EXERCISE 1

1.) Solve the Laplace-Poisson equation $L(u) = f$ inside a disk of radius a with Neumann boundary conditions—normal derivative is zero on the boundary. As the right-hand side, use the characteristic function of a disk of radius $b < a$; that is, f is 1 inside the disk of radius b and zero outside. Organize your exposition as follows:

(a) (30 Points) Present a clear derivation of the symbolic solution. Remember that clarity is not directly proportional to the length of exposition. In fact, suppress rote algebra and calculus so as not to distract the reader from the logic of the derivation

(b) (20 Points) Plot the (truncated series) solution for $b = 1$ and $a = 2, 4, 8, 16$. Comment on the plots: do they conform with you intuition?

(c) (10 Points) If $b = 1$ is held fixed and $a \to \infty$, what happens to the solution? Does the limit exist? If "Yes," try to find it.

Solution (Part 3): Now, after revisiting this after a couple of years and re-doing the calculation, I have been able to, without a doubt, complete this exercise in its entirety. I was on the right track, but my second solution is incorrect, more incorrect than the first solution. But the idea was correct. When we project f onto the appropriate space, we obtain the following graph in Figure 1, where Figure 3a show f (left) and Figure 3b shows f expressed as an element of $L^2(\Omega)$ (right). To understand how we arrive at the correct solution, we re-examine the derivation of the solution from the first solution. First, we understand that there are two PDEs that we need to solve: the first being the general solution to the homogeneous PDE, which we previously believed was equal to zero, and the particular solution to the nonhomogeneous PDE, which in general is

$$u_2(r, \theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m\big(\sqrt{\lambda_{mn}}r\big)(A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)), \tag{4}$$

both found in equation (2) with the homogeneous equation on the left and the nonhomogeneous equation on the right. Now, my error arose in the second solution by thinking the general solution in equation (4) was the projection of f onto $L^2(\Omega)$, which is incorrect. The above is, as we know from the first attempt of the solution, the solution to the Helmholtz equation. Since we know from the first solution that the first eigenvalue $\lambda_{01} = 0$, we know that the right-hand side of equation (1) is not in the image of $L^2(\Omega)$, that is $f \notin im(L^2(\Omega))$. So, in order to solve equation (1), we must, as I correctly described in solution 2, project f onto $L^2(\Omega)$. To project f onto this particular space, we know that, through the Helmholtz equation and its solution,

$$\Delta u = -\lambda u = f.$$

meaning that

$$f = -\lambda_{mn} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m \left(\sqrt{\lambda_{mn}} r \right) \left(A_{mn} \cos(m\theta) + B_{mn} \sin(m\theta) \right).$$

However, before continuing with the projections, if we compute the first few terms of the summation in m for equation (4), we find that the only valid value for m is $m = 0$ and that the other values of m result in zeros. Thus, we obtain

$$f = -\lambda_{0n} \sum_{n=1}^{\infty} A_n J_0 \left(\sqrt{\lambda_{0n}} r \right), \tag{5}$$

which means

$$u_2(r, \theta) = \sum_{n=1}^{\infty} A_n J_0 \left(\sqrt{\lambda_{0n}} r \right). \tag{6}$$

Further, since we know the space is defined by linear combinations of zero-order Bessel functions, we can write f as a linear combination of them,

$$f = \sum_{n=1}^{\infty} C_n J_0 \left(\sqrt{\lambda_{0n}} r \right), \tag{7}$$

Now, we may project onto the space in equation (7) in order to express f as an element of the space. To project f onto the space, we compute the inner product of both sides of equation (7) with $J_0 \left(\sqrt{\lambda_{0n}} r \right)$ in order to compute the coefficients C_n,

$$\left\langle f, J_0 \left(\sqrt{\lambda_{0n}} r \right) \right\rangle = C_n \left\langle J_0 \left(\sqrt{\lambda_{0n}} r \right), J_0 \left(\sqrt{\lambda_{0n}} r \right) \right\rangle,$$

