Homework 1 - Math 152

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September 2018

Contents

1. Homework 1

1

When first looking at this exercise, I was unsure on how I would compute this numerically, so one of my peers helped me with the code. Then, to understand what the code meant (the nested for loops), I went through the code, line by line, and figured out what it did, especially because I had not worked with iterators in Matlab. The nested for-loops iterate over the values of `i` and provide values for the area of a parallelogram for each `i`. Afterward, the average of the areas are computed for each `i`. Then, the results are plotted. The plot has an interesting behavior and converges to some value. The conjecture I have about the limiting value of the average as the number of parallelograms increases to infinity is this: for vectors in a finite-dimensional vector space V of dimension n, as the number of the areas of parallelograms spanned by elements of V increases, the average of those areas approaches $n-1$. I used two different approaches to attempt to prove this conjecture, but the math was more than I'd anticipated and the method was insufficient. The approaches I used were: (i) defining the cross-product for two vectors $\mathbf{u}, \mathbf{v} \in \mathbf{R}^3$ as the product is the unique vector $(\mathbf{u} \times \mathbf{v}) \in \mathbf{R}^3$ such that

$$\forall \mathbf{w} \in \mathbf{R}^3 \quad (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \det[\mathbf{u}\ \mathbf{v}\ \mathbf{w}], \tag{1}$$

where the right-hand side of equation (1) is the determinant of a matrix whose columns are the vectors $\mathbf{u}, \mathbf{v},$ and $\mathbf{w}$, respectively. Although in two dimensions the determinant gives the area of a parallelogram spanned by two vectors, in three dimensions the determinant gives the volume of a parallelepiped spanned by three vectors. As a result, in dimensions higher than 2, using this method doesn't work to provided areas of parallelograms. There were interesting behaviors, though (that I will include in the homework project). (ii) Generalizing the cross-product for any two vectors $\mathbf{u}, \mathbf{v}$ with n components such that

$$\begin{aligned} |\mathbf{u} \times \mathbf{v}| &= \sqrt{u^2 v^2 \sin^2(\theta)} \\ &= \sqrt{u^2 v^2 (1 - \cos^2(\theta)} \\ &= \sqrt{u^2 v^2 - (\mathbf{u} \cdot \mathbf{v})^2}, \end{aligned}$$

where θ is obtained from the dot-product of $\mathbf{u}$ and $\mathbf{v}$. Using this method to attempt to prove the conjecture was difficult and involved many summations that I tried to simplify, which did not go very well. This method did allow me to confirm my conjecture numerically. Here are the plots.

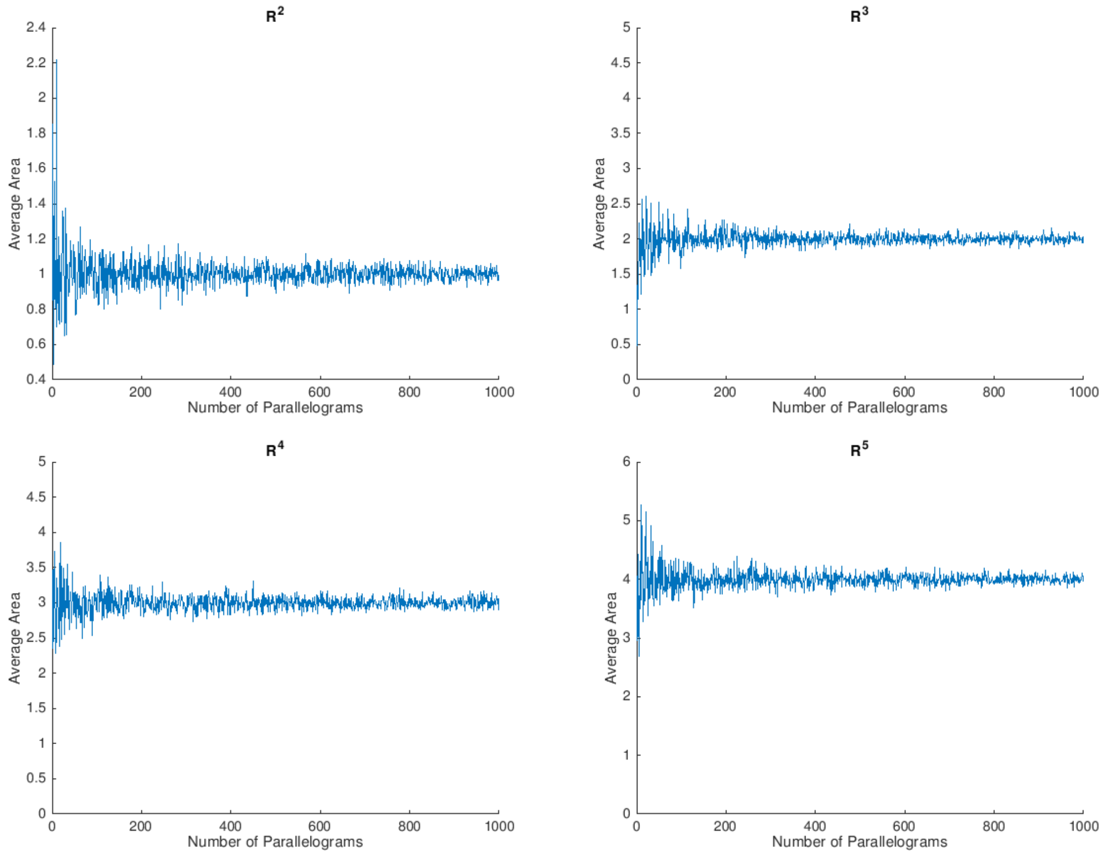


Figure 1: Plots of the average area as the number of parallelograms increase.

```
%% Exercise 1
n = 1000;
A = ones(1,n);
for i = 1:n
    a = 0;
    for j = 1:i
        u = randn(3,1);
        v = randn(3,1);
        a = a + sqrt(((norm(u)^2)*(norm(v)^2) - (dot(u,v)^2)));
    end
    A(i) = a/i;
end
figure
hold on
plot(1:n,A);
title('R^{3}')
xlabel('Number of Parallelograms');
ylabel('Average Area');
ylim([0 5]);

% To obtain the other plots, change the number of components in u and v
```

2

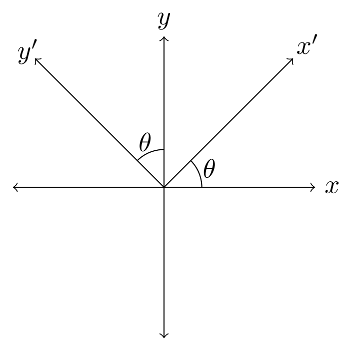


Figure 2: Rotation about the origin.

