A Boolean Matrix Iteration in Timetable Construction

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ABSTRACT

The annoying experience in timetable construction is that usually a complete timetable cannot be found without violating or diminishing some preconditions, even if the problem is theoretically solvable. Neither the control of the Hall conditions by Gotlieb's process of reducing availabilities nor the application of elaborate exchange operations guarantees a solution. In this paper an iteration of elementary implications is described which is expected to improve this situation, if applied in the final period of construction. In the course of these investigations, some formulas on Boolean matrices are derived, and a Galois connection between sets of Boolean vectors and Boolean matrices is exhibited.

1. FORMAL DESCRIPTION OF THE TIMETABLE PROBLEM

The construction of timetables for universities and high schools is the well-known problem of bringing together participants (several teachers, several classes, lecture halls) at suitable times in such a way that fundamental as well as special requirements are satisfied. We start with a definition of the timetable problem which includes the concept of meets introduced in [5, 6]. The formulation is in terms of Boolean vectors $x \in \mathbf{B}^N$, $\mathbf{B} = \{O, L\}$. Details regarding Boolean formalism are summarized in Sec. 2.

DEFINITION 1.1. The 6-tuple $\mathbf{T} = (\mathbf{P}, \mathbf{M}, \mathbf{H}, p, a, h)$ is called a *timetable problem (in reduced form)* if

(i) $\mathbf{P}$ is a finite set of *participants*,

(ii) $\mathbf{M}$ is a finite set of *meets*,
(iii) $\mathbf{H}$ is a finite set of *hours*,
(iv) $p:\mathbf{M}\rightarrow\mathcal{P}(\mathbf{P})\backslash\{\varnothing\}$ assigns to every meet M those participants that have *to meet (to participate)* in M,
(v) using the abbreviation $\mathbf{F}:=\mathbf{M}\times\mathbf{H}$, $a\in\mathbf{B}^{\mathbf{F}}$ is a Boolean vector which describes the common *availabilities* of the meets and their participants,
(vi) $h\in\mathbf{B}^{\mathbf{M}}$ determines whether an hour is required for a meet or not,
(vii) the trivial assumption $\bigwedge_M\left[(\bigvee_H a(M,H))\Rightarrow h(M)\right]$ is fulfilled.

The relation between meets and participants will be expressed by the inverse m of p as well:

$$m:\mathbf{P}\rightarrow\mathcal{P}(\mathbf{M}); \qquad m(P):=\{M\in\mathbf{M}|P\in p(M)\}.$$

Sometimes we call (M,H) with $a(M,H)$ an assignment of meet M to hour H and restrict ourselves to the subspace $\mathbf{A}:=\{(M,H)|a(M,H)\}\subset\mathbf{F}$ of possible assignments.

The definition just given is slightly more general than that used by other authors in that it is not restricted to exactly two participants for a meet and in that we may prescribe availabilities not only for the participants but also for a meet. The restriction in Definition 1.1 (vi) of providing only one hour for a meet is shown in Sec. 6 to include the interesting cases by appropriate modification.

DEFINITION 1.2. A Boolean vector $s\in\mathbf{B}^{\mathbf{F}}$ is called a *state* of the timetable problem $\mathbf{T}=(\mathbf{P},\mathbf{M},\mathbf{H},p,a,h)$ if the following fundamental requirements are fulfilled:

(i) $s\subset a$,
(ii) $\bigwedge_M\bigwedge_{M'}\bigwedge_H\,[M\neq M'\wedge p(M)\cap p(M')\neq\varnothing\Rightarrow\overline{s(M,H)}\vee\overline{s(M',H)}]$,
(iii) $\bigwedge_M\bigwedge_H\bigwedge_{H'}\,[H\neq H'\Rightarrow\overline{s(M,H)}\vee\overline{s(M,H')}]$.

If the state s is represented as an element of $\mathbf{B}^{\mathbf{A}}$, condition (i) is superfluous. (i) and Definition 1.1 (vii) imply $\bigwedge_M\left[(\bigvee_H s(M,H))\Rightarrow h(M)\right]$.

DEFINITION 1.3. A state s is called a *solution* of $\mathbf{T}$ if

$$\bigwedge_M\left[h(M)\Rightarrow\bigvee_H s(M,H)\right].$$

Every participant P, whose assignments are selected by $\rho_P\in\mathbf{B}^{\mathbf{F}}$,

$$\rho_P(M,H):=(M\in m(P))\wedge a(M,H),$$

is available at $\nu(P)$ hours and is involved in $\mu(P)$ hours, where

$$\nu(P):=\left|\left\{H\in\mathbf{H}\middle|\bigvee_M\rho_P(M,H)\right\}\right|,\qquad \mu(P):=|\{M\in m(P)|h(M)\}|.$$

In order to avoid the discussion of problems which are *trivially unsolvable*, we postulate $\mu(P)\leqslant\nu(P)$ for all $P\in\mathbf{P}$.

For further unification, we are led to a normalization process which is of use in the theory of timetables.

DEFINITION 1.4. $\mathbf{T}^*=(\mathbf{P},\mathbf{M}^*,\mathbf{H},p^*,a^*,h^*)$ is called the *normalized form* of the problem $\mathbf{T}=(\mathbf{P},\mathbf{M},\mathbf{H},p,a,h)$ if for every $P\in\mathbf{P}$ a set $\mathbf{M}_P:=\{M_P^i|\,\mu(P)<i\leqslant\nu(P)\}$ of new meets has been generated together with the following extensions:

(i) $\mathbf{M}^*:=\mathbf{M}\cup\bigcup_{P\in\mathbf{P}}\mathbf{M}_P$,
(ii) $p^*|_{\mathbf{M}}:=p$; $p^*|_{\mathbf{M}_P}:\equiv\{P\}$ for all P,
(iii) $a^*\in\mathbf{B}^{\mathbf{M}^*\times\mathbf{H}}$, with

$$a^*(M,H):=\begin{cases} a(M,H) & \text{if}\quad M\in\mathbf{M},\\ \bigvee_{M'}\rho_P(M',H) & \text{if}\quad M\in\mathbf{M}_P\quad\text{for some } P,\end{cases}$$

(iv) $h^*|_{\mathbf{M}}:=h$; $h^*|_{\mathbf{M}_P}:\equiv L$.

A problem $\mathbf{T}$ will be called *normalized* if $\mathbf{T}^*=\mathbf{T}$, i.e., $\nu=\mu$.

This introduction of $\nu(P)-\mu(P)$ new meets for P in a canonical way preserves solvability and assures that in a solution every participant is involved in a meet during each of his available hours.

LEMMA 1.5. *For a state s of a normalized problem we have*

$$s\text{ is a solution}\quad\Leftrightarrow\quad\bigwedge_P\bigwedge_H\left[\left(\bigvee_M\rho_P(M,H)\right)\Rightarrow\left(\bigvee_{M\in m(P)}s(M,H)\right)\right].$$

Proof. Due to Definition 1.2 (iii), a state s can be viewed as a partial mapping from $\mathbf{M}$ to $\mathbf{H}$. Following Definition 1.2 (ii), its restriction to $m(P)\subset\mathbf{M}$ is injective for all P. A further restriction of this partial mapping to its domain X_P, $X_P:=\{M\in m(P)|\bigvee_H s(M,H)\}\subset\{M\in m(P)|h(M)\}=:Y_P$, yields for all P an injective mapping from X_P to $Z_P:=\{H\in\mathbf{H}|\bigvee_M\rho_P(M,H)\}$. For a normalized problem we have $|Y_P|=\mu(P)=\nu(P)=|Z_P|$. Now it is obvious from Definition 1.3 that s is a solution exactly if $|X_P|=|Y_P|$ for all P.

On the other hand, $|X_P|=|Z_P|$ if and only if the mapping from X_P to Z_P is surjective, which in turn is equivalent to the existence of an inverse mapping from Z_P to X_P. The above right-hand part expresses that these inverse mappings exist for all P. ■

It should be noted that a normalized problem $\mathbf{T}=(\mathbf{P},\mathbf{M},\mathbf{H},p,a,h)$ together with a state s gives rise to another normalized problem $\mathbf{T}_s$.

DEFINITION 1.6. $\mathbf{T}_s=(\mathbf{P},\mathbf{M},\mathbf{H},p,a_s,h_s)$ is called the *residual problem* of $\mathbf{T}$ in state s if

(i) $a_s(M,H):=a(M,H)\wedge\overline{\bigvee_{M'\in m(p(M))} s(M',H)}\wedge\overline{\bigvee_{H'\neq H} s(M,H')}$,
(ii) $h_s(M):=h(M)\wedge\overline{\bigvee_H s(M,H)}$.