where the inner product $\langle \cdot, \cdot \rangle$ is the L^2 inner product

$$\langle h, g \rangle = \int_0^{2\pi} \int_0^a h(r) g(\theta) \ r dr d\theta,$$

and the sum is not actually omitted, but rather implied. In solving for C_n, we find that

$$C_n = \frac{\left\langle f, J_0 \left(\sqrt{\lambda_{0n}} r \right) \right\rangle}{\left\langle J_0 \left(\sqrt{\lambda_{0n}} r \right), J_0 \left(\sqrt{\lambda_{0n}} r \right) \right\rangle}.$$

Now that we have an expression for the right-hand side that resides in the space $L^2(\Omega)$, we can equate equations (5) and (7),

$$-\lambda_{0n} \sum_{n=1}^{\infty} A_n J_0 \left(\sqrt{\lambda_{0n}} r \right) = f = \sum_{n=1}^{\infty} C_n J_0 \left(\sqrt{\lambda_{0n}} r \right).$$

By taking the inner product of both sides again with $J_0\left(\sqrt{\lambda_{0n}}r\right)$, we find that

$$-\lambda_{0n}A_n\left\langle J_0\left(\sqrt{\lambda_{0n}}r\right), J_0\left(\sqrt{\lambda_{0n}}r\right)\right\rangle = C_n\left\langle J_0\left(\sqrt{\lambda_{0n}}r\right), J_0\left(\sqrt{\lambda_{0n}}r\right)\right\rangle.$$

Since inner products return scalars, we can simplify the inner products on each side and we find that

$$A_n = -\frac{C_n}{\lambda_{0n}}.$$

Thus the solution for the nonhomogeneous problem is

$$u_2(r,\theta) = -\sum_{n=1}^{\infty}\frac{C_n}{\lambda_{0n}}J_0\left(\sqrt{\lambda_{0n}}r\right).$$

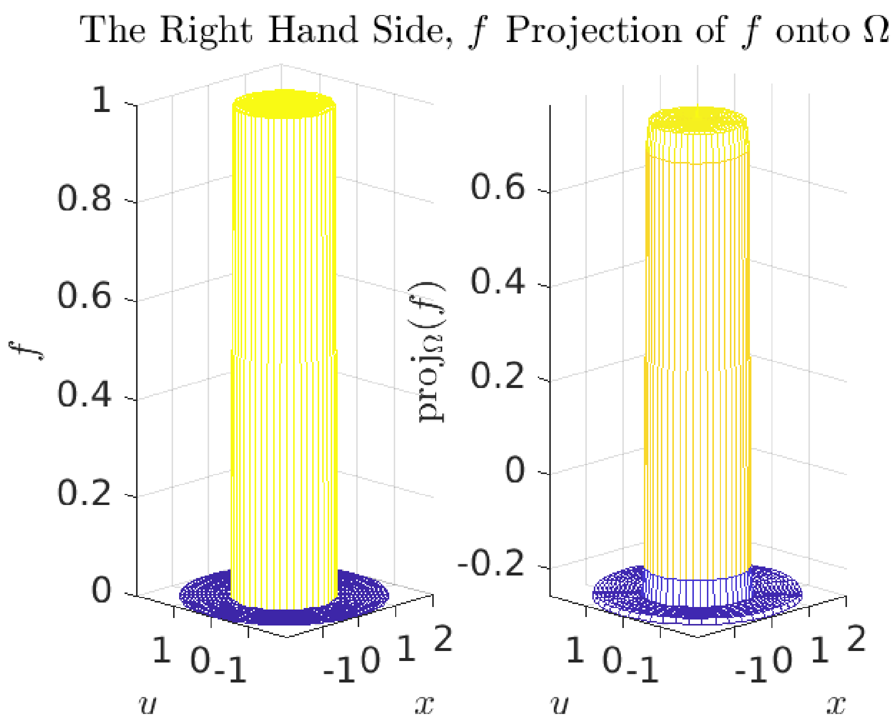
But what about the solution to the homogeneous PDE? Is it equal to zero? Certainly not. The reason being that the first eigenvalue is still zero, that is $\lambda_{01} = 0$. What does this mean? This means that the homogeneous PDE has a non-trivial solution. And the only solution for the homogeneous problem is $u_1(r,\theta) = K$, where K is a constant. We can check by substituting $u_1 = K$ into the PDE and we find that indeed the boundary value problem is satisfied. This also means that the sum goes from 2 to ∞. And we expect this result due to the spherical symmetry of the system. Thus the full solution is

