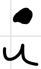
(pronounced "kleek")

<u>Def</u> A <u>complete graph</u> or <u>clique</u> is an undirected graph $G = (V, E)$ s.t.

$$\forall u, v \in V \quad u \neq v \Rightarrow \{u, v\} \in E$$

The clique on n nodes is denoted K_n.

<u>ex</u> K_1 

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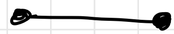
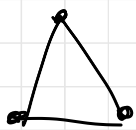
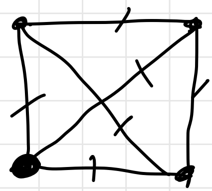
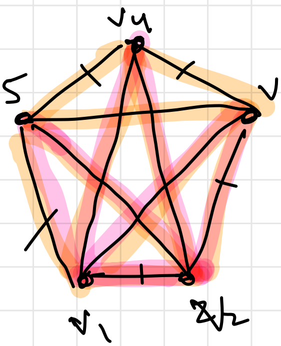
K_2  1

K_3  3

K_4  6

K_5 

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<u>Q</u> What is the relationship btwen $n = |V|$ and $m = |E|$ for K_n?

<u>A</u> # of ways to choose 2 nodes out of n

sum of natural #s less than n

$$\frac{n(n-1)}{2}$$

<u>Claim</u> K_n has $\frac{n(n-1)}{2}$ edges.

<u>Proof # 1</u> We give a way to count the edges and show that it gives $\frac{n(n-1)}{2}$.

label the nodes $v_1, v_2, \ldots, v_n$. Starting w/ v_1, count the uncounted edges and add to the total.

v_1 has $n-1$ uncounted edges
v_2 has $n-2$ uncounted edges
$\vdots$
v_{n-1} has 1 uncounted edge
v_n has 0 uncounted edges

$$|E| = (n-1) + (n-2) + \cdots 1 + 0 = \frac{n(n-1)}{2}$$

<u>Proof #2</u> In K_n every node has deg. $n-1$. So,

$$\sum_{v \in V} \deg(v) = \sum_{v \in V} (n-1) = \underline{n(n-1)}$$

But by the handshaking lemma,

$$\sum_{v \in V} \deg(v) = 2|E|$$

$$n(n-1) = 2|E|$$

$$\frac{n(n-1)}{2} = |E| = m$$

Proof #3 Let $P(n)$ denote that K_n has $\frac{n(n-1)}{2}$ edges. We prove $\forall n \geq 1 : P(n)$ using induction over n.

Base Case: $n=1$.

K_1 • has 0 edges.

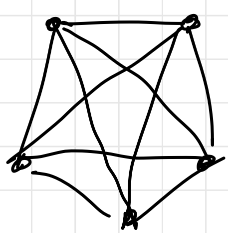
$\frac{1(1-1)}{2} = 0$ so $P(1)$ holds.

Inductive case: we WTS $\forall n \geq 2 : P(n-1) \Rightarrow P(n)$

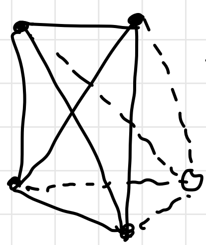
Assume $P(n-1)$. That is, assume K_{n-1} has $\frac{(n-1)(n-2)}{2}$ edges.

Now, consider an arbitrary clique K_n. Let K_n' be the graph created by removing one node and all its incident edges from K_n.

(example: K_5 :



K_5'



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Note that $K_n' = K_{n-1}$

Goal: # edges $K_n = \frac{n(n-1)}{2}$

edges of K_n = # of edges of K_{n-1} + # of edges have to add to K_{n-1} to get K_n

IH

$$= \frac{(n-1)(n-2)}{2} \qquad (n-1)$$

$$= \frac{n^2-3n+2}{2} + \frac{2(n-1)}{2}$$

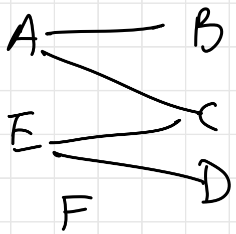
$$= \frac{n^2-3n+2+2n-2}{2}$$

$$= \frac{n^2-n}{2}$$

$$= \frac{n(n-1)}{2}$$

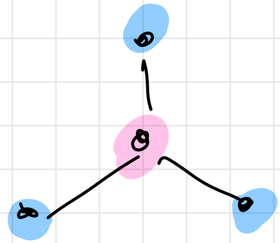
Def A <u>bipartite graph</u> $G = (L \cup R, E)$ s.t. $L \cap R = \emptyset$ (L, R disjoint) and $E \subseteq \{\{l, r\} : l \in L, r \in R\}$

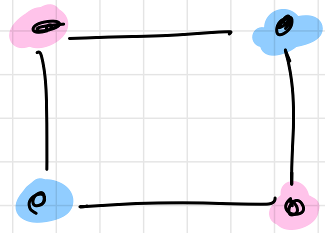
ex

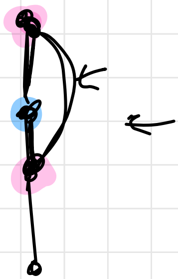


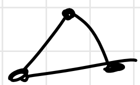
$L = \{A, E\}$

$R = \{B, C, D\}$







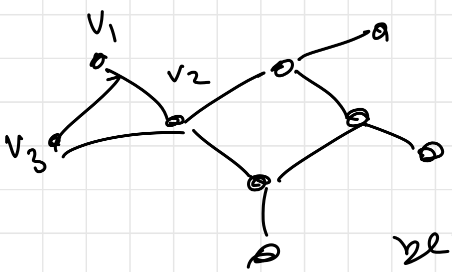


Claim If G contains a Δ (K_3), then it is not bipartite.

Proof Aiming for a contradiction, suppose that G contains a Δ and it is bipartite.

Let v_1, v_2, v_3 be the nodes of the Δ.

(example:



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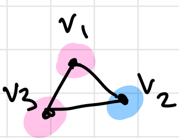

because we would reorder v_1, v_2, v_3

Without loss of generality, suppose $v_1 \in L$ and $v_2 \in R$. Since $v_2 \in R$, $v_3 \in L$. But there is an edge from v_1 to v_3 and both are in R, which contradicts that G is bipartite.

Q other dir?