THEOREM 1.7. *If* $\mathbf{T}$ *is a normalized problem in state* s, *then the residual problem* $\mathbf{T}_s$ *is either normalized or trivially unsolvable.*

Proof. We have $\mu(P)=\nu(P)$ for all P in problem $\mathbf{T}$, and we execute the transition from $\mathbf{T}$ to $\mathbf{T}_s$ step by step, assigning only one meet M to the hour H with $s(M,H)$ at a time. This decreases $\mu(P)$ by 1 and $\nu(P)$ by at least 1. Therefore we have $\mu_s(P)-\nu_s(P)\geqslant\mu(P)-\nu(P)=0$, which either indicates trivial unsolvability by $\mu_s(P)>\nu_s(P)$ for at least one P or indicates that $\mathbf{T}_s$ is normalized. ■

2. ELEMENTARY IMPLICATIONS IN TIMETABLE PROBLEMS

We start with some remarks on Boolean matrices and hypergraphs. The set $\mathbf{B}^{N_1\times N_2}$ of Boolean matrices is a complementary, distributive $(\vee,\wedge,\bar{\ })$-lattice, i.e., a Boolean algebra. For $A\in\mathbf{B}^{N_1\times N}$, $B\in\mathbf{B}^{N\times N_2}$ we define the product matrix $AB\in\mathbf{B}^{N_1\times N_2}$ by

$$(AB)(i,k):=\bigvee_{j\in N} A(i,j)\wedge B(j,k).$$

This multiplication is $\vee$-distributive, but instead of $\wedge$-distributivity only

$$A(B\wedge C)\subset AB\wedge AC$$

is valid, using the symbol $\subset$ for elementwise implication $\Rightarrow$. The zero matrix, the universal matrix and in the case of $N_1=N_2$ the identity matrix are denoted by O, L, I respectively. The following lemmas are important for the manipulation of Boolean matrices. Similar formulas are developed and applied in [2,4,7,8].

LEMMA 2.1. *The matrix implications*

$$\overline{A\bar{B}}\,C \subset \overline{A\,\overline{BC}} \quad \text{and} \quad A\,\overline{\bar{B}C} \subset \overline{\overline{AB}\,C}$$

hold for arbitrary Boolean matrices A, B, C.

Proof. $\overline{A\bar{B}}\,C \subset \overline{A\overline{BC}}$ is obviously true if C has less than two nonvanishing entries. Let C_i always denote a matrix with exactly i entries L and assume the theorem to be valid for $i \leqslant n$. Then we have $C_{n+1} = C_n \vee C_1$, and by induction

$$\overline{A\bar{B}}\,C_{n+1} = \overline{A\bar{B}}\,(C_n \vee C_1) = \overline{A\bar{B}}\,C_n \vee \overline{A\bar{B}}\,C_1 \subset \overline{A\,\overline{BC_n}} \vee \overline{A\,\overline{BC_1}}$$

$$\subset \overline{A\,\overline{BC_{n+1}}} \vee \overline{A\,\overline{BC_{n+1}}} = \overline{A\,\overline{BC_{n+1}}}\,. \qquad \blacksquare$$

LEMMA 2.2. *For arbitrary Boolean matrices A, B, C we have*

(i) $A \subset B^{\mathsf{T}} \Leftrightarrow I \subset \overline{\bar{B}A}$,
(ii) A has no vanishing column $\Leftrightarrow \overline{\bar{B}A} \subset BA$,
(iii) $AB \subset C \Leftrightarrow \bar{C}B^{\mathsf{T}} \subset \bar{A} \Leftrightarrow A^{\mathsf{T}}\bar{C} \subset \bar{B}$,
(iv) $(AL \wedge B)C = AL \wedge BC$, $(LA^{\mathsf{T}} \wedge B)C = B(C \wedge AL)$.

The universal matrices in (iv) *may be of different dimensions.*

Proof. The verification of (iv) is straightforward.

(i) $A \subset B^{\mathsf{T}} \Rightarrow \bar{B} \subset \bar{A}^{\mathsf{T}} \Rightarrow \bar{B}A \subset \bar{A}^{\mathsf{T}}A \subset \bar{I}$,

$I \subset \overline{\bar{B}A} \Rightarrow \overline{B^{\mathsf{T}}} \subset \bar{B}^{\mathsf{T}}\overline{\bar{B}A} \subset \overline{\overline{\bar{B}^{\mathsf{T}}B}A} \subset \overline{\bar{\bar{I}}A} \subset \overline{IA} = \bar{A}$.

(ii) $L = LA = (\bar{B} \vee B)A = \bar{B}A \vee BA \Leftrightarrow \overline{\bar{B}A} \subset BA$.

(iii) $AB \subset C \Leftrightarrow I \subset \overline{\overline{C^{\mathsf{T}}}AB} = \overline{\overline{\overline{\overline{C^{\mathsf{T}}}A}}B} \Leftrightarrow B \subset (\overline{\overline{C^{\mathsf{T}}}A})^{\mathsf{T}} \Leftrightarrow \overline{C^{\mathsf{T}}}A \subset \overline{B^{\mathsf{T}}} \Leftrightarrow$

$I \subset \overline{B\overline{C^{\mathsf{T}}}A} = \overline{\overline{\overline{B\overline{C^{\mathsf{T}}}}}A} \Leftrightarrow A \subset \overline{B\overline{C^{\mathsf{T}}}}^{\mathsf{T}} = \bar{C}B^{\mathsf{T}}$. $\blacksquare$

The description of the timetable problem is facilitated by the notion of a hypergraph; cf. [1].

DEFINITION 2.3. The triple $G=(V,E,\omega)$ is called a *hypergraph*, if V is a set of vertices, E is a set of (generalized) edges and $\omega : E \to \mathfrak{P}(V)\setminus\{\varnothing\}$ is the incidence function. A choice $\sigma \in \mathbf{B}^E$ of some edges is called a *matching* in G if

$$\bigwedge_{e\in E}\bigwedge_{e'\in E}\left(\left[e\neq e' \wedge \omega(e)\cap\omega(e')\neq\varnothing\right]\Rightarrow \overline{\sigma(e)}\vee\overline{\sigma(e')}\right).$$

In Boolean matrix notation this condition is expressed by

$$K\sigma \subset \bar{\sigma},$$

if $K \in \mathbf{B}^{E\times E}$ is the edge-adjacency matrix of G, i.e.,

$$K(e,e') := (e\neq e')\wedge(\omega(e)\cap\omega(e')\neq\varnothing).$$

The matching σ is called *perfect* if $\bigwedge_{P\in V}\bigvee_{e\in\omega^{-1}(P)}\sigma(e)$.

A description of the timetable problem $\mathbf{T}=(\mathbf{P},\mathbf{M},\mathbf{H},p,a,h)$ can be given by the family G_H, $H\in\mathbf{H}$ of hypergraphs $G_H=(\mathbf{P},\mathbf{A}_H,p_H)$, $\mathbf{A}_H := \{M\in\mathbf{M}\,|\,a(M,H)\}$ and $p_H(M) := p(M)$, which is equivalent to the original definition if $h\equiv L$. To a state s of $\mathbf{T}$ [cf. Definition 1.2 (ii)] corresponds a family $s_H\in\mathbf{B}^{\mathbf{A}_H}$, $H\in\mathbf{H}$ of matchings in G_H with $s_H(M) := s(M,H)$, which are perfect matchings in the case of a solution of a normalized problem with $\mu(P)=\nu(P)=|\mathbf{H}|$. Examples are shown in Figs. 1, 3 and 6.

Pursuing this idea, we try to describe a state s as a matching of a hypergraph, combining conditions (ii) and (iii) of Definition 1.2 into a single matrix implication $\Phi s\subset\bar{s}$. Due to Theorems 2.8 and 4.3, we define Φ in a manner more intricate than needed at the moment.

DEFINITION 2.4. Using $\rho_P\in\mathbf{B}^{\mathbf{F}}$, $P\in\mathbf{P}$, and the Boolean matrix $\psi\in\mathbf{B}^{\mathbf{F}\times\mathbf{F}}$,

$$\psi(M,H,M',H') := (M\neq M')\wedge(H=H'),$$

we introduce the *elementary forbidding matrices (elementary implications)* $\varphi,\varphi_P,\Phi\in\mathbf{B}^{\mathbf{F}\times\mathbf{F}}$ of the timetable problem $\mathbf{T}=(\mathbf{P},\mathbf{M},\mathbf{H},p,a,h)$ by

$$\varphi(M,H,M',H') := (M=M')\wedge(H\neq H')$$

and

$$\varphi_P := \left(\rho_P L^{\mathsf{T}}\wedge L\rho_P^{\mathsf{T}}\wedge\psi\right)\vee\left(L\bar{\rho}_P^{\mathsf{T}}\wedge\varphi\right),$$

a linear combination of φ and ψ, and

$$\Phi := \varphi \vee \bigvee_P \varphi_P = \varphi \vee \bigvee_P \left(\rho_P L^{\mathsf{T}} \wedge L\rho_P^{\mathsf{T}} \wedge \psi\right).$$

For practical purposes it is possible to restrict these matrices to $\mathbf{B}^{\mathbf{A}\times\mathbf{A}}$.

