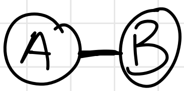
Intro to Graphs

Def An undirected graph $G = (V, E)$ is a non-empty set of vertices (nodes) V and a set $E = \{ \{u, v\} : u, v \in V \}$ of edges joining pairs of nodes.

ex

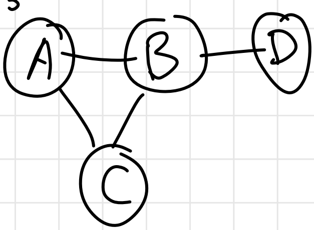
$V = \{A\}$
$E = \emptyset$



$V = \{A, B\}$

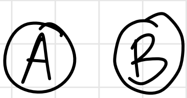
$E = \{ \{A, B\} \}$

G_3:



$V = \{A, B, C, D\}$

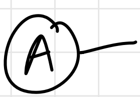
$E = \{ \{A, B\}, \{B, D\}, \{B, C\}, \{A, C\} \}$



$V = \{A, B\}$

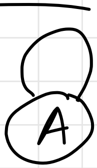
$E = \emptyset$

non-ex



all edges need 2 endpoints

Q



is this a graph? yes. $V = \{A\}$
$E = \{ \{A, A\} \}$
$= \{ \{A\} \}$

real-world examples

- Facebook friends

alice — bob

catherine

nodes: people
edge: 2 people are Facebook friends

- blood relationships

<u>Q</u> what property (or properties) would a mathematical relation need to have to be represented as an undirected graph?

ideas: symmetric $a \rightleftarrows b$ $\quad a - b$

reflexive a

self loops are equivalent when directed or undirected

$c \quad c$

<u>Def</u> A <u>directed graph</u> $G = (V, E)$ has a set of vertices V and edges $E \subseteq V \times V = \{(u,v) : u, v \in V\}$ so that edges are directed from one vertex to another.

Note: relations on a single set and directed graphs are the same!

ex (A)

$A \to B$ $\quad V = \{A, B\}$ $\quad E = \{(A,B)\}$

$\neq$

$A \leftarrow B$ $\quad V = \{A, B\}$ $\quad E = \{(B,A)\}$

ordered pair
tuple
list
array

undirected:

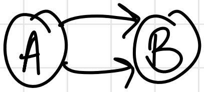
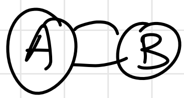
$A - B$ $\quad E = \{\{A,B\}\}$

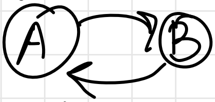
set

real-world example

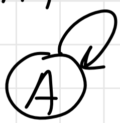
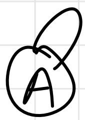
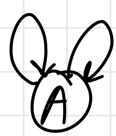
Twitter followers

Def A graph is simple if it contains no parallel edges or self-loops.

parallel edges:  

note that  has no parallel edges

$(A,B) \neq (B,A)$

self-loops:   

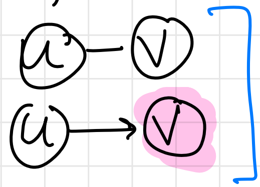
Example 11.3: Self-loops and parallel edges.

Suppose that we construct a graph to model each of the following phenomena. In which settings do self-loops or parallel edges make sense?

1. A social network: nodes correspond to people; (undirected) edges represent friendships.
2. The web: nodes correspond to web pages; (directed) edges represent links.
3. The flight network for a commercial airline: nodes correspond to airports; (directed) edges denote flights scheduled by the airline in the next month.
4. The email network at a college: nodes correspond to students; there is a (directed) edge $\langle u, v \rangle$ if u has sent at least one email to v within the last year.

	self-loops	parallel edges
social network	no	no
the web	yes	yes
flight network	no	yes
email network	yes	no

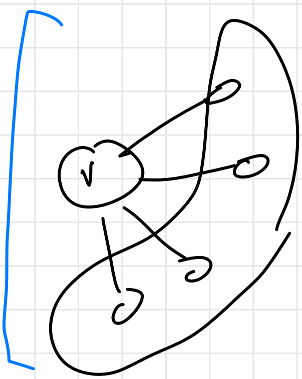
Def Let $e = \{u, v\}$ or (u, v)



