Last time

An algorithm is efficient if it does qualitatively better than brute force on every input.

2^n or worse , an^b (polynomial) = efficient

Poll:
Does every computational problem have a polynomial time algorithm?

1. yes
2. ~~no~~

432

$P = NP$

$P \neq NP$

Are we satisfied? No

How to define runtime:

1. level of detail

→ big O notation

2. which inputs?

best case ~~X~~ → one bad input = unusable

average case → probability distribution over inputs

worst case → "for all inputs"

How many primitive operations does the algorithm take?

Algorithm 1

```
Input: integer array A of length n
for i = 1, 2, …, n :
    total = 0
    for j = 1, 2, …, n:
        total = total + A[i]
    B[i] = total
```

assign i 1 prim op

assign total 1 p.o.

5 prim ops n times

inner loop

assign j 1 prim op

retrive total, A[i] 2, + 1, assign total 1

retrive total assign B[i] 2 p.o.

2 + 5n + 2 per outer loop

Assume the following

- *1 operation for basic arithmetic operations*
- *1 operation for variable assignment*
- *1 operation for variable retrieval*

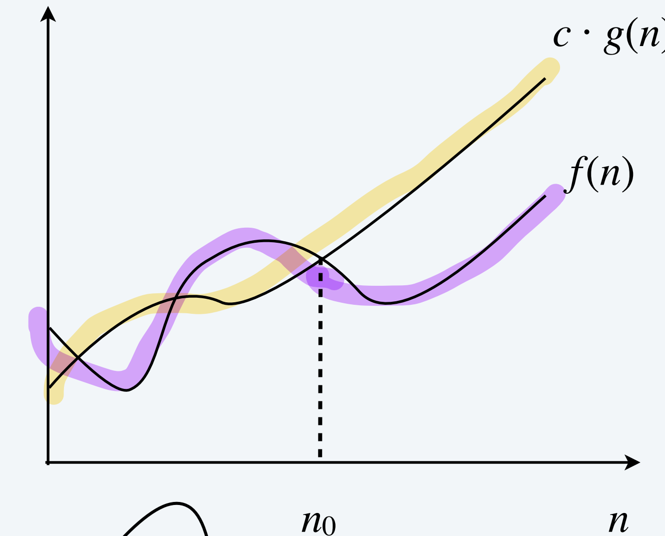
runtime = # primitive ops for input of size n

$T(n) = n(5n+4) = 5n^2 + 4n$

Big O notation

upper bound

Definition of big O: $f(n)$ is $O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.



ex $5n^2+4n$ is $O(n^2)$.

We need to show that there exist c and n_0 s.t.
$5n^2+4 \leq cn^2$ for all $n \geq n_0$.

how about $c=10$ and $n_0=0$?

Show that $5n^2+4n \leq 10n^2$ for $n \geq 0$.

Note that for all $n \geq 0$, $n^2 \geq n$.

So $5n^2+4n \leq 5n^2+4n^2 = 9n^2 \leq 10n^2$

could I have chosen $c=4$, $n_0=0$?

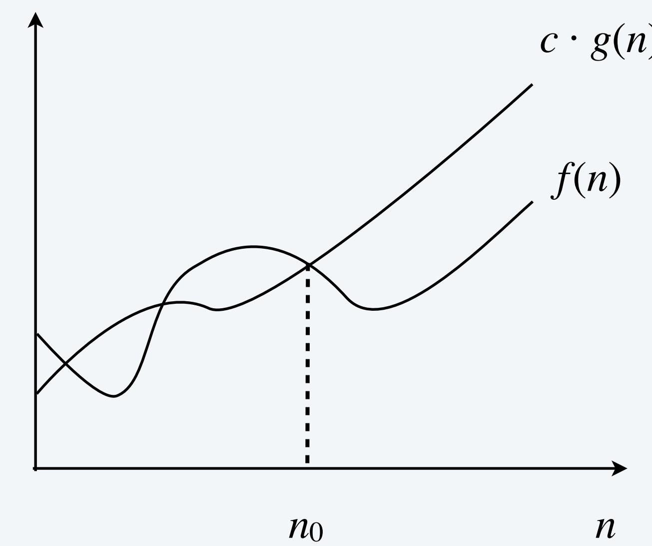
is $5n^2+4n \leq 4n^2$ for all $n \geq 0$?

could I have chosen $c=9$, $n_0=100$?