The universal vector L in Definition 2.4 is an element of $\mathbf{B}^{\mathbf{F}}$. Obviously the matrices φ, ψ, Φ are symmetric. Elementwise formulations are easily derived, e.g., $\rho_P L^{\mathsf{T}}(M,H,M',H') = M \in m(P) \wedge a(M,H)$ and

$$\bigvee_P \left(\rho_P L^{\mathsf{T}} \wedge L\rho_P^{\mathsf{T}} \wedge \psi\right)(M,H,M',H') = (M \neq M' \wedge H = H' \wedge p(M) \cap p(M') \neq \varnothing$$
$$\wedge a(M,H) \wedge a(M',H')).$$

THEOREM 2.5. *For a Boolean vector $s \in \mathbf{B}^{\mathbf{F}}$ with $s \subset a$ we have*

$$s \text{ is a state} \quad \Leftrightarrow \quad \Phi s \subset \bar{s}.$$

Proof.

$$(\Phi s \subset \bar{s}) = \left(\Phi \subset \overline{ss^{\mathsf{T}}}\right) = \left[\left(\varphi \subset \overline{ss^{\mathsf{T}}}\right) \wedge \left(\left\{ \bigvee_P \rho_P L^{\mathsf{T}} \wedge L\rho_P^{\mathsf{T}} \wedge \psi \right\} \subset \overline{ss^{\mathsf{T}}}\right)\right].$$

The two constituents are evaluated separately:

$$\left(\varphi \subset \overline{ss^{T}}\right) = \left(\bigwedge_{M,H,M',H'} \left[M = M' \wedge H \neq H' \Rightarrow \overline{s(M,H)} \vee \overline{s(M',H')}\,\right]\right)$$

Definition 1.2 (iii),

and using the preceding formula,

$$\left(\bigvee_P \left\{\rho_P L^{\mathsf{T}} \wedge L\rho_P^{\mathsf{T}} \wedge \psi\right\} \subset \overline{ss^{\mathsf{T}}}\right) = \bigwedge_{M,H,M',H'} \left[M \neq M' \wedge H = H' \wedge p(M) \cap p(M') \neq \varnothing\right.$$
$$\left.\wedge a(M,H) \wedge a(M',H') \Rightarrow \overline{s(M,H)} \vee \overline{s(M',H')}\,\right]$$

= Definition 1.2 (ii),

since $\bar{a} \subset \bar{s}$. ∎

Following Theorem 2.5, a state s can be interpreted as a matching in the hypergraph $((\mathbf{P}\times\mathbf{H})\cup\mathbf{M},\mathbf{A},w)$ with $w((M,H)) := (p(M)\times\{H\})\cup\{M\}$.

The assumption that s is a solution if and only if $\Phi s=\bar{s}$ is rejected by the following counterexample.

EXAMPLE 2.6. The normalized timetable problem given by the hypergraphs of Fig. 1 has $\varphi,\varphi_{P_1},\varphi_{P_2},\varphi_{P_3}$ as elementary forbidding matrices which are shown in Fig. 2 by indexing the entries of $\Phi\in\mathbf{B}^{\mathbf{A}\times\mathbf{A}}$ with $x,1,2,3$ respectively $(h\equiv L)$. For the state s indicated in Figs. 1 and 2 we have $\Phi s=\bar{s}$ in spite of the fact that s is no solution.

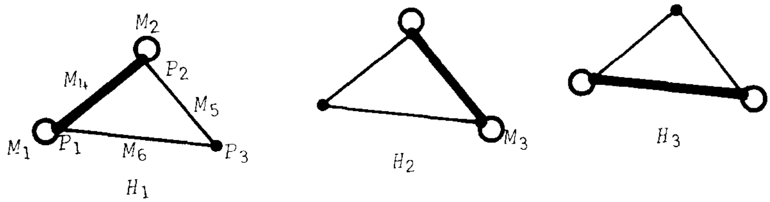

FIG. 1. Solvable timetable problem **T** in state s (dark lines).

$$
\begin{array}{r}
M_1\\ M_2\\ H_1, M_4\\ M_5\\ M_6\\ \hline M_2\\ M_3\\ H_2, M_4\\ M_5\\ M_6\\ \hline M_1\\ M_3\\ H_3, M_4\\ M_5\\ M_6
\end{array}
\left[\begin{array}{ccccc|ccccc|ccccc}
 & & 1 & & 1 & & & & & & x & & & & \\
 & & 2 & 2 & & x & & & & & & & & & \\
1 & 2 & & 2 & 1 & & & x & & & & & x & & \\
 & 2 & 2 & & 3 & & & & x & & & & & x & \\
1 & & 1 & 3 & & & & & & x & & & & & x \\ \hline
 & x & & & & & & 2 & 2 & & & & & & \\
 & & & & & & & & 3' & 3 & & x & & & \\
 & & x & & & 2 & & & 2 & 1 & & & x & & \\
 & & & x & & 2 & 3 & 2 & & 3 & & & & x & \\
 & & & & x & & 3 & 1 & 3 & & & & & & x \\ \hline
x & & & & & & & & & & & & 1 & & 1 \\
 & & & & & & x & & & & & & & 3 & 3 \\
 & & x & & & & & x & & & 1 & & & 2 & 1 \\
 & & & x & & & & & x & & & 3 & 2 & & 3 \\
 & & & & x & & & & & x & 1 & 3 & 1 & 3 &
\end{array}\right]
\left[\begin{array}{c}
\\ \\ L\\ \\ \\ \hline \\ \\ \\ L\\ \\ \hline \\ \\ \\ \\ L
\end{array}\right]
=
\left[\begin{array}{c}
L\\ L\\ \\ L\\ L\\ \hline L\\ L\\ L\\ \\ L\\ \hline L\\ L\\ L\\ L\\ \\
\end{array}\right]
$$

FIG. 2. Elementary forbidding matrices of **T** with $\Phi s=\bar{s}$.

Nevertheless, at least one direction of the preceding assumption is true, as we will state without proof:

REMARK 2.7. The states s characterized by $\Phi s=\bar{s}$ are exactly the maximal elements (with respect to $\subset$) of the set of states. In particular, we have

$$s \text{ is a solution } \Rightarrow \Phi s = \bar{s}.$$

Solutions are characterized by

THEOREM 2.8. *Let s be a state of the timetable problem* T *with $h \equiv L$.*

(i) *s is a solution $\Leftrightarrow$ $\varphi s = \bar{s}$.*

If T *is normalized, then*

(ii) *s is a solution $\Leftrightarrow$ $\bigwedge_P(\varphi_P^{\mathsf{T}} s = \bar{s})$.*

Proof. We have $\Phi s \subset \bar{s}$, and therefore $\varphi s \subset \Phi s \subset \bar{s}$, $\varphi_P^{\mathsf{T}} s \subset \Phi^{\mathsf{T}} s = \Phi s \subset \bar{s}$.

(i) $(\bar{s} \subset \varphi s)$

$$= (L = [I \vee \varphi] s)$$

$$= \bigwedge_{M,H} \bigvee_{M^*,H^*} [(M = M^* \wedge H = H^*) \vee (M = M^* \wedge H \neq H^*)] \wedge s(M^*, H^*)$$

$$= \bigwedge_{M} \bigvee_{H^*} s(M, H^*) = \text{Definition 1.3 with } h \equiv L.$$

(ii) Lemma 1.5 is used and transferred to a matrix implication:

$$s \text{ solution} = \bigwedge_P \bigwedge_H \left[\left(\bigvee_M \rho_P(M,H) \right) \Rightarrow \left(\bigvee_{M^*} M^* \in m(P) \wedge s(M^*, H) \right) \right]$$

$$= \bigwedge_P \bigwedge_H \bigwedge_M \left[\rho_P(M,H) \Rightarrow \left(\bigvee_{M^*} M^* \in m(P) \wedge s(M^*, H) \right) \right]$$

$$= \bigwedge_P \bigwedge_{M,H} \left[\rho_P(M,H) \Rightarrow \left(\bigvee_{M^*,H^*} H = H^* \wedge M^* \in m(P) \wedge s(M^*,H^*) \wedge a(M^*,H^*) \right) \right]$$

$$= \bigwedge_P \bigwedge_{M,H} [\rho_P(M,H) \Rightarrow ((I \vee \psi)(s \wedge \rho_P))(M,H)]$$

$$= \bigwedge_P [\rho_P \subset (I \vee \psi)(s \wedge \rho_P)]$$

$$= \bigwedge_P \left(L = [\rho_P \wedge (I \vee \psi)(s \wedge \rho_P)] \vee \overline{\rho_P}\right)$$

$$= \bigwedge_P \left(L = [\rho_P \wedge (I \vee \psi)(s \wedge \rho_P)] \vee \left[\overline{\rho_P} \wedge (I \vee \varphi) s\right]\right), \qquad \text{using (i)},$$

$$= \bigwedge_P \left(\bar{s} \subset \varphi_P^{\mathsf{T}} s\right),$$

since

$$(\bar{s} \subset \varphi_P^{\mathsf{T}} s) = \left(L = \left[I \vee \varphi_P^{\mathsf{T}}\right] s\right) = \left(L = \left[(I \wedge \rho_P L^{\mathsf{T}}) \vee \left(I \wedge \overline{\rho_P L^{\mathsf{T}}}\right) \vee \varphi_P^{\mathsf{T}}\right] s\right)$$

$$= \left(L = \left[(I \wedge \rho_P L^{\mathsf{T}} \wedge L\rho_P^{\mathsf{T}}) \vee (I \wedge \overline{\rho_P} L^{\mathsf{T}}) \vee (\rho_P L^{\mathsf{T}} \wedge L\rho_P^{\mathsf{T}} \wedge \psi) \vee (\overline{\rho_P} L^{\mathsf{T}} \wedge \varphi)\right] s\right)$$

$$= \left(L = \left[\rho_P L^{\mathsf{T}} \wedge L\rho_P^{\mathsf{T}} \wedge (I \vee \psi)\right] s \vee \left[\overline{\rho_P} L^{\mathsf{T}} \wedge (I \vee \varphi)\right] s\right)$$

$$= \left(L = \left[\rho_P \wedge (I \vee \psi)(s \wedge \rho_P)\right] \vee \left[\bar{\rho}_P \wedge (I \vee \varphi) s\right]\right),$$

applying Lemma 2.2 (iv) twice. ■

We state the definition of the kernel of a digraph in Boolean notation.

