Homework Project - Math 57

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Contents

1. Homework 1

1

```
b = bulb1(:,1); % Retrieves the first column of data, time.
a = bulb1(:,2); % Retrieves the second column of data, room temperature.
w = bulb1(:,3); % Retrieves the third column of data, bulb temperature.

figure
hold on
plot(b,a,'b','LineWidth',2);
plot(b,w,'r-',b,w,'ro','LineWidth',2);
xlabel('Time (sec)');
ylabel('Temperature (F)');
title('Cooling Bulb Data');
legend('Air','Bulb');
grid on
hold off

w0 = w(1);
aavg = mean(a);
z = -log((w - aavg)/(w0 - aavg));
k = (z'*t)/(t'*t);
w120 = a + (w0 - aavg)*exp(-k*120);
h = polyfit(t,z,1);
fittedt = linspace(min(t),max(t),100);
fittedz = polyval(h,fittedt);

figure
hold on
plot(t,z,'ro','LineWidth',2);
plot(fittedt,fittedz,'b-','LineWidth',2);
xlabel('time(sec)');
ylabel('y(non-dim.');
title('Best Fit Line');
legend('Data','Best Fit');
hold off
```

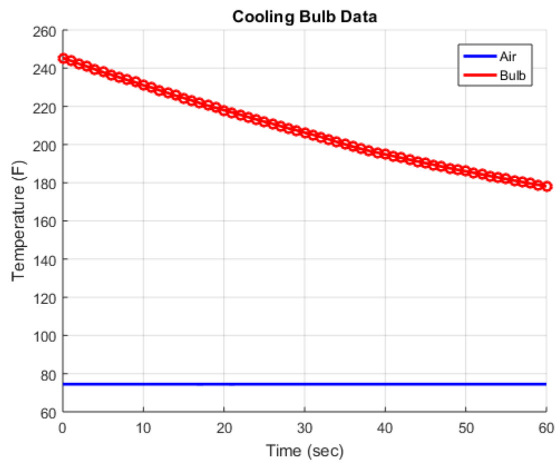


Figure 1:

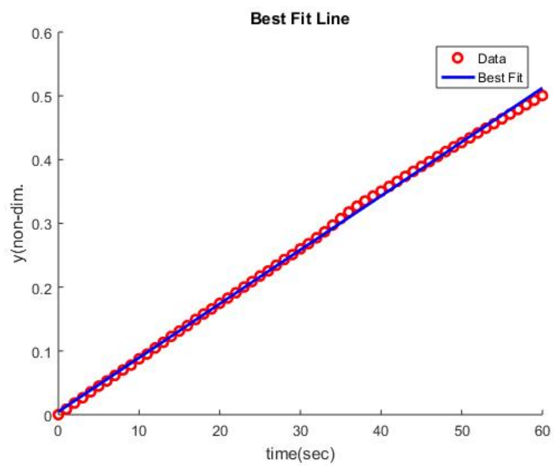


Figure 2:

2

For this exercise, in order to find the Maclaurin series of $e^{-x^2/2}$, I substituted $-\frac{x^2}{2}$ into the Taylor series expansion for e^x and obtained

$$y = 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \frac{x^8}{384}. \tag{1}$$

```
% Exercise 2
x = [0:.1:2]; % Interval from 0 to 2 with increments of 0.1
y = exp(-(x.^2)/2); % Original exponential function
y1 = 1 - ((x.^2)/2) + ((x.^4)/8) - ((x.^6)/48) + ((x.^8)/384); % Taylor approximation

figure
hold on
plot(x,y,'b-','LineWidth',2);
plot(x,y1,'ro','LineWidth',2);
xlabel('x');
```

```
ylabel('y');
title('Plot for Exercise 2');
legend('Original Function','Taylor Expansion');
grid on
hold off
```

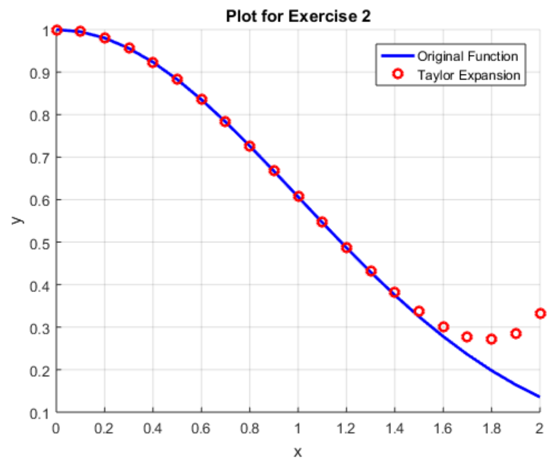


Figure 3:

I find this graph particularly intriguing because it conveys how the polynomial approximation of the exponential function behaves compared to the exponential itself. Seeing the graphs helps provide context to what is really going on with a Taylor approximation of a function: a function is being approximated by a power series such that the series mimics the behavior of the function around a given point (in this case the point is zero) and the series behaves more like the function with each successive derivative that is taken at that point. This is a very beautiful, yet extremely subtle detail.

3

For exercise 3, I first simplified $x(t) = t(3 - \ln(t)) + 2\ln(t) + 4$ by distribution so that when I substituted it into the second order differential equation, each term could be differentiated individually, and therefore easily differentiated with respect to t. After substituting x into the ODE, I found $f(t) = 2\ln(t)$. I performed the computation several times to make sure that the calculus and, sometimes more importantly, the algebra was correct.

4

For exercise 4, I remember my initial reaction from a year ago upon reading the handout: “What am I supposed to do with THESE?!” Surely enough I didn’t recognize that these were the elements of a Maclaurin series of order three beginning at $n = 0$. I used the derivatives in the the Taylor series expansion centered at zero and obtained

$$x(t) = \sum_{n=0}^{2} \frac{f^{(n)}(0)}{n!}(x)^n = 1 + 2t - \frac{3t^2}{2!}. \tag{2}$$

I found the function evaluated at $t = 1$ to be $x(t) = \frac{3}{2}$. For a series approximation of a function, estimations are generally better when there are more terms, assuming that the series converges.

5

For this exercise, I found the amplitude of $y = 2\cos(3t) - \sin(3t)$ by using the first derivative test. This method works because the derivative test provides the value of the independent variable at which the rate of change of the function is zero, which I found to be $t = -\tan^{-1}(-1/2)$. At this t, the function either occupies a local maximum or local minimum. In this case, it is a local maximum, which also happens to be the global maximum of the harmonic, and ultimately the amplitude as well. I then found the phase shift using algebra. Because both of the functions y and $y_1 = A\cos(3t + \phi)$ should be equal, these equations can be equated and any time t can be used for substitution in order to find ϕ; I used $t = 0$, which was a simple choice. After the algebra, I found the amplitude to be $A = 2.24$ and the phase shift to be $\phi = 0.464$ radians, which is the same value of t that was used to find the amplitude. Thus,

$$y_1 = 2.24\cos(3t + 0.464). \tag{3}$$

```
% Exercise 5
b = [0:.1:5];
u = 2.*cos(3.*b) - sin(3.*b);
u1 = 2.236067.*cos(3.*b + 0.463647);//

figure
hold on
plot(b,u,'ro','LineWidth',2);
plot(b,u1,'b-','LineWidth',2);
xlabel('b');
ylabel('u');
title('The Joy of Sinusoids: Exercise 5');
legend('y = 2cos(3t) - sin(3t)','y = 2.24cos(3t + 0.464)');
grid on
hold off
```

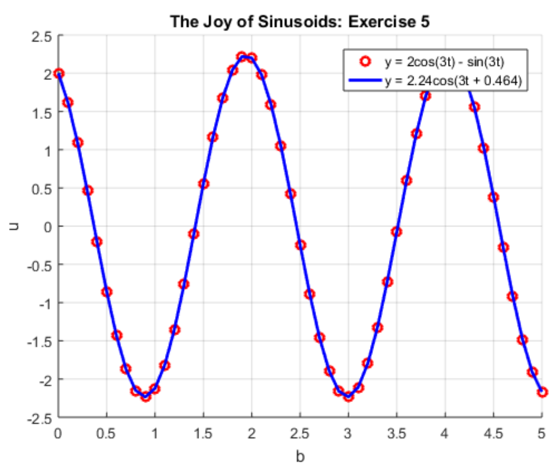


Figure 4:

6

For this exercise, as N becomes large, the way in which the graph changes is such that the phase shift decreases. As N approaches infinity, I think that the phase shift will eventually approach zero, causing the peaks to become infinitely close.

```
c = [0:100];
f3 = sind(c) + (sind(3.*c)/3) + (sind(5.*c)/5);
f6 = f3 + (sind(7.*c)/7) + (sind(9.*c)/9) + (sind(11.*c)/11);
f9 = f6 + (sind(13.*c)/13) + (sind(15.*c)/15) + (sind(17.*c)/17);
f12 = f9 + (sind(19.*c)/19) + (sind(21.*c)/21) + (sind(23.*c)/23);

figure
hold on
plot(c,f3,'r-','LineWidth',2);
plot(c,f6,'b-','LineWidth',2);
plot(c,f9,'m-','LineWidth',2);
plot(c,f12,'g-','LineWidth',2);
xlabel('c');
ylabel('f');
title('Square Waves: Exercise 6');
legend('N = 3','N = 6','N = 9','N = 12');
grid on
hold off
```

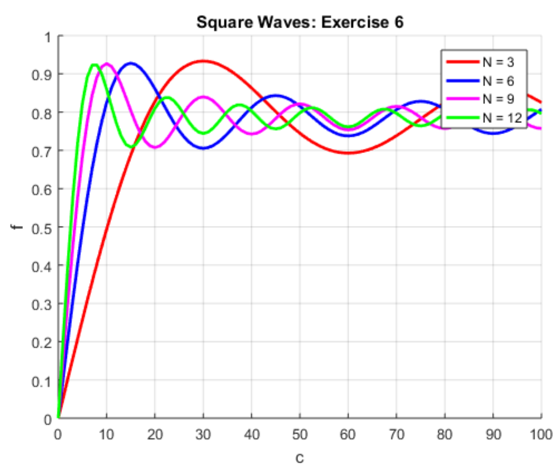


Figure 5:

7

In the partial fraction decomposition of

$$\frac{s^2-1}{s(s+1)(s+2)}, \tag{4}$$

I multiplied both sides of the equation by $s(s+1)(s+2)$, and proceeded to solve the equation. To find a, I set $s=0$, leaving only the a term. To find b, I set $s=-1$ to isolate b, and for c, I set $s=-2$ to do the same thing. I found the coefficients to be $a=\frac{1}{2}$, $b=-2$, and $c=\frac{5}{2}$.

```
% Exercise 7
s = [0:10];
v = ((s.^2) + 1)./(s.*(s + 1).*(s + 2));
v1 = ((1/2)./(s)) + ((-2)./(s + 1)) + ((5/2)./(s + 2));//

figure
hold on
plot(s,v,'ro','LineWidth',2);
plot(s,v1,'b-','LineWidth',2);
xlabel('s');
ylabel('v');
title('Partial Fraction Decomposition: Exercise 7');
legend('Original Function','Decomposed Function');
grid on
hold off
```

One thing that I find puzzling is how partial fraction decomposition works; my Calculus II professor did not elaborate on how it works, yet at the same time, I should have been more curious and inquisitive to ask why it does work.

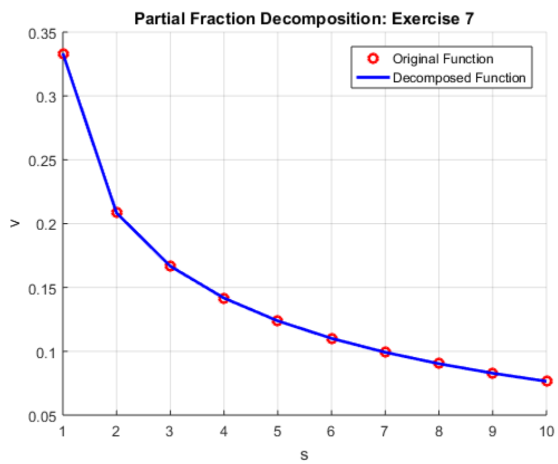


Figure 6:

8

In this exercise, the Laplace transform was computed for $f = t^2$, $f = \sin(t)$, and $f = 1$ for $t < 1$; otherwise $f = 0$. For $\mathcal{L}[t^2]$, the improper integral was calculated by implementing integration by parts twice to ultimately achieve

$$\frac{2}{s^2} \lim_{h\to\infty} \int_0^h e^{-st} dt. \tag{5}$$

After further calculations, the answer that I obtained was $\frac{2}{s^3}$. It is important to specify the domain of the parameter s; for parts a, b, and c, $s > 0$. If s is negative, then the evaluations from the previous steps and the last improper integral would be impossible, for the exponential portion would approach $-\infty$, which wouldn't make sense for the evaluation. For b, the same process was used. However, because the result from the second pass of integration by parts returns the original integral

$$\lim_{h\to\infty}\int_0^h \sin(t)e^{-st}dt = -\frac{1}{s^2}\lim_{h\to\infty}\cos(t)e^{-st}\Big|_0^h - \frac{1}{s^2}\lim_{h\to\infty}\int_0^h \sin(t)e^{-st}dt, \tag{6}$$

the original integral was added to both sides, algebra was performed, the calculus was done (evaluating the improper integral), and the result that was obtained is $\mathcal{L}[\sin(t)] = \frac{1}{s^2+1}$. For the piece-wise function, the Laplace transform was computed for each parameter and what was obtained was

$$\mathcal{L}[f] = \begin{cases} \frac{1}{s}, & s > 0 \\ 0, & otherwise \end{cases}. \tag{7}$$

These were validated with a table in which Laplace transforms were applied to multiple functions.