$5n^2+4n \leq 9n^2$ for all $n \geq 100$.

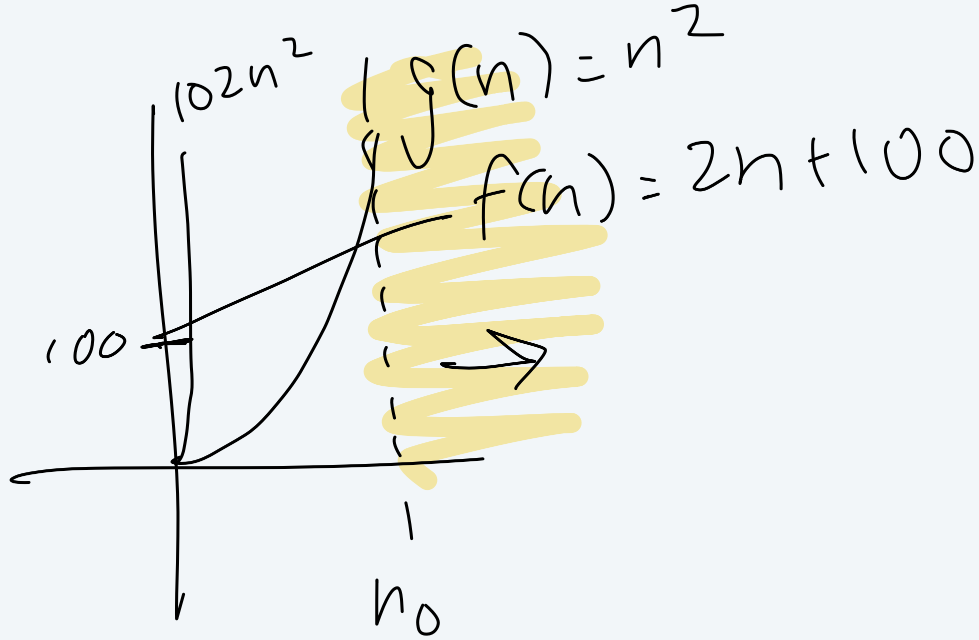
Big O notation: another example

Definition of big O: $f(n)$ is $O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.



Claim: $2n + 100$ is $O(n^2)$.

To show $2n+100$ is $O(n^2)$, we have to show that there exist constants $c>0$ and $n_0 \geq 0$ s.t. $2n+100 \leq cn^2$ for all $n \geq n_0$.



try $c = 102$
$n_0 = 1$

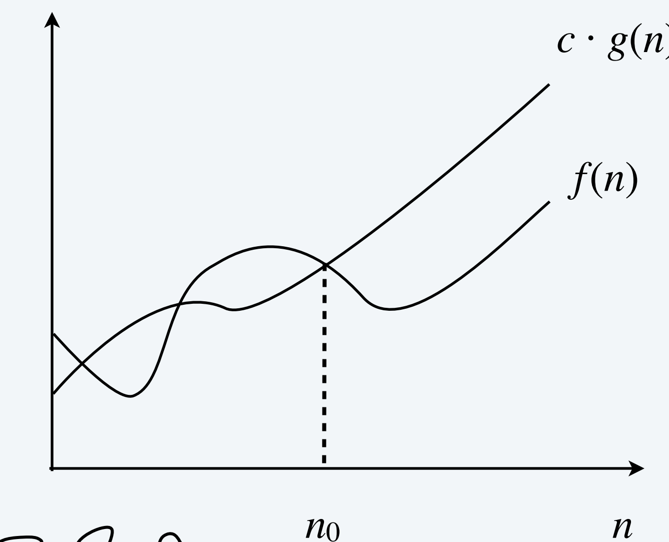
need to show:
$2n+100 \leq 102n^2$
for all $n \geq 1$

$c=1$?
140 400
need $2n+100 \leq n^2$
for all $n \geq 20$

when $n=1$:
$2+100 \leq 102 \cdot 1^2$ ✓

Big O notation: another example $\leq$

Definition of big O: $f(n)$ is $O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.

