Final Assessment Game: 40 Pages Later

Okay, technically 39 pages...

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Exercise 1: Divergence Theorem Validation

The Divergence Theorem states that the integral of the normal component of any field across the boundary of an arbitrary curve is equal to the integral of the flux density (divergence) of the field over the interior enclosed by the curve. Mathematically, this can be stated as follows:

$$\int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} \; dS = \int_{\Omega} \operatorname{div}(\mathbf{F}) \; dV. \tag{1}$$

In $\mathbb{R}^3$, equation (1) can be written as

$$\int_{\partial\Omega} F_x dy \wedge dz + F_y dz \wedge dx + F_z dx \wedge dy = \int_{\Omega} \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dx \wedge dy \wedge dz, \tag{2}$$

where the left hand side of equation (2) represents the integral of the flux-form of the field $\mathbf{F}$ and the right side of the equation represents the the integral of the volume-form of the field, which is its divergence. To confirm the divergence theorem for the unit cube and the field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, each face of the unit cube was parameterized in terms of u and v, where $\{u \in \mathbb{R} \mid 0 \leq u \leq 1\}$ and $\{v \in \mathbb{R} \mid 0 \leq v \leq 1\}$. The parameterized faces are given in Figure 1,

Face 1: $x = u, \; y = v, \; z = 1$ Face 2: $x = u, \; y = 1, \; z = v$
Face 3: $x = 1, \; y = 1, \; z = v$ Face 4: $x = u, \; y = v, \; z = 0$
Face 5: $x = u, \; y = 0, \; z = v$ Face 6: $x = 0, \; y = u, \; z = v$,

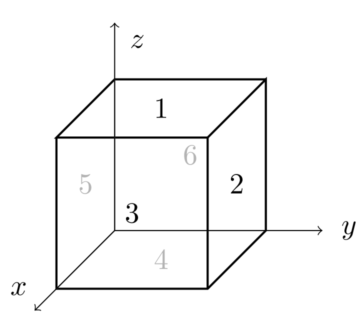

Figure 1: The parameterization of the unit cube in three dimensions.

where the first three faces are the three outer faces of the cube and the last three faces are the xy, xz, and yz planes, respectively. After parameterizing the faces, I applied the divergence theorem to compute the flux across the cube. For the first computation, using the left hand side of equation (2), because flux is additive, the total flux was expressed as the sum of the fluxes through the individual faces. For the second computation, using the right hand side of equation (2), the divergence of the field was taken and integrated over the volume.

Computation 1:

$$\text{Faces 1, 2, and 3:}\;\; \Phi_1 = \Phi_2 = \Phi_3 = \int_0^1 \int_0^1 du dv = 1$$

$$\text{Faces 4, 5, and 6:}\;\; \Phi_4 = \Phi_5 = \Phi_6 = \int_0^1 \int_0^1 0\; du dv = 0$$

$$\boxed{\Phi_{total} = \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}\; dS = \sum_{i=1}^{6} \Phi_i = 1 + 1 + 1 + 0 + 0 + 0 = 3.}$$

Computation 2:

$$\mathrm{div}(\mathbf{F}) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 1 + 1 + 1 = 3$$

$$\boxed{\Phi_{total} = \int_{\Omega} \mathrm{div}(\mathbf{F})\; dV = \int_{\Omega} 3\; dV = 3 \int_{\Omega} dV = 3.}$$

As per both computations 1 and 2, the total flux is 3 and the divergence theorem is confirmed. Further, the computation is confirmed from the plot of the field and the unit cube, provided in Figure 2. For the faces representing the xy, xz, and yz planes, the normal component of the field is actually tangent to those faces. Thus, the normal component of the field is zero, resulting in zero flux.

