Discrete Structures (CSCI 246)

Homework 2

Purpose & Goals

The following problems provide practice relating to:

- direct proofs, proof by cases, and proofs by counter-example,
- mathematical definitions (rational, absolute value, divisibility, sets, etc.), and
- the problem solving process.

Submission Requirements

- **Type or clearly hand-write your solutions** into a pdf format so that they are legible and professional. Submit your pdf to Gradescope. **Illegible, non-pdf, or emailed solutions will not be graded.**
- Each problem should start on a new page of the document. When you submit to Gradescope, associate each page of your submission with the correct problem number. Please post in Discord if you are having any trouble using Gradescope.
- Try to model your formatting off of the proofs from lecture and/or the textbook.
- Submit to Gradescope early and often so that last-minute technical problems don't cause you any issues. Only the latest submission is kept. Per the syllabus, assignments submitted within 24 hours of the due date will receive a 25% penalty and assignments submitted within 48 hours will receive a 50% penalty. After that, no points are possible.

Academic Integrity

- You may work with your peers, but **you must construct your solutions in your own words on your own.**
- Do not search the web for solutions or hints, post the problem set, or otherwise violate the course collaboration policy or the MSU student code of conduct.
- Violations (regardless of intent) will be reported to the Dean of Students, per the MSU student code of conduct, and you will receive a 0 on the assignment.

Tips

- Answer each problem to the best of your ability. Partial credit is your friend!
- Read the hints for where to find relevant examples for each problem.
- Refer to the problem solving and homework tips guide.
- It is not a badge of honor to say that you spent 5 hours on a single problem or 15 hours on a single assignment. Use your time wisely and get help (see "How to Get Help" below).

How to Get Help

When you are stuck and need a little or big push, **please ask for help!**

- Timebox your effort for each problem so that you don't spend your life on the problem sets. (See the problem solving tips guide for how to do this effectively.)
- Post in Discord. If you're following the timebox guide, you can post the exact statement that you produced after spending 20 minutes being stuck.
- Come to office hours or visit the CS Student Success Center.

Problem 1 (12 points)

(a) (4 points) Construct a truth table for $(p \wedge q) \Rightarrow p$.

Grading Notes. This rubric is straightforward.

(1) You need one row per truth assignment.

(1) You need at least a column for the final answer. You should have at least one scratch work column in case you get something wrong and want partial credit.

(2) Your final column is correct. If you didn't show any work and you get this wrong, you will lose these points.

(b) (2 points) Is $(p \wedge q) \Rightarrow p$ a tautology? Why or why not?

Grading Notes. This rubric is straightforward.

(1) You correctly state whether this is a tautology.

(1) You correctly explain your choice.

(c) (4 points) Construct a truth table for $(\neg(p \wedge (\neg q))) \Rightarrow (p \vee q)$. Show at least 3 columns of scratch work (your choice).

Grading Notes. This rubric is straightforward.

(1) You need one row per truth assignment.

(1) You need at least a column for the final answer. You should have at least one scratch work column in case you get something wrong and want partial credit.

(2) Your final column is correct. If you didn't show any work and you get this wrong, you will lose these points.

(d) (2 points) Is $(\neg(p \wedge (\neg q))) \Rightarrow (p \vee q)$ a tautology? Why or why not?

Grading Notes. This rubric is straightforward.

(1) You correctly state whether this is a tautology.

(1) You correctly explain your choice.

Problem 2 (12 points)

Let A and B be sets.

(a) (9 points) Prove that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

Note. Recall that $\mathcal{P}(A)$ denotes the power set of A, and is defined as the set of all subsets of A. That is, $\mathcal{P}(A) = \{X : X \subseteq A\}$.

Hint. We saw a similar proof about power sets in the lectures about sets.

Grading Notes. While a detailed rubric cannot be provided in advance as it gives away the solution details, the following is a general idea of how points are distributed for this problem. We give partial credit where we can.

- (7) **Correctness.** If your proof is not correct, this is where you'll get docked.
  - (5) Regardless of how you formulate your proof, somewhere you'll need certain facts without which the proof wouldn't work. E.g., if it weren't true that the sum of two integers is an integer, would your proof fail? If so, then that is a fact I need to see stated somewhere.
  - (2) The order of these facts must make sense, so that you're not inferring something before you have all the facts to infer it. E.g., you cannot use the fact that the sum of two integers is integer if you don't already know that you have two integers to begin with.
- (2) **Communication.** We need to see a mix of notation and intuition, preferably in the "column" format with the statements in mathematical notation on the left and the reasons on the right. If you skip too many steps at once, or we cannot follow your proof, or if your proof is overly wordy or confusing, this is where you'll get docked.

(b) (3 points) Is it true that $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$?

Hint: Remember that two sets C and D are equal if and only if we have containment in both directions; that is, if $C \subseteq D$ and $D \subseteq C$. Since we already proved that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$, this question is really asking whether containment is true in the other direction. That is, it is asking whether it is true that $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.

Grading Notes. While a detailed rubric cannot be provided in advance as it gives away the solution details, the following is a general idea of how points are distributed for this problem.