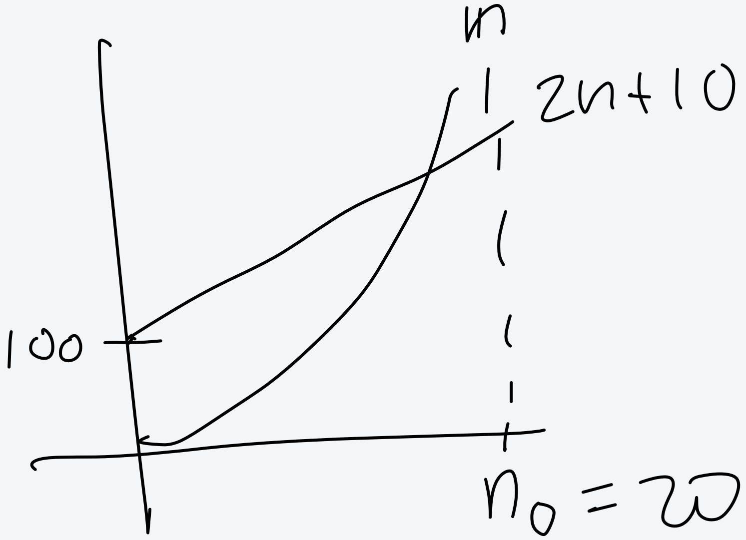


Claim: $2n + 100$ is $O(n^2)$.

Proof:

choose $c = 1$ and $n_0 = 20$. Then, we have $2n + 100 \leq n^2$ for all $n \geq 20$.

see picture.



Claim: 10 is $O(2^n)$.

is $5n+4$ $O(n)$?

how do I prove that

$5n^2+4$ is not $O(n)$?

To prove true: give $c>0$, $n_0 \geq 0$ s.t.

$5n^2+4 \leq cn$ for all $n \geq n_0$.

To prove false ($5n^2+4$ is not $O(n)$):

show that there are no $c>0$, $n_0 \geq 0$

s.t. $5n^2+4 \leq cn$ for all $n \geq n_0$.

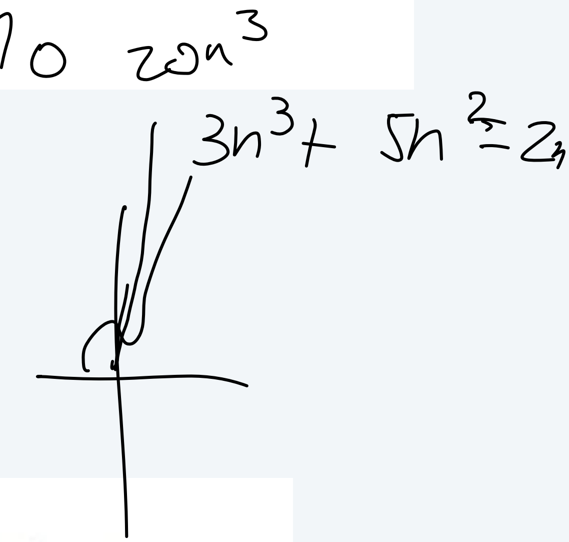
Let's do some examples together

1. Prove that $3n^3 + 5n^2 - 2n = O(n^3)$ using the definition of big O. That is, you must give a c and an n_0 and show that they fit the definition, either by reasoning in words or by drawing a graph.