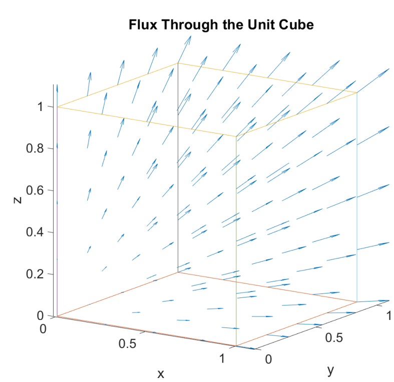

Figure 2: A plot of the flux of the field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ through the unit cube.

```
%% Exam 1 Exercise 1

% The Field
h = 0.01;
[x,y,z] = meshgrid(0:h:1);
u = x;
v = y;
w = z;

% The faces of the cube
% Flux through face 1
t = 0:h:1;
xt1 = t;
yt1 = t;
zt1 = ones(size(t));
ut1 = interp3(x,y,z,u,xt1,yt1,zt1);
vt1 = interp3(x,y,z,v,xt1,yt1,zt1);
wt1 = interp3(x,y,z,w,xt1,yt1,zt1);
fnorm1 = wt1;
Flux1 = sum(fnorm1*h);

% Face 2
t = 0:h:1;
xt2 = t;
yt2 = ones(size(t));
zt2 = t;
ut2 = interp3(x,y,z,u,xt2,yt2,zt2);
vt2 = interp3(x,y,z,v,xt2,yt2,zt2);
wt2 = interp3(x,y,z,w,xt2,yt2,zt2);
fnorm2 = yt2;
Flux2 = sum(fnorm2*h);

% Face 3
t = 0:h:1;
xt3 = t;
yt3 = t;
zt3 = zeros(size(t));
ut3 = interp3(x,y,z,u,xt3,yt3,zt3);
vt3 = interp3(x,y,z,v,xt3,yt3,zt3);
wt3 = interp3(x,y,z,w,xt3,yt3,zt3);
fnorm3 = zt3;
Flux3 = sum(fnorm3*h);

% Face 4
t = 0:h:1;
xt4 = t;
yt4 = zeros(size(t));
zt4 = t;
ut4 = interp3(x,y,z,u,xt4,yt4,zt4);
vt4 = interp3(x,y,z,v,xt4,yt4,zt4);
wt4 = interp3(x,y,z,w,xt4,yt4,zt4);
fnorm4 = yt4;
Flux4 = sum(fnorm4*h);

% Face 5
t = 0:h:1;
xt5 = ones(size(t));
yt5 = t;
zt5 = t;
ut5 = interp3(x,y,z,u,xt5,yt5,zt5);
vt5 = interp3(x,y,z,v,xt5,yt5,zt5);
wt5 = interp3(x,y,z,w,xt5,yt5,zt5);
fnorm5 = xt5;
Flux5 = sum(fnorm5*h);

% Face 6
t = 0:h:1;
xt6 = zeros(size(t));
yt6 = t;
zt6 = t;
ut6 = interp3(x,y,z,u,xt6,yt6,zt6);
vt6 = interp3(x,y,z,v,xt6,yt6,zt6);
wt6 = interp3(x,y,z,w,xt6,yt6,zt6);
fnorm6 = xt6;
Flux6 = sum(fnorm6*h);

% Total Flux
Flux = Flux1 + Flux2 + Flux3 + Flux4 + Flux5 + Flux6;


% Divergence through the interior
divF = divergence(u,v,w);
divcube1 = divF(x <= 1);
divcube2 = divF(y <= 1);
divcube3 = divF(z <= 1);
flux_from_div = (sum(divcube1) + sum(divcube2)...
                + sum(divcube3))*h^2/3;

hold on
quiver3(x,y,z,u,v,w);
X = [0;1;1;0;0];
Y = [0;0;1;1;0];
Z = [0;0;0;0;0];
plot3(X,Y,Z);   % draw a square in the xy plane with z = 0
plot3(X,Y,Z+1); % draw a square in the xy plane with z = 1
set(gca,'View',[-28,35]); % set the azimuth and...
                %elevation of the plot
 for k=1:length(X)-1
     plot3([X(k);X(k)],[Y(k);Y(k)],[0;1]);
 end
axis equal

Flux =

   3.030000000000002

flux_from_div =

   3.090902999992486
```