For a vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ in $\mathbf{R}^2$ defined in the standard basis, if a rotation about the origin of θ degrees is performed on the coordinate system to provide the transformed vector $\mathbf{v} = a'\mathbf{i}' + b'\mathbf{j}'$, then the primed components can be described by the following matrix

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a' \\ b' \end{bmatrix}, \tag{2}$$

where the first column of the matrix denotes the transformation of the x-component and the second column denotes the transformation of the y-component. The matrix can be obtained by examining the axes of the rotated coordinate-system in Figure 2. The x'-axis can be described in terms of the x-axis that has an x-component of $\cos(\theta)$ and a y-component of $\sin(\theta)$. For the y'-axis, it can be described such that the x-component is $-\sin(\theta)$, using trigonometric identities and the fact that the rotation is in the $-\mathbf{i}$ direction, and the y-component is $\cos(\theta)$. Thus, equation (2) describes a counter-clockwise rotation (positive rotation) about the origin, in the standard basis.

3

For a disk with a given radius r, the position of each point on the disk is generally given by

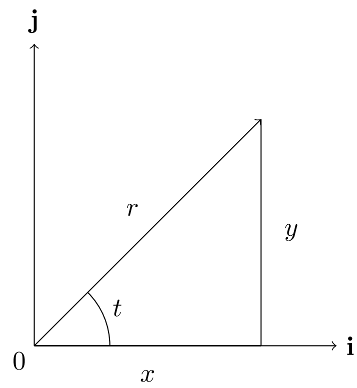
$$\mathbf{r}(t) = \begin{bmatrix} x_0 + r\cos(t) \\ y_0 + r\sin(t) \end{bmatrix},$$

where the point (x_0, y_0) denotes the center of the arbitrary disk, and the velocity of every point on the disk is given by

$$\frac{d}{dt}[\mathbf{r}(t)] = \mathbf{v}(t) = \frac{d}{dt}\begin{bmatrix} x_0 + r\cos(t) \\ y_0 + r\sin(t) \end{bmatrix} = \begin{bmatrix} -r\sin(t) \\ r\cos(t) \end{bmatrix},$$

producing a very important result: the velocity of any point on any disk is independent of the center of that disk, i.e. its position in space. Thus, because the velocity of a rotating object is the product of the radius and the angular velocity ω, then

$$\mathbf{v}(t) = r\omega \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}.$$



To further simplify the expression for the velocity field, using the figure above, $\sin(t)$ and $\cos(t)$ can be expressed as y/r and x/r, respectively, to ultimately obtain a general equation to describe the velocity of any point on the disk,

$$\mathbf{v}(t) = \omega \begin{bmatrix} -y \\ x \end{bmatrix},$$

for any given ω. Figure 3 is a plot of the velocity field of a rotating disk with and angular speed of $\omega = 1$ rad/s.

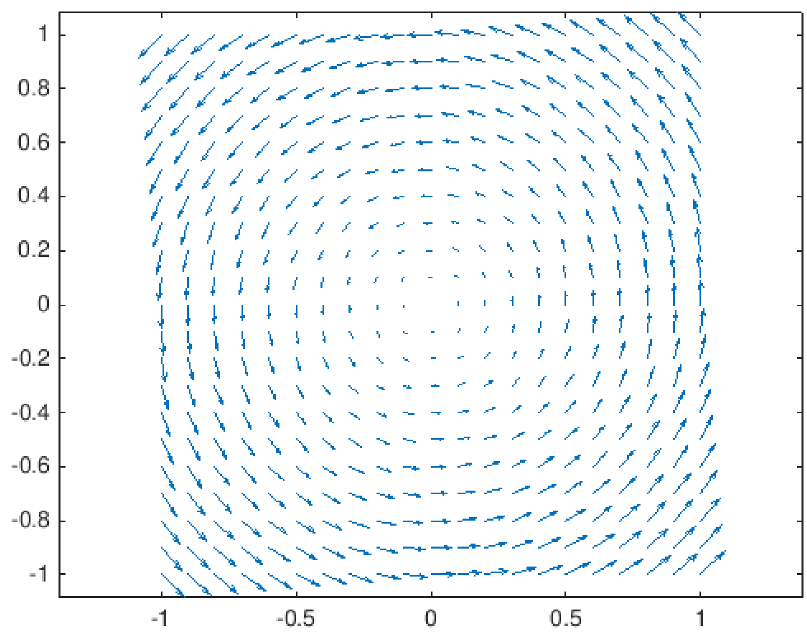


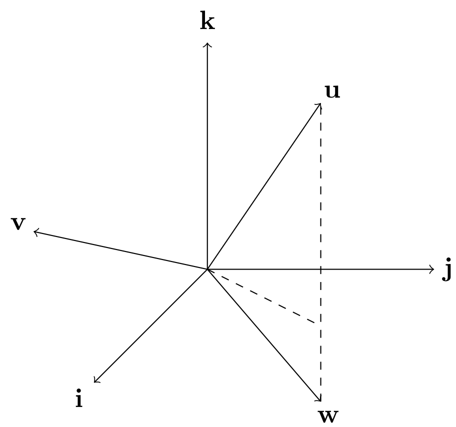
Figure 3: The velocity field of any rotating disk.

```
%% Exercise 3

[x,y] = meshgrid(-1:0.1:1,-1:0.1:1);
```

```
w = 1;
u = -w.*y;
v = w.*x;

figure
quiver(x,y,u,v);
axis equal
```

4

As an extension of Exercise 3, to calculate the velocity field of a rotating sphere about an arbitrary axis $\mathbf{u}$, I used numerical methods because I was unsure of how I would compute the velocity field analytically (it was a mess). I did have a few ideas though. I first tried using the rotational matrices to transform the standard basis in $\mathbf{R}^3$ to an arbitrary basis. In doing so, I tried to define the rotational matrices in terms of an arbitrary vector, which was indeed a challenging exercise. I was able to find some derivations of the overall matrix online, but I wanted to perform the computations myself, which I did until I got stuck.



Then, one of my peers suggested that to obtain an arbitrary basis, take the cross-product $\mathbf{k} \times \mathbf{u} = \mathbf{v}$, and then take the cross-product $\mathbf{u} \times \mathbf{v} = \mathbf{w}$ for three mutually orthogonal vectors $\mathbf{v}, \mathbf{w}, \mathbf{u}$.