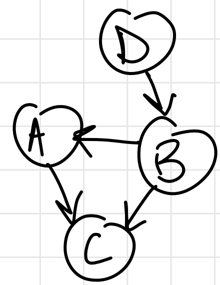
- nodes u, v are <u>adjacent</u> or <u>neighbors</u>
- in a directed graph, v is an <u>out-neighbor</u> of u and u is an <u>in-neighbor</u> of v
- u, v are <u>endpoints</u> of e
- u, v are <u>incident</u> to e

Let v be a node in a simple undirected graph.

$$\text{degree}(v) = \deg(v) = d(v) = \#\text{ of neighbors of } v$$
$$= \left| \{ u \in V : \underbrace{\{v, u\}}_{\text{or } \{u,v\}} \in E \} \right|$$



$\deg(v) = 4$



for directed graphs, $\text{indeg}(v) = \#$ of in-neighbs of v

$\text{outdeg}(v) = \#$ of out-neighbors of v

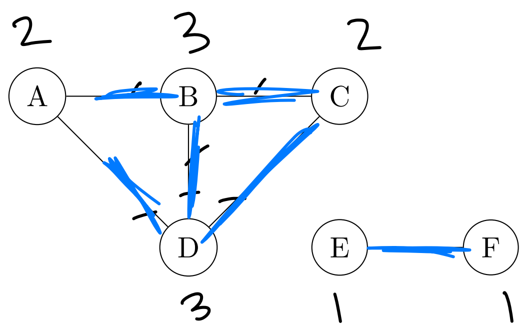
Proofs about graphs

Discrete Structures (CSCI 246)

in-class activity

Names: ____

1. For each of the two graphs, label each node v with $deg(v)$, and give $\sum_{v \in V} deg(v)$, the total degree of the graph, and $|E|$, the number of edges in the graph.

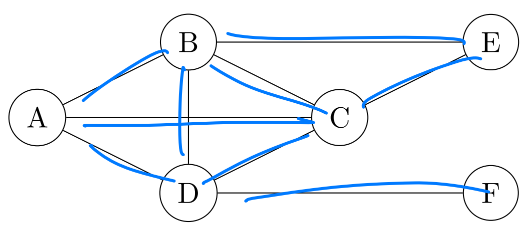


$$\sum_{v \in V} deg(v) = 2+3+2+3+1+1 = 12$$

$$|E| = 6$$

$$|E| = \frac{1}{2} \sum_{v \in V} deg(v)$$

$$2|E| = \sum_{v \in V} deg(v)$$



9 edges

total degree: 18

2. Can you give a conjecture about the relationship between $\sum_{v \in V} deg(v)$ and and $|E|$?

Theorem 11.8 "Handshaking Lemma"

Let $G = (V, E)$ be a simple undirected graph. Then

$$\sum_{v \in V} deg(v) = 2|E|.$$

Proof Let $G = (V,E)$ be an undirected graph. Notice that every edge is connected to exactly 2 nodes, meaning that it contributes 1 to the degree of 2 nodes.

So $\sum_{v \in V} deg(v) = 2|E|$.

Corollary

fact that follows simply from a previous theorem/lemma

Let n_{odd} denote the number of nodes whose degree is odd. Then n_{odd} is even.

Proof Aiming for a contradiction, suppose n_{odd} is odd.

Note that

$$\underbrace{\sum_{v \in V} deg(v)}_{\substack{\text{this is } 2|E|, \\ \text{which is even}}} = \underbrace{\sum_{\substack{v \in V: \\ deg(v) \text{ is odd}}} deg(v)}_{\substack{\text{this must be odd, because} \\ \text{sum of odd \# of odds is odd}}} + \underbrace{\sum_{\substack{v \in V: \\ deg(v) \text{ is even}}} deg(v)}_{\substack{\text{this must be even,} \\ \text{because sum evens is even}}}$$

even = odd + even, ↑

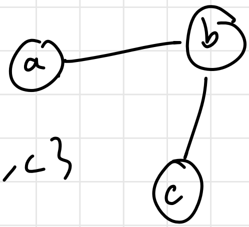
a contradiction!

So n_{odd} must be even. $\square$

"kleek"

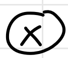
Def A complete graph or clique is an undirected graph $G=(V,E)$ s.t.

$$\forall u,v \in V: \quad u \neq v \implies \{u,v\} \in E$$

is  a clique?

$V=\{a,b,c\}$

No. Consider nodes a, c. $a \neq c$, but $\{a,c\} \in E$

is  a clique?

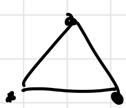
yes.