Exercise 2: Derivation of the Divergence in Polar Coordinates

For the following field defined as follows

$$
\begin{aligned}
\mathbf{F} &= f(r,\theta)\mathbf{e}_r + g(r,\theta)\mathbf{e}_\theta \\
\mathbf{e}_r &= \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j} \\
\mathbf{e}_\theta &= -\sin(\theta)\mathbf{i} + \cos(\theta),
\end{aligned}
$$

in order to compute the divergence in terms of r and θ, it was useful to first express them in terms of both x and y. This was done according to Figure 3. After rewriting r and θ in terms of both x and y, the field was written using the standard basis and then the divergence of the field was taken. The computation is below, which included the cancellation of many terms.

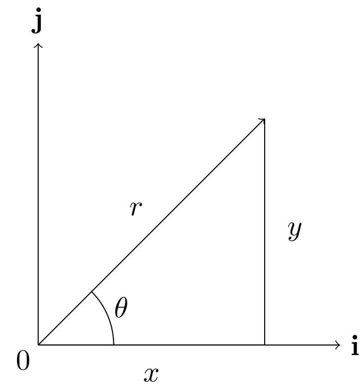

Figure 3: Using the above illustration to rewrite the radius and the angle of rotation in terms of both x and y coordinates.

Computation:

$$
r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right)
$$

$$
\mathbf{F} = f(r,\theta)\mathbf{e}_r + g(r,\theta)\mathbf{e}_\theta = [f(r,\theta)\cos(\theta) - g(r,\theta)\sin(\theta)]\mathbf{i} + [f(r,\theta)\sin(\theta) - g(r,\theta)\cos(\theta)]\mathbf{i}
$$

$$
\operatorname{div}(\mathbf{F}) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = \frac{\partial F_x}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial F_x}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial F_y}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial F_y}{\partial \theta}\frac{\partial \theta}{\partial y}
$$

$$
\boxed{\operatorname{div}(\mathbf{F}) = \frac{\partial f}{\partial r} + \frac{f}{r} + \frac{1}{r}\frac{\partial g}{\partial \theta}}
$$

To confirm this expression, the three examples used, and the resulting computations, are

$$
\begin{gathered}
\text{Example 1:} \quad f(r,\theta) = r^2, \quad g(r,\theta) = \theta^2 \\
\operatorname{div}[\mathbf{F}(r,\theta)] = 2r + r + \frac{2\theta}{r} = \frac{3r^2 + 2\theta}{r} \\
\operatorname{div}[\mathbf{F}(x,y)] = \frac{3(x^2+y^2) + 2\arctan\left(\frac{y}{x}\right)}{(x^2+y^2)^{1/2}} \\
\text{Example 2:} \quad f(r,\theta) = \frac{r}{\theta}, \quad g(r,\theta) = \frac{\theta}{r} \\
\operatorname{div}[\mathbf{F}(r,\theta)] = \frac{2r^2 + \theta}{r^2\theta} \\
\operatorname{div}[\mathbf{F}(x,y)] = \frac{2(x^2+y^2) + \arctan\left(\frac{y}{x}\right)}{(x^2+y^2)\arctan\left(\frac{y}{x}\right)} \\
\text{Example 3:} \quad f(r,\theta) = \sin(r+\theta), \quad g(r,\theta) = \cos(r+\theta) \\
\operatorname{div}[\mathbf{F}(r,\theta)] = \cos(r+\theta) \\
\operatorname{div}[\mathbf{F}(x,y)] = \cos\left[(x^2+y^2)^{1/2} + \arctan\left(\frac{y}{x}\right)\right]
\end{gathered}
$$

where r and θ can be written in terms of x and y to confirm the expression for the divergence in polar coordinates. The same is true x and y written in terms of r and θ.