DEFINITION 2.9. Let $D = (V, \Gamma)$ be a digraph with vertex set V and associated matrix $\Gamma \in \mathbf{B}^{V \times V}$. Then $\kappa \in \mathbf{B}^V$ is called a *kernel* of D if $\Gamma\kappa = \bar{\kappa}$.

As a consequence of Remark 2.7 and Theorem 2.8 (i) we have

COROLLARY 2.10. *A maximal state of the timetable problem* $\mathrm{T} = (\mathrm{P}, \mathrm{M}, \mathrm{H}, p, a, h)$ *with* $h \equiv L$ *is a kernel of the digraph* $D_\Phi := (\mathrm{F}, \Phi)$ *and a solution of* T *is a kernel of* D_Φ *and* $D_\varphi := (\mathrm{F}, \varphi)$ *simultaneously.*

3. IMPLICATION STRUCTURES

The elementary implications of Definition 2.4 have been introduced in order to produce other implications by iteration. In this section we will derive the formal mechanism.

DEFINITION 3.1. For an arbitrary subset $\mathbf{S}$ of $\mathbf{B}^N$ we define the *implication matrices* $E_{\mathbf{S}}, F_{\mathbf{S}}, C_{\mathbf{S}} \in \mathbf{B}^{N \times N}$ of $\mathbf{S}$ by

$$E_{\mathbf{S}}(i,k) := \bigwedge_{x \in \mathbf{S}} x(i) \Rightarrow x(k),$$

$$F_{\mathbf{S}}(i,k) := \bigwedge_{x \in \mathbf{S}} x(i) \Rightarrow \overline{x(k)},$$

$$C_{\mathbf{S}}(i,k) := \bigwedge_{x \in \mathbf{S}} \overline{x(i)} \Rightarrow x(k).$$

$E_{\mathbf{S}}$ and $F_{\mathbf{S}}$ are called *enforcing* and *forbidding matrices* respectively. We say that $E_{\mathbf{S}}, F_{\mathbf{S}}, C_{\mathbf{S}}$ describe the *implication structure of* $\mathbf{S}$.

DEFINITION 3.2. For an arbitrary triple of Boolean matrices $E, F, C \in \mathbf{B}^{N \times N}$ with $E \supset I$ we call the set $\mathbf{S}_{E,F,C} \subset \mathbf{B}^N$ given by

$$\mathbf{S}_{E,F,C} := \{x \in \mathbf{B}^N | E^{\mathsf{T}} x = x, Fx \subset \bar{x}, C\bar{x} \subset x\}$$

the set of *solutions* of the triple E, F, C.

We will develop some interesting relations between implication matrices and their solutions.

THEOREM 3.3. *Let* $\mathbf{S}$ *and* E, F, C *be as defined above. Then*

(i) $\mathbf{S} \subset \mathbf{S}_{E_{\mathbf{S}}, F_{\mathbf{S}}, C_{\mathbf{S}}}$,
(ii) $E \subset E_{\mathbf{S}_{E,F,C}}$, $F \subset F_{\mathbf{S}_{E,F,C}}$, $C \subset C_{\mathbf{S}_{E,F,C}}$.

Proof.

(i) Let x be an element of $\mathbf{S}$. Then $I \subset E_{\mathbf{S}}$ implies $x \subset E_{\mathbf{S}}^{\mathsf{T}} x$; conversely $(E_{\mathbf{S}}^{\mathsf{T}} x)(i) = \bigvee_j E_{\mathbf{S}}^{\mathsf{T}}(i,j) \wedge x(j) = \bigvee_j \{\bigwedge_{y \in \mathbf{S}}^{\mathsf{T}} y(j) \Rightarrow y(i)\} \wedge x(j) \Rightarrow (\bigvee_j \{x(j) \Rightarrow x(i)\} \wedge x(j)) \Rightarrow (\bigvee_j x(i)) = x(i)$. The rest is proved analogously.

(ii) For every $x \in \mathbf{S}_{E,F,C}$ we have $E^{\mathsf{T}} x = x$ by definition. Obviously, if $x(i)$ holds $x(k)$ will hold, assumed $E(i,k)$ is valid. Therefore $E(i,k) \Rightarrow E_{\mathbf{S}_{E,F,C}}(i,k)$. ■

The following theorem is easily proved. It demonstrates that there is a Galois connection between subsets $\mathbf{S} \subset \mathbf{B}^N$ and triples E, F, C of matrices with $E \supset I$.

THEOREM 3.4. *Assume* $\mathbf{Q}, \mathbf{R}, \mathbf{S} \subset \mathbf{B}^N$ *and* $E, E^*, F, F^*, C, C^* \in \mathbf{B}^{N \times N}$. *Then*

(i) $\mathbf{R} \subset \mathbf{S} \Rightarrow E_{\mathbf{S}} \subset E_{\mathbf{R}}$, $F_{\mathbf{S}} \subset F_{\mathbf{R}}$, $C_{\mathbf{S}} \subset C_{\mathbf{R}}$,
(ii) $I \subset E \subset E^*$, $F \subset F^*$, $C \subset C^* \Rightarrow \mathbf{S}_{E^*, F^*, C^*} \subset \mathbf{S}_{E,F,C}$,
(iii) $\mathbf{Q} = \mathbf{R} \cap \mathbf{S} \Rightarrow E_{\mathbf{Q}} \supset E_{\mathbf{R}} \vee E_{\mathbf{S}}$, $F_{\mathbf{Q}} \supset F_{\mathbf{R}} \vee F_{\mathbf{S}}$, $C_{\mathbf{Q}} \supset C_{\mathbf{R}} \vee C_{\mathbf{S}}$,
(iv) $\mathbf{Q} = \mathbf{R} \cup \mathbf{S} \Rightarrow E_{\mathbf{Q}} = E_{\mathbf{R}} \wedge E_{\mathbf{S}}$, $F_{\mathbf{Q}} = F_{\mathbf{R}} \wedge F_{\mathbf{S}}$, $C_{\mathbf{Q}} = C_{\mathbf{R}} \wedge C_{\mathbf{S}}$.

Since this is a Galois connection, the operations

$$\mathbf{S} \rightarrow \mathbf{S}_{E_{\mathbf{S}}, F_{\mathbf{S}}, C_{\mathbf{S}}}$$

and

$$E, F, C \rightarrow E_{\mathbf{S}_{E,F,C}}, F_{\mathbf{S}_{E,F,C}}, C_{\mathbf{S}_{E,F,C}}$$

are closure operations.

In the following theorem and its corollary we state some fundamental properties of implication matrices. For convenience, we suppress the index $\mathbf{S}$ in $E_{\mathbf{S}}, F_{\mathbf{S}}, C_{\mathbf{S}}$.

THEOREM 3.5. *If E, F, C are implication matrices, then*

(i) $F = F^{\mathsf{T}}$, $C = C^{\mathsf{T}}$,
(ii) $I \subset E = EE$,
(iii) $EF = F$, $CE = C$,
(iv) $FC \subset E$.

Proof. Observing $(a \Rightarrow b) \Leftrightarrow (\bar{b} \Rightarrow \bar{a})$, the assertions of (i) follow immediately from the definition. $I \subset E$, because $a \Rightarrow a$ is always true, and therefore $E \subset EE$, $F \subset EF$, $C \subset CE$. The opposite inclusions can be proved following the pattern

$$(EF)(i,k) = \bigvee_j E(i,j) \wedge F(j,k) = \bigvee_j \bigwedge_{x \in \mathbf{S}} \big(x(i) \Rightarrow x(j)\big) \wedge \big(x(j) \Rightarrow \overline{x(k)}\,\big)$$

$$\Rightarrow \bigvee_j \bigwedge_{x \in \mathbf{S}} \big(x(i) \Rightarrow \overline{x(k)}\,\big) = F(i,k),$$

thus establishing (ii), (iii) and (iv). ∎

COROLLARY 3.6. *If E, F, C are implication matrices then*

(i) $E\bar{C} = \bar{C}$, $\bar{F}E = \bar{F}$, $\bar{E}E^{\mathsf{T}} = \bar{E} = E^{\mathsf{T}}\bar{E}$,
(ii) $C\bar{C} \subset \bar{E}$, $\bar{E}C \subset \bar{F}$,
(iii) $\bar{F}F \subset \bar{E}$, $F\bar{E} \subset \bar{C}$.

Proof. Lemma 2.2 (iii) is applied to the results of Theorem 3.5; for instance

$$FC \subset E \quad \Leftrightarrow \quad \bar{E}C \subset \bar{F} \quad \Leftrightarrow \quad F\bar{E} \subset \bar{C}.$$

$E \supset I$ yields the stated equalities. ∎

Theorem 3.5 (ii) means that $E \wedge E^{\mathsf{T}}$ is an equivalence relation. Obviously, the implication matrices depend only on the equivalence classes of N modulo $E \wedge E^{\mathsf{T}}$, i.e., with $N^* := N/_{E \wedge E^{\mathsf{T}}}$ E, F, C can be considered to be matrices in $\mathbf{B}^{N^* \times N^*}$. All elements of a class are either simultaneously contained in a solution or simultaneously not contained in a solution. The most interesting classes are the class of pseudo-elements and the class of tight elements.

The structure of the definition of implication matrices leads us to call i a *tight* element with respect to $\mathbf{S}$ if $C_{\mathbf{S}}(i,i)$ holds, because of $C_{\mathbf{S}}(i,i)=\bigwedge_{x\in\mathbf{S}}x(i)$, and to call i a *pseudo*-element if $F_{\mathbf{S}}(i,i)$ holds, because of $F_{\mathbf{S}}(i,i)=\bigwedge_{x\in\mathbf{S}}\overline{x(i)}$; otherwise i is called a *flexible* element. We say that the implication structure is flexible if there are only flexible elements. A flexible implication structure is characterized by $I\subset\bar{F}\wedge\bar{C}$.

