

Recursively Defined Structures

A string is either: happy

- nothing
- a symbol followed by a string
 \nwarrow character
 element of alphabet Σ ,
 some non-empty set

A string is either:

- \rightarrow • ϵ (empty string)
- \rightarrow • $a \cdot x$ where $a \in \Sigma$ and x is a string

$$\begin{aligned}\text{happy} &= h \cdot \text{appy} \\ &= h \cdot \underline{a} \cdot \text{ppy} \\ &= h \cdot a \cdot p \cdot \text{py} \\ &= h \cdot a \cdot p \cdot p \cdot y \\ &= h \cdot a \cdot p \cdot p \cdot y \cdot \epsilon\end{aligned}$$

happy ϵ

Σ^* = set of all strings over Σ

happy $\in \{a, b, c, \dots, x, y, z\}^*$

0110 $\in \{0, 1\}^*$

notice: Σ^* is infinite as long as $|\Sigma| \geq 1$

however, $x \in \Sigma^*$ is finite

length function — " # of symbols "

let w be a string.

$$|w| := \begin{cases} 0 & \text{if } w = \epsilon \\ 1 + |x| & \text{if } w = ax \end{cases}$$

$$|\text{happy}| = 1 + |\text{appy}|$$

$$= 1 + 1 + |\text{ppy}|$$

:

$$= 1 + 1 + 1 + 1 + 1 + |\epsilon|$$

$$= 5 + 0$$

$$= 5$$

Concatenation— "paste one after another"

let w, z be strings.

$$w \bullet z := \begin{cases} z & \text{if } w = \varepsilon \\ a(x \bullet z) & \text{if } w = ax \end{cases}$$

$$\begin{aligned} \text{foot} \bullet \text{ball} &= f(\text{oot} \bullet \text{ball}) \\ &= f(o(\text{of} \bullet \text{ball})) \\ &= f(o(o(t \bullet \text{ball}))) \\ &= f(o(o(t(\varepsilon \bullet \text{ball})))) \\ &= f \cdot o \cdot o \cdot t \text{ ball} \end{aligned}$$

Theorem for any string w , $w \bullet \varepsilon = w$.

Proof

let w be an arbitrary string.

Assume that for all strings x smaller than w , $x \bullet \varepsilon = x$. (inductive hypothesis)

Case 1: $w = \varepsilon$.

$$w \bullet \varepsilon = \varepsilon \bullet \varepsilon$$

$$= \varepsilon$$

$$= w$$

because $w = \varepsilon$

by def. of \bullet

by $w = \varepsilon$.

case 2: $w = ax$.

$$\begin{aligned} w \cdot \varepsilon &= a(x \cdot \varepsilon) \\ &= ax \\ &= w \end{aligned}$$

by def. of \cdot
by IH
by $w = ax$

Therefore, $w \cdot \varepsilon = w$.

Theorem for all strings w and z ,

$$|w \cdot z| = |w| + |z|.$$

Proof Let w, z be arbitrary strings.

Assume that for all x smaller than w ,
 $|x \cdot z| = |x| + |z|$.

Case 1: $w = \varepsilon$

$$\begin{aligned} |w \cdot z| &= |\varepsilon \cdot z| \\ &= |z| \\ &= 0 + |z| \\ &= |\varepsilon| + |z| \\ &= |w| + |z| \end{aligned}$$

because $w = \varepsilon$
by def. of \cdot
math
by def. of $||$
because $w = \varepsilon$

Case 2: $w = ax$

Therefore, $|w \cdot z| = |w| + |z|$.