$$\boxed{u(r,\theta) = u_1 + u_2 = K - \sum_{n=2}^{\infty}\frac{C_n}{\lambda_{0n}}J_0\left(\sqrt{\lambda_{0n}}r\right).}$$

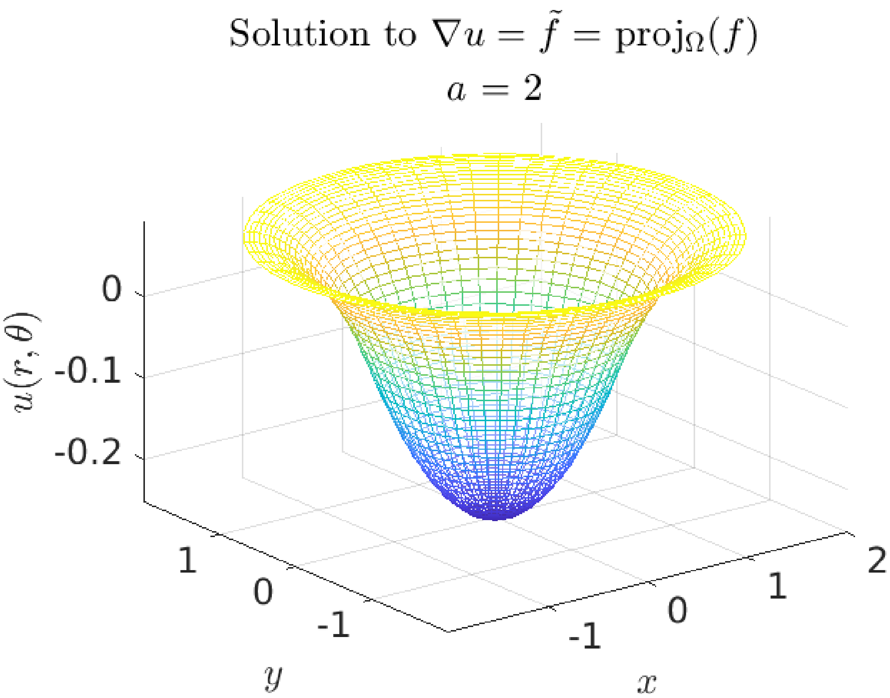
Further, in the limit as $a \to \infty$ this result means that the average of the right-hand side over the region is equal to zero

$$\bar{f} = \frac{1}{\mathrm{vol}(\Omega)}\int_{\Omega} f\ d\Omega = \frac{b^2}{a^2} = 0.$$

Thus $C_1 = \bar{f} = \lambda_{01}\cdot K = 0$. This is true for the Poisson kernel. Physically, we can visualize the solution as the electric potential of a 2-dimensional point charge in the limiting case.

