

# Recursively Defined Structures

A string is either: **happy**

- nothing
- a symbol followed by a string

↖ character  
element of alphabet  $\Sigma$ ,  
some non-empty set

A string is either:

- •  $\epsilon$  (empty string)
- •  $a \cdot X$  where  $a \in \Sigma$  and  $X$  is a string

$$\text{happy} = h \cdot \text{appy}$$

$$= h \cdot a \cdot \text{ppy}$$

$$= h \cdot a \cdot p \cdot \text{py}$$

$$= h \cdot a \cdot p \cdot p \cdot y$$

$$= h \cdot a \cdot p \cdot p \cdot y \cdot \epsilon$$

$$\text{happy} \epsilon$$

$\Sigma^*$  = set of all strings over  $\Sigma$   
 $\text{happy} \in \{\text{a}, \text{b}, \text{c}, \dots, \text{x}, \text{y}, \text{z}\}^*$   
 $\underline{0110} \in \{\text{0}, \text{1}\}^*$

Notice:  $\Sigma^*$  is infinite as long as  $|\Sigma| \geq 1$   
however,  $x \in \Sigma^*$  is finite

length function - " # of symbols "

Let  $w$  be a string.

$$|w| := \begin{cases} 0 & \text{if } w = \epsilon \\ 1 + |x| & \text{if } w = ax \end{cases}$$

$$\begin{aligned} |\text{happy}| &= 1 + |\text{appy}| \\ &= 1 + 1 + |\text{ppy}| \\ &\vdots \\ &= 1 + 1 + 1 + 1 + |\epsilon| \\ &= 5 + 0 \\ &= 5 \end{aligned}$$

Concatenation - "paste one after another"

Let  $w, z$  be strings.

$$w \bullet z := \begin{cases} z & \text{if } w = \epsilon \\ a(x \bullet z) & \text{if } w = ax \end{cases}$$

$$\text{foot} \bullet \text{ball} = f(\text{o} \circ \text{ot} \bullet \text{ball})$$

$$= f(\text{o} (\underline{\text{o}} \text{f} \bullet \underline{\text{ball}}))$$

$$= f(\text{o} (\text{o} (\text{o} (\text{t} \bullet \text{ball}))))$$

$$= f(\text{o} (\text{o} (\text{o} (\text{t} (\epsilon \bullet \text{ball})))))$$

$$= f \cdot \text{o} \cdot \text{o} \cdot \text{t ball}$$

Theorem for any string  $w$ ,  $w \bullet \epsilon = w$ .

Proof

let  $w$  be an arbitrary string.

Assume that for all strings  $x$  smaller than  $w$ ,  $x \bullet \epsilon = x$ . (Inductive hypothesis)

Case 1:  $w = \epsilon$ .

$$w \bullet \epsilon = \underline{\epsilon} \bullet \epsilon \quad \text{because } w = \epsilon$$

$$= \epsilon \quad \text{by def. of } \bullet$$

$$= w \quad \text{by } w = \epsilon.$$

case 2:  $w = ax$ .

$$\left\{ \begin{aligned} w \bullet z &= a(x \bullet z) && \text{by def. of } \bullet \\ &= ax && \text{by I H} \\ &= w && \text{by } w = ax \end{aligned} \right.$$

Therefore,  $w \bullet z = w$ .

Theorems for all strings  $w$  and  $z$ ,

$$\underline{|w \bullet z| = |w| + |z|}$$

Proof Let  $w, z$  be arbitrary strings.

Assume that for all  $x$  smaller than  $w$ ,  
 $|x \bullet z| = |x| + |z|$ .

Case 1:  $w = \epsilon$

$$\begin{aligned} |w \bullet z| &= |\epsilon \bullet z| && \text{because } w = \epsilon \\ &= |z| && \text{by def of } \bullet \\ &= 0 + |z| && \text{What} \\ &= |\epsilon| + |z| && \text{by def. of } || \\ &= |w| + |z| && \text{because } w = \epsilon \end{aligned}$$

Case 2:  $w = ax$

Therefore,  $|w \bullet z| = |w| + |z|$ .