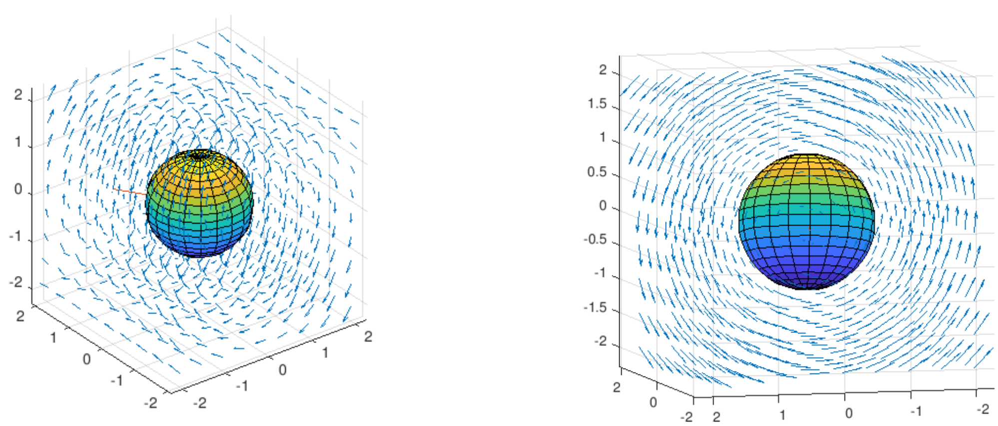


Figure 4: The plot of the velocity field of a rotating sphere about an arbitrary axis.

After generating these random vectors, a meshgrid was made, for which each point in the grid was taken into the new basis using a change of basis matrix, whose columns are the vectors $\mathbf{v}, \mathbf{w}, \mathbf{u}$ in that order. This was the purpose of the first nested for-loop. Then, the second nested for-loop takes the points in the new basis and computes them back into the standard basis, again using the change of basis matrix. Lastly, the plot was made. Both figures are of the same sphere. The left one is the original and the right one was oriented to show the velocity field around the arbitrary vector. Interestingly enough, the field resembles that for a rotating disk, which is expected.

```
%% Exercise 4

h = [0 0 1];
u = randn(1,3);
u = u./norm(u);

v = cross(h,u);
v = v./norm(v);

n = cross(u,v);
n = n./norm(n);

w = 0.2;
r = 1;
[x,y,z] = meshgrid(-r-1:r/2:r+1);

f1 = -w.*r.*y;

A = [v' n' u'];

for i = 1:length(f1);
    for j = 1:length(f1);
        for k = 1:length(f1);
            f = (A\[x(i,j,k);y(i,j,k);z(i,j,k)])';
            dv(i,j,k) = f(1,1);
            dn(i,j,k) = f(1,2);
            du(i,j,k) = f(1,3);
        end
    end
end

f1 = -w.*r.*dn;
f2 = w.*r.*dv;
f3 = 0*f1;

for i = 1:length(f1)
    for j = 1:length(f1)
        for k = 1:length(f1)
            f = (A*[f1(i,j,k);f2(i,j,k);f3(i,j,k)])';
            fv(i,j,k) = f(1,1);
            fn(i,j,k) = f(1,2);
            fu(i,j,k) = f(1,3);
        end
    end
end
```

```
[a,b,c] = sphere;
surf(a*r,b*r,c*r);
hold on
quiver3(x,y,z,fv,fn,fu);
plot3([0 u(1)*(r+1)],[0 u(2)*(r+1)],[0 u(3)*(r+1)]);
axis equal
```

5

```
%% Exercise 5

% The Field
h = 0.01;
a = 1;
b = 1;
r = 1;
[x,y] = meshgrid(-5:h:5);
z = peaks(x,y);
[u,v] = gradient(z,h,h);

% Flux across boundary
t = 0:h:2*pi;
xt = a + r*cos(t);
yt = b + r*sin(t);
ut = interp2(x,y,u,xt,yt);
vt = interp2(x,y,v,xt,yt);
Fnorm = ut.*cos(t) + vt.*sin(t);
flux = sum(Fnorm*h*r);

% Divergence through the interior
divF = divergence(x,y,u,v);
divcircle = divF((x-a).^2 + (y-b).^2 <= r^2);
flux_from_div = sum(divcircle)*h^2;

flux =

   -5.3079

flux_from_div =

   -5.3010
```

6

To compute the flux through a circle of radius r centered at (x_0, y_0), I parameterized the circle, and defined the parameterization, the unit normal, and the field as functions of time. Then, for various values of (x_0, y_0) and r, I computed the integral of the normal component of the field for various values. For the circles that included the origin, the flux was always 2π, and the circles not including the origin, the flux returned machine epsilon, meaning that the flux was zero. For the circles through the origin, the flux was always π, suggesting that due to radial symmetry of the field, only half of the field exits the circle.