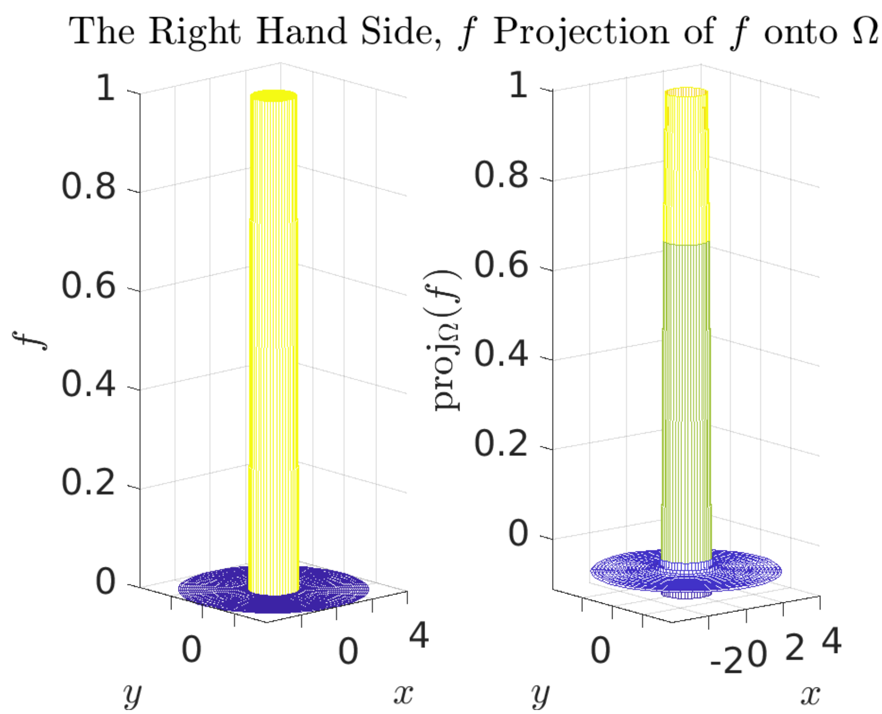


(a)

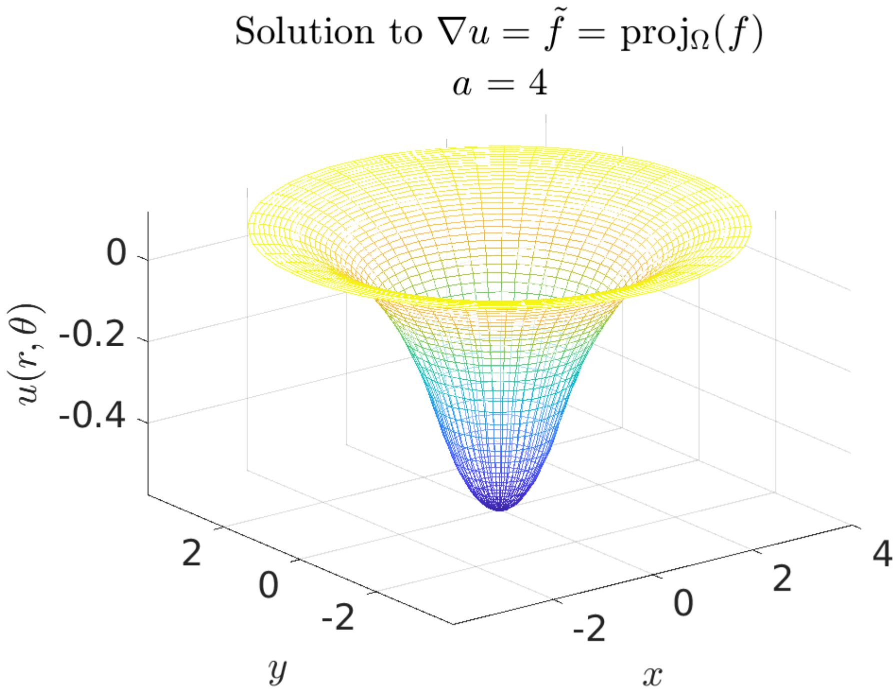


(b)

Figure 1: The correct projection of f onto $L^2(\Omega)$ and the correct solution for $a = 2$.