Exercise 3: Circulation Density

The circulation density of a vector field $\mathbf{F}$ over some closed curve $\partial\Omega$ is defined as such:

$$
\operatorname{curl}(\mathbf{F}) \cdot \mathbf{n} = \lim_{\text{Area}(\Omega)\to 0^+} \frac{\int_{\partial\Omega} \mathbf{F}\cdot\mathbf{t}\; ds}{\text{Area}(\Omega)}, \tag{3}
$$

where $\mathbf{n}$ is the normal vector around which the tangential component of the field, $\mathbf{F}\cdot\mathbf{t}$, is circulating and ds is the line element. Let $\operatorname{circ}(\mathbf{F})$ be the shorthand notation for circulation density of the field $\mathbf{F}$.

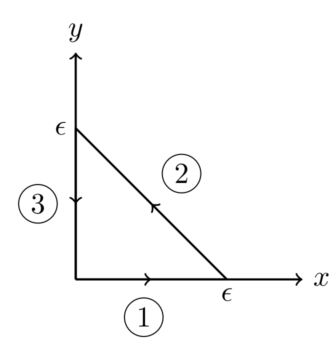

Figure 4: A triangle with legs of length ϵ and counterclockwise orientation.

For the field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ over the triangle with legs of length ϵ, as depicted in Figure 4, the circulation density was computed by first parameterizing the curve in terms of t such that $\{t \in \mathbb{R} \mid 0 \leq t \leq \epsilon\}$. Then using equation (3), the circulation density was calculated and was validated using Stokes' Theorem and the equation for circulation density of a field in $\mathbb{R}^2$.

Computation:

$$\text{Along path } ①\text{:} \quad x = t, \quad y = 0, \quad \mathbf{F} = t\mathbf{j}, \quad \mathbf{t} = \mathbf{i}, \quad ds = dt$$

$$\text{circ}(\mathbf{F})_1 = \lim_{\epsilon \to 0^+} \frac{\displaystyle\int_{\partial T} \mathbf{F} \cdot \mathbf{t}\ ds}{\frac{1}{2}\epsilon^2} = \lim_{\epsilon \to 0^+} \frac{\displaystyle\int_0^\epsilon 0\ ds}{\frac{1}{2}\epsilon^2} \overset{\text{(H)}}{=} 0$$

$$\text{Along path } ②\text{:} \quad x = t, \quad y = -t + \epsilon, \quad \mathbf{F} = (t - \epsilon)\mathbf{i} + t\mathbf{j}, \quad \mathbf{t} = \frac{-\mathbf{i} + \mathbf{j}}{\sqrt{2}}, \quad ds = \sqrt{2}dt$$

$$\text{circ}(\mathbf{F})_2 = \lim_{\epsilon \to 0^+} \frac{\displaystyle\int_{\partial T} \mathbf{F} \cdot \mathbf{t}\ ds}{\frac{1}{2}\epsilon^2} = \lim_{\epsilon \to 0^+} \frac{\displaystyle\int_0^\epsilon \epsilon\ dt}{\frac{1}{2}\epsilon^2} = 2$$

$$\text{Along path } ③\text{:} \quad x = 0, \quad y = t, \quad \mathbf{F} = t\mathbf{j}, \quad \mathbf{t} = -\mathbf{j}, \quad ds = dt$$

$$\text{circ}(\mathbf{F})_3 = \lim_{\epsilon \to 0^+} \frac{\displaystyle\int_{\partial T} \mathbf{F} \cdot \mathbf{t}\ ds}{\frac{1}{2}\epsilon^2} = \lim_{\epsilon \to 0^+} \frac{\displaystyle\int_0^\epsilon 0\ ds}{\frac{1}{2}\epsilon^2} \overset{\text{(H)}}{=} 0$$

$$\boxed{\text{circ}(\mathbf{F})_{\text{total}} = \sum_{i=1}^{3} \text{circ}(\mathbf{F})_i = 0 + 2 + 0 = 2}$$