A rearrangement of elements in the ordering pseudo-, flexible, tight establishes a canonical decomposition of the implication matrices and a flexible implication substructure.

Theorem 3.7. *By simultaneous permutations of rows and columns the implication matrices E,F,C of $\mathbf{S}$ decompose into the standard form*

$$E=\begin{bmatrix} L & L & L\\ O & E_0 & L\\ O & O & L\end{bmatrix},\qquad F=\begin{bmatrix} L & L & L\\ L & F_0 & O\\ L & O & O\end{bmatrix},\qquad C=\begin{bmatrix} O & O & L\\ O & C_0 & L\\ L & L & L\end{bmatrix},$$

and E_0,F_0,C_0 describe the flexible implication substructure *of* $\mathbf{S}$.

Proof. We verify only the last columns of F and C. If k is a tight element, i.e., $\bigwedge_{x\in\mathbf{S}}x(k)$ holds, then $C(i,k)=\bigwedge_{x\in\mathbf{S}}x(i)\vee x(k)$ holds for all i, and $F(i,k)=\bigwedge_{x\in\mathbf{S}}\overline{x(i)}\vee\overline{x(k)}=\bigwedge_{x\in\mathbf{S}}\overline{x(i)}=F(i,i)$, which proves these two sample cases. ■

In the case of a flexible implication structure, i.e., $E=E_0$, $F=F_0$, $C=C_0$, further inequalities are deduced from $I\subset\bar{F}\wedge\bar{C}$.

Theorem 3.8. *Let E,F,C describe a flexible implication structure. Then*

(i) $E\vee E^{\mathsf{T}}\subset\bar{F}\wedge\bar{C}$,
(ii) $E^{\mathsf{T}}E\subset\bar{F}$, $FE\subset\bar{E}^{\mathsf{T}}$,
(iii) $EE^{\mathsf{T}}\subset\bar{C}$, $CE^{\mathsf{T}}\subset\bar{E}$, $CC\subset\bar{F}$.

The implication structure of a timetable problem will fulfill additional formulas, since in Corollary 3.6 (iii) equalities hold.

Remark 3.9. Let E,F,C describe a flexible implication structure with $\bar{F}F=\bar{E}$ and $F\bar{E}=\bar{C}$. Then

(i) $\bar{F}\subset\bar{E}F\wedge\bar{E}\bar{E}^{\mathsf{T}}$, $E\subset FF$, $C\subset FE$, $\bar{E}\subset F\bar{C}$,
(ii) $\bar{E}^{\mathsf{T}}\subset\bar{C}\bar{E}$, $E\subset\bar{E}^{\mathsf{T}}\bar{E}^{\mathsf{T}}$, $\bar{C}\subset\bar{E}^{\mathsf{T}}\bar{E}$,
(iii) $\bar{C}\vee\bar{F}\subset\bar{E}\bar{E}$, $\bar{E}\subset\bar{F}\bar{E}^{\mathsf{T}}$.

4. THE IMPLICATION STRUCTURE OF THE TIMETABLE PROBLEM

Applying the investigations of Sec. 3 to the timetable problem $\mathbf{T} = (\mathbf{P}, \mathbf{M}, \mathbf{H}, p, a, h)$, we are led to consider the set

$$\mathsf{S} := \{ s | s \text{ is a solution of } \mathbf{T} \} \subset \mathsf{B}^{\mathbf{F}}$$

and the implication matrices $E, F, C \in \mathsf{B}^{\mathbf{F} \times \mathbf{F}}$ of S:

$$E(M,H,M',H') = \bigwedge_{s \in \mathsf{S}} s(M,H) \Rightarrow s(M',H'),$$

$$F(M,H,M',H') = \bigwedge_{s \in \mathsf{S}} s(M,H) \Rightarrow \overline{s(M',H')},$$

$$C(M,H,M',H') = \bigwedge_{s \in \mathsf{S}} \overline{s(M,H)} \Rightarrow s(M',H').$$

We say that E, F, C describe the implication structure of $\mathbf{T}$ and remember that we have called (M,H) *tight* if $C(M,H,M,H)$ holds, *pseudoavailable* if $F(M,H,M,H)$ holds, and *flexible* otherwise. Furthermore we call $\mathbf{T}$ flexible if all (M,H) are flexible.

The implication matrices E, F describe relations such as "If an assignment (M,H) is contained in any solution s, then the assignment (M',H') will also (will not) be contained in s".

We mention some connections between the implication matrices E, F, C of $\mathbf{T}$ and the elementary forbidding matrices φ, φ_P, Φ of $\mathbf{T}$. Obviously, we have $\varphi, \varphi_P \subset \Phi \subset F$, and because of $E \subset \overline{\bar{F}F}$ [see Corollary 3.6(iii)],

$$E \subset \overline{\bar{F}\Theta} \qquad \text{for all} \quad \Theta \subset F$$

is valid. Since $\overline{\bar{F}\Theta}$ is monotone decreasing in Θ, and since we are interested in $E = \overline{\bar{F}\Theta}$, we will look for sufficiently large matrices $\Theta \subset F$ in $\overline{\bar{F}\Theta}$.

THEOREM 4.1. *If E, F, C describe the implication structure of a timetable problem $\mathbf{T}$ with $h \equiv L$, we have*

$$E = \overline{\bar{F}\varphi}, \qquad C = \overline{\bar{E}^{\mathbf{T}}\varphi}.$$

Proof. $\overline{\bar{F}\varphi}(M, H, M', H') = \bigwedge_{M^*,H^*}(\bigwedge_s \bar{s}(M, H) \vee \bar{s}(M^*, H^*)) \vee \bar{\varphi}(M^*,H^*,M',H') = \bigwedge_s \bar{s}(M,H) \vee \bigwedge_{M^*,H^*} \bar{s}(M^*,H^*) \vee \bar{\varphi}(M^*,H^*,M',H') = \bigwedge_s \bar{s}(M,H) \vee \varphi^{\mathsf{T}} s(M',H') = \bigwedge_s \bar{s}(M,H) \vee s(M',H') = E(M,H,M',H')$, using Theorem 2.8. The other equality is obtained analogously. ■

The following corollary assures that in the case of a flexible timetable problem all the formulas of Sec. 3 hold.

COROLLARY 4.2. *The implication structure of a timetable problem with $h \equiv L$ is determined by F, because*

$$E = \overline{\bar{F}F}\,, \qquad C = \overline{\bar{E}^{\mathsf{T}}F}\,.$$

Proof. Because of Corollary 3.6 (iii) we have only to show $\supset$. $\varphi \subset F$ implies $E = \overline{\bar{F}\varphi} \supset \overline{\bar{F}F}$ and $C = \overline{\bar{E}^{\mathsf{T}}\varphi} \supset \overline{\bar{E}^{\mathsf{T}}F}$. ■

For the other elementary forbidding matrix φ_P, $E \subset \overline{\bar{F}\varphi_P}$ is trivially valid, whereas the opposite inclusion is not true in general. It is necessary to restrict to normalized problems.

THEOREM 4.3. *If E, F, C describe the implication structure of a normalized timetable problem with $h \equiv L$, we have for all $P \in \mathrm{P}$*

$$E = \overline{\bar{F}\varphi_P}\,, \qquad C = \overline{\bar{E}^{\mathsf{T}}\varphi_P}\,.$$

The *proof* is completely analogous to that of Theorem 4.1. ■

To illustrate some aspects of our inquiries we give an example of a timetable problem with $h \equiv L$.

EXAMPLE 4.4. For the normalized timetable problem **T** given by the hypergraphs of Fig. 3, the implication matrices E, F are shown in Fig. 4.