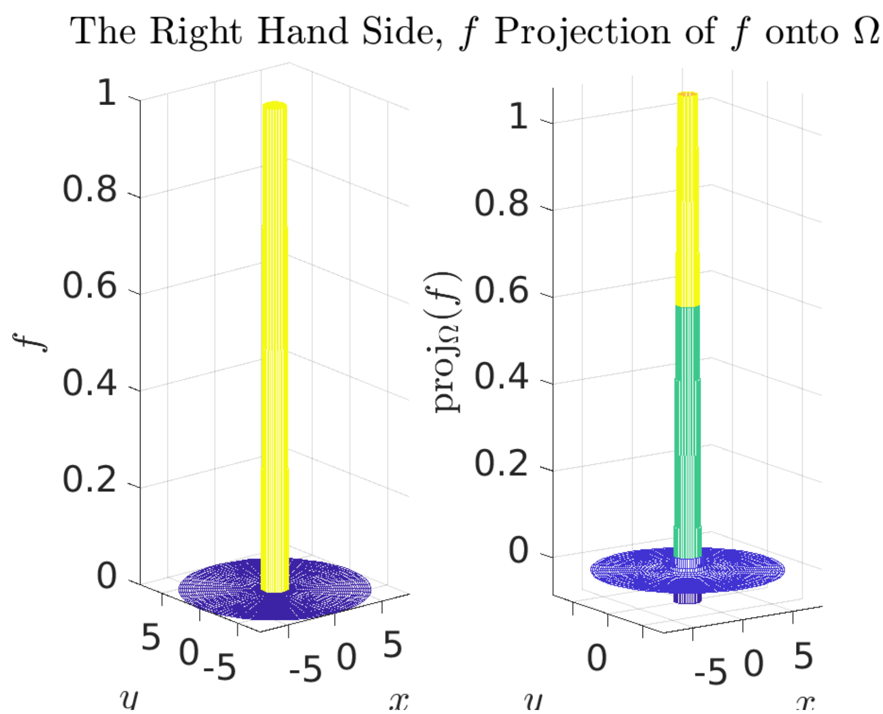


(a)

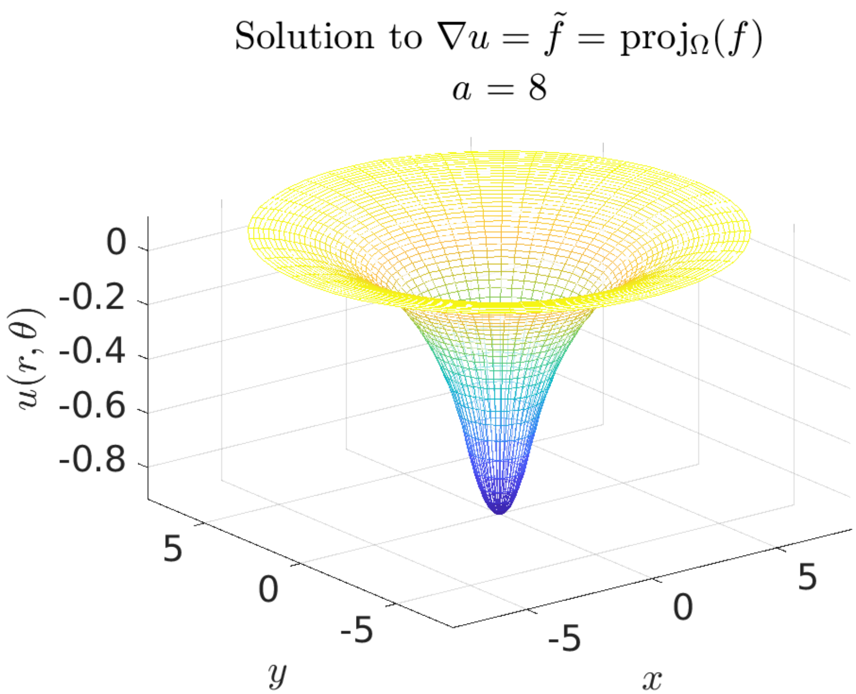


(b)

Figure 2: The right-hand side for $a = 4$.