Correction: In looking at the piece-wise function I did not realize, and this is where the rethinking comes in, that

$$f = \begin{cases} 1, & \text{for } t < 1 \\ 0, & otherwise \end{cases}, \tag{8}$$

is the same as

$$f = \begin{cases} 1, & \text{for } t < 1 \\ 0, & t \geq 1 \end{cases}. \tag{9}$$

What's more, the definition of the Laplace transform is that the integral is evaluated between 0 and ∞, so the intervals for integration are $[0,1)$ and $[1,\infty)$. So, the new Laplace transform for f becomes

$$\mathcal{L}[f] = \int_0^\infty f(t)e^{-st}dt = \lim_{h\to 1}\int_0^h e^{-st}dt + \int_1^\infty 0 \cdot dt. \tag{10}$$

So, the second half of the Laplace transform goes to zero and the evaluation of the first half of the transform produces

$$\mathcal{L}[f] = \lim_{h\to\infty} -\frac{e^{-st}}{s}\Big|_0^h. \tag{11}$$

Evaluating this limit produces

$$\mathcal{L}[f] = \frac{1-e^{-s}}{s}, \qquad s \neq 0. \tag{12}$$

Before validating with a calculator, it would be important to validate with mathematical logic and reasoning. So, in order to properly integrate a piece-wise function, the integration must be carried out for the correct interval, which is in this case $0 \leq t < 1$ and $1 \leq t < \infty$. This would cause the integral to be separated into a sum of two different integrals, each defined by the bounds of their respective intervals. The integrand of each integral must then contain the piece of the piece-wise function for which the respective bounds are defined, so $f = 1$ for the interval of [0,1) and $f = 0$ for $[1,\infty)$. Lastly, the integral must be evaluated correctly based on what is known about derivatives, anti-derivatives, and the methods of, ahem, Dr. Chain. Doing this guarantees that equation (12) is the solution. This solution was then validated with an online Laplace transform calculator.

9

For this exercise, I computed the derivative of $y = e^{-\cos(t)}$ and obtained
$y' = \sin(t)e^{-\cos(t)}$ through the implementation of the chain rule, from the notorious Dr. Chain. In MATLAB, attempting to plot this result, along with the plot of the MATLAB computation of the derivative, would have

probably been the death of me had I not left the library Sunday night. The reason for my grief lay with the fact that when MATLAB takes derivatives, the original vector is shortened by one element. This provides an inconsistency when trying to plot because the vector dimensions don't agree between the independent variable vector and the derivative vector. Fortunately, I was able to find some code online that aided in my endeavor to plot the derivative I calculated and the derivative Matlab calculated. Of course, my difficulty in trying to plot the data was due to how unfamiliar I still am concerning Matlab.

```
d = [0:.1:10];
g = exp(-cos(d));
g2 = sin(d).*exp(-cos(d));
dd = diff(d);
dg = diff(g);
derivative = dg./dd;

figure
hold on
plot(d(1:end-1),derivative,'ro','LineWidth',2);
plot(d,g2,'b-','LineWidth',2);
xlabel('d');
ylabel('Derivative of g');
title('Derivative Plot: Exercise 9');
legend('Matlab Derivative','Calculated Derivative');
grid on
hold off
```

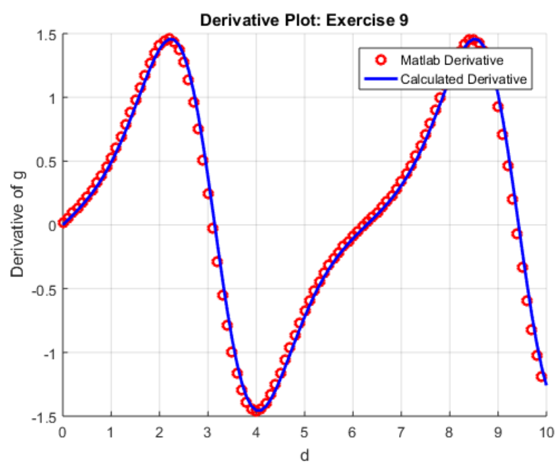


Figure 7:

10

In order to properly construct a model of a dissolving spherical capsule, I must first understand what is at play, mathematically. It is given that the sphere is losing volume at a rate that is proportional to its surface area. So, let v be the volume of the sphere and let a be the surface area of the sphere. Interestingly enough, there are two important considerations: 1.) there are already formulas for the volume and surface area of a sphere and 2.) both are functions of the radius of the sphere. As a result of consideration (2), however, therein lies another interesting caveat. The sphere loses volume as time progresses, so this must suggest that volume and, thus, surface area are both implicit functions of time via the radius, $r(t)$. Further, if the volume decreases with time, being a function of the radius, then it must follow that the radius also decreases with time. Now would be a good time to construct a simple ODE for a dissolving sphere as such,

$$\frac{dv}{dr} = -ka. \tag{13}$$

Equation (8) says that the sphere's volume decreases as the radius decreases in a manner that is proportional to its surface area. Unfortunately, this is not the whole story, for it leaves out the most important parameter: time. To account for this, I must revisit the beloved Dr. Chain and apply the chain rule to the left-hand side of the equation. This will result in an equation that looks something like this:

$$\frac{dv}{dr} \cdot \frac{dr}{dt} = -ka. \tag{14}$$

This describes how the volume changes with time, which is exactly what I'm looking for. The next step would be substitute the equations for volume and surface area into the ODE to make a bit more sense of what is going on, so

$$\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) \cdot \frac{dr}{dt} = -k\big(4\pi r^2\big). \tag{15}$$

Differentiating the volume with respect to r and further simplifying gives this result

$$\frac{dr}{dt} = -k. \tag{16}$$

This means that the volume decreases with respect to time at a constant rate. To find $r(t)$, dt must be multiplied by both sides so as to separate the variables and both sides are then integrated to produce a linear equation

$$r(t) = -kt + C. \tag{17}$$

It is clear that if the first derivative is taken to validate the proposed solution, the result will be that of equation (16), thus concluding the mathematical construction for a dissolving spherical capsule.

11

This part is related to the quiz. I am quite frustrated at my inability to critically and analytically think about what I was doing, considering I tutored Calculus I last fall, especially when I emphasized critical thinking during my tutoring sessions. After receiving my quiz back, I sat and really thought about what tangent line approximation really is, why it is used and how it can be used to approximate crazy values like $\sqrt{4.1}$. Thinking for a good 10 minutes led me to slopes; not derivatives and rates of change, but rather simple slopes like those from slope intercept form, which is ultimately the form in which the tangent line approximation is utilized. What is a slope at its core? It is a ratio that conveys how the dependent variable changes when the independent variable changes by some amount. Expanding this idea to calculus, derivatives are simply limits of the ratio that slope normally gives, as the independent variable undergoes a very finitely small change (approaches zero; infinitely small change makes no sense). Let's examine the formula for slope:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}, \quad \text{given two points} \quad (x_1, y_1) \quad \text{and} \quad (x_2, y_2). \tag{18}$$

Multiplying both sides by $(x_2 - x_1)$ gives the point-slope form of an equation. From this form alone, it can be surmised that through some algebraic manipulation, it is totally possible to solve for y_2 if the slope is known and if (x_1, y_1) is given. Extending this to calculus, slope becomes the derivative, (x_1, y_1) becomes the point at which the derivative is computed, y_2 becomes the crazy value of the function that is desired, and Δx is the change that prompts the function to move from y_1 to y_2. And so, a calculus problem is now reduced to an algebra problem; however, much of the beauty lies in the calculus, which is why tangent line approximation is actually a very useful and powerful tool, although extremely simple in principle. In examining the calculus analog of the point-slope form, the equation for the tangent line approximation at $x = a$,

$$f(a) = f(x_0) + f'(x_0)(x - a), \tag{19}$$

it is clear that this is the calculus analog of the algebraic equation for point-slope form. Where the real beauty lies is in the fact that the equation for the tangent line approximation is exactly the same as a first order Taylor approximation centered at a! Taylor approximations for a function are supposed to mimic the function, and the more terms there are in the expansion, the more the Taylor polynomial behaves like the original function. So, to answer the first two questions in the first paragraph, the tangent line approximation, being a first degree Taylor polynomial, is used to approximate values of functions [1]. To answer the third question, I will use the quiz as an example, using proper mathematical exposition. To begin, let y be a function of x such that $y = \sqrt{x}$. In order to approximate $\sqrt{4.1}$, there are a few things that are needed: 1.) two points (part of one is given) and 2.) slope, also known as the derivative. The derivative is the most obvious to find, which is $y' = \frac{1}{2\sqrt{x}}$. To find the two points, it may not be as obvious. For the first point, it would be desirable to have a value that is within the neighborhood the desired x value, 4.1, and it would be preferred if it could be used in the derivative formula with ease. The reason for these two requirements is because approximations are better when Δx is small [2] and making sure that computing the derivative with it is easy makes life a bit better. So, the x value of the first point would be $x = 4$. This is a good candidate because it is only 0.1 away from the desired value and substituting it into y' gives a value of 0.25 for the slope, an easily computed derivative. To obtain the y value, substitute 4 into the original function to get $y = 2$. For the second point, the x value is already given. The only thing left to find is $y = \sqrt{4.1}$, for which the tangent line approximation will be used. Now we have all we need to construct the slope formula in order to approximate $y(4.1)$, which is

$$y' = 0.25 \approx \frac{y - 2}{4.1 - 4}. \tag{20}$$

Solving for y gives

$$y \approx 2 + 0.25(4.1 - 4) = 2.025. \tag{21}$$

[1] Of course there are other important applications.

[2] This follows from how the derivative is better approximated with smaller and smaller values of Δx.

2. Homework 2

"*A good teacher can inspire hope, ignite the imagination, and instill a love of learning.*" - Brad Henry

1

As found in Homework 1, the equation for the radius as a function of time was determined to be

$$r(t) = -kt + C.$$

This equation, however, only serves as the general formula for the radius as a function of time. In order to find the equation that would best fit the data, the initial value of the data must be considered. So, at the time $t = 0$, the value that is given by the data is 33.510322. This value, though, is the volume of the capsule, so the radius must be found using the equation for the volume of a sphere and some algebraic manipulation. As a result, the radius of the capsule was determined to be r(0) = 2.0000, as computed by hand and confirmed through Matlab. The next step, and the most important, is to transform equation (16) into a matrix-vector equation in order to solve for the proportionality constant. Let A be the 61x2 matrix such that

$$A = \begin{bmatrix} C & t_0 \\ C & t_1 \\ \vdots & \vdots \\ C & t_{60} \end{bmatrix}, \tag{22}$$

let x be the column vector

$$x = \begin{bmatrix} 1 \\ k \end{bmatrix}, \tag{23}$$

and let r be the column vector of the capsule data, keeping in mind that it should be the radius of the capsule at time t instead of its volume

$$r = \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_{60} \end{bmatrix}. \tag{24}$$

```
% Exercise 1.
% Construct Vandermonde matrix using capsule data and equation for radius.
A = ones(length(capsule),2);
A(:,1) = nthroot(capsule(1,2).*(3/(4*pi)),3);
A(:,2) = capsule(:,1);

% Cube root of volume data to obtain the radius.
r = nthroot(capsule(:,2).*(3/(4*pi)),3);

% Solve the System Ax = r.
x = linsolve(A,r);
```

Now equation 1 has the form $Ax = r$: a simple linear equation in one variable. To solve for x, the `linsolve` command must be used. Doing so produces

$$x = \begin{bmatrix} 1.0000 \\ -0.0200 \end{bmatrix}. \tag{25}$$

```
x =

    1.0000
   -0.0200
```

