$P(x)$ $\quad x \in \mathcal{U}$ $\quad$ isEven(x)

$\swarrow \quad \searrow$ $\qquad\qquad$ $x \in \mathbb{Z}$

T $\quad$ F

Quantifiers

$\forall$ for all, universal quantifier

$\forall x \in S : P(x)$ "for all x in S, P(x) is true"

T iff P(x) is T for every $x \in S$

$\exists$ there exists, existential quantifier

$\exists x \in S : P(x)$ "there exists x in S s.t. P(x) is true"

T iff P(x) is T for some (≥ 1) $x \in S$.

<u>Def</u> A fully quantified expression in predicate logic is a <u>theorem</u> iff it is T for every possible meaning of its predicates (akin to a tautology)

<u>Thm</u> (3.39) Let S be any set. $\forall S$ $\forall x \in S : [P(x) \vee \neg P(x)]$

implied $\forall P$

<u>ex</u> $P(x) = $ isEven(x), $S = \mathbb{Z}$.

$\forall x \in \mathbb{Z} : [\text{isEven}(x) \vee \neg \text{isEven}(x)]$

Pf For any $x \in S$, $P(x)$ is defined, and $P(x) = T$ or $P(x) = F$ — def. of predicate

For any $x \in S$, $P(x) \vee \neg P(x)$ — def of $\vee, \neg$

$\forall x \in S : [P(x) \vee \neg P(x)]$ — def of $\forall$

Non-Thm (3.40)

$$\left[[\forall x \in S : P(x)] \vee [\forall x \in S : \neg P(x)] \right]$$

note: implied $\forall S, \forall P$

disproof

$$[\underbrace{\forall x \in \mathbb{Z} : isEven(x)}_{\text{consider } x=3}] \vee [\underbrace{\forall x \in \mathbb{Z} : \neg isEven(x)}_{\text{consider } x=2}]$$

Def Fully quantified expressions φ and ψ are logically equivalent ($\varphi \equiv \psi$, $\varphi \Leftrightarrow \psi$) iff "$\varphi \Leftrightarrow \psi$" is a theorem – that is, they have the same meaning under every interpretation of predicates.

Thm (3.41) $\neg [\forall x \in S : P(x)] \Leftrightarrow [\exists x \in S : \neg P(x)]$

ex to disprove $\forall x \in \mathbb{Z} : isEven(x)$, we found $x \in S : \neg isEven(x)$ ($x = 3$)

This theorem explains why disproof by counterexample works!

Intuition behind proof:

let $S = \{s_1, s_2, s_3, \dots\}$. Then:

$\neg[\forall x \in S : P(x)]$ given

$\cong \neg[P(s_1) \wedge P(s_2) \wedge P(s_3) \dots]$ def of $\forall$

$\cong \neg P(s_1) \vee \neg P(s_2) \vee \neg P(s_3) \vee \dots$ de Morgan's Law

$\cong \exists x \in S : \neg P(x)$ def of $\exists$ $\square$

suppose $S = \emptyset$. $P(x)$ is generic.

$\neg[\forall x \in S : P(x)]$ $[\exists x \in S : \neg P(x)]$

F F

Thm (3.42) $\neg[\exists x \in S : Q(x)] \Leftrightarrow [\forall x \in S : \neg Q(x)]$

Pf let $P(x) = \neg Q(x)$.

$\neg[\forall x \in S : P(x)] \Leftrightarrow [\exists x \in S : \neg P(x)]$ 3.41

$\forall x \in S : P(x) \Leftrightarrow \neg[\exists x \in S : \neg P(x)]$ negating both sides

$\forall x \in S : \neg Q(x) \Leftrightarrow \neg[\exists x \in S : Q(x)]$ subs. $\square$

ex $\neg(\exists x \in \mathbb{R} : x^2 + 1 = 0) \equiv \forall x \in S : x^2 + 1 \neq 0$

Thm (3.43) For $S \neq \emptyset$, $[\forall x \in S : P(x)] \Rightarrow [\exists x \in S : P(x)]$

"if it's true for all, it's true for one"

"if everybody's doing it, then somebody's doing it" ex $\underline{\forall x \in \mathbb{Z} : isEven(2x)} \Rightarrow \underline{\exists x \in \mathbb{Z} : isEven(2x)}$

Pf (direct)

Assume $\forall x \in S : P(x)$. WTS $\exists x \in S : P(x)$.