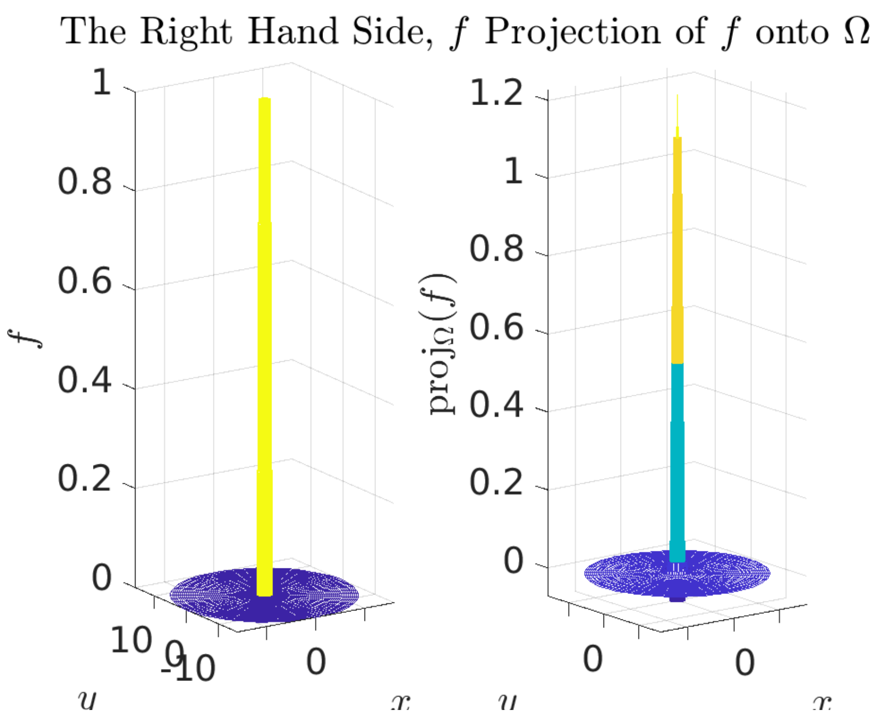


(a)

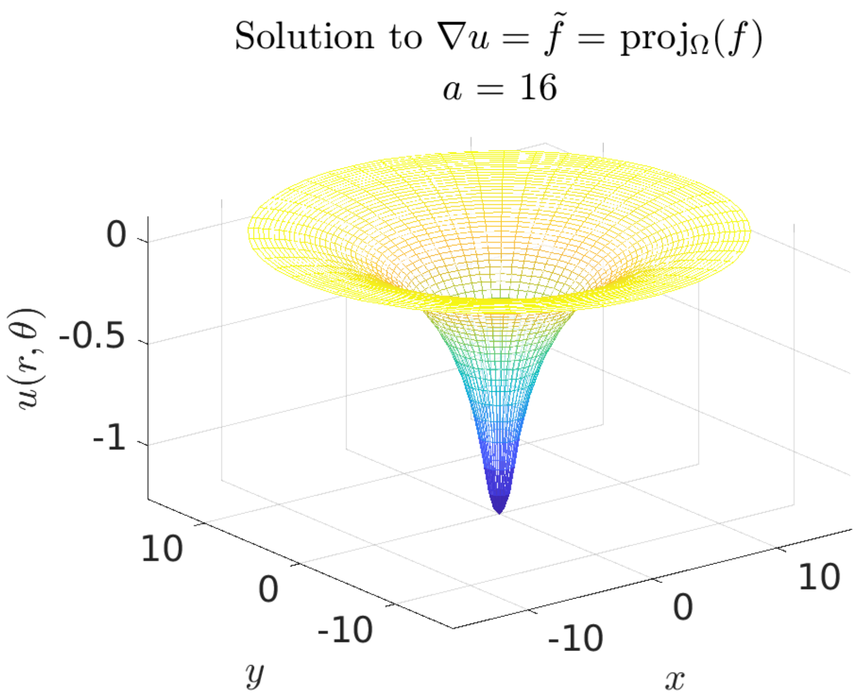


(b)

Figure 3: The right-hand side for $a = 8$.



(a)



(b)

Figure 4: The right-hand side for $a = 16$.