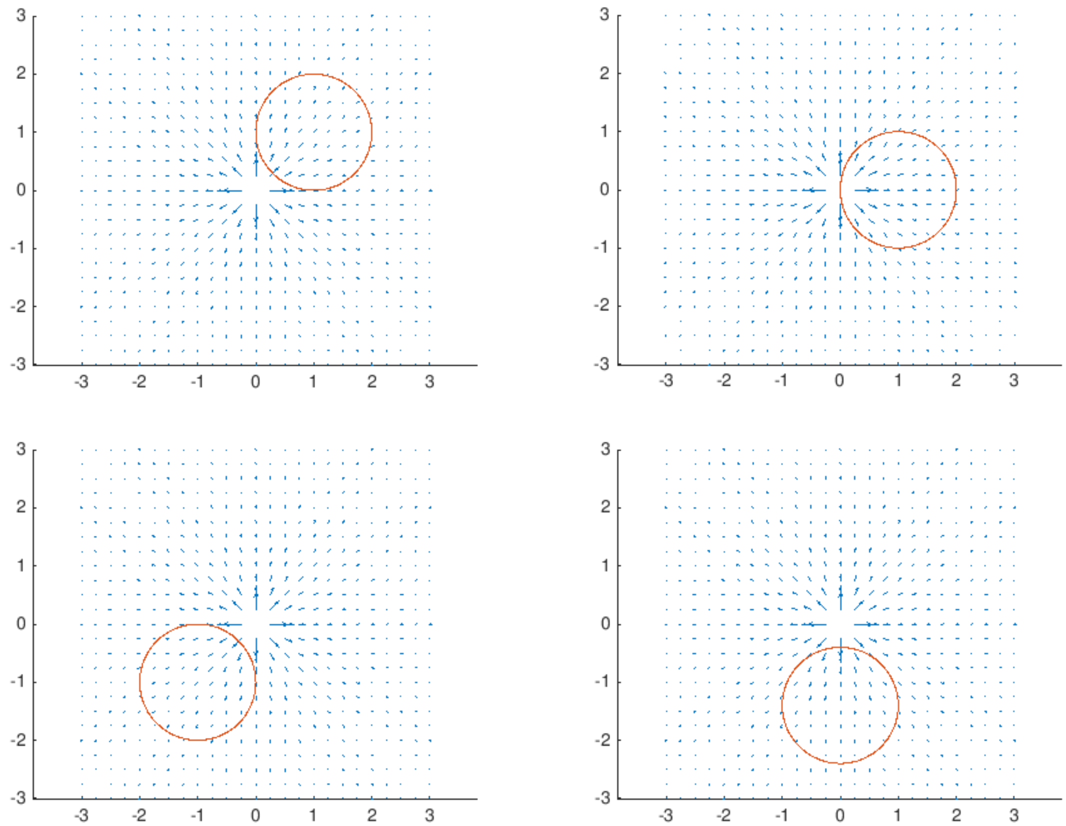


Figure 5: Plots of the flux through circles of radius 1.

```
%% Exercise 6

x0 = 2;
y0 = 5;
r = 10;
x = @(t) x0 + r.*cos(t);
y = @(t) y0 + r.*sin(t);
n = @(t) [cos(t);sin(t)];
F = @(t) [x(t)./(x(t).^2 + y(t).^2); y(t)./(x(t).^2 + y(t).^2)];
Fn = @(t) F(t)'*n(t)*r;

area = integral(Fn,0,2*pi,'ArrayValued',true);

[a,b] = meshgrid(-3:.25:3,-3:.25:3);
u = a./(a.^2 + b.^2);
v = b./(a.^2 + b.^2);

figure
hold on
quiver(a,b,u,v);
plot(x(0:0.01:2*pi),y(0:0.01:2*pi));
```

`axis equal`

2. Exam 1

Exercise 1

The Divergence Theorem states that the integral of the normal component of any field across the boundary of an arbitrary curve is equal to the integral of the flux density (divergence) of the field over the interior enclosed by the curve. Mathematically, this can be stated as follows:

$$\int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}\ ds = \int_{\Omega} \mathrm{div}(\mathbf{F})\ dV, \tag{3}$$

where $\partial\Omega$ represents the boundary of an orientable manifold, such that an n-manifold is a space that resembles $\mathbf{R}^n$ at each point. In this case, the the manifold is a 1-manifold, representing curves that are (for this example) circles of radius r centered at (a, b)[1], oriented so that the outward normal $\mathbf{n}$ gives positive flux[2]. Further, Ω is the interior of the manifold, ds is an element of the arc length of the curve, and dV is an element of the volume. In order to verify equation (1) for the field $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j}$ and the circle of radius r centered at (a, b) in the field, I reparameterized the circle as a function of t and redefined the field in terms of the reparameterized quantities such that

$$\begin{aligned} x(t) &= a + r\cos(t) \\ y(t) &= b + r\sin(t) \\ \mathbf{F} &= (a + r\cos(t))^3\mathbf{i} + (b + r\sin(t))^3\mathbf{j}, \end{aligned}$$

where $x(t), y(t)$ give the points on the circle, and are defined on the interval $0 \leq t \leq 2\pi$. The reparameterized quantities can be constructed from the figure below.

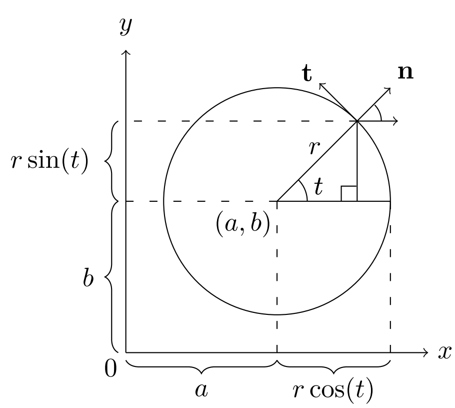


Figure 6: Parameterizing the circle as a function of t.

Further, both the unit normal $\mathbf{n}$ to the circle and the arc length ds can also be determined from the diagram, using corresponding angles and the formula for arc length ($s = rt$), respectively[3]:

[1] The values of a, b, and r are constants such that $a, b, r \in \mathbf{R}^1$.

[2] I'm certain that there is a more rigorous way to define the orientation, so I will not attempt to do so.

[3] The element of arc length ds was initially computed such that a small change in arc length Δs can be represented as $\Delta s = \sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2}$. The argument of the square root can be further written as a difference quotient, where both x and y are evaluated at two different times, values that are finitely close together such that their difference is small.

$$\mathbf{n} = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$
$$ds = \frac{\partial s}{\partial r}dr + \frac{\partial s}{\partial t}dt = tdr + rdt = rdt,$$

where the differential was applied to the arc length. If the figure was not present, the unit normal can still be determined usnig the unit tangent, whose derivation is computed in EXERCISE 4. So, because the unit normal is orthogonal to the unit tangent, its components can be determined from the unit tangent. Thus, in order to compute the flux of the field through the circle, the dot product $\mathbf{F} \cdot \mathbf{n}$ must be taken, to which the differential of the arc length will be multiplied, and the integral can be calculated:

$$\int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}\; ds = \int_0^{2\pi} \Big[\cos(t)(a + r\cos(t))^3 + \sin(t)(b + r\sin(t))^3 \Big]\; rdt = 3\pi r^2 \bigg(a^2 + b^2 + \frac{\pi r^2}{2} \bigg).$$

Next, the flux density of the field over the interior of the circle needs to be computed. Flux density, also called the divergence, is defined as the flux per volume such that

$$\mathrm{div}(\mathbf{F}) = \lim_{\mathrm{Vol}(\Omega)\to 0} \frac{\displaystyle\int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}\; ds}{\mathrm{Vol}(\Omega)} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \tag{4}$$

where $\mathrm{div}(\mathbf{F})$ is commonly denoted as $\nabla \cdot \mathbf{F}$, and the ∇ (del) operator is a two-component vector given by $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}$ in $\mathbf{R}^2$. Thus, for the field in question,

$$\nabla \cdot \mathbf{F} = 3x^2 + 3y^2 = 3(a + r\cos(t))^2 + 3(b + r\sin(t))^2.$$

Now, to construct the right hand side of equation (1), the limits and variables of integration need to be determined. In Cartesian coordinates, the limits of integration are $a - r\cos(t) \leq x \leq a + r\cos(t)$ and $b - r\sin(t) \leq y \leq b + r\sin(t)$, with the variables of integration being $dxdy$. Using Cartesian coordinates, however, produces a very length and difficult integral, so it is natural to convert from the Cartesian coordinates to the polar coordinates. The flux density is already expressed in terms of polar coordinates, so the limits and variables of integration must be determined. Although it is trivial to determine the limits of integration without heavy computation, I believe it would be useful to discover them in the context of differential forms. In treating dx and dy as differential forms, then their product is really $dx \wedge dy$, which is a 2-form on $\mathbf{R}^2$, which takes the vector $(a + r\cos(t), b + r\sin(t))$ as an input to both dx and dy. In doing so, the result is

$$dx\wedge dy = \bigg(\frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial t}dt \bigg) \wedge \bigg(\frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial t}dt \bigg) = \bigg(\frac{\partial x}{\partial r}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial r} \bigg) dr\wedge dt = r\big(\cos^2(t) + \sin^2(t)\big) dr\wedge dt = r\; dr\wedge dt,$$

where in taking the differential with respect to both x and y, the differential applied to the constants a and b returns 0 and the radius is not assumed to be constant, as was the case for the flux. Thus, because the integral of the divergence of the field is over the interior of a curve, for a circle centered at (a, b) with a radius R, its interior ranges from $0 \leq r \leq R$. For t, the interval of $[0, 2\pi]$ remains. Now, the integral of the divergence of $\mathbf{F}$ over the interior of the circle can be constructed and computed:

Then, by multiplying both terms by $\Delta t/\Delta t$, factoring the common Δt^2 out of the root, and taking the limit as the small finite time difference approaches zero, the familiar expression for arc length becomes apparent: $ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}dt$.

$$\int_{\Omega} \nabla \cdot \mathbf{F}\, dV = 3\int_0^{2\pi}\int_0^R \left[(a + r\cos(t))^2 + (b + r\sin(t))^2\right] r dr dt$$
$$= 3\pi R^2\left(a^2 + b^2 + \frac{\pi R^2}{2}\right).$$

So, when $R = r$, then the Divergence Theorem is verified.

(c)

```
%% Exercise 5

% The Field
h = 0.01;
a = 1;
b = 1;
r = 1;
[x,y] = meshgrid(-5:h:5);
[x,y] = meshgrid(-5:h:5);
u = x.^3;
v = y.^3;

% Flux across boundary
t = 0:h:2*pi;
xt = a + r*cos(t);
yt = b + r*sin(t);
ut = interp2(x,y,u,xt,yt);
vt = interp2(x,y,v,xt,yt);
Fnorm = ut.*cos(t) + vt.*sin(t);
flux = sum(Fnorm*h*r);

% Divergence through the interior
divF = divergence(x,y,u,v);
divcircle = divF((x-a).^2 + (y-b).^2 <= r^2);
flux_from_div = sum(divcircle)*h^2;

flux =

   23.6168

flux_from_div =

   23.5614
```

Commentary

With the (very slight) knowledge of differential forms, the computation of these integrals is much quicker, considering I originally did these calculations by hand because my laptop doesn't support Maple. Not only do differential forms streamline the computation, they provide me with a better understanding of not only what Flux and Divergence are in $\mathbf{R}^2$, but also that I can utilize methods that make analytic computation quicker while appreciating the difficulty of the computations in the absence of, or rather the lack of knowledge of, differential forms.

Exercise 2

A constant field $\mathbf{F}$ in the plane has the form $\mathbf{F} = a\mathbf{i} + b\mathbf{j}$, where a and b are constants such that $a, b \in \mathbf{R}$. As a result, the flux density, defined in equation (2), and the circulation density,

$$\text{circulation density}(\mathbf{F}) = \lim_{\text{Vol}(\Omega)\to 0} \frac{\displaystyle\int_{\partial\Omega} \mathbf{F}\cdot\mathbf{t}\; ds}{\text{Vol}(\Omega)} = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}, \tag{5}$$

are both zero. So, in order to determine if a non-constant field has the same properties, the following conditions need to be met

$$\begin{aligned} \text{circulation density}(\mathbf{F}) &= \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0 \\ \frac{\partial g}{\partial x} &= \frac{\partial f}{\partial y} \qquad (4) \\ \nabla\cdot\mathbf{F} &= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0 \\ \frac{\partial f}{\partial x} &= -\frac{\partial g}{\partial y}. \qquad (5) \end{aligned}$$

Initially, I guessed a field $\mathbf{F} = 2x\mathbf{i} - 2y\mathbf{j}$, which does indeed satisfy the above conditions. However, I want to systematically determine the general form of a field; such treatment requires that equations (4) and (5) be treated as partial differential equations

$$\frac{\partial f}{\partial y} = \alpha = \frac{\partial g}{\partial x} \tag{6}$$

$$\frac{\partial f}{\partial x} = \beta = -\frac{\partial g}{\partial y}, \tag{7}$$

where α and β are constants such that $\alpha, \beta \in \mathbf{R}$. Because each PDE is equal to a constant, solving them is quite trivial, with the solutions of equation (6) being

$$\begin{aligned} \frac{\partial f}{\partial y} &= \alpha,\; f(x,y) = \alpha y + \eta(x) \\ \frac{\partial g}{\partial x} &= \alpha,\; g(x,y) = \alpha x + \xi(y). \end{aligned}$$

The functions $\eta(x)$ and $\xi(y)$ are obtained from the solutions of equation (7), where $\eta(x) = \beta x$ and $\xi(y) = -\beta y$. So, the general solutions of the PDE's ultimately comprise the general components of the field where both circulation density and flux density are zero:

$$\begin{aligned} \mathbf{F} &= f(x,y)\mathbf{i} + g(x,y)\mathbf{j} \\ &= (\beta x + \alpha y)\mathbf{i} + (\alpha x - \beta y)\mathbf{j}. \qquad (8) \end{aligned}$$

It is clear to see that the field that was guessed has the form of equation (8) where $\alpha = 0$ and $\beta = 2$. Thus, any non-constant field with the form of equation (8) will satisfy the conditions of equations (6) and (7).

Commentary

I felt that this exercise was very straight forward.

Exercise 3

I was completely stumped by this exercise because I did not understand what it meant for f to be a function in the plane. It meant that f was a function of x and y[4]. With that knowledge, this exercise was fairly simple and took me less than 20 minutes to compute. So, in order to determine whether or not if the limits exist, the variables must first be defined. Let the circle of radius r centered at (a, b) and the unit normal be defined in a manner similar to that in Exercise 1, i.e., reparameterized in x and y, using Figure 1 to determine the unit normal $\mathbf{n}$, from which the unit tangent $\mathbf{t} = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j}$ can be obtained, and setting $ds = r dt$. Next, the function can be rewritten in terms of the reparameterized variables and the integral can be constructed as such

$$\begin{aligned} X(f) &= \lim_{r \to 0^+} \frac{\displaystyle\int_C f \ \mathbf{t} \ ds}{\pi r^2} \\ &= \frac{1}{\pi} \lim_{r \to 0^+} \left[-\left(\frac{\int_0^{2\pi} f(a + r\cos(t), b + r\sin(t)) \sin(t) dt}{r} \right) \mathbf{i} + \left(\frac{\int_0^{2\pi} f(a + r\cos(t), b + r\sin(t)) \cos(t) dt}{r} \right) \mathbf{j} \right] \end{aligned}$$

where the integral of the vector $f \ \mathbf{t}$ is the vector of the integrals for each component. Next, in order for the limit to exist, L'Hôspital's rule needs to be applied, but even before that occurs, we must be convinced that the numerator is zero. If the integral for each component is taken with respect to t, then what is returned is the vector whose components are products of trigonometric functions. The fact that the vector is evaluated over a period of 2π suggests that the numerator is zero, allowing for the use of L'Hôspital's rule. As a result of applying the rule and taking the limit, the integral becomes

$$\frac{1}{\pi} \left[-\left(\int_0^{2\pi} \left(\frac{\partial f}{\partial x}(a, b) \cos(t) \sin(t) + \frac{\partial f}{\partial y}(a, b) \sin^2(t) \right) dt \right) \mathbf{i} + \left(\int_0^{2\pi} \left(\frac{\partial f}{\partial x}(a, b) \cos^2(t) + \frac{\partial f}{\partial y}(a, b) \cos(t) \sin(t) \right) dt \right) \mathbf{j} \right].$$