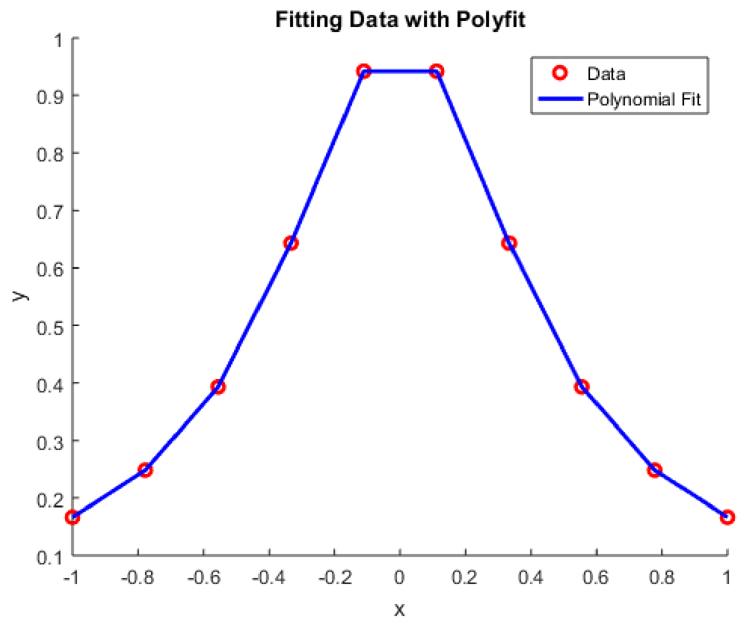
This makes makes sense, as the first component of the vector has a value of 1, which is expected. Now, the equation of the radius as a function of time becomes

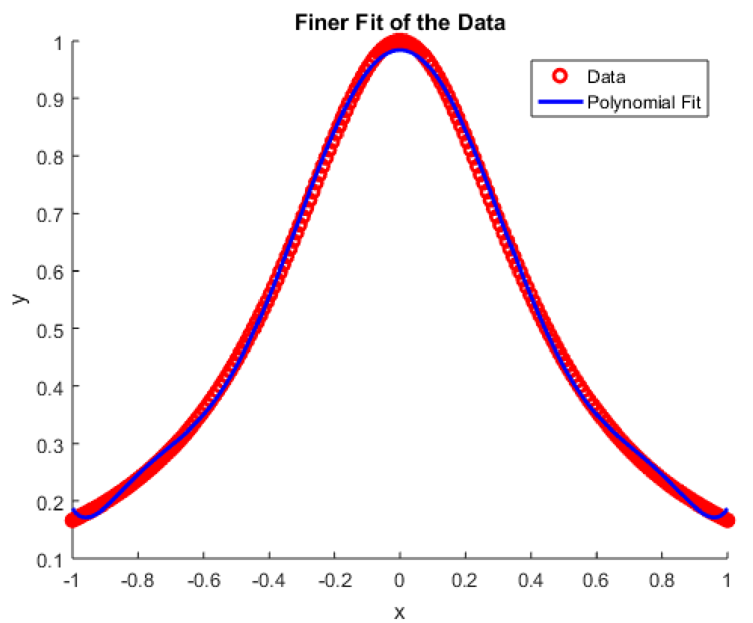
$$r(t) = -0.0200t + 2.0000. \tag{26}$$

To solve for the time at which the capsule completely dissolves, set $r(t) = 0$, perform some algebra, and obtain the value of $t = 100$ seconds.

2

```
% Exercise 2.
x = linspace(-1,1,10);
y = 1./(1+5*x.^2);
z = polyfit(x,y,9);
w = polyval(z,x);

figure
hold on
plot(x,y,'ro','LineWidth',2);
plot(x,w,'b-','LineWidth',2);
xlabel('x');
ylabel('y');
legend('Data','Polynomial Fit');
title('Fitting Data with Polyfit');
hold off

% Finer plot.
xx = linspace(-1,1,200);
yy = 1./(1+5*xx.^2);
zz = polyfit(xx,yy,9);
ww = polyval(zz,xx);

figure
hold on
plot(xx,yy,'ro','LineWidth',2);
plot(xx,ww,'b-','LineWidth',2);
xlabel('x');
ylabel('y');
legend('Data','Polynomial Fit');
title('finer Fit of the Data');
hold off
```



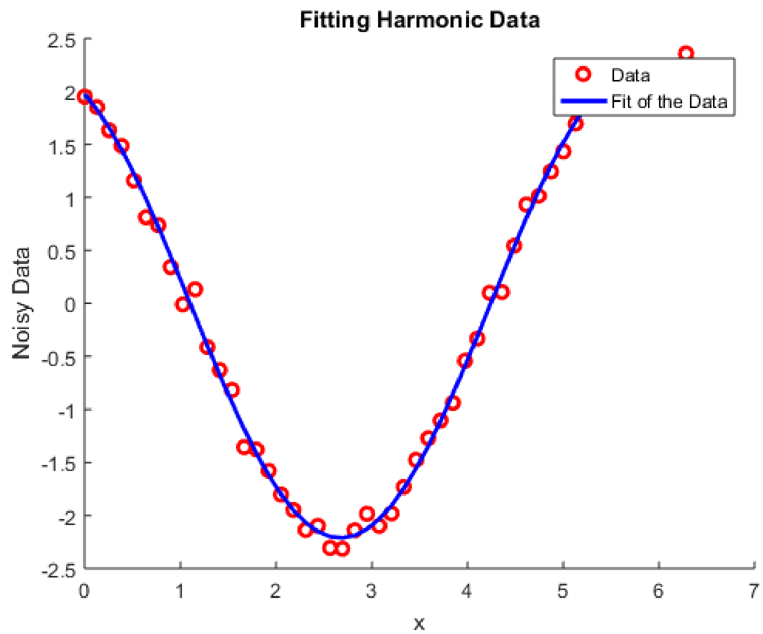


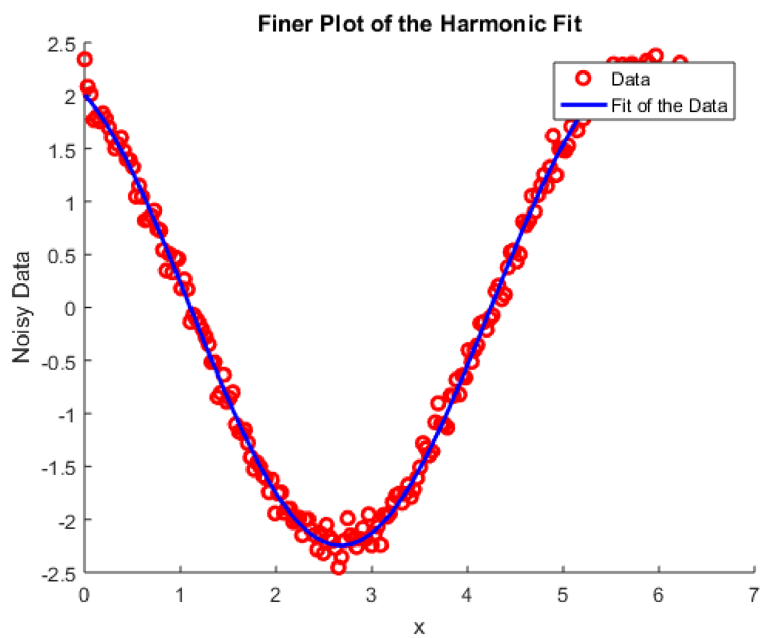
These graphs are interesting because the above plot conveys the fit as exactly matching the data, yet in the second plot the data clearly does not match the data exactly. This demonstrates how polynomials can be fit to data. The important consideration is understanding that the degree of the polynomial affects the fit for the data.

3

```
% Exercise 3.
% A simple harmonic with some noise.
x = linspace(0,2*pi,50);
```

```
y = 2*cos(x) - sin(x) + .1*randn(size(x));

% Create the Vandermonde matrix using y.
B = zeros(50,2);
B(:,1) = cos(x);
B(:,2) = sin(x);

% Solve the system.
p = linsolve(B,y'); % y is transposed because it produces a row vector.
q = B*p;

figure
hold on
plot(x,y,'ro','LineWidth',2);
plot(x,q,'b-','LineWidth',2);
xlabel('x');
ylabel('Noisy Data');
legend('Data','Fit of the Data');
title('Fitting Harmonic Data');
hold off

% Finer plot.
xx = linspace(0,2*pi,200);
yy = 2*cos(xx) - sin(xx) + .1*randn(size(xx));
BB = zeros(200,2);
BB(:,1) = cos(xx);
BB(:,2) = sin(xx);
pp = linsolve(BB,yy');
qq = BB*pp;

figure
hold on
plot(xx,yy,'ro','LineWidth',2);
plot(xx,qq,'b-','LineWidth',2);
xlabel('x');
ylabel('Noisy Data');
legend('Data','Fit of the Data');
title('Finer Plot of the Harmonic Fit');
hold off
```


Fitting Harmonic Data
Data
Fit of the Data
Noisy Data
x
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
1
2
3
4
5
6
7


Finer Plot of the Harmonic Fit
Data
Fit of the Data
Noisy Data
x
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
1
2
3
4
5
6
7

4

Upon first trying to solve this exercise, there was some struggle. I first applied matrix multiplication to the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{27}$$

and multiplied it by itself to obtain the following product

$$\begin{bmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{bmatrix}. \tag{28}$$

I then set these equal to zero and struggled with the algebra. I did initially make progress: I found that for such a matrix, $a = -d$, from manipulating the equation in the first row and the first column. After that, though, I was not able to figure where else to go. Even though I had all the tools I needed in order to obtain a satisfactory solution, I did not think of all of the possible ways I could manipulate the equations I found. Eventually, I looked online, not at the solution but rather at an equation that was an important stepping stone in the process: the determinant for a 2x2 matrix. The determinant for a 2x2 matrix is defined as $ad - cb$. After looking at this equation, I realized that I did not have to completely substitute $a = -d$ for $a^2 + bc = 0$, but rather can only substitute one of the a terms to obtain $-ad + bc = 0$, which can be arranged as $ad = bc$. Thus, if $a = -d$, then it must be true that $b = -c$; otherwise $ad = bc$ would be a false statement. Further, in order for $ad = bc$, it also must be true that $a = b = -c = -d$. Thus any matrix that contains these parameters satisfies the conditions such that $X^2 = 0$. Just as a quick example, consider the following matrix

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}.$$

Multiplying this matrix by itself will produce a 2x2 zero matrix. The matrix could also take the form of

$$\begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}.$$

This was validated with MATLAB code. If I were to rethink the process by which I initially attempted to solve this exercise, I would convince myself that, as long as it is mathematically correct, I can make any substitution, whether the substitution is complete or partial. This would ultimately transform the way I think about and perceive the algebraic manipulation of equations and expressions, which is extremely useful and applicable, especially in the disciplines for which I am studying.

5

I struggled with this exercise for the same reasons as in the last one and also needed some assistance, but I did learn from the struggles I had. At first, I attempted to solve the system algebraically, which was indeed a difficult task. I then proceeded to look for guidance, and the suggestion I found was to set one of the variables in the matrix equal to some value and then proceed to solve for the remaining variables such that the conditions are satisfied. This further exemplifies how I should have been more aware of the tools I have to use when analyzing a mathematics exercise. Although I had found this suggestion, my desire for understanding had not been satisfied, so I decided to look deeper. The more I thought and stared at the exercise, the more I realized that I approached the exercise blindly. I initially assumed that a non-zero matrix was a matrix that did not have any components equal to zero—I'm sure glad that I was wrong. I proceeded to search the definition of a non-zero matrix and found that a non-zero matrix is any matrix where at least one of the components is not equal to zero, otherwise the result would be the zero matrix. With my new understanding of the exercise, and the suggestion that I found online, I proceeded to attempt to address the exercise. Upon squaring equation (27), equation (28) was obtained and I set equation (28) equal to equation (27) to obtain

$$\begin{bmatrix} a^2+bc=a & ab+bd=b \\ ca+dc=c & cb+d^2=d \end{bmatrix}.$$

In solving this system, I set c = 0 and started from here. In doing so, I obtained

$$\begin{bmatrix} a^2=a & ab+bd=b \\ 0=0 & d^2=d \end{bmatrix}.$$

In examining $ab+bd=b$, I rearranged the equation to give $b[(a+d)-1]=0$. Using the zero-product property, it can be concluded that either $b=0$, $a+d=1$, or both. In analyzing the first case, if $b=0$, then it can be concluded that $a=0$ or $a=1$; this is also true for d. Here, if $a+d=1$, then $a=1$ and $d=0$[3]. However, for this case, although condition (ii) would be satisfied, condition (i) is not satisfied. So this must mean that it is not always true that $a+b=1$. The only other option to try is $a=d=1$. In this case, both conditions are satisfied and the matrix that is obtained is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

further suggesting that $a+b\neq 1$. This matrix is known as a 2x2 identity matrix, which I am sure will have great importance in this course and in future mathematical applications I am sure to encounter.

6

This exercise also was a struggle for the same reasons as the previous two exercises; however, I was able to solve this exercise completely because of one important principle, which I will eventually acknowledge. In the first handout, one of the more important concepts that you provided—of course all the concepts presented are of great value—was that for any form of modeling, start with a simple model first and then proceed to add more complex elements along the way. While thinking of this exercise, I remembered that concept and realized that it can be applied to exercises that are not necessarily related to modeling. In essence, the purpose of the exercise is to model a matrix such that when multiplied by a vector, the product is the vector, but interchanged. And this brings me to the important principle: to apply what I learn! Although very simple, at times it is very deceptively difficult to do so, depending on how the material that was learned is perceived and retained. At first, I tried to solve this system symbolically, which was extremely difficult. However, when I applied what I had learned from the first handout, about keeping it simple, I found the matrix that satisfied the conditions. For the matrix vector system,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} y \\ x \end{bmatrix}.$$

I let $a=0$ and $d=0$. In doing so, the new equation becomes

$$\begin{aligned} by &= y \\ cx &= x. \end{aligned}$$

In order for this to be true, it must follow that $b=c=1$. This result produces the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For the MATLAB validation, I input this matrix as `D = [0 1; 1 0]` and multiplied it by various column vectors of the form `v = [x y]'`. The input of `D*v` returned `[y x]'`, confirming my solution.