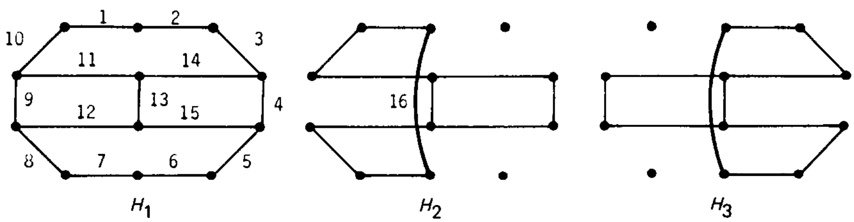

FIG. 3. **Mincut-flexible but not flexible timetable problem T.**

	M 1	L O L O L O L O O O L L O O O	O L O L L O O L O O L	L O O L L O O O L L O	
	M 2	O L O O O L O L O L O O O L L	L L L O O L L O O O O	O L L O L O O L O O L	
	M 3	L O L O L O L O O O L L O O O	O L O L L O O L O O L	L O O L L O O O L L O	
	M 4	L L L L L L L L L L L L L L L	L L L L L L L L L L L	L L L L L L L L L L L	
	M 5	L O L O L O L O O O L L O O O	O L O L L O O L O O L	L O O L L O O O L L O	
	M 6	O L O O O L O L O L O O O L L	L L L O O L L O O O O	O L L O L O O L O O L	
	M 7	L O L O L O L O O O L L O O O	O L O L L O O L O O L	L O O L L O O O L L O	
H1	M 8	O L O O O L O L O L O O O L L	L L L O O L L O O O O	O L L O L O O L O O L	
	M 9	L L L L L L L L L L L L L L L	L L L L L L L L L L L	L L L L L L L L L L L	
	M10	O L O O O L O L O L O O O L L	L L L O O L L O O O O	O L L O L O O L O O L	
	M11	L O L O L O L O O O L L O O O	O L O L L O O L O O L	L O O L L O O O L L O	
	M12	L O L O L O L O O O L L O O O	O L O L L O O L O O L	L O O L L O O O L L O	
	M13	L L L L L L L L L L L L L L L	L L L L L L L L L L L	L L L L L L L L L L L	
	M14	O L O O O L O L O L O O O L L	L L L O O L L O O O O	O L L O L O O L O O L	
	M15	O L O O O L O L O L O O O L L	L L L O O L L O O O O	O L L O L O O L O O L	
	M 1	O L O O O L O L O L O O O L L	L L L O O L L O O O O	O L L O L O O L O O L	
	M 4	O O O O O O O O O O O O O O O	O L O O O O O O O O O	O O O O L O O O O O O	
	M 7	O L O O O L O L O L O O O L L	L L L O O L L O O O O	O L L O L O O L O O L	
	M 8	L O L O L O L O O O L L O O O	O L O L L O O L O O L	L O O L L O O O L L O	E
	M10	L O L O L O L O O O L L O O O	O L O L L O O L O O L	L O O L L O O O L L O	
H2	M11	O L O O O L O L O L O O O L L	L L L O O L L O O O O	O L L O L O O L O O L	
	M12	O L O O O L O L O L O O O L L	L L L O O L L O O O O	O L L O L O O L O O L	
	M13	L O L O L O L O O O L L O O O	O L O L L O O L O O L	L O O L L O O O L L O	
	M14	L L L L L L L L L L L L L L L	L L L L L L L L L L L	L L L L L L L L L L L	
	M15	L L L L L L L L L L L L L L L	L L L L L L L L L L L	L L L L L L L L L L L	
	M16	L O L O L O L O O O L L O O O	O L O L L O O L O O L	L O O L L O O O L L O	
	M 2	L O L O L O L O O O L L O O O	O L O L L O O L O O L	L O O L L O O O L L O	
	M 3	O L O O O L O L O L O O O L L	L L L O O L L O O O O	O L L O L O O L O O L	
	M 5	O L O O O L O L O L O O O L L	L L L O O L L O O O O	O L L O L O O L O O L	
	M 6	L O L O L O L O O O L L O O O	O L O L L O O L O O L	L O O L L O O O L L O	
	M 9	O O O O O O O O O O O O O O O	O L O O O O O O O O O	O O O O L O O O O O O	
H3	M11	L L L L L L L L L L L L L L L	L L L L L L L L L L L	L L L L L L L L L L L	
	M12	L L L L L L L L L L L L L L L	L L L L L L L L L L L	L L L L L L L L L L L	
	M13	O L O O O L O L O L O O O L L	L L L O O L L O O O O	O L L O L O O L O O L	
	M14	L O L O L O L O O O L L O O O	O L O L L O O L O O L	L O O L L O O O L L O	
	M15	L O L O L O L O O O L L O O O	O L O L L O O L O O L	L O O L L O O O L L O	
	M16	O L O O O L O L O L O O O L L	L L L O O L L O O O O	O L L O L O O L O O L	

	M 1	O L O L O L O L L L O O L L L	L O L O O L L O L L O	O L L O O L L L O O L	
	M 2	L O L L L O L O L O L L L O O	O O O L L O O L L L L	L O O L O L L O L L O	
	M 3	O L O L O L O L L L O O L L L	L O L O O L L O L L O	O L L O O L L L O O L	
	M 4	L L L L L L L L L L L L L L L	L L L L L L L L L L L	L L L L L L L L L L L	
	M 5	O L O L O L O L L L O O L L L	L O L O O L L O L L O	O L L O O L L L O O L	
	M 6	L O L L L O L O L O L L L O O	O O O L L O O L L L L	L O O L O L L O L L O	
	M 7	O L O L O L O L L L O O L L L	L O L O O L L O L L O	O L L O O L L L O O L	
H1	M 8	L O L L L O L O L O L L L O O	O O O L L O O L L L L	L O O L O L L O L L O	
	M 9	L L L L L L L L L L L L L L L	L L L L L L L L L L L	L L L L L L L L L L L	
	M10	L O L L L O L O L O L L L O O	O O O L L O O L L L L	L O O L O L L O L L O	
	M11	O L O L O L O L L L O O L L L	L O L O O L L O L L O	O L L O O L L L O O L	
	M12	O L O L O L O L L L O O L L L	L O L O O L L O L L O	O L L O O L L L O O L	
	M13	L L L L L L L L L L L L L L L	L L L L L L L L L L L	L L L L L L L L L L L	
	M14	L O L L L O L O L O L L L O O	O O O L L O O L L L L	L O O L O L L O L L O	
	M15	L O L L L O L O L O L L L O O	O O O L L O O L L L L	L O O L O L L O L L O	
	M 1	L O L L L O L O L O L L L O O	O O O L L O O L L L L	L O O L O L L O L L O	
	M 4	O O O L O O O O L O O O L O O	O O O O O O O O L L O	O O O O O L L O O O O	
	M 7	L O L L L O L O L O L L L O O	O O O L L O O L L L L	L O O L O L L O L L O	
	M 8	O L O L O L O L L L O O L L L	L O L O O L L O L L O	O L L O O L L L O O L	F
	M10	O L O L O L O L L L O O L L L	L O L O O L L O L L O	O L L O O L L L O O L	
H2	M11	L O L L L O L O L O L L L O O	O O O L L O O L L L L	L O O L O L L O L L O	
	M12	L O L L L O L O L O L L L O O	O O O L L O O L L L L	L O O L O L L O L L O	
	M13	O L O L O L O L L L O O L L L	L O L O O L L O L L O	O L L O O L L L O O L	
	M14	L L L L L L L L L L L L L L L	L L L L L L L L L L L	L L L L L L L L L L L	
	M15	L L L L L L L L L L L L L L L	L L L L L L L L L L L	L L L L L L L L L L L	
	M16	O L O L O L O L L L O O L L L	L O L O O L L O L L O	O L L O O L L L O O L	
	M 2	O L O L O L O L L L O O L L L	L O L O O L L O L L O	O L L O O L L L O O L	
	M 3	L O L L L O L O L O L L L O O	O O O L L O O L L L L	L O O L O L L O L L O	
	M 5	L O L L L O L O L O L L L O O	O O O L L O O L L L L	L O O L O L L O L L O	
	M 6	O L O L O L O L L L O O L L L	L O L O O L L O L L O	O L L O O L L L O O L	
	M 9	O O O L O O O O L O O O L O O	O O O O O O O O L L O	O O O O O L L O O O O	
H3	M11	L L L L L L L L L L L L L L L	L L L L L L L L L L L	L L L L L L L L L L L	
	M12	L L L L L L L L L L L L L L L	L L L L L L L L L L L	L L L L L L L L L L L	
	M13	L O L L L O L O L O L L L O O	O O O L L O O L L L L	L O O L O L L O L L O	
	M14	O L O L O L O L L L O O L L L	L O L O O L L O L L O	O L L O O L L L O O L	
	M15	O L O L O L O L L L O O L L L	L O L O O L L O L L O	O L L O O L L L O O L	
	M16	L O L L L O L O L O L L L O O	O O O L L O O L L L L	L O O L O L L O L L O	

FIG. 4. Implication matrices E, F of **T**.

It can be shown that full rows and full columns in E are indicators for pseudoavailable and tight assignments respectively. Therefore, α is the class of pseudoavailable assignments and δ is the class of tight assignments. In the problem of Example 4.4 there are exactly the two solutions $\beta \cup \delta$ and $\gamma \cup \delta$.

The equivalence relation $E \wedge E^{\mathrm{T}}$ together with the partial order on the classes is demonstrated by Fig. 5. We write (i,j) instead of (M_i, H_j).

$$
\begin{tikzcd}
 & \beta \arrow[dr] & \\
\alpha \arrow[ur] \arrow[rr] \arrow[dr] & & \delta \\
 & \gamma \arrow[ur] &
\end{tikzcd}
\qquad
\begin{aligned}
&\alpha = \{(4,1),(9,1),(13,1),(14,2),(15,2),(11,3),(12,3)\}\\
&\beta = \{(1,1),\dots\}\\
&\gamma = \{(2,1),\dots\}\\
&\delta = \{(4,2),(9,3)\}
\end{aligned}
$$

FIG. 5. Partial order and equivalence classes determined by E.

In [5, 6] the timetable problem of Example 4.4 was shown to be mincut-flexible without being flexible, and therefore it is a short counterexample to the Csima-Gotlieb conjecture. Since for this problem E, F are evaluated by Algorithm 5.1 we have an example that the technique of Boolean matrix iteration can be more powerful than Gotlieb's process of reducing the availability array. On the other hand, trivial examples can be found for which the iteration fails to produce results, whereas Gotlieb's method succeeds. Pursuing some considerations of Sec. 3 we add the following remark.