In applying trigonometric identities and evaluating the integral, what is obtained is

$$X(f) = \lim_{r \to 0^+} \frac{\displaystyle\int_C f \ \mathbf{t} \ ds}{\pi r^2} = -\frac{\partial f}{\partial y}(a, b) \ \mathbf{i} + \frac{\partial f}{\partial x}(a, b) \ \mathbf{j}.$$

In applying the same logic to the computation of the integral with the unit normal, the integral is equal to the gradient of f,

$$Y(f) = \lim_{r \to 0^+} \frac{\displaystyle\int_C f \ \mathbf{n} \ ds}{\pi r^2} = \frac{\partial f}{\partial x}(a, b) \ \mathbf{i} + \frac{\partial f}{\partial y}(a, b) \ \mathbf{j}.$$

Thus, both limits exist.

[4] Thank you very much, professor, for telling me what this meant.

Exercise 4

Given that $\phi(x,y) = \int_\Gamma \mathbf{F}\cdot\mathbf{t}\; ds$ where Γ is a straight line segment from $(0,0)$ to (x,y) and $\mathbf{F}$ is a field $\mathbf{F} = f(x,y)\mathbf{i} + g(x,y)\mathbf{j}$, whose circulation density is zero, in order to prove that $\mathbf{F} = \nabla\phi$, it would be useful to begin by reparameterizing x and y as functions of t. For the simplest reparameterization[5], let x and y be linear in t, suggesting that $x(t) = xt$ and $y(t) = yt$, with t defined in the interval $0 \le t \le 1$. So, let $\phi(x(0), y(0))$ be equal to $(0,0)$. Now, the reparameterized quantities can be written as a position vector $\mathbf{r}(t) = xt\mathbf{i} + yt\mathbf{j}$ and the field can be rewritten as $\mathbf{F} = f(xt,yt)\mathbf{i} + g(xt,yt)\mathbf{j}$. The unit tangent $\mathbf{t}$ and the arc length ds can be obtained from $\mathbf{r}'(t)$ such that

$$\mathbf{t} = \frac{\mathbf{r}'(t)}{\mid \mathbf{r}'(t) \mid} = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}}$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{x^2+y^2}dt.$$

The relation between $\phi(x,y)$ and the integral of the tangential component of the field along Γ can be expressed as

$$\begin{aligned}\phi(x,y) &= \int_\Gamma \big(f(xt,yt)\mathbf{i} + g(xt,yt)\mathbf{j}\big)\cdot\left(\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}\right)\sqrt{x^2+y^2}dt \\ &= \int_0^1 \big(f(xt,yt)x + g(xt,yt)y\big)dt.\end{aligned} \tag{9}$$

In differentiating both sides with respect to x, equation (9) becomes

$$\frac{\partial\phi}{\partial x} = \int_0^1 \left(\frac{\partial f}{\partial x}(xt,yt)xt + f(xt,yt) + \frac{\partial g}{\partial x}(xt,yt)yt\right)dt,$$

where the argument of the integral is obtained by invoking the chain rule[6]. Because circulation density is zero, then $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$, giving

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= \int_0^1 \left(\frac{\partial f}{\partial x}(xt,yt)xt + f(xt,yt) + \frac{\partial f}{\partial y}(xt,yt)yt\right)dt \\ &= \int_0^1 \frac{d}{dt}\Big[f(xt,yt)t\Big]dt = f(x,y) - f(0,0) = f(x,y),\end{aligned}$$

using the Fundamental Theorem of Calculus to conclude the computation. When this process is repeated for $\partial\phi/\partial y$, then the result is $\partial\phi/\partial y = g(x,y)$. Using the vector definition of ∇, as defined in Exercise 1, and applying it to $\phi(x,y)$, then

$$\nabla\phi(x,y) = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} = f(x,y)\mathbf{i} + g(x,y)\mathbf{j} = \mathbf{F},$$

which was to be demonstrated.

[5]This choice of reparameterization was inspired by one of your previous Vector Analysis handouts. After the reparameterization, I decided to perform the computation myself.

[6]By Dr. Chain...

Commentary

This exercise was difficult because I don't have a solid understanding of reparameterization. I am not sure, though, why the first time I took the exam, I did not take the partial derivative of ϕ.

Exercise 5

Gauss's Law, in conjunction with the Principle of Superposition, states that the normal component $\mathbf{n}$ of the electric field $\mathbf{E}$, called the electric flux, through an arbitrary closed surface is equal to the total amount of charge enclosed in that surface divided by some constant, which can be written as such

$$\int_S \mathbf{E} \cdot \mathbf{n} \, dS = \frac{Q}{\epsilon_0}, \tag{10}$$

where Q is the total charge enclosed in the surface and ϵ_0 is the permittivity of free space. In assuming that the field generated by a single charge is radially symmetric, then Coulomb's Law in $\mathbf{R}^2$ and in $\mathbf{R}^n$ can be formulated. For the 2-dimensional case, because radial symmetry is assumed for each charge, and because the Principle of Superposition states that the total electric field is the sum of the individual electric fields of particles, then for an arbitrary circle of radius r that encloses a number of charges q_i, then the electric flux is constant through the circle. Thus, in treating $\mathbf{E}$ as a constant, it can be pulled out of the integral. Further, the integral of the normal component of the circle over the entire circle produces the surface area of the circle, i.e., its circumference in this case. Thus, equation (10) becomes

$$\int_S \mathbf{E} \cdot \mathbf{n} \, dS = E \int_S \mathbf{n} \, dS = E(2\pi r) = \frac{Q}{\epsilon_0}$$
$$E = \frac{Q}{2\pi\epsilon_0 r}.$$

which is Coulomb's Law in 2-dimensions. For the n-dimensional case, then Coulomb's law is given by

$$E = \frac{Q}{S(r)\epsilon_0},$$

where $S(r)$ is the surface area of an n-dimensional sphere[7], given by

$$S(r)_{n-1} = \frac{2\pi^{n/2} r^{n-1}}{\Gamma\left(\frac{n}{2}\right)}.$$

Commentary

I can confidently say that I over-thought this exercise. Also, I did attempt to derive the general formula for the surface area of a sphere in n-dimensions and it was much more difficult than I expected.