[3] a could also equal 0 and $d=1$.

7

This exercise begins as a Calculus III problem and is quickly reduced to a Calculus I exercise. Because I am looking for a point on the line $2x + 3y = 1$ that is closest to the origin, this means that the distance from the origin to the point (x, y) must be minimized. So, let f be a function of the distance between two points

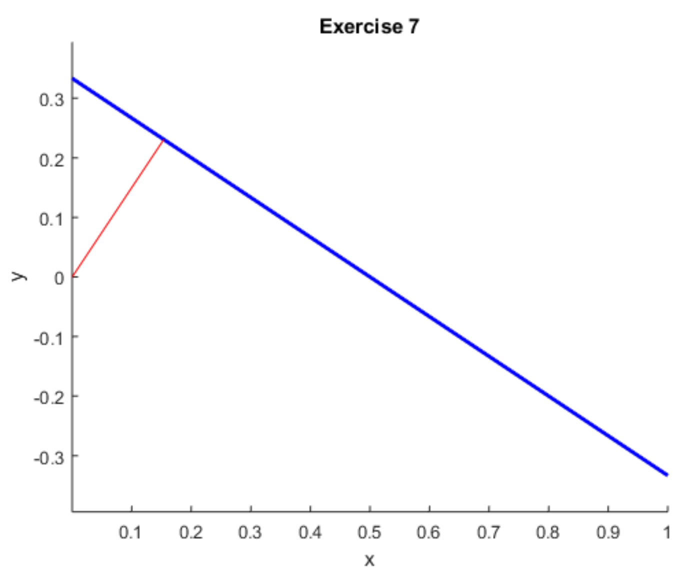
$$f(x,y) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Because $(x_1, y_1) = (0, 0)$ and because $y = 1/3 - 2x/3$, equation (18) becomes

$$\begin{aligned} f\left(x, \frac{1}{3} - \frac{2x}{3}\right) = f(x) &= \sqrt{x^2 + \left(\frac{1}{2} - \frac{2x}{3}\right)^2} \\ &= \sqrt{\frac{13}{9}x^2 - \frac{4}{9}x + \frac{1}{4}}. \end{aligned}$$

Now, the minimum of f can be determined using the first derivative test from Calculus I. Computing the derivative and setting it equal to zero gives an x-value of $x = \frac{2}{13}$. Substituting this into $2x + 3y = 1$ gives $y = \frac{3}{13}$.

```
% Exercise 7
C = linspace(0,1,50);
f = (1/3) - (2/3)*C;

figure
hold on
plot(C,f,'b-','LineWidth',2);
plot([0 (2/13)],[0 (3/13)],'r-');
xlabel('x');
ylabel('y');
title('Exercise 7');
axis equal
hold off
```



3. Homework 3

"A true teacher would never tell you what to do. But he would give you the knowledge with which you could decide what would be best for you to do."
- Christopher Pike[4]

1

In order to solve the following IVP

$$\frac{dQ}{dt} = \frac{1}{RC}(CE(t) - Q),\ Q(0) = Q_0,\ E(t) = a + bt, \tag{29}$$

it would be useful to understand what it means. The ODE conveys that the current of an RC-circuit $I = dQ/dt$ is directly proportional to the difference between the electromotive force $E(t)$ times the capacitance $Cand$ the charge Q, with the proportionality constant being the RC delay, the inverse of the product of the resistance R and the capacitance. This ODE is first order and non-separable due to the presence of $E(t)$, which is a known, explicit function of time. Now that the ODE has been characterized, it can be accurately and efficiently analyzed. Firstly, let $(RC)^{-1}$ be k. Because the ODE is non-separable, substitution must be used so as to produce a separable equation. After two substitutions, a bit of calculus, and some algebra, the relation that is obtained is of the form

$$\frac{dv}{dt} = -kv, \tag{30}$$

which is indeed a separable ODE whose solution is a decaying exponential. After integration, re-substitution, and the implementation of the initial condition, the explicit solution that is obtained is

$$Q = CE(t) - RC^2b + \left(Q_0 + RC^2b - Ca\right)e^{-t/RC}. \tag{31}$$

Combining equations (29) and (31) validates the solution. I remember when I initially took this course, I had a lot of trouble solving this IVP. In fact, I think it took me three or four times until I arrived at the solution. However, this time it only took me one pass. I think the difference between last spring and this semester is that I am finally understanding the importance of what I am studying and I am attaching meaning to what I am doing, in addition to using what I've learned in the past, applying that knowledge now. For example, when solving the IVP this time, I realized after integrating that I can't choose C for the integration constant because it is being used to denote the capacitance, so I used D as the constant. Even though this is a minor detail, it would drastically alter the solution of the IVP. Further, although I obtained this solution symbolically, I wanted to further validate through Matlab as a programming exercise. I will do some further analysis in my free time to study the behavior of RC-circuits. In the meantime, here is my code:

```
%% Exercise 1
Q0 = rand;
a = rand;
b = rand;
R = rand;
C = rand;
T = 3;
dQdt = @(t,Q) (1/(R*C))*((C*a) + (C*b*t) - Q); % rhs of the ODE
[t,Q] = ode45(dQdt,[0 T],Q0);
```

[4] These quotes are meant to reflect the manner in which I approach the homework each week. One important thing I've learned this passed week is that the difference between knowledge and wisdom is that knowledge is only knowing. It is not until what is known is understood and applied; then it becomes wisdom.

```
Q1 = C*(a + (b*t)) - (R*(C^2)*b) - ((C*a) - (R*(C^2)*b) - (Q0))*exp(-t/(R*C));

figure
hold on
plot(t,Q,'ro','LineWidth',2);
plot(t,Q1,'b-','LineWidth',2);
xlabel('time (s)');
ylabel('Charge Q (C)');
legend('Matlab Solution', 'Symbolic Solution');
hold off
```

Random data was generated and I honestly chose the prettiest plot of the random ones that were generated. However, it is clear that the symbolic solution was indeed correct. Here is the plot of the solution.

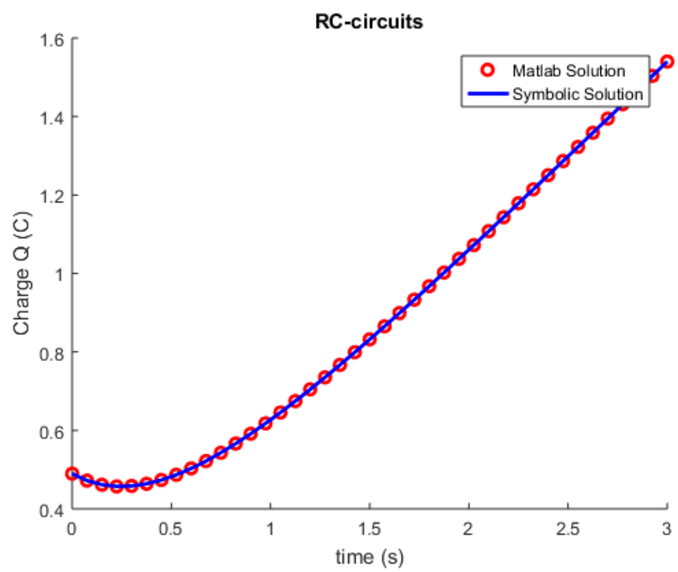


Figure 8: RC-circuit

2

$$\frac{dx}{dt} = f(t) - x, \;\; x(0) = 0. \tag{32}$$

(a) $f(t) = t^2$

In order to solve an IVP with the form of equation (32), one could either employ substitution or Taylor's theorem[5]. Because $f(t)$ is a monomial, it would be more efficient to use substitution. In doing so, two substitutions are performed and what is obtained was

$$\frac{dv}{dt} = 2 - v, \tag{33}$$

which is a variant of equation (30), also producing a decaying exponential. After some calculus, re-substitution, and the application of the initial condition, the solution that was obtained was

$$x = t^2 - 2t + 2 - 2e^{-t}. \tag{34}$$

[5] I am certain there are more methods with which this IVP can be solved; for example, using Matlab

Combining equations (32) and (34) validates the solution symbolically. When solving this IVP, I noticed that I had added a k to the exponential, subconsciously indexing the solution to the cooling light bulb. It is nice to be able to look at what I am doing and to notice when something is amiss. Further, this emphasizes the importance of treating each problem as its own exercise, not basing the solution of one problem on another. Here is the code:

```
%% Exercise 2 part a
dxdt = @(t,x) (t.^2) - x;
x0 = 0;
T = 5;
[t,x] = ode45(dxdt,[0 T],x0);
xsymb = (t.^2) - (2*t) + 2 - 2*exp(-t);

figure
hold on
plot(t,x,'ro','LineWidth',2);
plot(t,xsymb,'b-','LineWidth',2);
xlabel('t');
ylabel('x');
legend('Matlab Solution','Symbolic Solution');
title('Exercise 2(a)');
hold off
```

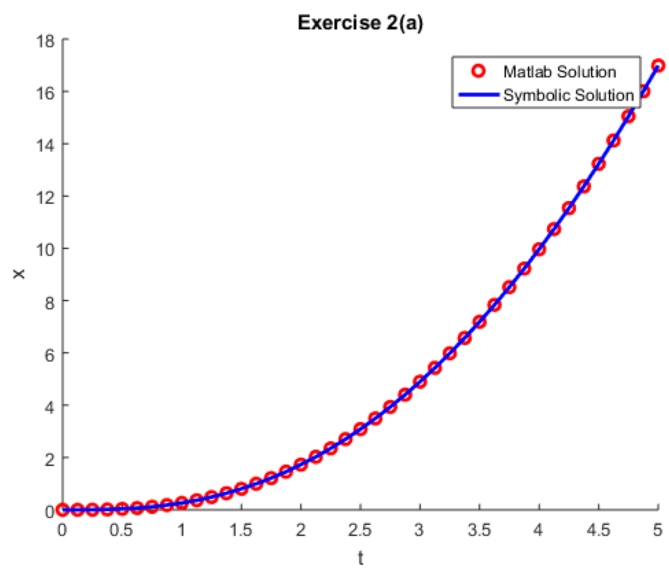


Figure 9: Exercise 2(a)

(*b*) $f(t) = t^3$

This exercise was performed using the same methods as in part (a). The differences between this part and part (a) are that an extra substitution is used and $f(t)$ is cubed instead of squared. The solution that is found is

$$x = t^3 - 3t^2 + 6t - 6 + 6e^{-t}, \tag{35}$$

and was validated by substituting it into equation (32).

```
%% Exercise 2 part b
dxdt = @(t,x) (t.^3) - x;
x0 = 0;
T = 5;
[t,x] = ode45(dxdt,[0 T],x0);
xsymb = (t.^3) - (3*(t.^2)) + (6*t) - 6 + 6*exp(-t);

figure
hold on
plot(t,x,'ro','LineWidth',2);
plot(t,xsymb,'b-','LineWidth',2);
xlabel('t');
ylabel('x');
legend('Matlab Solution','Symbolic Solution');
title('Exercise 2(b)');
hold off
```

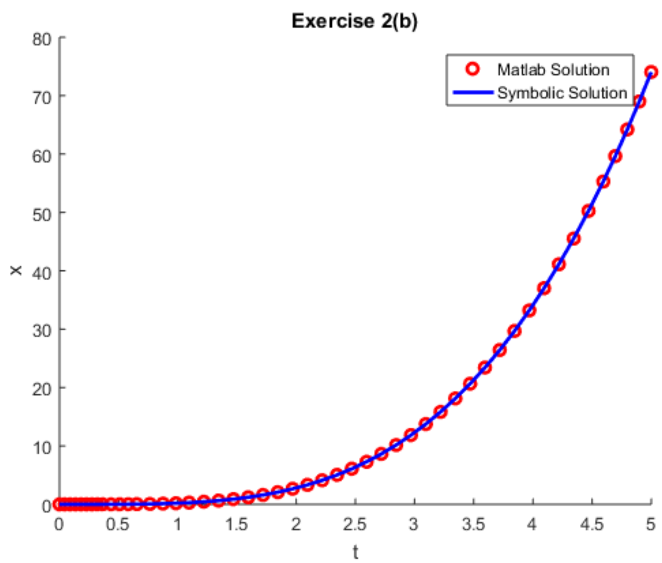


Figure 10: Exercise 2(b)