Validation:

$$\int_{\partial T} \mathbf{F} \cdot \mathbf{t}\ ds = \int_T \text{circ}(\mathbf{F})\ dA$$

$$\int_{\partial T} \mathbf{F} \cdot \mathbf{t}\ ds = \epsilon^2$$

$$\text{circ}(\mathbf{F}) = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 1 - (-1) = 2$$

$$\int_T \text{circ}(\mathbf{F})\ dA = \int_T 2\ dA = 2\int_T dA = 2\left(\frac{1}{2}\epsilon^2\right) = \epsilon^2$$

Exercise 4: Poynting's Theorem

Computation:[1]

$$\sigma = \frac{1}{2}\left(\epsilon_0\ (\mathbf{E}\cdot\mathbf{E}) + \frac{1}{\mu_0}(\mathbf{B}\cdot\mathbf{B})\right)$$

$$\begin{aligned}
X &= \frac{\partial\sigma}{\partial t} + \operatorname{div}\left(\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\right)\\
&= \frac{\partial}{\partial t}\left[\frac{1}{2}\epsilon_0\ (\mathbf{E}\cdot\mathbf{E}) + \frac{1}{2\mu_0}(\mathbf{B}\cdot\mathbf{B})\right] + \frac{1}{\mu_0}\operatorname{div}\left(\mathbf{E}\times\mathbf{B}\right)\\
&= \frac{1}{2}\epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{E}) + \frac{1}{2\mu_0}\frac{\partial}{\partial t}(\mathbf{B}\cdot\mathbf{B}) + \frac{1}{\mu_0}\left[\mathbf{B}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{B})\right]\\
&= \epsilon_0\ \mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} + \frac{1}{\mu_0}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} + \frac{1}{\mu_0}\left[-\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} - \mathbf{E}\cdot\left(\mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\right]\\
&= \epsilon_0\ \mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} + \frac{1}{\mu_0}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} - \frac{1}{\mu_0}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} - \mathbf{E}\cdot\mathbf{J} - \epsilon_0\ \mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}\\
&= \boxed{-\mathbf{E}\cdot\mathbf{J}}
\end{aligned}$$

Exercise 5: Scalar Potential

For the following field

$$\mathbf{F} = \left(1 + R^2\frac{y^2-x^2}{(x^2+y^2)^2}\right)\mathbf{i} - R^2\frac{2xy}{(x^2+y^2)^2}\mathbf{j}, \quad x^2+y^2 > R$$

is potential because the circulation density is zero. Thus,

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} = R^2\frac{2y(3x^2-y^2)}{(x^2+y^2)^3}.$$

This suggests that there is indeed a function ϕ such that $\mathbf{F} = \nabla\phi$. The function I found was

$$\phi = x + R^2\frac{x}{x^2+y^2} + C,$$

where the function is not defined everywhere. However, it is defined for the domain of definition of $\mathbf{F}$. This can be seen from the plots in Figure 5. The radius of the circle is always larger than the values of x and y for which the function is undefined. The only parameter the radius changes is the height and depth of the undefined region. Thus, the function provided satisfies the condition $\mathbf{F} = \nabla\phi$.

