need to see stated somewhere.

   (1) The order of these facts must make sense, so that you're not inferring something before you have all the facts to infer it. E.g., you cannot use the fact that sum of two integers is an integer if you don't already know that you have two integers to begin with.

   (2) **Communication**. We need to see a mix of mathematical notation and intuation, prefarably in the column format with the notation on the left and the reasoning on the right. If you skip too many steps at once, or we cannot follow your proof, or if your solution is overly wordy or confusing, this is where you'll get docked.

   (b) (4 points) Consider the similar claim that if v and z are rational numbers, then v/z is a rational number. This claim is false.

   Use a disproof by counterexample to disprove this claim. That is, provide a counterexample and a short explanation as to why it is a counterexample.

   Then *clearly* label the earliest step in your proof of (a) that fails for this claim, and give a short explanation as to why that step does not follow from the previous under this new claim.

   Grading Notes. Note that this problem gives you practice disproving a statement by counterexample, and critically reading your own proofs.

   (c) (2 points) Clearly state the converse of the statement from (a).

   Hint: We formulated the converse of *if-then* statements in the lecture on direct proofs and disproofs by example. Do the same here.

   Hint: If something is *not* both red and an apple then it is either not red or not on apple (or, neither red nor an apple, which is implicit in "or").

   Grading Notes. This one is short: the 2 points are for correctly formulating the exact converse of the original statement.

   (d) (4 points) Is the converse you constructed in (c) true? If so, provide a short direct proof; if not, provide a counterexample and a short explanation as to why it is a counterexample.

   Hint: It's tricky when you don't know whether you're proving something true or false! We did examples of this in the first week and a half of lectures. Usually the way to do this is to try to prove the claim false; if you can't find a counterexample, then figuring out why you can't might lead you to a proof.

   Grading Notes. While a detailed rubric cannot be provided in advance as it gives away the solution details, the following is a general idea of how points are distributed for this problem.

   (1) Correctly decide if the statement is true or false.

(3) Correctly prove your claim.
   - A proof requires clearly stated facts and explanations in the column format with notation on the left and explanation on the right.
   - A disproof requires a clearly stated counterexample along with an explanation of why the counterexample is a counterexample.

Note that if you incorrectly think the statement is false when it is true or vice-versa, partial credit will be sparse. Try to check that you have the right claim before proceeding too far.

2. Problem 2 (12 points)

   Let v, z be real numbers such that $|v| \leq |z|$. Give a direct proof by cases that $\frac{|v+z|}{2} \leq |z|$.

   Hint: You may use the definition of absolute value on any real number w:

   - $|w| = w$ for $w \geq 0$, and
   - $|w| = -w$ for $w \leq 0$.

   Hint: We saw a direct proof by cases in the lecture on proof by cases and intro to sets. You should first try to understand that proof before attempting this one. Don't forget to apply the problem solving process by trying out the claim on a few examples to get a sense for the result.

   Grading Notes. While a detailed rubric cannot be provided in advance as it gives away the solution details, the following is a general idea of how points are distributed for this problem. If you can get at least partway, we give partial credit where we can.

   (10) **Correctness**. If your proof is not correct, this is where you'll get docked.

   - (2) Regardless of how you formulate your proof, you will need clearly labeled, exhaustive cases.
   - (7) Regardless of how you formulate your proof, somewhere you'll need certain facts without which your proof wouldn't work. For example, if it weren't true that the sum of two integers is an integer, would your proof fail? If so, then that is a fact I need to see stated somewhere.
   - (1) The order of these facts must make sense, so that you're not inferring something before you have all the facts to infer it. E.g., you cannot use the fact that sum of two integers is an integer if you don't already know that you have two integers to begin with.

   (2) **Communication**. We need to see a mix of mathematical notation and intuation, prefarably in the column format with the notation on the left and the reasoning on the right. If you skip too many steps at once, or we cannot follow your proof, or if your solution is overly wordy or confusing, this is where you'll get docked.

3. Problem 3 (12 points)

   Give a direct proof that $\{3v : v \in \mathbb{Z}\} \cap \{z \in \mathbb{Z} : 10|z\} \subseteq \{6v : v \in \mathbb{Z}\} \cap \{z \in \mathbb{Z} : 15|z\}$.

   Hint: We saw direct proofs involving sets and divisibility in the first full lecture on sets. Try to understand those proofs before attempting this one. Don't forget to apply the problem solving process by making sure you understand all notation and trying the claim out on examples first to get a sense for the result.

   Grading Notes. While a detailed rubric cannot be provided in advance as it gives away the solution details, the following is a general idea of how points are distributed for this problem. If you can get at least partway, we give partial credit where we can.

   (10) **Correctness**. If your proof is not correct, this is where you'll get docked.

   - (9) Regardless of how you formulate your proof, somewhere you'll need certain facts without which your proof wouldn't work. For example, if it weren't true that the sum of two integers is an integer, would your proof fail? If so, then that is a fact I need to see stated somewhere.
   - (1) The order of these facts must make sense, so that you're not inferring something before you have all the facts to infer it. E.g., you cannot use the fact that sum of two integers is an integer if you don't already know that you have two integers to begin with.

   (2) **Communication**. We need to see a mix of mathematical notation and intuation, prefarably in the column format with the notation on the left and the reasoning on the right. If you skip too many steps at once, or we cannot follow your proof, or if your solution is overly wordy or confusing, this is where you'll get docked.

4. Problem 4 (12 points)

   Give a direct proof that for any sets A, B, C, $(B - A) \cap (C - A) = (B \cap C) - A$.

   Hint: We saw direct proofs involving set properties in the second full lecture on sets. Try to understand those proofs before attempting this one. Don't forget to apply the problem solving process by making sure you understand all notation and trying the claim out on examples first to get a sense for the result.

   Hint: Recall that, given two sets X and Y, to prove that $X = Y$, it is almost always easier to split it intoi two proofs: one that $X \subseteq Y$ and the other that $Y \subseteq X$. While there is a way to prove this without doing so, you are less likely to miss steps and explanations (and thus points) if you *do* split it apart. Up to you.

   Grading Notes. While a detailed rubric cannot be provided in advance as it gives away the solution details, the following is a general idea of how points are distributed for this problem. If you can get at least partway, we give partial credit where we can.

   (10) **Correctness**. If your proof is not correct, this is where you'll get docked.

   - (8) Regardless of how you formulate your proof, somewhere you'll need certain facts without which your proof wouldn't work. For example, if it weren't true that the sum of two integers is an integer, would your proof fail? If so, then that is a fact I need to see stated somewhere.
   - (2) The order of these facts must make sense, so that you're not inferring something before you have all the facts to infer it. E.g., you cannot use the fact that sum of two integers is an integer if you don't already know that you have two integers to begin with.

   (2) **Communication**. We need to see a mix of mathematical notation and intuation, prefarably in the column format with the notation on the left and the reasoning on the right. If you skip too many steps at once, or we cannot follow your proof, or if your solution is overly wordy or confusing, this is where you'll get docked.