(*c*) $f(t) = t^2 + t^3$

This exercise was also solved using substitution. Like part (b), three substitutions were used and the solution that is acquired is

$$x = t^3 - 2t^2 + 4t - 4 + 4e^{-t}, \tag{36}$$

also validated using equation (32).

```
%% Exercise 2 part c
dxdt = @(t,x) (t.^2) + (t.^3) - x;
x0 = 0;
T = 5;
[t,x] = ode45(dxdt,[0 T],x0);
xsymb = (t.^3) - (2*(t.^2)) + (4*t) - 4 + 4*exp(-t);

figure
hold on
plot(t,x,'ro','LineWidth',2);
plot(t,xsymb,'b-','LineWidth',2);
xlabel('t');
ylabel('x');
legend('Matlab Solution','Symbolic Solution');
title('Exercise 2(c)');
hold off
```

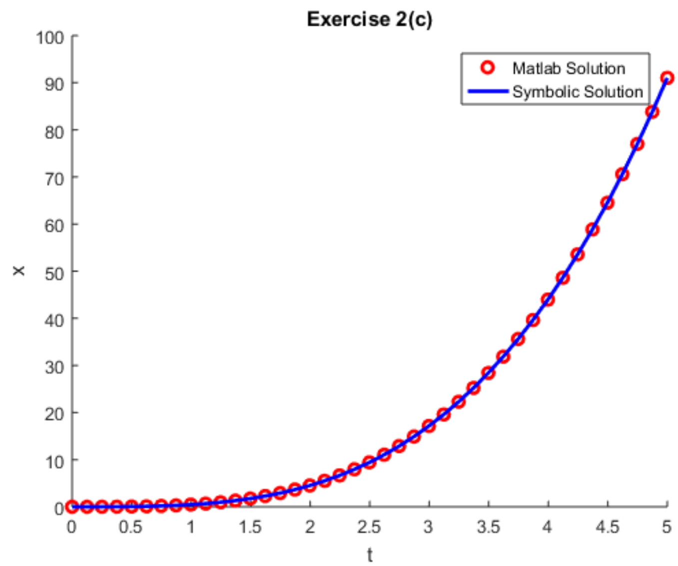


Figure 11: Exercise 2(c)

There are several commonalities between the solutions of (a), (b), and (c). Each solution is a combination of a polynomial and a decaying exponential. What's more, the degree of the polynomial in the solution is the same as the degree of $f(t)$. Some other interesting similarities are that the signs of the polynomial alternate, with the highest degree term always being positive, and that the constant is the negative of the zeroth degree term of the polynomial. I'm sure that if $f(t)$ had been negative, then the highest degree term in the solution would also be negative, ultimately reversing the signs of the other terms present in the solution. Lastly, it is worth noting that the degree of the monomial is directly proportional to the number of substitutions required to obtain a separable equation, due to the derivative that needs to be taken for each substitution. For the general case of

$$f = a_0 + a_1 t + a_2 t^2 + \cdots + a_N t^N,$$

it can be said that if f is an nth degree polynomial, then the general solution of an IVP of the form of equation (32) will contain an alternating nth degree polynomial combined with a decaying exponential.

3

```
%% Exercise 3
dxdt = @(t,x) t - x;
x0 = 1;
[t,x] = ode45(dxdt,[0 10],x0);
y = t;

figure
hold on
plot(t,y,'b-','LineWidth',1);
plot(t,x,'r-','LineWidth',1);

% Finding the point of intersection
h = @(t) t - t - 1 + 2*exp(-t);
Zero1 = fzero(h,.5);
plot(Zero1,h(Zero1),'ro');
hold off

Zero1 =

    0.6931
```

The point of intersection takes place at $t = 0.6931$, which looks suspiciously like the value of the natural logarithm evaluated at 2. Upon further investigation, I found that evaluating the natural log at 2 gives the same result as using the `long` command and then running the code for `Zero1`:

```
>> log(2)

ans =

   0.693147180559945

>> format long
>> Zero1

Zero1 =

   0.693147180559945
```

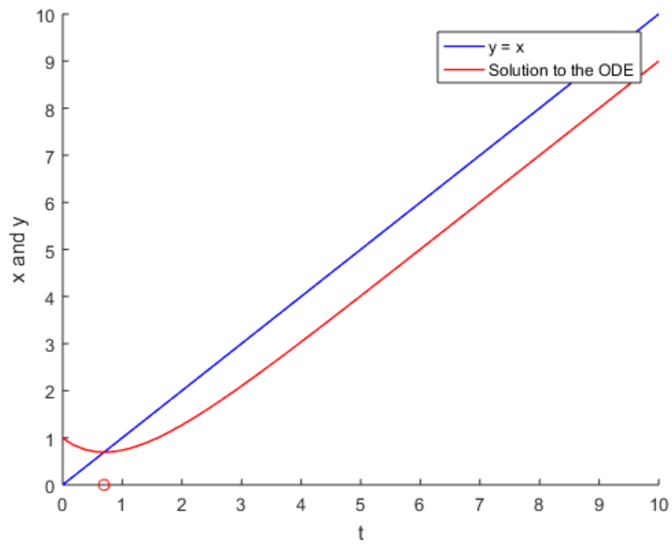


Figure 12: Caption

When first doing the exercise, I only expected three digits to be correct, but I am surprised to see that the values are nearly exact. What does this mean? Well, as I was just thinking about what this meant, I was able to visualize the first derivative evaluated at zero. When the solution to IVP (37) is set equal to zero, what is obtained is $e^t = 2$, which is, in essence, after taking the log of both sides, the natural log evaluated at 2. In further analysis, another method to obtain the point of intersection is to set $x = t$.

I really enjoy thinking of different methods of solving the same problem; it allows me to exercise my mind in testing my understanding of the problem.

4

(a) Approximate $\int_0^1 e^{-x^2}$correct to three decimals.

In order to approximate the integral, the ideal method would be to apply Taylor's theorem must to determine the degree of the polynomial that will be used. In doing so, properly, the degree of the polynomial is determined is $n = 5$. However, although I understand the important qualitative and quantitative usefulness of using Taylor approximations, I need to understand how to utilize them in a useful manner. Because I know that I must use Taylor approximations, I substituted $-x^2$ into the Taylor expansion of e^x and guessed the number of terms I would need, which was $n = 5$. After performing integration with the polynomial approximation of e^{-x^2}, the value of the approximation is $0.746729\ldots$, compared to the calculated value of 0.746824, which is accurate to three decimal places.

(b) Approximate IVP (37) using a Taylor expansion.

To approximate

$$\frac{dx}{dt} = t - x, \; x(0) = 1, \tag{37}$$

using a fourth degree polynomial, Taylor's theorem is used. Since the first derivative is already provided through the IVP, the second, third, and fourth derivatives can be taken. Further, in order to accurately apply Taylor's theorem, the polynomial must be centered at a point. An optimal point around which to center the polynomial would be at $t = 0$ because the value of x is provided at that point. Thus, the solution that is obtained for x is

$$x \approx 1 - t + t^2 - \frac{t^3}{3} + \frac{t^4}{12}. \tag{38}$$

```
%% Exercise 4 part b
dxdt = @(t,x) t - x;
x0 = 1;
[t,x] = ode45(dxdt,[0 3],x0);
x1 = 1 - t + (t.^2) - ((t.^3)/3) + ((t.^4)/12);

figure
hold on
plot(t,x,'ro');
plot(t,x1,'b-');
xlabel('t');
ylabel('x');
legend('Matlab Solution','Taylor Approximation');
title('Ode45 vs. Taylor Approximation');
hold off
```

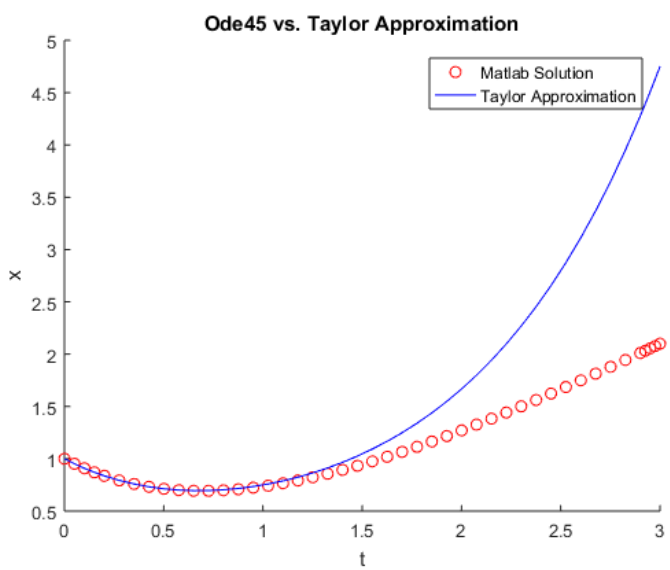


Figure 13: Caption

It is clear from the graph that the approximation quickly diverges from the actual solution. This makes sense because the solution is only an approximation. If the Taylor approximation were to go to infinity, then the approximation would line up nearly perfectly with the actual solution of the ODE.

5

In solving

$$\frac{dx}{dt} + x = f(t), \ x(0) = 5, \tag{39}$$

such that

$$f(t) = \begin{cases} 2t, & \text{when } 0 \leq t \leq 1 \\ 2, & \text{for } t > 1 \end{cases},$$

it is always important to characterize the equation. Equation (39) is a first order, linear differential equation whose right-hand side is a piece-wise defined function. In solving the ODE on the interval from $[0, 1]$, substitution was used and the solution obtained is

$$x = 2t - 2 + 3e^{-t},$$

For the interval of $(1, \infty)$, separation of variables was implemented and the solution is

$$x = 2 - Ce^{-t}.$$

Up to this point, I did not know how to continue with the exercise, so I wanted to use MATLAB to investigate. However, after struggling with MATLAB for about an hour, I could not figure out what I needed to do. I did plot x, f, and dx/dt by hand and with Desmos graphing software to observe a rough approximation of their behavior. What I found was that f is indeed piece-wise continuous, but both x and dx/dt are both discontinuous. However, in thinking about these outcomes, the graph of dx/dt is not accurate if it is not correctly evaluated with the piece-wise components. Some questions I have:

1. What would be the initial condition for the second component of the ODE and why?

4. Homework 4

1. Taylor Approximation of $y = e^{-t}\sin(t)$

In attempting to understand the properties of Taylor polynomials, I used two methods to compute the sixth degree Taylor polynomial of the function y, shown in the section heading.

Method 1. For this first method, I treated both the decaying exponential and the sinusoid of y as individual functions and found their respective sixth degree Taylor polynomials. Let p_1 denote the Taylor polynomial of the decaying exponential and let p_2 denote the Taylor polynomial of the sinusoid. Because the Taylor polynomial is defined as a polynomial approximation of a function such that the polynomial matches the function and its derivatives at the center of expansion, p_1 and its first six derivatives must match e^{-t} and its first six derivatives at the center of expansion, which in this exercise is $t_0 = 0$. This process was also performed for p_2 and $\sin(t)$. After determining both p_1 and p_2, I took their product in order to obtain $T_6(t)$, the sixth degree Taylor polynomial of y, which I found was

$$y \approx T_6(t) = t - t^2 + \frac{t^3}{3} - \frac{t^5}{30} + \frac{t^6}{90}\ . \tag{40}$$

In order to validate equation (40), I used Matlab and produced the following code and plot.

```
%% Exercise 1
t = linspace(0,2*pi,500);
y = exp(-t).*sin(t);
y1 = t;
y2 = y1 - (t.^2);
y3 = y2 + ((t.^3)/3);
y4 = y3 - ((t.^5)/30);
y5 = y4 + ((t.^6)/90);

figure
hold on
plot(t,y,'r-','LineWidth',2);
plot(t,y1,'b-');
plot(t,y2,'b-');
plot(t,y3,'b-');
plot(t,y4,'b-');
plot(t,y5,'b-');
xlabel('t');
ylabel('x');
legend('x = e^{- t} sin(t)');
title('Taylor Polynomials of x');
ylim([-0.4 0.4]);
```

Method 2. In this method, I directly computed the first six derivatives of y by hand and matched y and its derivatives to the Taylor polynomial of degree six, here denoted by x. Theoretically, this outcome and the result found in method 1 should produce the same result. However, what I obtained was

$$y \approx x(t) = t - t^2 + \frac{t^3}{3} + \frac{t^5}{30}, \tag{41}$$

which is drastically different from equation (40), not only in that equation (41) has a positive fifth term but also because the sixth term is completely missing. As a result of this discrepancy, this most likely suggested an error in my computation, and this was indeed the case. When taking the fifth derivative of y, I made

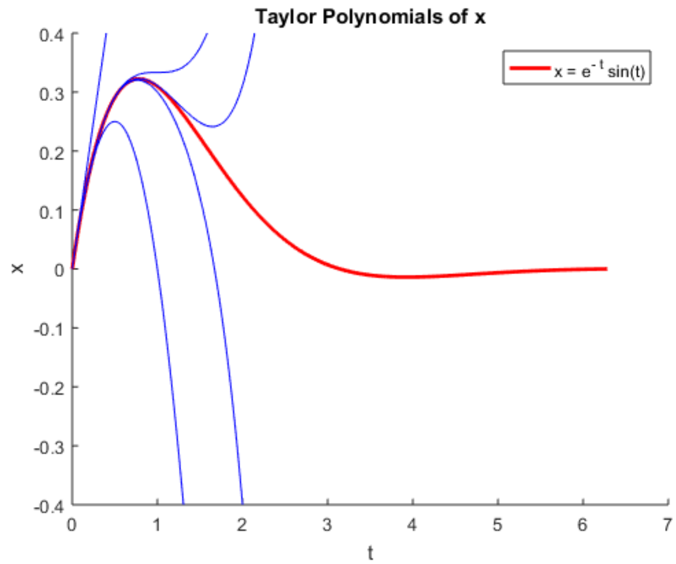


Figure 14: The curves in blue are the first several Taylor polynomials of y.

a sign error, and as a result, the sixth derivative was incorrect as well, having a value of zero. This is an important point of discussion because it conveys the significance of checking my work and understanding that small mishaps can lead to catastrophic mistakes in mathematics. Further, this confirms that methods 1 and 2 both produce the same output. This is a result of how for a function $f(t) = g(t) \cdot h(t)$, the Taylor polynomial of the product of $g(t)$ and $h(t)$ is the product of the Taylor polynomials for $g(t)$ and $h(t)$, and even still, this is an application of the general distributive nature of the product of polynomials, which is pretty cool.