- (1) Correctly decide if the statement is true or false.
- (3) Correctly prove your claim.
  - – A proof requires clearly stated facts and explanations in the column format with notation on the left and explanation on the right.
  - – A disproof requires a clearly stated counterexample along with an explanation of why the counterexample is a counterexample.

Note that if you incorrectly think the statement is false when it is true or vice-versa, partial credit will be sparse. Try to check that you have the right claim before proceeding too far.

Problem 3 (15 points)

(a) (2 points) Recall that for sets A and B, $A \subseteq B$ is equivalent to the statement if $x \in A$, then $x \in B$.

What is the contrapositive of this statement (written as an if-then statement)?

Hint: We discuss the contrapositive in the proof by contrapositive lecture.

Grading Notes. For this problem, you get 2 points if you correctly state the contrapositive in the requested format.

(b) (13 points) Let S, T, and W be sets such that $S \cap T \subseteq W$ and suppose that $t \in T$. Use a **proof by contradiction** to show that $t \in \overline{S \setminus W}$.

Hint: Since this claim involves relationships amongst sets, I recommend drawing a Venn diagram to help visualize the claim.

Hint: Double check that you correctly negate the necessary propositions before you attempt the proof, and ensure that you are doing a proof by contradiction. You can see examples in the proof by contradiction lecture.

Grading Notes. While a detailed rubric cannot be provided in advance as it gives away the solution details, the following is a general idea of how points are distributed for this problem.

(11) **Correctness.** If your proof is not correct, this is where you'll get docked.

- (7) Regardless of how you formulate your proof, somewhere you'll need certain facts without which the proof wouldn't work. E.g., if it weren't true that the sum of two integers is an integer, would your proof fail? If so, then that is a fact I need to see stated somewhere.
- (1) The order of these facts must make sense, so that you're not inferring something before you have all the facts to infer it. E.g., you cannot use the fact that the sum of two integers is integer if you don't already know that you have two integers to begin with.
- (3) You must use a proof by contradiction, which clearly states that it is a proof by contradiction, states what the contradictory assumption is, finds a contradiction, and clearly states what and where that contradiction is.

(2) **Communication.** We need to see a mix of notation and intuition, preferably in the "column" format with the statements in mathematical notation on the left and the reasons on the right. If you skip too many steps at once, or we cannot follow your proof, or if your proof is overly wordy or confusing, this is where you'll get docked.

Problem 4 (12 points)

Consider the following claim:
Claim. Let n and c be positive integers. If c divides n^2, then c divides n.

(a) (3 points) The following proof by contrapositive of the claim is faulty:

Bogus proof that if c divides n^2 then c divides n:

We want to prove that if c divides n^2, then c divides n. We prove the contrapositive of this statement. That is, **if c does not divide n, then c does not divide n^2**. Suppose that c does not divide n. Then

(1)	$n = ck + d$ for some $k \in \mathbb{Z}, d \in \{1, 2, \ldots, c-1\}$	by def. of not divisible by c
(2)	$n^2 = (ck+d)^2$ for some $k \in \mathbb{Z}, d \in \{1, 2, \ldots, c-1\}$	by algebra
(3)	$n^2 = c^2k^2 + 2ckd + d^2$ for some $k \in \mathbb{Z}, d \in \{1, 2, \ldots, c-1\}$	by algebra
(4)	$n^2 = c(ck^2 + 2kd) + d^2$ for some $k \in \mathbb{Z}, d \in \{1, 2, \ldots, c-1\}$	by algebra
(5)	c does not divide n^2	since $d^2 \geq 1$

What is the first line (1-5) of the proof that contains a logic error and what is that error? Give a counter example to demonstrate the error.

Grading Notes. This rubric is straightforward.

(1) Correctly state the faulty line.
(2) Correctly explain why the line is faulty by providing an example (and explanation) for why it is faulty.

(b) (9 points) Consider the following modified claim:

Modified claim. Let n and c be positive integers such that c is not a factor of any perfect square (strictly) less than c^2. If c divides n^2 then c divides n.

Prove the modified claim **by contrapositive** by fixing (and adding to) the proof above.

Note: A perfect square is an integer m such that $m = p^2$ for some integer p.

Grading Notes. While a detailed rubric cannot be provided in advance as it gives away the solution details, the following is a general idea of how points are distributed for this problem. We give partial credit where we can.

(7) **Correctness.** If your proof is not correct, this is where you'll get docked.
   (5) Regardless of how you formulate your proof, somewhere you'll need certain facts without which the proof wouldn't work. E.g., if it weren't true that the sum of two integers is an integer, would your proof fail? If so, then that is a fact I need to see stated somewhere.
   (1) The order of these facts must make sense, so that you're not inferring something before you have all the facts to infer it. E.g., you cannot use the fact that the sum of two integers is integer if you don't already know that you have two integers to begin with.
   (1) You must use a proof by contrapositive, which clearly states that it is a proof by contrapositive, states what the contrapositive is, then proceeds to prove that contrapositive statement. Note that a proof by contradiction is not a proof by contrapositive.

(2) **Communication.** We need to see a mix of notation and intuition, preferably in the "column" format with the statements in mathematical notation on the left and the reasons on the right. If you skip too many steps at once, or we cannot follow your proof, or if your proof is overly wordy or confusing, this is where you'll get docked.