[1] I had some help from Griffith's Electrodynamics

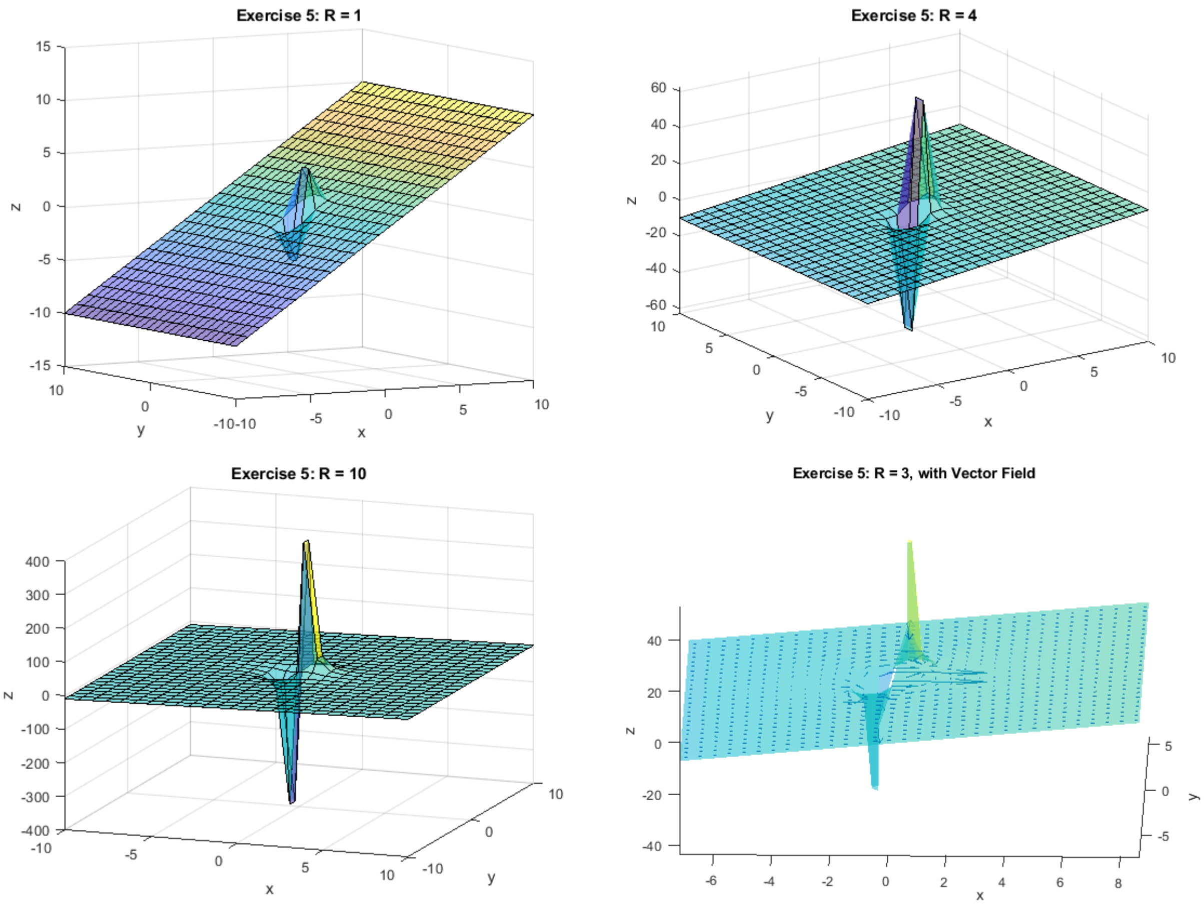

Figure 5: Plots of ϕ with different radii. The last plot includes the field.

```
h=0.5;
r = 3; % Radius
[x,y] = meshgrid(-10:h:10);

% The Field
u = 1 + (r^2).*((y.^2 - x.^2)./(x.^2 + y.^2).^2);
v = -r^2 * ((2.*x.*y)./(x.^2 + y.^2).^2);
w = zeros(size(u));

% The function phi
phi =  x + (r^2 * x./((x.^2 + y.^2).^2));

figure
hold on
s = surf(x,y,phi);
quiver3(x,y,phi,u,v,w);
alpha 0.5;
s.EdgeColor = 'none';
```

Exercise 6: Derivation of the Laplacian in Generalized Coordinates

For this exercise, I used three different methods to compute the Laplacian in generalized coordinates, and each method produced the same result, suggesting that the derivation was a success. The first two methods are essentially identical, yet the way in which the solution was obtained was slightly different. The third method utilized the Riemannian structure and the Hodge Star operator, whereas the first two methods completely excluded the use of either. I don't have much else to say about this exercise except that I absolutely enjoyed thinking about the mathematics and not only performing the derivation, but also validating the derivation with two different methods.