2. Finding the Interval of Convergence

The interval of convergence for a power series $a_n = c_n(t-a)^n$ is an interval consisting of the values of t for which the series converges, where c_n are the coefficients of the series, a is a number, and t denotes time. In order to determine the interval of convergence for the following power series

$$J(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2 4^2 6^2} + \frac{t^8}{2^2 4^2 6^2 8^2} - \cdots , \tag{42}$$

the general formula for the coefficients of the series must be determined. When first attempting to find the general formula, it was really difficult because there is no definite method for determining the coefficients. The only way to do so is to think about how the series behaves, and based on its behavior, the general formula can be obtained. I first successfully described the general behavior of the numerator and found it could be represented as $(-1)^n t^{2n}$, where n is the index of the series and runs from $n = 0$ to $n = \infty$, $(-1)^n$ conveys the alternating behavior of the series, and t^{2n} denotes the even powers of t. Determining the general term for the denominator was much more difficult because (1) I do not recall ever finding the general term for a series, maybe except for very simple ones, and (2) the denominator behaves in a way I was not at first able to understand. In trying to find some direction on how to determine the general term of something that resembled the denominator, I found some very interesting sequences from number theory. One such sequence was the concatenating even sequence, which is almost exactly like the denominator of equation (42), but is different in that each term is not squared. I found this sequence attributed to a mathematician named Florentin Smarandache. However, when I furthered my research, I found that he renamed a function, called the Kempner Function, to the Smarandache Function, and many individuals attribute this function to Smarandache due to his "rediscovery" of it, so I feel a bit suspicious of Smarandache, but I digress. After looking at all of these sequences, and realizing that they were of no real use in my struggle with the homework, I could not find anything that would lead me in the direction of determining the general formula for the denominator of J; so, I instead struggled with the math and thought about different ways I could determine the general formula. I first tried using Taylor polynomials to see if I could find something useful. However, the only thing I found was that the odd terms were equal to zero, which is quite obvious from equation (42). I then decided to only work with the even terms and I began to find my way. I eventually found that the denominator of J was $[2(n!)]^2$, which was very close to the solution. For some strange reason, when trying to validate, I rationalized that $[2 \cdot (3 \cdot 2 \cdot 1)]^2 = 2304$. It didn't occur to me that squaring the product of two and three factorial would equal sixty-four, and I initially thought something was wrong with my calculator. But when I came to my senses, I realized my solution needed revising. In order to solve for the general formula for the denominator of J, I used only the first four terms and expressed them in terms of n, so I knew that $n = 4$ and had

$$(2n)^2(2n-2)^2(2n-4)^2(2n-6)^2.$$

After expanding and factoring out a 2^2 from each term, I found that

$$2^2n^22^2(n^2-2n+1)2^2(n^2-4n+1)2^2(n^2-6n+9).$$

In examining the previous equation, I saw that if I collected all the powers of two, I would obtain 2^8, and knowing that $n = 4$, I finally determined that instead of multiplying $(n!)^2$ by 2, I needed to multiply by 2^{2n}. After validating several times, and ensuring that $[2 \cdot (3 \cdot 2 \cdot 1)]^2 \neq 2304$, I found that the general formula for the series $J(t)$ was

$$J(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^24^2} - \frac{t^6}{2^24^26^2} + \frac{t^8}{2^24^26^28^2} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{[2^n(n!)]^2} \ . \tag{43}$$

In order to find the interval of convergence of equation (43), I applied the ratio test and found that

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|-\frac{t^2}{4(n+1)^2}\right| = \frac{t^2}{4}\cdot\lim_{n\to\infty}\left|-\frac{1}{(n+1)^2}\right| = \frac{t^2}{4}\cdot 0 = 0.$$

Because $L = 0$, $J(t)$ is absolutely convergent, and therefore convergent, for all values of t, suggesting that its interval of convergence is $(-\infty, \infty)$. This was validated by the following MATLAB code and plots.

```
%% Exercise 2
t = 85;
n = 100;
den = ((2.^([0:n])).*factorial([0:n])).^2;
num = (((-1).^([0:n])).*(t.^(2.*[0:n])));
series = num./den;

subplot(1,2,1);
plot(cumsum(series),'r-');
title('t = -85');

subplot(1,2,2);
plot(cumsum(series),'r-');
title('t = 85');
```

For the plot, I used the same value of t to show that the $J(t)$ is absolutely convergent. In substituting different values of t in the code, what is observed is that the translation of the graph to either the left or the right. Doing this exercise really helped me understand the processes required to analyze a series effectively.

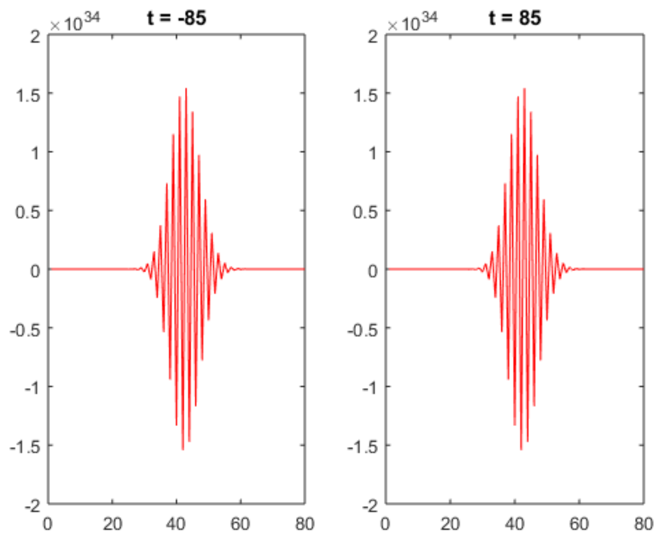


Figure 15: Both plots are the same because t is squared.

Of course, there are many other methods for doing so, like the many different convergence tests. Further, I appreciate what series are and how they are used in many different applications, like how power series are used to approximate functions and how Fourier series can be used to analyze sound and other types of frequency-based phenomena, like that shown in figure 15.

3. Approximating the Value of a Series

Because series are infinite objects, which are impossible to use in numerical computations, it is often useful to approximate the value of a series, assuming that it converges to some value. In order to approximate the following series to four decimal spaces

$$a = \sum_{n=1}^{\infty} \frac{1}{n^4}, \tag{44}$$

I needed to find a value for n so that the error, or remainder of the series, is less than or equal to 0.000005 such that

$$E_n = a - a_n = a_{n+1} \leq 0.000005, \tag{45}$$

where a_n is an approximation of a with N terms and a_{n+1} is the error staring from the $N+1$-th term to ∞. Equation (45) can also be expressed as

$$E_n = \sum_{n=1}^{\infty} \frac{1}{n^4} - \sum_{n=1}^{N} \frac{1}{n^4} = \sum_{n=N+1}^{\infty} \frac{1}{n^4} \leq 0.000005.$$

Another important concept to understand is what exactly is meant by "approximate a to four decimals." In my understanding, what it means is that the approximation of the series must be a value such that the four significant figures after the decimal place match the value of the series without changing, upon the addition of more terms to the approximation. Additionally, I chose 0.000005 instead of 0.00001 for my error bound because the former value is half-way in between the latter value and 0.00000099, meaning this would be a much better approximation. With this in mind, I applied a corollary to the integral test in order to determine the number of terms needed to approximate the series to four decimal places because a is positive, decreasing, and a convergent p-series. In doing so, I obtained

$$\lim_{h\to\infty}\sum_{n=N+1}^{h}\frac{1}{n^4}=\lim_{h\to\infty}\int_{N+1}^{h}\frac{1}{x^4}dx\leq 0.000005,$$

where the argument in the integral is a function $f(x) = 1/x^4$ that takes the indices n as inputs. In applying the integral test, the relation between the number of indices and the error bound I obtained was $n \geq 40$. Thus, at least forty terms are required for an approximation accurate to four decimals. This was validated using MATLAB with the code and plots provided below.

```
%% Exercise 3
n = 10000000;
n1 = 40;
den = ([1:n]).^(4);
den1 = ([1:n1]).^(4);
series = 1./den;
series1 = 1./den1;
error = sum(series) - sum(series1);
x = 0:40;
y = (pi^4)/90;


figure
hold on
plot(cumsum(series1),'b-','LineWidth',2);
plot(x,y*ones(size(x)),'ro','LineWidth',2);

>> sum(series1)

ans =

   1.082318217436113 % Sum of $a$ with 40 terms.

>> sum(series)

ans =

   1.082323233378306 % Sum of $a$ with 1000 terms.

>> sum(series)

ans =

   1.082323233710861 % Sum of $a$ with 10000000 terms.

>> y

y =

   1.082323233711138 % The limit as n approaches infinity
```

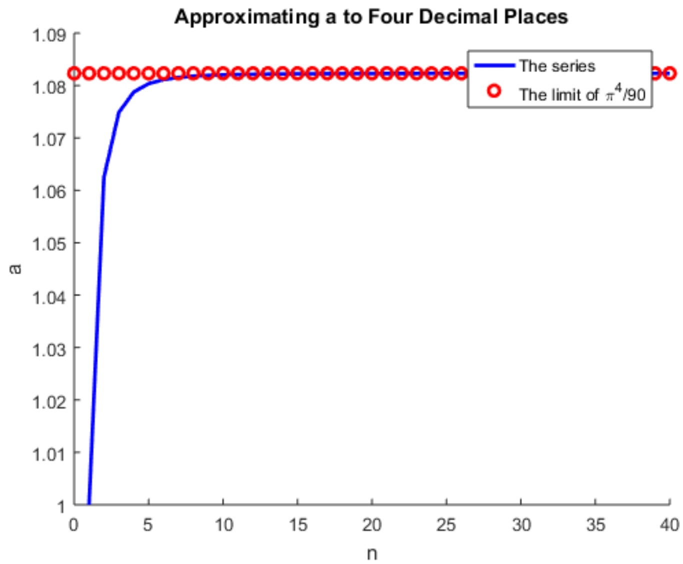


Figure 16: It is quite interesting to so the series numerically converging to some limiting value.

When first trying to do this exercise, I had a lot of trouble understanding how to solve it because I did not remember the different methods used to approximate the value of a series and what's worse is even if I did remember them, I most likely wouldn't have known how to apply them correctly. So, I decided to relearn all of the different convergence tests and the methods used to approximate series. In doing so, I have gained a better understanding of how to approximate series and why approximating series is important. In addition, I needed to research what it meant to approximate something to x decimal places; however, there weren't very many good explanations, but one did suggest that when approximating to four decimal places, approximate to 0.000005 instead of 0.00001 for a better approximation, which is what I did. Otherwise, my explanation above is indeed how I perceive an approximation accurate to four decimal places.

