Randomness + probability uses in CS:

- randomized algorithms
- data structures using randomness
- modeling real-world phenomena

But first, we need to learn to count!

Sum rule: If $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$.

Product rule: The number of pairs (x, y) with $x \in A$, $y \in B$ is $|A| \cdot |B|$.

$$|A \times B| = |A| \cdot |B|$$

ex A restaurant has 2 lunch specials.

1. soup or salad
2. soup and salad

If A = set of soups = {chicken noodle, ...}
B = set of salads = {caesar, ...}

How many possibilities are there for

1: $|A \cup B| = |A| + |B|$

2: $|A \times B| = |A| \cdot |B|$

More general product rule:

$$|A_1 \times A_2 \times A_3 \times \cdots \times A_k| = |A_1| \cdot |A_2| \cdot |A_3| \cdots |A_k|$$

ex How many 32-bit string are there?

$$\underbrace{010\cdots001}_{\text{32-bit string}}$$

2^{32} by the generalized product rule.

$$|\underbrace{\{0,1\}\times\{0,1\}\times\{0,1\}\times\cdots\times\{0,1\}}_{\text{32 times}}|$$

$$=\underbrace{|\{0,1\}|\cdot|\{0,1\}|\cdot|\{0,1\}|\cdots|\{0,1\}|}_{\text{32 times}}=2^{32}$$

ex How many MAC addresses are there?

(you don't need to evaluate the value)

16^{12}

$$\underbrace{12:AC:D9:03:F9:7B}_{\text{12 digits}}$$

$$|\underline{\{0,1,2,\ldots,9,A,B,\ldots,F\}}\times\{0,1,2,\ldots,9,A,B,\ldots,F\}\times\cdots|$$

per pair: 16^2 possibilities.

$$(16^2)^6=16^{12}$$

Inclusion-Exclusion rule:

$|A \cup B| = |A| + |B| - |A \cap B|$ → 0 if empty

ex let $O = \{1,3,5,7,9\}$ and $P = \{2,3,5,7\}$
what is $|O \cup P|$?

$|O \cap P| = |\{3,5,7\}| = 3$

$|O \cup P| = |O| + |P| - |O \cap P| = 5 + 4 - 3 = 6$

(double check: $O \cup P = \{1,2,3,5,7,9\}$)

0123, 7980, 0111

ex How many invalid PINs are there?

hint: it's btwn 150 and 250

let S denote the set of PINs starting w/ 3 repeated digits.

eg 1110 $\quad |S| = 10^2 = 100$
2223
3332

let E denote the set of PINs ending w/ 3 repeated digits.

eg 0111 $\quad |E| = 100$
2333

$S \cap E$: all digits same

$|S \cap E| = 10$

$|S \cup E| = |S| + |E| - |S \cap E| = 100 + 100 - 10$
$= 190$

Def Given some random process, the <u>sample space</u> S is the set of all possible <u>outcomes</u>.

A <u>probability function</u> $Pr : S \to \mathbb{R}$ describes the fraction of the time that $s \in S$ occurs.

$f : A \to B$
$f(a) = b$

$Pr[s]$

$$\sum_{s \in S} Pr[s] = 1 \checkmark$$

$$Pr[s] \geq 0 \quad \forall s \in S \checkmark$$

<u>ex</u>

flipping a fair coin

$S = \{ \text{heads}, \text{tails} \}$

$Pr[\text{heads}] = 0.5 \checkmark$
$Pr[\text{tails}] = 0.5 \checkmark$

$$\sum_{s \in S} Pr[s] = Pr[\text{heads}] + Pr[\text{tails}] = 0.5 + 0.5 = 1 \checkmark$$

drawing a card

$S = \{ 2 \text{ clubs}, 3 \text{ clubs}, \dots \}$

$Pr[s] = \frac{1}{52} \quad \forall s \in S$

flipping 2 fair coins

$S = \{(H,H), (H,T), (T,H), (T,T)\}$

each has $Pr[s] = 0.25$