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}} \underset{v}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{b}{\begin{bmatrix} u \\ v \end{bmatrix}}$$

Computation 1:

$$\nabla^2 f(u,v) = \star\, d \star df$$

$$u = a_{11}x + a_{12}y, \quad du = a_{11}dx + a_{12}dy$$

$$v = a_{21}x + a_{22}y, \quad dv = a_{21}dx + a_{22}dy$$

$$\star du = a_{11} \star dx + a_{12} \star dy = a_{11}dy - a_{12}dx$$

$$\star dv = a_{21} \star dx + a_{22} \star dy = a_{21}dy - a_{22}dx$$

$$df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv$$

$$\star df = \frac{\partial f}{\partial u} \star du + \frac{\partial f}{\partial v} \star dv = \frac{\partial f}{\partial u}(a_{11}dy - a_{12}dx) + \frac{\partial f}{\partial v}(a_{21}dy - a_{22}dx)$$

$$d \star df = \left(\frac{\partial^2 f}{\partial u^2}du + \frac{\partial^2 f}{\partial u \partial v}dv\right) \wedge (a_{11}dy - a_{12}dx) + \left(\frac{\partial^2 f}{\partial v \partial u}du + \frac{\partial^2 f}{\partial v^2}dv\right) \wedge (a_{21}dy - a_{22}dx)$$

Substitute the du's and dv's and then simplify and collect terms to obtain

$$d \star df = \left[\left(a_{11}^2 + a_{12}^2\right)\frac{\partial^2 f}{\partial u^2} + 2\left(a_{11}a_{21} + a_{12}a_{22}\right)\frac{\partial^2 f}{\partial u \partial v} + \left(a_{21}^2 + a_{22}^2\right)\frac{\partial^2 f}{\partial v^2}\right] dx \wedge dy$$

$$\boxed{\star d \star df = \left(a_{11}^2 + a_{12}^2\right)\frac{\partial^2 f}{\partial u^2} + 2\left(a_{11}a_{21} + a_{12}a_{22}\right)\frac{\partial^2 f}{\partial u \partial v} + \left(a_{21}^2 + a_{22}^2\right)\frac{\partial^2 f}{\partial v^2}}$$

Computation 2:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$dx \wedge dy = \frac{1}{\det(A)} du \wedge dv$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}\right)dy$$

dx and dy were defined in terms of the x and y coordinates obtained from the matrix-vector system.

$$dx = \frac{a_{22}du - a_{12}dv}{\det(A)}, \quad dy = \frac{-a_{21}du + a_{11}dv}{\det(A)}$$

$$\star df = \frac{\partial f}{\partial x} \star dx + \frac{\partial f}{\partial y} \star dy = \frac{\partial f}{\partial x}dy - \frac{\partial f}{\partial y}dx$$

$$= -\left[\left(\frac{a_{11}a_{21} + a_{12}a_{22}}{\det(A)}\right)\frac{\partial f}{\partial u} + \left(\frac{a_{21}^2 + a_{22}^2}{\det(A)}\right)\frac{\partial f}{\partial v}\right]du + \left[\left(\frac{a_{11}^2 + a_{12}^2}{\det(A)}\right)\frac{\partial f}{\partial u} + \left(\frac{a_{11}a_{21} + a_{12}a_{22}}{\det(A)}\right)\frac{\partial f}{\partial v}\right]dv$$

$$d \star df = \left[\left(\frac{a_{11}^2 + a_{12}^2}{\det(A)}\right)\frac{\partial^2 f}{\partial u^2} + 2\left(\frac{a_{11}a_{21} + a_{12}a_{22}}{\det(A)}\right)\frac{\partial^2 f}{\partial u \partial v} + \left(\frac{a_{21}^2 + a_{22}^2}{\det(A)}\right)\frac{\partial^2 f}{\partial v^2}\right]du \wedge dv$$

$$= \left[\left(a_{11}^2 + a_{12}^2\right)\frac{\partial^2 f}{\partial u^2} + 2\left(a_{11}a_{21} + a_{12}a_{22}\right)\frac{\partial^2 f}{\partial u \partial v} + \left(a_{21}^2 + a_{22}^2\right)\frac{\partial^2 f}{\partial v^2}\right]\frac{1}{\det(A)}du \wedge dv$$

$$\boxed{\star d \star df = \left(a_{11}^2 + a_{12}^2\right)\frac{\partial^2 f}{\partial u^2} + 2\left(a_{11}a_{21} + a_{12}a_{22}\right)\frac{\partial^2 f}{\partial u \partial v} + \left(a_{21}^2 + a_{22}^2\right)\frac{\partial^2 f}{\partial v^2}}$$