REMARK 4.5. If $\mathbf{S}$ is the set of states of a timetable problem with $h \equiv L$, then $\mathbf{S}$ is closed with respect to the closure operation

$$\mathbf{S} \rightarrow \mathbf{S}_{E_{\mathbf{S}}, F_{\mathbf{S}}, C_{\mathbf{S}}}.$$

Proof. Theorem 2.5 is used first to derive $\Phi \subset F_{\mathbf{S}}$. If $\Phi(i,k)$ holds, it is easy to show that the validity of $x(i)$ in a solution x implies $\overline{x(k)}$. Therefore $F_{\mathbf{S}}(i,k)$ must hold. Then $x \in \mathbf{S}_{E_{\mathbf{S}}, F_{\mathbf{S}}, C_{\mathbf{S}}} \in \mathbf{B}^{\mathrm{F}}$ implies $\bar{x} \supset F_{\mathbf{S}} x \supset \Phi x$, and therefore x is a state. ∎

5. ITERATION OF ELEMENTARY IMPLICATIONS

The goal of timetable construction is to obtain at least one solution. Therefore it would be unsound to try it, working with the implication matrices, if these themselves were evaluated from the set of all solutions.

Nevertheless there are situations where it is profitable to approximate the implication matrices by an iterative technique starting with the elementary implication matrices.

The main idea of this iteration for normalized problems is based heavily on the local implication property of Theorems 4.1 and 4.3:

Let Θ be an elementary forbidding matrix either of type φ or of type φ_P; the equation $E=\overline{\overline{F}\Theta}$, written in the form

$$\bigwedge_{M^*,H^*}\left[\Theta(M^*,H^*,M',H')\Rightarrow F(M,H,M^*,H^*)\right]=E(M,H,M',H'),$$

can then be interpreted as: "If an assignment (M,H) forbids all assignments (M^*,H^*) which elementarily forbid the assignment (M',H'), then the assignment (M',H') is enforced by (M,H)." Therefore, the effectiveness of the algorithm does crucially depend on the frequent occurrence of the situation in which an assignment forbids all assignments but one of a participant P_o at hour H_o.

Before formulating the iteration, we recall the concepts of the transitive and the symmetric closure of Boolean matrices. A matrix $E\in\mathbf{B}^{N\times N}$ is said to be transitive if $E^2\subset E$; in the case $I\subset E$ we can define the *transitive closure* of E to be $E^{\text{t.c.}}:=E^{|N|-1}$. There are well-known techniques for abbreviating the evaluation of $E^{\text{t.c.}}$. The *symmetric closure* of a matrix F is $F^{\text{s.c.}}:=F\wedge F^{\mathsf{T}}$.

ALGORITHM 5.1. *Let φ,φ_P,Φ be the elementary forbidding matrices of a normalized timetable problem* T *with $h\equiv L$ and with implication matrices E,F,C. Take the matrices $E_0:=I$ and $F_0:=\Phi$ as initial values in the iteration*

$$E_{n+1}:=\left[E_n\vee\overline{\overline{F}_n\varphi}\vee\bigvee_P\overline{\overline{F}_n\varphi_P}\right]^{\text{t.c.}},$$

$$F_{n+1}:=\left[E_{n+1}F_n\right]^{\text{s.c.}}.$$

Then the following assertions are valid:

(i) *The iteration yields monotone sequences*

$$E_0\subset E_1\subset E_2\subset\cdots\quad\text{and}\quad F_0\subset F_1\subset F_2\subset\cdots.$$

We denote the resulting matrices by E_∞, F_∞.

(ii) *Together with the matrix*

$$C_\infty := \overline{\bar{E}_\infty^{\mathsf T} F_\infty}$$

we have

$$E_\infty \subset E, \qquad F_\infty \subset F, \qquad C_\infty \subset C.$$

(iii) *The matrices* $E_\infty, F_\infty, C_\infty$ *fulfill the relations of implication matrices stated in Theorem* 3.5 *and Corollary* 3.6 *and the special relations of Theorem* 4.1, *Corollary* 4.2 *and Theorem* 4.3, *i.e.*,

$$E_\infty = \overline{\bar{F}_\infty \varphi} = \overline{\bar{F}_\infty \varphi_P} = \overline{\bar{F}_\infty F_\infty}\,, \qquad C_\infty = \overline{\bar{E}_\infty^{\mathsf T} \varphi} = \overline{\bar{E}_\infty^{\mathsf T} \varphi_P}\,.$$

Proof. (i) and (ii) are proved by induction using Theorems 4.1 and 4.3. In the proof of (iii) we will write φ_* for convenience instead of φ and E, F, C instead of $E_\infty, F_\infty, C_\infty$. Introducing $J := \mathbf{P} \cup \{*\}$, we can formulate $E_{n+1} = [E_n \vee \bigvee_{j \in J} \bar{F}_n \varphi_j]^{\mathrm{t.c.}}$. We make use of Lemmas 2.1 and 2.2. $I \subset E = E^2$, and $F = F^{\mathsf T}$ is valid because of the construction. $I \subset E \Rightarrow F \subset EF$, and since the opposite inclusion is valid by construction, we have

$$EF = F. \tag{1}$$

$I \subset E^{\mathsf T} \Rightarrow C \subset E^{\mathsf T} C$ and $E^{\mathsf T} C = E^{\mathsf T} \overline{\bar{E}^{\mathsf T} F} \subset \overline{E^{\mathsf T} \bar{E}^{\mathsf T} F} = \overline{\bar{E}^{\mathsf T} F} = C$ imply

$$CE = C. \tag{2}$$

$\varphi_j \subset F^{\mathsf T} \Rightarrow I \subset \overline{\bar{F} \varphi_j} \Rightarrow E \subset E \overline{\bar{F} \varphi_j} \subset \overline{E \bar{F} \varphi_j} = \overline{\bar{F} \varphi_j}$. This yields, together with the opposite inclusion (which is true by construction),

$$E = \overline{\bar{F} \varphi_j}\,. \tag{3}$$

$EF \subset F \Leftrightarrow \bar{F} F \subset \bar{E}$ and $\varphi \subset F \Rightarrow \overline{\bar{F} F} \subset \overline{\bar{F} \varphi} = E$ imply

$$\overline{\bar{F} F} = E. \tag{4}$$

From (1) and (4) we derive

$$FC = F\overline{\overline{E^{\mathsf{T}}}F} \subset \overline{\overline{FE^{\mathsf{T}}}F} = \overline{\bar{F}F} = E, \qquad \text{i.e.,} \quad FC \subset E. \tag{5}$$

From $\varphi_j \subset F$ and the definition of C,

$$C \subset \overline{\overline{E^{\mathsf{T}}}\varphi_j} \tag{6}$$

follows, and from (3) we get $C^{\mathsf{T}} = \overline{F\bar{E}} = \overline{F\bar{F}\varphi_j} = \overline{\bar{E}^{\mathsf{T}}\varphi_j}$, i.e.,

$$C^{\mathsf{T}} = \overline{\bar{E}^{\mathsf{T}}\varphi_j}\,. \tag{7}$$

(6) and (7) imply $C^{\mathsf{T}} \supset C$; hence $C \supset C^{\mathsf{T}}$, and therefore

$$C = C^{\mathsf{T}}. \tag{8}$$

This changes (7) to

$$C = \overline{\overline{E^{\mathsf{T}}}\varphi_j}\,.$$

∎

We remark that the determination of E_{n+1} in the algorithm is not stated in the form $E_{n+1} = [\overline{\bar{F}_n\Phi}]^{\text{t.c.}}$, since the lack of $\wedge$-distributivity of Boolean matrix multiplication may cause smaller resulting matrices. The initial values of the iteration which we had chosen were

$$E_0 := I, \qquad F_0 := \Phi.$$

In practice we look for solutions which, in addition to the general requirements of Definition 1.2, fulfill some other special requirements. The set $\mathbf{S}$ of solutions is then diminished to $\mathbf{S}^* \subset \mathbf{S}$ with implication matrices [see Theorem 3.4 (i)] related by $E_{\mathbf{S}} \subset E_{\mathbf{S}^*}, F_{\mathbf{S}} \subset F_{\mathbf{S}^*}, C_{\mathbf{S}} \subset C_{\mathbf{S}^*}$.

In the next section we will see that the special requirements usually occurring are relations between assignments of an enforcing or of a mutual forbidding type, which subsume to the formalism of the definition of implication matrices. In other words we assume the *special requirements* to be determined by two matrices $R_e, R_f \in \mathbf{B}^{\mathrm{F}\times\mathrm{F}}$ with the meaning

$$\text{if}\quad R_e(M,H,M',H'),\quad \text{then}\quad \bigwedge_{s\in \mathbf{S}^*} s(M,H) \Rightarrow s(M',H'),$$
$$\text{if}\quad R_f(M,H,M',H'),\quad \text{then}\quad \bigwedge_{s\in \mathbf{S}^*} s(M,H) \Rightarrow \overline{s(M',H')}\,.$$

Hence the inclusions $R_e \subset E_{\mathbf{S}^*}$, $R_f \subset F_{\mathbf{S}^*}$ are valid, and we start the iteration with

$$E_0 := I \vee R_e, \qquad F_0 := \Phi \vee R_f,$$

thus obeying the special requirements. We will then still have (i) and (ii) of Algorithm 5.1.

The iteration algorithm, applied to the elementary implication matrices of the timetable problem of Example 4.4, results with $E_0 = I$, $F_0 = \Phi$ in the matrices E, F given in Fig. 4, which are indeed the implication matrices. To show the limited power of our technique of Boolean matrix iteration, we modify this problem slightly.