$3n^3 + 5n^2 - 2n \le cn^3$ for all $n \ge n_0$

$c = 20$, $n_0 = 0$.

$3n^3 + 5n^2 - 2n \le 20n^3$ for all $n \ge 0$.



2. How would you prove that $3n^3 + 5n^2 - 2n \ne O(n^2)$? Don't do it, just tell me how you would.

Def of big O: $f(n)$ is $O(g(n))$ if there exist $c > 0$, $n_0 \ge 0$ s.t. $f(n) \le cg(n)$ for all $n \ge n_0$.

Back at
9:45

1. (8 points) Suppose you have algorithms with the following runtimes. (Assume these are exact running times as a function of the input size n, not a asymptotic running times.) How much slower do these algorithms get when you double the input size?

(a) n

$$\frac{\text{time for double}}{\text{time}} = \frac{(2n)}{n} = 2$$

(b) $n \log n$

$$\frac{(2n \log(2n))}{n \log n} = \frac{2n \log 2 + 2n \log n}{n \log n}$$

$$= \frac{2\cancel{n}\overset{1}{\log 2}}{\cancel{n}\log n} + \frac{2\cancel{n \log n}}{\cancel{n \log n}} = \frac{2}{\log n} + 2$$

$\log_2 2 = 1$

for large n, 2

2. (7 points) Recall the definition of big O:

$f(n)$ is $O(g(n))$ if there exist positive constants n_0, c such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.

$f(n)$ $\leftarrow g(n)$

Using this definition, prove that $2n + 5$ is $O(n^2)$.

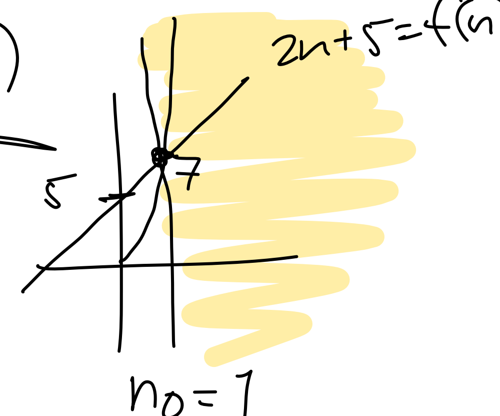
Give a c, n_0 $\quad c = 7, n_0 = 1$

Notice that

$2(1) + 5 = 7$ and $7 \cdot (1)^2 = 7$ $\quad 7n^2$

So we have $f(n_0) \leq c \cdot g(n_0)$

$\forall n \geq n_0$



Consider $c = 7$, $n_0 = 1$.

Notice that $2n + 5 \leq 7 \cdot n^2$

for all $n \geq 1$.

Summary

When we talk about the runtime of an algorithm, we always mean:

(1) there is a $f(n)$ expressing the # of primitive operations an alg. takes on an input of size n (ex $5n\log n + 2n + 25$)

(2) give $g(n)$ such that $f(n)$ is $O(g(n))$

ex (not exactly what we want: $5n\log n + 2n + 25$ is $O(2^n)$)