3. Homework 2

Exercise 1

Sphere centered at the origin:

[7] Haber, Howard. (2011). *The volume and surface area of an n-dimensional hypersphere.* Retrieved October 25, 2018, from http://scipp.ucsc.edu/ haber/ph116A/volume_11.pdf

```
%% Parameterize a Sphere and Flux

% Parameterization
theta = linspace(0,2*pi);
phi = linspace(0,pi);
[t,p] = meshgrid(theta,phi); % create meshgrid of angles
r = 1;
x0 = 1; % x0 -> y0 make up the initial point
y0 = 0;
z0 = 0;
x = @(t,p) x0 + r*sin(p)*cos(t); % parameterization of sphere
y = @(t,p) y0 + r*sin(p)*sin(t);
z = @(t,p) z0 + r*cos(p);
norm = @(t,p) sqrt(x(t,p).^2 + y(t,p).^2 + z(t,p).^2);

% Electric field evaluated on the sphere
Ex = @(t,p) (1/(4*pi))*x(t,p)./((norm(t,p)).^3);
Ey = @(t,p) (1/(4*pi))*y(t,p)./((norm(t,p)).^3);
Ez = @(t,p) (1/(4*pi))*z(t,p)./((norm(t,p)).^3);
dydz = @(t,p) -(r^2)*cos(t)*sin(p)^2;
dxdz = @(t,p) (r^2)*sin(t)*sin(p)^2;
dxdy = @(t,p) -(r^2)*sin(p)*cos(p);
flux_form = @(t,p) Ex(t,p).*dydz(t,p) - Ey(t,p).*dxdz(t,p) + Ez(t,p).*dxdy(t,p);
flux = dblquad(flux_form,0,2*pi,0,pi);

flux =

   1.0000
```

Sphere centered at (1,0,0) (on the boundary):

```
flux =

   0.5000
```

Sphere centered at (10,0,0) (charge outside of the surface):

```
flux =

  1.0809e-07
```

Exercise 2

Ellipse centered at the origin:

```
%% Parameterize Ellipsoid and Flux in R3

% Parameterization
theta = linspace(0,2*pi);
phi = linspace(0,pi);
[t,p] = meshgrid(theta,phi); % create meshgrid of angles
a = 2;
```

```
b = 6;
c = 3;
x0 = 0; % x0 -> y0 make up the initial point
y0 = 0;
z0 = 0;
x = @(t,p) x0 + a*sin(p)*cos(t); % parameterization of ellipsoid
y = @(t,p) y0 + b*sin(p)*sin(t);
z = @(t,p) z0 + c*cos(p);
norm = @(t,p) sqrt(x(t,p).^2 + y(t,p).^2 + z(t,p).^2);

% Electric field evaluated on the ellipsoid
Ex = @(t,p) (1/(4*pi))*x(t,p)./((norm(t,p)).^3);
Ey = @(t,p) (1/(4*pi))*y(t,p)./((norm(t,p)).^3);
Ez = @(t,p) (1/(4*pi))*z(t,p)./((norm(t,p)).^3);
dydz = @(t,p) -(b*c)*cos(t)*sin(p)^2;
dxdz = @(t,p) (a*c)*sin(t)*sin(p)^2;
dxdy = @(t,p) -(a*b)*sin(p)*cos(p);
flux_form = @(t,p) Ex(t,p).*dydz(t,p) - Ey(t,p).*dxdz(t,p) + Ez(t,p).*dxdy(t,p);
flux = dblquad(flux_form,0,2*pi,0,pi);

flux =

   1.0000
```

Ellipse centered at (2,0,0) (on the boundary):

```
flux =

  0.5000
```

Ellipse centered at (10,0,0) (charge outside the boundary):

```
flux =

   8.2605e-07
```

Exercise 3

So, I attempted to compute the flux where the charge was inside the Torus, but the value of the flux was not 1.000, suggesting that a partial derivative was taken incorrectly.

```
%% Flux through a Torus

theta = linspace(0,2*pi);
psi = linspace(0,2*pi);
[p,t] = meshgrid(psi,theta); % create meshgrid of angles
r = 2; % radius from the center of the torus to the center of the tube
rho = 1; % radius of the center of the tube
x0 = 2; % x0 -> z0 comprise initial point
y0 = 0;
z0 = 0;
x = @(t,p) (r + rho.*cos(t)).*cos(p) + x0; % parameterization of torus
```

```
y = @(t,p) (r + rho.*cos(t)).*sin(p) + y0;
z = @(t,p) rho.*sin(t) + z0;
norm = @(t,p) sqrt(x(t,p).^2 + y(t,p).^2 + z(t,p).^2);

% Electric field evaluated on the ellipsoid
Ex = @(t,p) (1/(4*pi))*x(t,p)./((norm(t,p)).^3);
Ey = @(t,p) (1/(4*pi))*y(t,p)./((norm(t,p)).^3);
Ez = @(t,p) (1/(4*pi))*z(t,p)./((norm(t,p)).^3);
dydz = @(t,p) -(rho.*cos(t)).*(r.*cos(p) + rho.*cos(t).*cos(p));
dxdz = @(t,p) -(rho*cos(t)).*(-r.*sin(p) - rho.*cos(t).*cos(p));
dxdy = @(t,p) -rho.*sin(t).*cos(p).*(r.*cos(p) + rho.*cos(t).*cos(p)) ...
        + rho.*sin(t).*sin(p).*(-r.*sin(p) - rho.*cos(t).*sin(p));
flux_form = @(t,p) Ex(t,p).*dydz(t,p) - Ey(t,p).*dxdz(t,p) + Ez(t,p).*dxdy(t,p);
flux = dblquad(flux_form,0,2*pi,0,2*pi);
```

4. Exam 2

Divergence Theorem

$$\text{Flux} = \int\limits_{\partial C} \mathbf{F} \cdot \mathbf{n}\ ds = \int\limits_{C} \text{div}(\mathbf{F})\ dA$$

Because each "point charge," located at an integer coordinate with charge $q = \epsilon_0$, enclosed in the curve

$$\begin{aligned} x(t) &= \frac{3}{2} + 3\sin(t) + 5\cos(2t) \\ y(t) &= 7\cos(t) \\ t &\in [0, 2\pi], \end{aligned}$$

should have the same amount of flux, then the total flux through the curve should be

$$\text{Total Flux} = \frac{1}{4\pi} \int_0^{2\pi} \left(\sum_{i=0}^{17} \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|} \right) \cdot \mathbf{n}\ ds,$$

where $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j}$, and $\mathbf{r}_i$ is the location of each point charge with integer coordinates. So, the number of points enclosed in the curve is equal to the total flux divided by the flux per point charge, which is 17 points.

Commentary

I tried to code it, but I couldn't.