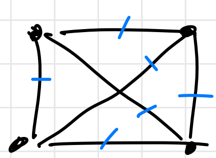
The clique on n nodes is denoted K_n.

examples:

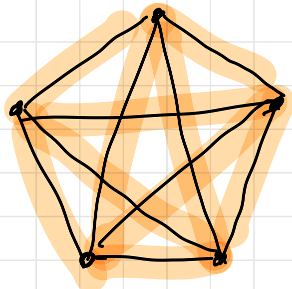
K_1  0

K_2  1

K_3  3

K_4  6

K_5



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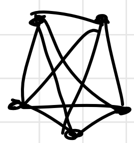
Q What is the relationship between $n = |V|$ and $m = |E|$ for K_n?

Conjectures:

$m = (n-1)!$? nope.

☆ $m = \sum_{i=1}^{n} (i-1) = 0 + 1 + 2 + 3 + \dots + (n-1)$

n	$\sum_{i=1}^{n}(i-1)$	m	K_n
1	$(1-1) = 0$	0	
2	$(1-1)+(2-1) = 1$	1	
3	$(1-1)+(2-1)+(3-1) = 0+1+2 = 3$	3	
5	$0+1+2+3+4 = 10$	10	

recall: $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

so $\sum_{i=1}^{n} (i-1) = \frac{n(n-1)}{2} = m$, the # edges in K_n.

<u>claim</u> K_n has $\frac{n(n-1)}{2}$ edges.

Proof #1 We give a way to count the edges and show that it gives $\frac{n(n-1)}{2}$.

Suppose we have a complete graph K_n. Label its nodes $v_1, v_2, \dots, v_n$. Starting with v_1, count the uncounted edges adjacent to v_1 and add the count to the total.

v_1 has $n-1$ uncounted edges
v_2 has $n-2$ uncounted edges
$\vdots$
v_{n-1} has 1 uncounted edge
v_n has 0 uncounted edges

$$m = |E| = 0 + 1 + 2 + \cdots + n-1 = \frac{n(n-1)}{2} \qquad \square$$

Proof #2 Let K_n be the complete graph on n nodes.

Note that every node has degree $n-1$.

$$\sum_{v \in V} \deg(v) = \sum_{v \in V} (n-1) = n(n-1)$$

But by the handshaking lemma,

$$\sum_{v \in V} \deg(v) = 2|E|.$$

$$n(n-1) = 2|E|$$

$$\frac{n(n-1)}{2} = |E| = m$$

$\square$

Proof #3 Let $P(n)$ denote that K_n has $\frac{n(n-1)}{2}$ edges.

We prove $\forall n \geq 1 : P(n)$ using induction over n.

Base case: $P(1)$ is true. That is, K_1 has $\frac{1(1-1)}{2} = 0$ edges. Yes, this is true!

Inductive case: We WTS $\forall n \geq 2 : P(n-1) \Rightarrow P(n)$

Assume $P(n-1)$. That is, assume

$$K_{n-1} \text{ has } \frac{(n-1)(n-1-1)}{2} = \frac{(n-1)(n-2)}{2} \text{ edges.}$$

Now, consider an arbitrary clique K_n. Let K_n' be the graph created by removing one node and all its edges. Note that $K_n' = K_{n-1}$.

Goal: # edges of $K_n = \frac{n(n-1)}{2}$.

edges of K_n = # of edges K_{n-1} + # of edges we have to add to K_{n-1} to get K_n

$$= \frac{(n-1)(n-2)}{2} + n-1$$

$$= \frac{n^2-3n+2}{2} \qquad \frac{2(n-1)}{2}$$

$$= \frac{n^2 - 3n + \cancel{2} + 2n - \cancel{2}}{2}$$

$$= \frac{n^2 - n}{2} = \frac{n(n-1)}{2}$$

We've proved the inductive case.

$$\underline{6 = 5}$$

$6 + x = 5 + x$ add x to both sides

$6 + x + y = 5 + x + y$ add y to both sides

$6 + y = 5 + y$ subtract x from both sides

$\underline{6 = 5}$ subtract y from both sides

$$\text{WTS} \sum_{i=0}^{n+1} i^2 = \frac{(n+1)((n+1)+1)(2(n+1))}{6}.$$

subs w/ ~~IH~~

$$\sum_{i=0}^{n+1} i^2 = \underbrace{\sum_{i=0}^{n} i^2}_{} + (n+1)^2$$

$$= \frac{n(n+1)(2n+1)}{6}$$