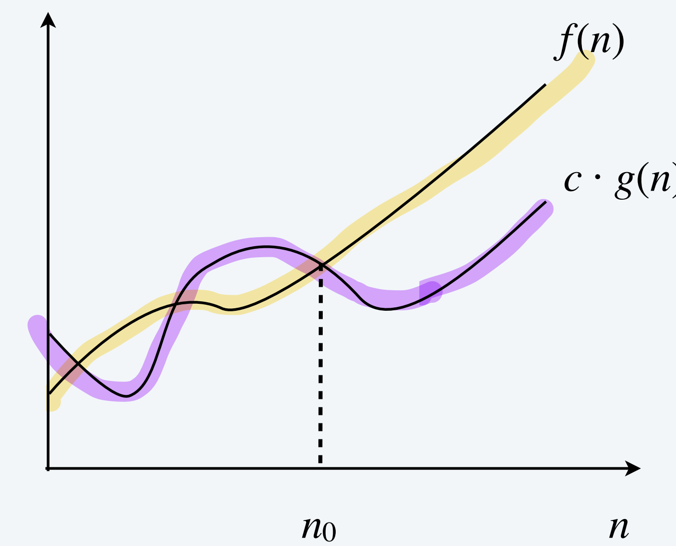
ex $5n\log n + 2n + 25$ is $O(n\log n)$

For $i = 1, 2, \ldots, n$
  For $j = i+1, i+3, \ldots, n$
    Add up array entries $A[i]$ through $A[j]$
    Store the result in $B[i, j]$
  Endfor
Endfor

Ω $\geq$

Big Omega notation

Lower bounds. $f(n)$ is $\Omega(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that $f(n) \geq c \cdot g(n)$ for all $n \geq n_0$.



Ex. $f(n) = 32n^2 + 17n + 1$.

- $f(n)$ is both $\Omega(n^2)$ and $\Omega(n)$.
- $f(n)$ is not $\Omega(n^3)$.

how would I prove $32n^2+17n+1$ is $\Omega(n^2)$?

give c, n_0 s.t. $32n^2+17n+1 \geq cn^2$ for all $n \geq n_0$.

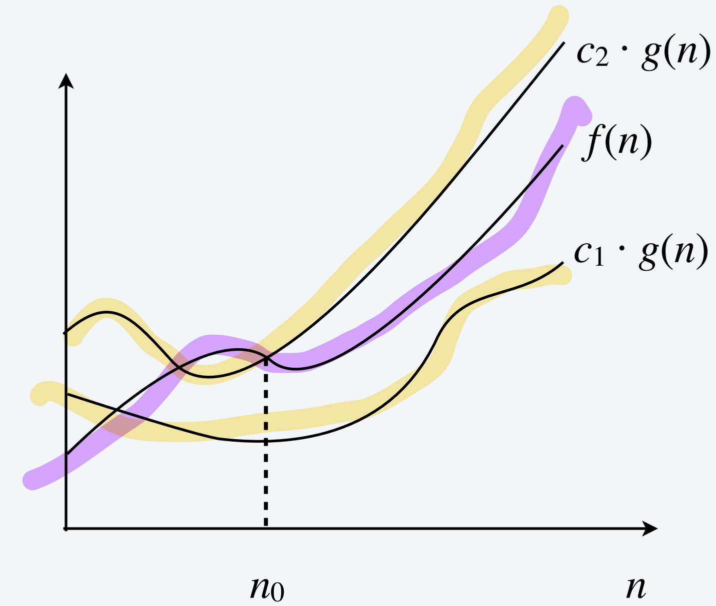
$32n^2+17n+1$ is not $\Omega(n^3)$

to prove:

argue that there are no c, n_0 s.t. $32n^2+17n+1 \geq cn^3$ for all $n \geq n_0$.

Big Theta notation

Tight bounds. $f(n)$ is $\Theta(g(n))$ if there exist constants $c_1 > 0$, $c_2 > 0$, and $n_0 \geq 0$ such that $0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ for all $n \geq n_0$.



Ex. $f(n) = 32n^2 + 17n + 1$.

- $f(n)$ is $\Theta(n^2)$.
- $f(n)$ is neither $\Theta(n)$ nor $\Theta(n^3)$.

f(n) is Θ(g(n)) if
f(n) is O(g(n)) and f(n) is Ω(g(n))

Is $f(n) = 32n^2 + 17n + 5$...

1. $\Theta(n)$
2. $\Theta(n^2)$
3. $\Theta(n^3)$
4. All three
5. $\Theta(n^2)$ and $\Theta(n^3)$

f(n) is $\Theta(g(n))$ if both $\Omega(g(n))$ and $O(g(n))$

$g(n)$	$O(g(n))$? ($\geq$)	$\Omega(g(n))$? ($\leq$)
n	no	yes
n^2	yes	yes
n^3	yes	no

Multiplication by a constant

back at :20

Suppose I have runtime $f(n)$ and I know it is $O(g(n))$.

ex $f(n) = 3n^2$

$g(n) = n^3$

for all b,

$b \cdot 3n^2$ is $O(n^3)$?

