Claim (example 4.12) Let n be any int. Then $n \cdot (n+1)^2$ is even.

terms: integer ✓

even → divisible by 2
→ $x/2$ is an integer
→ $x = 2c$ for an integer c
→ $x \bmod 2 = 0$

examples:

n	$n(n+1)^2$	$n(n+1)^2$ even?
0	$0(1) = 0$	T
3	$3 \cdot (3+1)^2 = 48$	T
-2	$-2(-2+1)^2 = -2$	T

easy special cases:

n is even. so n times anything is even!
n is odd. so $n+1$ is even.

wait! that covers everything.

<u>Proof</u> Consider 2 cases.

Case 1: n is even.

Statement	reasoning
$n = 2c$ for int. c	by def. of even
$n(n+1)^2 = 2c(n+1)^2$	by substitution
$c(n+1)^2$ is an int.	sum, product of ints is int
$n(n+1)^2$ is even	we gave a way to write it as $2k$ for integer k ($k = c(n+1)^2$)

case 2: n is odd.

statement	reasoning
$n+1$ is even	n is odd
$n+1 = 2c$ for integer c	def. of even
$n(n+1)^2 = n(2c)^2 = 2n2c^2$	substitution, algebra
$n2c^2$ is integer	product of ints is int
$n(n+1)^2$ is even	by def. of even

Since n is either even or odd, and in either case $n(n+1)^2$ is even, $n(n+1)^2$ even. $\boxtimes$

Proof by cases: if it is useful, split your claim into cases.
- prove claim in each case
- ensure that cases are exhaustive (cover all possibilities in original claim)

<u>claim</u> (example 4.13) Let x be a real number. Then $-|x| \le x \le |x|$.

terms: absolute value $|x| = \begin{cases} -x & \text{if } x<0 \\ x & \text{if } x \ge 0 \end{cases}$

examples:

x	$\lvert x\rvert$	$-\lvert x\rvert$	$-\lvert x\rvert \le x \le \lvert x\rvert$?	
2	2	-2	$-2 \le 2 \le 2$	T
-2	2	-2	$-2 \le -2 \le 2$	T

<u>Proof</u> There are two cases.

Case 1: $x \ge 0$.

statement	reasoning
$-x \le 0 \le x$	because $x \ge 0$
$-\lvert x\rvert = -x \le 0 \le x = \lvert x\rvert$	because $\lvert x\rvert = x$ when $x \ge 0$
$-\lvert x\rvert \le x \le \lvert x\rvert$	by algebra

Case 2: $x < 0$.

$x \le 0 \le -x$	because $x \le 0$
$-\lvert x\rvert = x \le 0 \le -x = \lvert x\rvert$	because $\lvert x\rvert = -x$ when $x<0$
$-\lvert x\rvert \le x \le \lvert x\rvert$	by algebra

Since every real number is either ≥ 0 or <0, the claim holds. $\boxtimes$

Def A *set* is a collection of distinct, unordered items called *elements*.

ex $D = \{0,1,2,3,\ldots,9\}$ has 10 elements
$\text{bits} = \{0,1\}$ has 2 elements
$\text{Bool} = \{\text{True}, \text{False}\}$ has 2 elements
$\mathbb{Z}$ = integers has ∞ elements
$\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Q}$ = rationals
$\mathbb{R}$ = reals
$V = \{a, e, i, o, u, y\}$ has 6 elts.
$\Sigma = \{a, b, c, \ldots, x, y, z\}$ has 26 elts.

Def Two set A, B are *equal* (denoted $A = B$) iff A and B contain exactly the same elts.

ex. $\{0,1\} = \{1,0\} = \{0,0,1\}$ (but we usually don't write down repeats)

Def We write $x \in S$ ($x \notin S$) iff x is in (not in) S.

ex. $0 \in \text{bits}$ $2 \notin \text{bits}$ $\pi \notin \mathbb{Z}$

Def The *cardinality* or *size* of a set S (denoted by $|S|$) is the number of distinct elements in S.

ex. $|\text{bits}| = 2$ $|\Sigma| = 26$

note: we don't consider infinity to be a number, so we don't write $|\mathbb{Z}| =$ anything. We just say "$\mathbb{Z}$ has infinite cardinality" or similar.

Q Can we have a set S such that (s.t.) $|S| = 0$?

Def The <u>empty set</u>, denoted $\{\}$ or $\emptyset$, is the set with no elements.

$|\emptyset| = 0$.

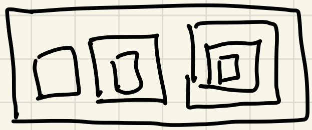
Note $\{\emptyset\} \neq \emptyset \rightarrow$ empty box

↳ box containing an empty box

$|\{\emptyset\}| = 1$

$F = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$ has 3 elements.

F is a box with 3 elements:



1. an empty box
2. a box containing an empty box
3. a box containing a box containing an empty box

Q If $A = B$ does $|A| = |B|$? yes, by substitution

Q Is the converse true? If $|A| = |B|$, does $A = B$?

Disproof by counter example:

Consider $A = \{5\}$, $B = \{c\}$.
$|A| = 1$ and $|B| = 1$. But $A \neq B$.