Computation 3:

$$ds^2 = dx^2 + dy^2$$

$$\begin{bmatrix} du & dv \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} du \\ dv \end{bmatrix} = \left(\frac{a_{11}^2 + a_{12}^2}{\det(A)^2}\right)dv^2 - 2\left(\frac{a_{11}a_{21} + a_{12}a_{22}}{\det(A)^2}\right)dudv + \left(\frac{a_{21}^2 + a_{22}^2}{\det(A)^2}\right)du^2$$

$$a = \left(\frac{a_{21}^2 + a_{22}^2}{\det(A)^2}\right), \quad b = -\left(\frac{a_{11}a_{21} + a_{12}a_{22}}{\det(A)^2}\right), \quad c = \left(\frac{a_{11}^2 + a_{12}^2}{\det(A)^2}\right)$$

$$g_{ij} = \begin{bmatrix} \left(\frac{a_{21}^2 + a_{22}^2}{\det(A)^2}\right) & -\left(\frac{a_{11}a_{21} + a_{12}a_{22}}{\det(A)^2}\right) \\ -\left(\frac{a_{11}a_{21} + a_{12}a_{22}}{\det(A)^2}\right) & \left(\frac{a_{11}^2 + a_{12}^2}{\det(A)^2}\right) \end{bmatrix}, \quad g^{ij} = \begin{bmatrix} \left(a_{11}^2 + a_{12}^2\right) & \left(a_{11}a_{21} + a_{12}a_{22}\right) \\ \left(a_{11}a_{21} + a_{12}a_{22}\right) & \left(a_{21}^2 + a_{22}^2\right) \end{bmatrix}$$

$$\star df = \frac{\partial f}{\partial u} \star du + \frac{\partial f}{\partial v} \star dv$$

$$\star du: \ du \wedge (\alpha du + \beta dv) = \left\langle \star du, \alpha du + \beta dv \right\rangle \frac{1}{\det(A)} du \wedge dv, \ \text{Let } \star du = \gamma du + \eta dv$$

$$\beta = \left\langle \gamma du + \eta dv, \alpha du + \beta dv \right\rangle \frac{1}{\det(A)}$$

$$\star du = -\left(\frac{a_{11}a_{21} + a_{12}a_{22}}{\det(A)}\right)du + \left(\frac{a_{11}^2 + a_{12}^2}{\det(A)}\right)dv,$$

$$\text{Repeat to obtain} \star dv = -\left(\frac{a_{21}^2 + a_{22}^2}{\det(A)}\right)du + \left(\frac{a_{11}a_{21} + a_{12}a_{22}}{\det(A)}\right)dv$$

$$d \star df = \left[\left(a_{11}^2 + a_{12}^2\right)\frac{\partial^2 f}{\partial u^2} + 2\left(a_{11}a_{21} + a_{12}a_{22}\right)\frac{\partial^2 f}{\partial u \partial v} + \left(a_{21}^2 + a_{22}^2\right)\frac{\partial^2 f}{\partial v^2}\right]\frac{1}{\det(A)}du \wedge dv$$

$$\boxed{\star d \star df = \left(a_{11}^2 + a_{12}^2\right)\frac{\partial^2 f}{\partial u^2} + 2\left(a_{11}a_{21} + a_{12}a_{22}\right)\frac{\partial^2 f}{\partial u \partial v} + \left(a_{21}^2 + a_{22}^2\right)\frac{\partial^2 f}{\partial v^2}}$$