Simple Forms

(a)

$$
\begin{aligned}
dx \wedge dy + dy \wedge dz + dz \wedge dx &= dx \wedge dy + dy \wedge dz + dz \wedge dx + 0 && (\text{by } \mathbf{v} + 0 = \mathbf{v}) \\
&= dx \wedge dy + dy \wedge dz + dz \wedge dx + dy \wedge dy && (\text{by } \alpha \wedge \alpha = 0) \\
&= dx \wedge dy + dy \wedge dz - dx \wedge dz - dy \wedge dy && (\text{by } \alpha \wedge \beta = -\beta \wedge \alpha) \\
&= dx \wedge (dy - dz) - dy \wedge (dy - dz) && (\text{by bilinearity of the wedge product}) \\
&= (dx - dy) \wedge (dy - dz) && (\text{by bilinearity of the wedge product}).
\end{aligned}
$$

Because every 1-form can be expressed as a linear combination of 1-forms where the basis of 1-forms is $dx, dy, dz \in \Lambda^1(\mathbf{R}^3)$ and the underlying field consists of 0-forms, or functions, the result $(dx - dy) \wedge (dy - dz)$ is the wedge product of two 1-forms where the functions are constant functions of the form $f(x, y, z) = 1$.
(b)

Every 1-form is a linear combination of the basis of 1-forms $dx, dy, dz \in \Lambda^1(\mathbf{R}^3)$, and every 2-form can be written as the wedge product of two 1-forms. Every 2-form is not simple, though, meaning that it cannot be factored into the wedge product of two 1-forms. As an example, the two form $\omega \in \Lambda^2(\mathbf{R}^3)$, where ω is

$$\omega = d\alpha \wedge d\beta + d\gamma \wedge d\eta,$$

and

$$
\begin{aligned}
\alpha &= \alpha_1 dx + \alpha_2 dy + \alpha_3 dz \\
\beta &= \beta_1 dx + \beta_2 dy + \beta_3 dz \\
\gamma &= \gamma_1 dx + \gamma_2 dy + \gamma_3 dz \\
\eta &= \eta_1 dx + \eta_2 dy + \eta_3 dz,
\end{aligned}
$$

are 1-forms. If α, β, γ, and η are unique 1-forms that are linearly independent, then ω cannot be fundamentally be factored into the wedge product of 1-forms because even ω written as a linear combination of the basis $dx \wedge dy, dy \wedge dz, dz \wedge dx \in \Lambda^2(\mathbf{R}^3)$ conveys that the only that it can be factored into the wedge product of 1-forms is if each 1-form is either a linear combination of the the other 1-forms or if they are multiples of one another.

Star Operator

(a)

$$d \star df = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) dx \wedge dy \wedge dz.$$

(b)

$$\omega_1 \wedge \star\omega_2 = (a_1 a_2 + b_1 b_2 + c_1 c_2) dx \wedge dy \wedge dz.$$

Finding the Field

The field associated with $-dH$ is

$$\mathbf{F} = -\frac{\partial H}{\partial q}\mathbf{i} + \frac{\partial H}{\partial p}\mathbf{j}.$$

$$
\begin{aligned}
d\phi = d\Big(\int_0^1 A(xt,yt,zt)tdt\Big) \wedge (ydz - zdy) + \Big(\int_0^1 A(xt,yt,zt)tdt\Big) \wedge d(ydz - zdy) + \\
d\Big(\int_0^1 B(xt,yt,zt)tdt\Big) \wedge (zdx - xdz) + \Big(\int_0^1 B(xt,yt,zt)tdt\Big) \wedge d(zdx - xdz) + \\
d\Big(\int_0^1 C(xt,yt,zt)tdt)\Big) \wedge (xdy - ydx) + \Big(\int_0^1 C(xt,yt,zt)tdt\Big) \wedge d(xdy - ydx) \\
= \Big[\frac{\partial}{\partial x}\Big(\int_0^1 A(xt,yt,zt)tdt\Big)dx + \frac{\partial}{\partial y}\Big(\int_0^1 A(xt,yt,zt)tdt\Big)dy + \frac{\partial}{\partial z}\Big(\int_0^1 A(xt,yt,zt)tdt\Big)dz\Big] \wedge (ydz - zdy) + \\
\Big(\int_0^1 2A(xt,yt,zt)tdt\Big)(dy \wedge dz) + \\
\Big[\frac{\partial}{\partial x}\Big(\int_0^1 B(xt,yt,zt)tdt\Big)dx + \frac{\partial}{\partial y}\Big(\int_0^1 B(xt,yt,zt)tdt\Big)dy + \frac{\partial}{\partial z}\Big(\int_0^1 B(xt,yt,zt)tdt\Big)dz\Big] \wedge (zdx - xdz) - \\
\Big(\int_0^1 2B(xt,yt,zt)tdt\Big)(dx \wedge dz) + \\
\Big[\frac{\partial}{\partial x}\Big(\int_0^1 C(xt,yt,zt)tdt\Big)dx + \frac{\partial}{\partial y}\Big(\int_0^1 C(xt,yt,zt)tdt\Big)dy + \frac{\partial}{\partial z}\Big(\int_0^1 C(xt,yt,zt)tdt\Big)dz\Big] \wedge (xdy - ydx) + \\
\Big(\int_0^1 2C(xt,yt,zt)tdt\Big)(dx \wedge dy).
\end{aligned}
$$

After performing the wedge products and observing the cancellations in the wedge product, there are 15 terms. In looking at the terms with $dx \wedge dz$, using the condition that $\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} = 0$, and interchanging differention and integration, what is obtained is:

$$
\begin{aligned}
\int_0^1 \frac{\partial A}{\partial x}(xt,yt,zt)yt^2dt - \int_0^1 \frac{\partial B}{\partial x}(xt,yt,zt)xt^2dt - \int_0^1 \frac{\partial B}{\partial z}(xt,yt,zt)zt^2dt + \int_0^1 \frac{\partial C}{\partial z}(xt,yt,zt)yt^2dt - \int_0^1 2B(xt,yt,zt)tdt \\
= -\int_0^1 \Big(\frac{\partial B}{\partial y}(xt,yt,zt)yt^2dt + \frac{\partial B}{\partial x}(xt,yt,zt)xt^2dt + \frac{\partial B}{\partial z}(xt,yt,zt)zt^2dt + 2B(xt,yt,zt)tdt\Big) \\
= -\int_0^1 \frac{d}{dt}(B(xt,yt,zt)t^2)dt = -(B(x,y,z) - B(0,0,0))dx \wedge dz = -B(x,y,z)dx \wedge dz.
\end{aligned}
$$

Of course, the wedges were always present, but in performing this with the $dy \wedge dz$ and $dx \wedge dy$ terms, it is confirmed that $d\phi = \omega = A(x,y,z)dy \wedge dz - B(x,y,z)dx \wedge dz + C(x,y,z)dx \wedge dy$.