So far:

whether something occurs

Now: how many?

ex how many times do we have to flip a coin to get 100 heads?

In a randomly sorted array, ~~for~~ how many slots is $A[i] < A[i+1]$?

def random variable

A random variable X assigns a numerical value to every outcome in the sample space S.

note: random variable is a bad name for this!

$X : S \to \mathbb{R}$

ex Suppose we flip a fair coin 3 times.

$S = \{H, T\}^3 = \{HHH, HHT, \dots\}$

$\Pr[x] = \frac{1}{8} \quad \forall x \in S$

let X = # heads
Y = # of consecutive T (from start)

$X(TTT) = 0$ $\quad$ $X(THH) = 2$
$Y(TTT) = 3$ $\quad$ $Y(THH) = 1$

ex Let S be the set of all English words.

Let $L :=$ # letters

$L(\text{computer}) = 8$

Def The expectation of a random variable X, denoted $E[X]$, is the average value of X.

$$E[X] = \sum_{x \in S} X(x) \cdot Pr[x].$$

$$E[X] = \sum_{\substack{y : \exists x \in S : \\ X(x) = y}} y \cdot Pr[X = y]$$

ex Counting heads in 3 coin flips

$$E[X] = \sum_{x \in S} X(x) \cdot Pr[x].$$

$$= X(HHH)\, Pr[HHH] + X(HHT)\, \overbrace{Pr[HHT]}^{\text{always } 1/8}$$
$$+ \cdots$$

$$= \frac{1}{8}\Big(X(HHH) + X(HHT) + X(HTH) + X(HTT)$$
$$+ X(THH) + X(THT) + X(TTH) + X(TTT)\Big)$$

$$= \frac{1}{8}(3 + 2 + 2 + 1 + 2 + 1 + 1 + 0)$$

$$= \frac{1}{8}(12) = \frac{12}{8} = 1.5$$

$$E[X] = \sum_{y \in \{0,1,2,3\}} y \cdot \Pr[X(x) = y]$$

$$= 0 \cdot \Pr[0 \text{ heads}] + 1 \cdot \Pr[1 \text{ head}] + 2 \cdot \Pr[2 \text{ heads}] + 3 \cdot \Pr[3 \text{ heads}]$$

$$= 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{3+6+3}{8} = \frac{12}{8}$$

<u>Theorem 10.19</u> Linearity of Expectation

let S be a sample space and $X: S \to \mathbb{R}$, $Y: S \to \mathbb{R}$ be any two random variables.

Then $E[X+Y] = E[X] + E[Y]$.

<u>ex</u> flip a p-biased coin ($\Pr[h] = p$) 10 times. what is the expected # of heads?

let S = all outcomes of the 10 flips

let $X(x)$ be # heads in outcome x.

$x = HTTHHTHHTT$, $X(x) = 5$

hard way: $\underbrace{E[X]}_{\#\text{heads}} = \sum_{y \in \{0,1,\dots,10\}} y \cdot \underbrace{\Pr[X = y]}_{\#\text{heads is } y} = 0 \cdot \_\_ + 1 \cdot \_\_ + 2 \cdot \_\_ + \dots$

easier way: let $X_1, X_2, \dots, X_{10}$ be random variables

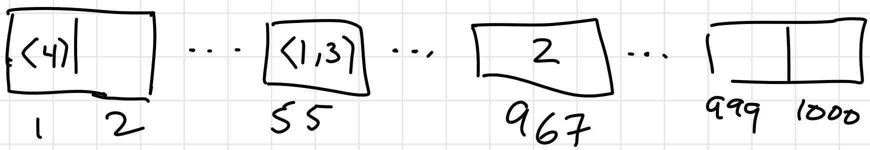
$$X_i = \begin{cases} 1 & \text{if } i\text{th flip H} \\ 0 & \text{otherwise} \end{cases}$$

So total # heads is $X_1 + X_2 + \cdots + X_{10}$

$$E[X] = E[X_1 + X_2 + \cdots + X_{10}] = \sum_{i=1}^{10} E[X_i] = \sum_{i=1}^{10} p = 10p.$$

ex Suppose we hash 1000 elements into a 1000-slot table using a random hash function, resolving collisions by chaining.

$h(1) = 55$
$h(2) = 967$
$h(3) = 55$
$h(4) = 1$



How many empty slots do we expect?

Let S be all outcomes: all ways of hashing 1000 elements into the table.

Let X be the random variable counting # of empty slots.

Q which x has $X(x) = 0$?

$X(x) = 999$?

is there an $x \in S$ s.t. $X(x) = 1000$?

$\Pr[X(x) = 5]$ is... hard to figure out.

$\Pr[\text{slot } i \text{ is empty}] =$

$= \Pr[\text{none of the 1000 elements hashes to } i]$

$= \Pr[\text{every element } j \in \{1, 2, \ldots, 1000\} \text{ hashes to slot other than } i]$

$$= \left(\frac{999}{1000}\right)^{1000} \approx 0.3678$$

let $X_i = \begin{cases} 1 & \text{if slot } i \text{ empty} \\ 0 & \text{if slot } i \text{ full} \end{cases}$

$$E\left[\sum_{i=0}^{1000} X_i\right] = \sum_{i=0}^{1000} E[X_i] = 1000 \cdot 0.3678 = 367.8.$$