4. Using Taylor Theory to Approximate Solutions to ODE

In addition to the already many uses of series and polynomials, they can also be used to approximate solutions to ODE, and this is a method that is guaranteed to work every time, if one has the patience. For the following IVP

$$\frac{dx}{dt} = \cos(t) + \sin(t) - x, \;\; x(0) = 0, \tag{46}$$

Taylor theory was applied in the same manner as for the first exercise. I let x be the sixth degree polynomial solution of the IVP. In order to treat equation (46) properly, though, x was substituted into the IVP and both $\cos(t)$ and $\sin(t)$ were defined in terms of their respective sixth degree polynomial approximations so that both sides of the equation were in terms of t. After this application, I obtained

$$a_1 + 2a_2t + 3a_3t^2 + 4a_4t^3 + 5a_5t^4 + 6a_6t^5 = \\ 1 + t - \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} - \frac{t^6}{720} - a_0 - a_1t - a_2t^2 - a_3t^3 - a_4t^4 - a_5t^5 - a_6t^6,$$

and equated the terms of the same power of t in order to obtain the coefficients of x, also using the initial condition, ultimately finding x to be

$$x(t) = t - \frac{t^3}{6} + \frac{t^5}{120} \approx \sin(t).$$

Thus, through the use of Taylor theory, I can confidently conclude that the solution to equation (46) is $x = \sin(t)$. This was confirmed and validated with MATLAB.

```
%% Exercise 4
t = linspace(0,2*pi,200);
dxdt = @(t,x) cos(t) + sin(t) - x;
x0 = 0;
T = 2*pi;
[t,x] = ode45(dxdt,[0 T],x0);
T6 = t - (t.^3)/6 + (t.^5)/120;

figure
hold on
plot(t,x,'ro','LineWidth',2);
plot(t,T6,'b-','LineWidth',2);
ylim([-1.5 1.5]);
xlabel('t');
ylabel('x');
legend('Numerical Solution','Taylor Polynomial Solution');
title('Plot for Exercise 4');
```

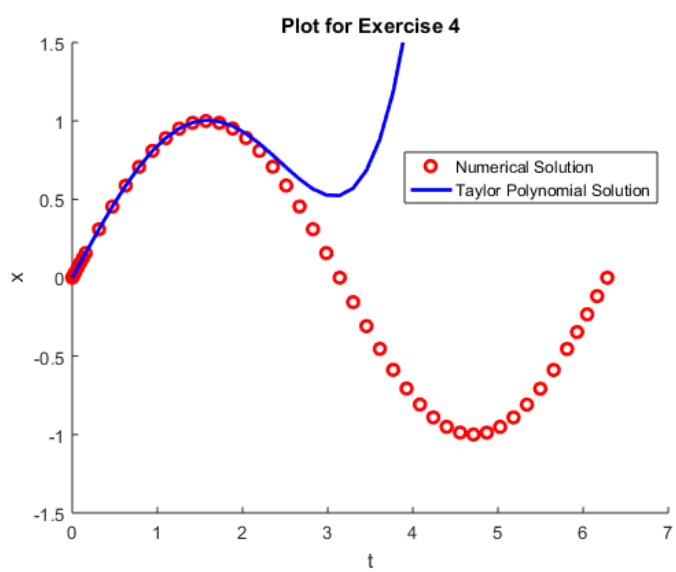


Figure 17: The solution is indeed $\sin(t)$, as confirmed with MATLAB.

I remember when I did this exercise last spring and my excitement upon discovering the solution is as great as it was when I first computed it.

5. More Applications of Taylor Theory to ODE

In this exercise, the following IVP

$$\frac{dx}{dt} = e^{-t}\sin(t) - x, \; x(0) = 0, \tag{47}$$

the same methods used in the previous exercise were used to solve this ODE. I first substituted the product of the decaying exponential and the sinusoid with the solution obtained in exercise 1 so that equation (47) was in one variable. Then, I substituted x, the polynomial of degree six, into the IVP and equated the terms that contained the same powers, using the initial condition to find the first term of x. What I obtained for x was

$$x = \frac{t^2}{2} - \frac{t^3}{2} + \frac{5t^4}{24} - \frac{t^5}{24} + \frac{t^6}{720},$$

and this was validated in Matlab.

```
%% Exercise 5
dxdt = @(t,x) exp(-t)*sin(t) - x;
x0 = 0;
T = 2*pi;
[t,x] = ode45(dxdt,[0 T],x0);
t1 = linspace(0,2*pi);
x1 = (t1.^2)/2;
x2 = x1 - (t1.^3)/2;
x3 = x2 + 5*(t1.^4)/24;
x4 = x3 - (t1.^5)/24;
x5 = x4 + (t1.^6)/720;

figure
hold on
plot(t,x,'ro','LineWidth',2);
plot(t1,x5,'b-','LineWidth',2);
ylim([-0.3 0.3]);
xlabel('t');
ylabel('x');
legend('Numerical Solution','Taylor Polynomial Solution');
title('Plot for Exercise 5');
```

The Taylor approximation of x is only reliable on the interval from zero to a little less than one and a half. Thus, the polynomial approximation of x is not that accurate and would be better with more terms, as most approximations are.

6. Even More Applications of Taylor Theory to ODE

This IVP

$$\frac{dx}{dt} = -x^2, \; x(0) = 1, \tag{48}$$

was my favorite IVP on the handout. In order to find x I utilized the same methods as in exercises 4 and 5, but because there is no function of t on the right-hand side of equation (48), besides x, I did not need to expand any additional terms; there was indeed a lot of expansion though. I decided to let x be a Taylor polynomial of degree six because I was interested in seeing the behavior of the solution, and what I found

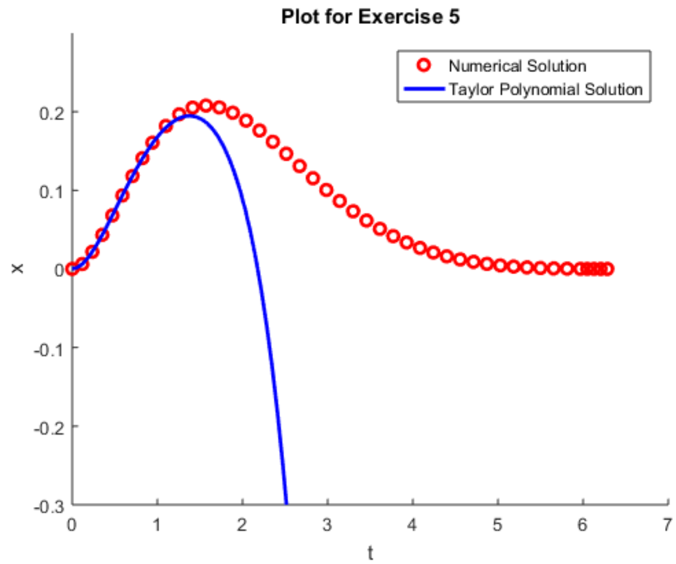


Figure 18:

was very interesting. I found that x was symmetric and it reminded me of Pascal's triangle and the binomial coefficients. As an example of its symmetry, the first four terms were

$$\begin{gathered} a_0^2+ \\ a_0a_1t + a_0a_1t+ \\ a_0a_2t^2 + a_1^2 + a_0a_2t^2+ \\ a_0a_3t^3 + a_1a_2^3 + a_1a_2t^3 + a_0a_3t^3+ \\ \cdots, \end{gathered}$$

and this symmetry continued throughout the expansion, which I thought was really neat. After equating the like terms and using the initial condition, I found x to be

$$x = 1 - t + t^2 - t^3 + t^4 - t^5 + t^6 - \cdots,$$

Which is a familiar series. When plotting x against the numerical solution to equation (48) obtained with `ode45`, the approximation is only accurate for values of t less than 1. Moreover, for x, as t increases, the polynomial solution diverges from the numerical solution. In testing this with negative values of t, this was also the case. when I further analyzed the system, I found that the interval on which the approximation is reliable is (-1,1) by changing the values of t for the series.

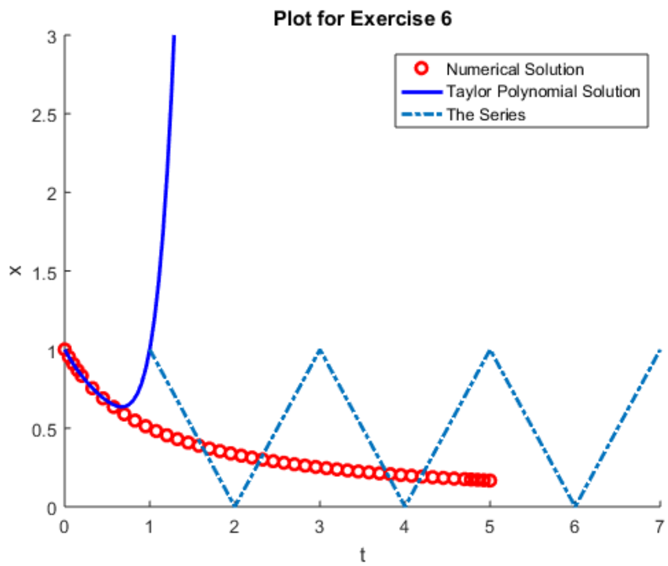


Figure 19: The numerical and polynomial solutions to equation (48).

```
%% Exercise 6
dxdt = @(t,x) -(x*x);
x0 = 1;
T = 5;
[t,x] = ode45(dxdt,[0 T],x0);
t1 = linspace(0,10,200);
x1 = 1;
x2 = x1 - t1;
x3 = x2 + t1.^2;
x4 = x3 - t1.^3;
x5 = x4 + t1.^4;
x6 = x5 - t1.^5;
x7 = x6 + t1.^6;
n = 6;
t2 = 1;
series = (-1).^(0:n).*(t2.^(0:n));

figure
hold on
plot(t,x,'ro','LineWidth',2);
plot(t1,x7,'b-','LineWidth',2);
plot(cumsum(series),'-.','linewidth',2);
ylim([0 3]);
xlabel('t');
ylabel('x');
legend('Numerical Solution','Taylor Polynomial Solution','The Series');
title('Plot for Exercise 6');
```

7. Questions

1. In seeing how approximations are better with more terms, is this why Simpson's rule for integration is such a good method for evaluating definite integrals, for I never understood why Simpson's rule works?

2. Who exactly is Florentin Smarandache and what contributions has he made to mathematics that weren't already made/discovered (I couldn't find very many things online)?

3. Even though I was able to code a series/sum and plotted it, is there a better and more efficient way to do so?

4. Is there a better and more concrete method to validate the approximation of a series?

5. Is there a special description of a function or a function that behaves as in equation (48), and if so, what is it and why is it symmetric?

5. Homework 6

1. ODE Investigation

$$\frac{dx}{dt} + 2x = \cos(3t), \ x(0) = 10. \tag{49}$$

Part a

```
%% ODE Investigation

t = linspace(0,2*pi,200);
dxdt = @(t,x) cos(3*t) - 2*x;
x0 = -5:1:5;
T = 2*pi;
[t,x] = ode45(dxdt,[0 T],x0);

figure
hold on
plot(t,x,'b-');
xlabel('t');
ylabel('x');
legend('Solutions to equation (49)');
```

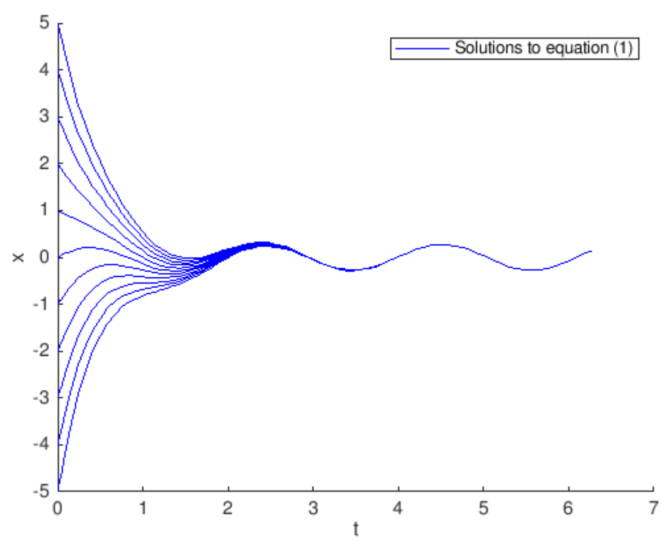


Figure 20: This is a graph of solutions to equation (49) with initial conditions varying from -5 to 5.

Observations of the ODE:

1. The starting position of each graph begins at different y-values, but it appears that they all converge to one solution, which looks like a harmonic.

Hypotheses:

1. "If only the initial condition is changed, then the only the beginning portion of the solution will be affected."

2. "If only the initial condition is changed, then the solution will after some time will approach that of a simple harmonic."

Part b

```
ind1 = t > 3;
t1 = t(ind1);
x1 = x(ind1);
A1 = [cos(3*t1) sin(3*t1)];
b1 = linsolve(A1,x1);
x2 = A1*b;

figure
hold on
plot(t1,x1,'b-');
plot(t1,x2,'ro');
xlabel('t > 3');
ylabel('x(t>3)');
title('The steady part of the solution');

b1 =

    0.1511
    0.2282
```

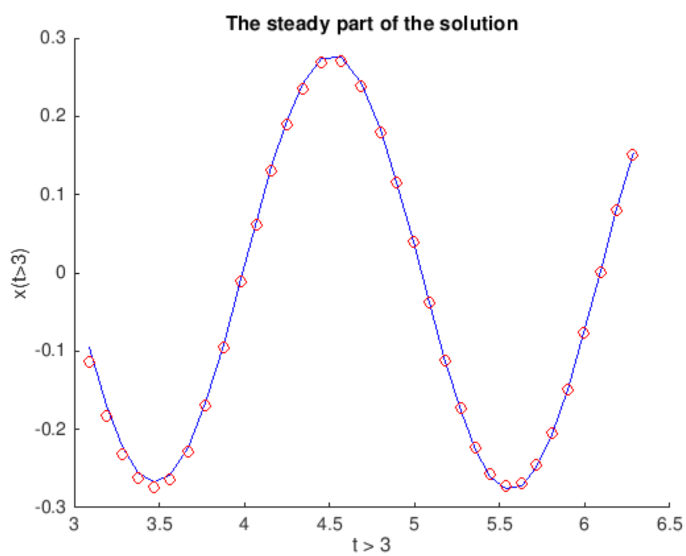


Figure 21: The steady part of the solution to equation (49).