Is $b \cdot f(n) = O(g(n))$ for all constants b?

① yes ② no

no? to prove, give a counterexample

$f(n) = 3n^2$

$g(n) = n^3$

$b = 2$

$b3n^2$ not $O(n^3)$.

$6n^2$ not $O(n^3)$. F

yes? there is some c, n_0 s.t. $6n^2 \leq cn^3$ for all $n \geq n_0$

✓

is 2^{n+1} $O(2^n)$?

notice that $2^{n+1} = 2^n 2$ so 2^{n+1} is $O(2^n)$

Is 2^{2n} $O(2^n)$?

hint: $2^{2n} = 2^{n+n} = 2^n 2^n$

$2^n \cdot f(n)$

non-constant

Asymptotically different functions

c — constant

$\log n$ — logarithmic

n — linear

$n \log n$ — linearithmic

n^2 — quadratic

n^3 — cubic

...

n^a — polynomial

2^n

3^n

...

b^n

$n!$

→ exponential
in efficient

$2n$ is not $O(n!)$

where does $\sqrt{n}$ fall ?

$\sqrt{n} = n^{1/2}$

$n^{\log n}$

Summary

Big O is ______ for functions

Big Omega is ________ for functions

Big Theta is _________ for functions

We use Big O/Big Omega/Big Theta in order to:

Is there a worst-case input here? (Or best case?)

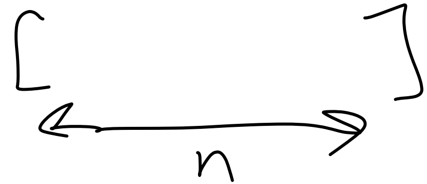
```
Algorithm 1
Input: integer array A of length n
for i = 1, 2, …, n :
    total = 0
    for j = 1, 2, …, n:
        total = total + A[i]
    B[i] = total
```

[1, 2, 3, 4, 5]

[5, 4, 3, 2, 8]

Worst-case: worst case for given n.

insertion sort:

for size n input 

if sorted: $\Theta(n)$

if reverse-sorted: $\Theta(n^2)$

worst-case runtime:

$f(n)$ such that alg takes $f(n)$ steps on worst-case input of size n.

Summary

When we talk about the runtime of an algorithm, we always mean:

Runtime of Gale-Shapley?

```
GALE–SHAPLEY (preference lists for hospitals and students)

INITIALIZE M to empty matching.
WHILE (some hospital h is unmatched and hasn't proposed to every student)
   s ← first student on h's list to whom h has not yet proposed.
   IF (s is unmatched)
      Add h–s to matching M.
   ELSE IF (s prefers h to current partner h′)
      Replace h′–s with h–s in matching M.
   ELSE
      s rejects h.

RETURN stable matching M.
```

hospitals' preference lists

	1st	2nd	3rd	4th	5th
Atlanta	Wayne	Val	Yolanda	Zeus	Xavier
Boston	Yolanda	Wayne	Val	Xavier	Zeus
Chicago	Wayne	Zeus	Xavier	Yolanda	Val
Detroit	Val	Yolanda	Xavier	Wayne	Zeus
El Paso	Wayne	Yolanda	Val	Zeus	Xavier

students' preference lists

	1st	2nd	3rd	4th	5th
Val	El Paso	Atlanta	Boston	Detroit	Chicago
Wayne	Chicago	Boston	Detroit	Atlanta	El Paso
Xavier	Boston	Chicago	Detroit	El Paso	Atlanta
Yolanda	Atlanta	El Paso	Detroit	Chicago	Boston
Zeus	Detroit	Boston	El Paso	Chicago	Atlanta