Example 5.2. In Fig. 6 a normalized timetable problem $\mathbf{T}$ is given, $(h \equiv L)$. It is derived from the problem of Example 4.4 by substituting the subproblem shown in Fig. 7 for selected assignments. From the matrices $E_\infty, F_\infty, C_\infty$ computed by Algorithm 5.1, starting with $E_0 = I$, $F_0 = \Phi$, we present in Fig. 8 only the partial order of E_∞, neglecting those equivalence classes that are totally disconnected from another. We have used (i,j) instead of (M_i, H_j); equivalent assignments are enclosed in boxes.

If we consider the solutions of the subproblem of $\mathbf{T}$, we conclude that $E(M,H_i,M',H_i)$ and $E(M',H_i,M,H_i)$ hold. Therefore this subproblem can be replaced by a simple assignment, resulting in the problem of Example 4.4 without affecting solvability. But the partial order of E_∞ indicates neither the enforcings $E(M,H_i,M',H_i)$ nor the pseudoavailable or tight assignments.

6. PRACTICAL ASPECTS

In practice some storage problems could arise in connection with a naive implementation of Algorithm 5.1. Of course, there is no need to make φ and φ_P, $P \in \mathbf{P}$, available as arrays. We discuss some special requirements as indicated in Sec. 1.

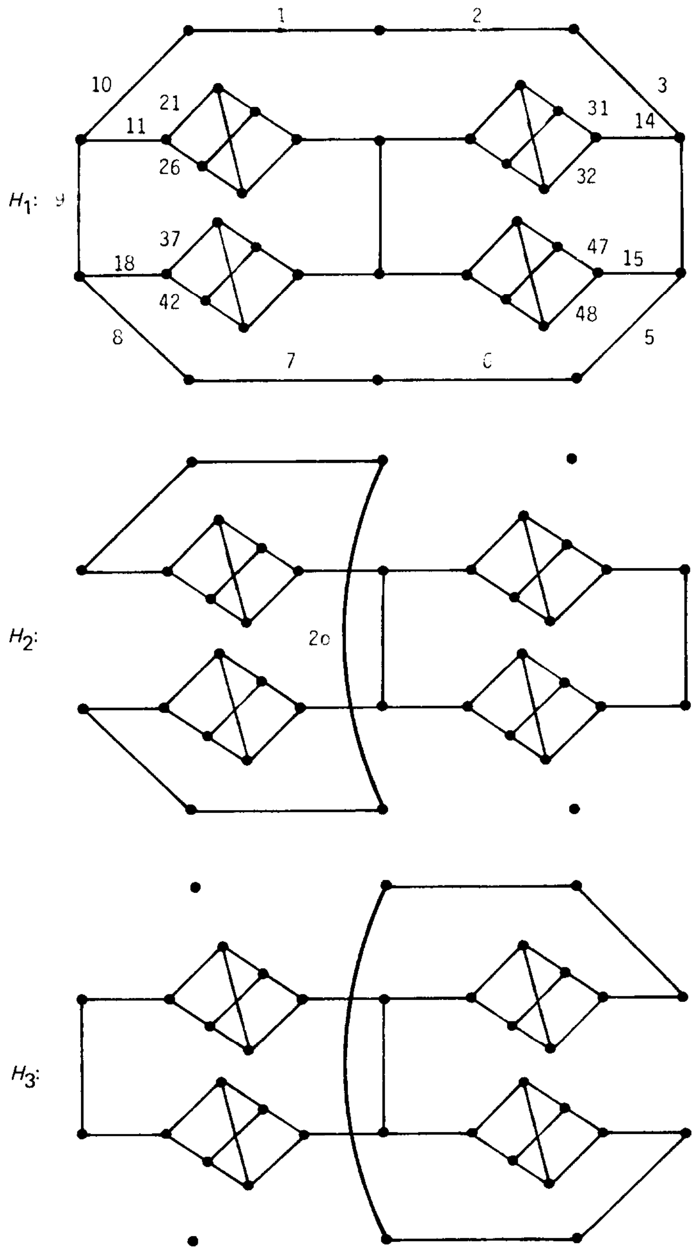

FIG. 6. Solvable, not flexible timetable problem T.

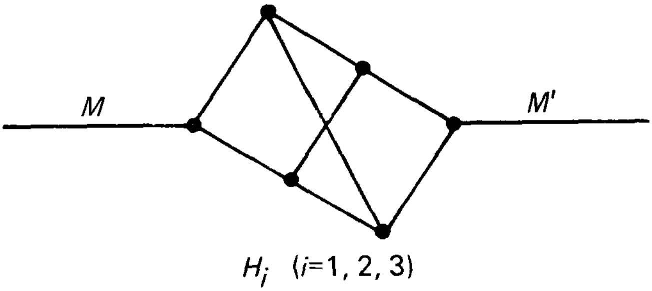

FIG. 7. Subproblem of T with $E(M, H_i, M', H_i)$.

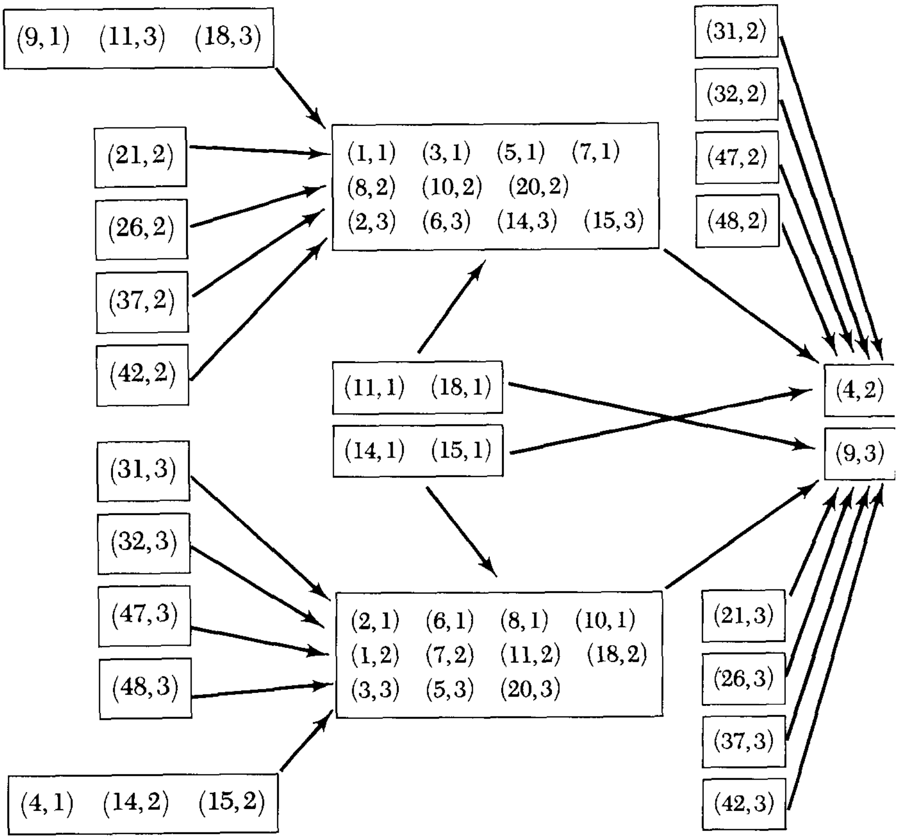

FIG. 8. The partial order of E_∞ (nontrivial connected component).

6.1. Special Requirements of the Forbidding Type

If we widen our concept and introduce a function $h: \mathbf{M} \rightarrow \mathbf{N}$, where $h(M)$ determines the number of hours required for a meet, then the condition $h(M) \leqslant 1$ has been imposed throughout this paper. This is not a vital restriction. A meet M with $h(M) > 1$ is mainly discussed in view of the *distribution problem* (cf. [5, 6]), in which the assignment of more than one of the $h(M)$ different lectures of this meet to the same *day* $d \subset \mathbf{H}$ is to be avoided. In connection with the iteration technique, the distribution problem can be solved by simply introducing $h(M)$ meets instead of the single meet M, and by entering the mutual forbiddings of the assignments of these $h(M)$ meets which belong to the same day d into the matrix R_f describing special requirements of the forbidding type.

There are several other requirements that result in a mutual forbidding of assignments and that can be added to the matrix R_f. For instance:

1. The noncompatibility condition for M and M', according to which an assignment of the meet M at an hour of day $d \subset \mathbf{H}$ is for some reason not compatible with the assignment of the meet M' at any hour of that day.

2. Meets M with only two lectures [i.e., $h(M) = 2$] must not be simultaneously assigned on Saturday and on Monday, because there would not be enough time for homework on the weekend.

6.2. Special Requirements of the Enforcing Type

There are two main requirements that can be formulated as enforcings of assignments:

1. If M is a meet with $h(M) = 2$ and if the two lectures of M are to be held consecutively, M should be split into two meets M, M' for which $R_e(M, H, M', H+1)$ and $R_e(M', H+1, M, H)$ hold.

2. A well-known requirement in timetable construction is that two different meets be assigned consecutively in order (for instance) to use preparations for a lecture in physics twice.

Enforcings of this type are added to the matrix R_e.

The whole theory of implications has been developed in order to replace mincut arguments and elaborate recursive exchange operations in situations where they tend to be ineffective and to try long-range planning in these cases. As was mentioned in Sec. 5, the effectiveness of the Algorithm 5.1 depends heavily on the given timetable problem. We think that in the final period of the construction of a timetable, if about 80% of the lectures are scheduled using other techniques, an analysis of the residual problem by the iteration of the implication matrices is practicable and may be profitable.

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