Part c

```
ind2 = t < 3;
t2 = t(ind2);
x3 = x(ind2);
A2 = [cos(3*t2) sin(3*t2)];
```

```
b2 = x3 - A2*b1;
c = log(b2);
p = polyfit(t2,c,1);

figure
hold on
plot(t2,c,'ro');
plot(t2,polyval(p,t2),'b-');
xlabel('t');
ylabel('log(t<3)');
title('The transient part of the solution');

p =

   -1.9921    2.2830


>> exp(p(2))

ans =

    9.8060
```

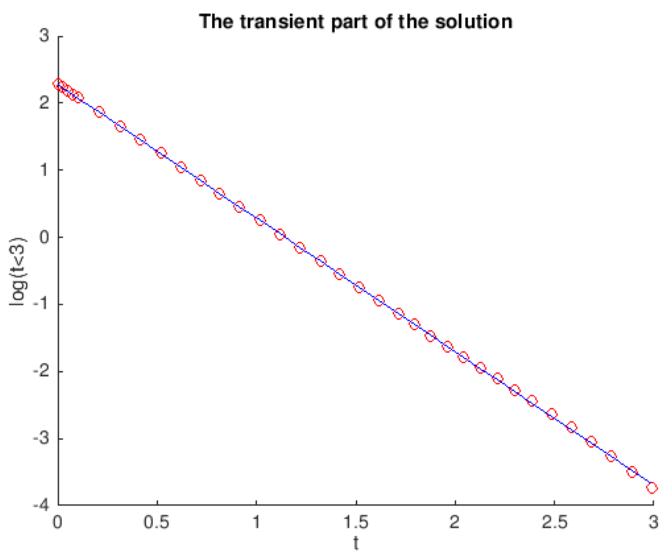


Figure 22: The transient portion of the solution to equation (49).

Conclusion: the solution is of the form

$$x = a\cos(3t) + b\sin(3t) + Ce^{kt} = 0.1511\cos(3t) + 0.2282\sin(3t) + 9.8060e^{-1.9921t}.$$

Part d

Symbolic solution:

$$x = \frac{128}{13}e^{-2t} + \frac{2}{13}\cos(3t) + \frac{3}{13}\sin(3t).$$

Numerical solution:

```
%% ODE Investigation

t = linspace(0,2*pi,200);
dxdt = @(t,x) cos(3*t) - 2*x;
x0 = 10;
T = 2*pi;
[t,x] = ode45(dxdt,[0 T],x0);
ind1 = t > 3;
t1 = t(ind1);
x1 = x(ind1);
A1 = [cos(3*t1) sin(3*t1)];
b1 = linsolve(A1,x1);
x2 = A1*b1;
ind2 = t < 3;
t2 = t(ind2);
x3 = x(ind2);
A2 = [cos(3*t2) sin(3*t2)];
b2 = x3 - A2*b1;
c = log(b2);
p = polyfit(t2,c,1);
x4 = b1(1)*cos(3*t) + b1(2)*sin(3*t) + exp(p(2))*exp(p(1)*t);
x5 = (128/13)*exp(-2*t) + (2/13)*cos(3*t) + (3/13)*sin(3*t);

figure
hold on
plot(t,x,'ro');
plot(t,x4,'b-');
plot(t,x5,'*');
xlabel('t');
ylabel('x');
legend('Numerical Solution','Investigative Solution','Symbolic Solution');
```

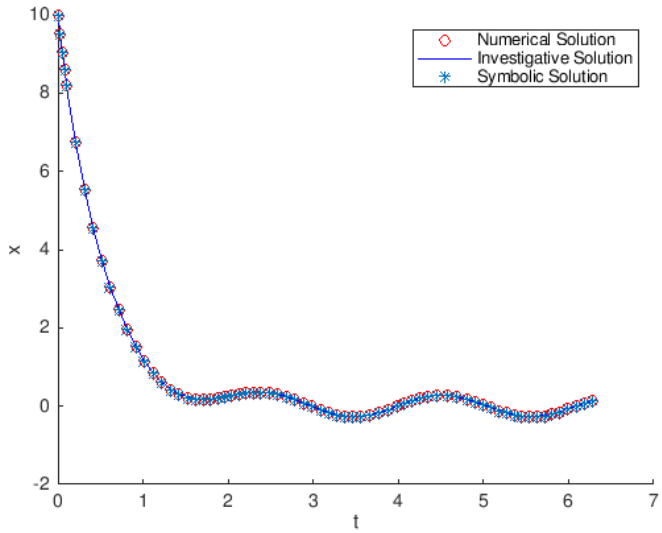


Figure 23: The numerical, investigative, and symbolic solutions to equation (49).

2. ODE Investigation 2

$$\frac{dx}{dt} + 3x = t, \;\; x(0) = 10. \tag{50}$$

Part a

```
%% ODE Investigation 2

t = linspace(0,10,200);
dxdt = @(t,x) t - 3*x;
x0 = -5:1:5;
T = 10;
[t,x] = ode45(dxdt,[0 T],x0);

figure
hold on
plot(t,x,'b-');
xlabel('t');
ylabel('x');
legend('Solutions to equation (50)');
```

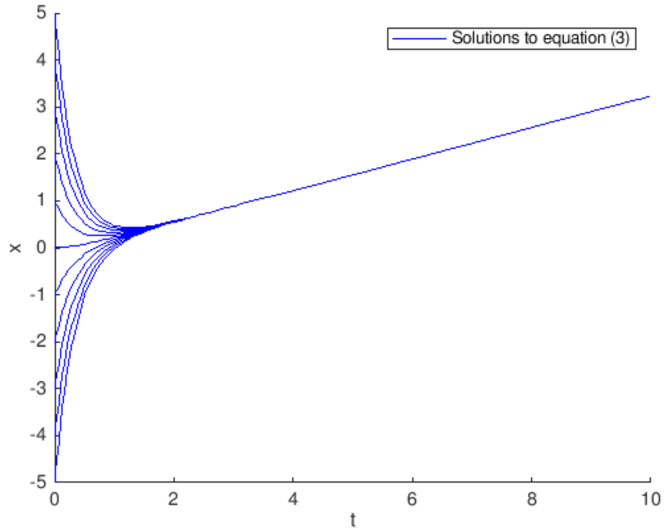


Figure 24: Solutions to equation (50) with initial conditions ranging from -5 to 5.

Observations of the ODE:

1. The starting position of each graph begins at different y-values, but it appears that they all converge to a linear relationship.

Hypotheses:

1. "If only the initial condition is changed, then the only the beginning portion of the solution will be affected."
2. "If only the initial condition is changed, then the solution will after some time will approach a linear relation."

Part b

```
ind1 = t > 2;
t1 = t(ind1);
x1 = x(ind1);
p = polyfit(t1,x1,1);
p1 = polyval(p,t1);

figure
hold on
plot(t1,x1,'b-');
plot(t1,p1,'ro');
xlabel('t > ');
ylabel('x(t>3)');
title('The steady part of the solution');

p =

    0.3338   -0.1150
```

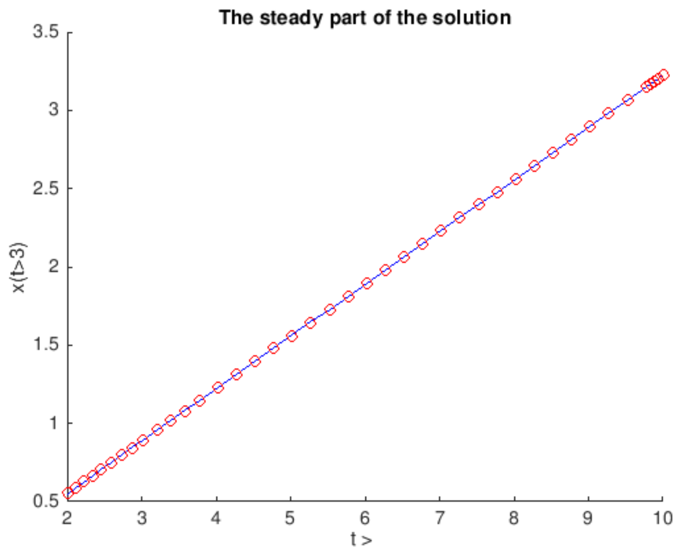


Figure 25: The steady part of the solution to equation (50).

Part c

```
ind2 = t < 2;
t2 = t(ind2);
x3 = x(ind2);
A = p(1)*t2 + p(2);
b = x3 - A;
c = log(b);
p2 = polyfit(t2,c,1);

figure
hold on
plot(t2,c,'ro');
plot(t2,polyval(p2,t2),'b-');
xlabel('t');
ylabel('log(t<3)');
title('The transient part of the solution');

p2 =

   -3.0697    2.3401

>> exp(p2(2))

ans =

   10.3828
```

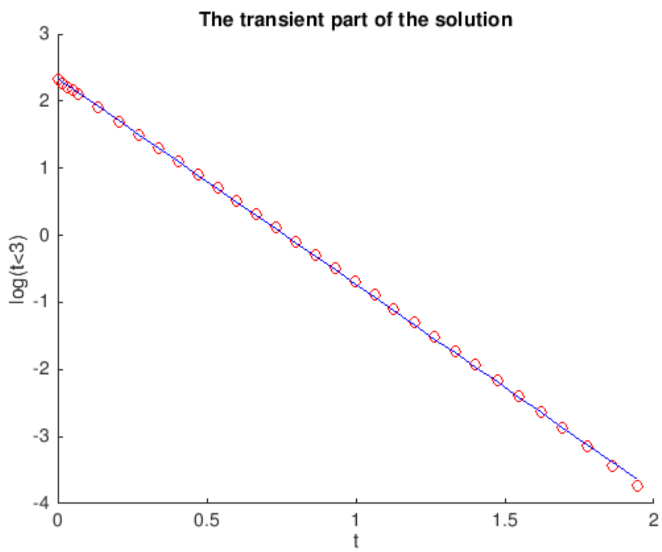


Figure 26: The transient portion of the solution.

Conclusion: the solution is of the form

$$x = a + bt + Ce^{kt} = -0.1150 + 0.3338t + 10.3828e^{-3.0697t}.$$

Part d

Symbolic solution:

$$x = -\frac{1}{9} + \frac{1}{3}t + \frac{91}{9}e^{-3t}.$$

Numerical solution:

```
%% ODE Investigation 2

t = linspace(0,10,200);
dxdt = @(t,x) t - 3*x;
x0 = 10;
T = 10;
[t,x] = ode45(dxdt,[0 T],x0);
ind1 = t > 2;
t1 = t(ind1);
x1 = x(ind1);
p = polyfit(t1,x1,1);
p1 = polyval(p,t1);
ind2 = t < 2;
t2 = t(ind2);
x3 = x(ind2);
A = p(1)*t2 + p(2);
b = x3 - A;
c = log(b);
p2 = polyfit(t2,c,1);
```

```
x4 = p(2) + p(1)*t + exp(p2(2))*exp(p2(1)*t);
x5 = (1/3)*t - (1/9) + (91/9)*exp(-3*t);

figure
hold on
plot(t,x,'ro');
plot(t,x4,'b-');
plot(t,x5,'*');
xlabel('t');
ylabel('x');
legend('Numerical Solution','Investigative Solution','Symbolic Solution');
```

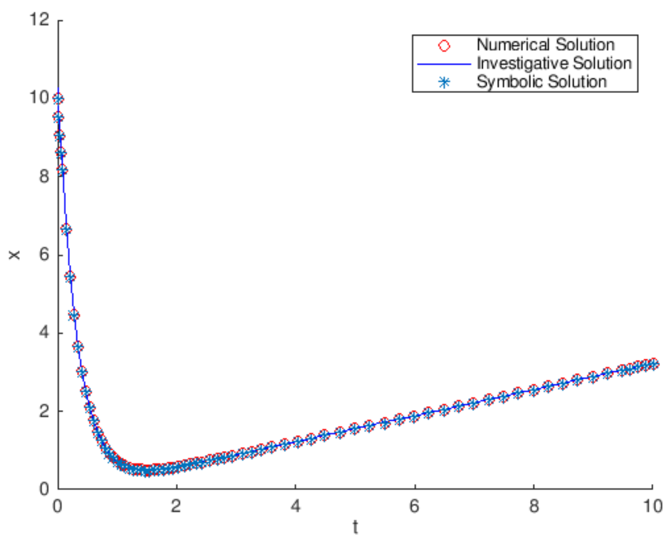


Figure 27: The numerical, investigative, and symbolic solutions to ODE (50).

6. Homework 7

1. The Fundamental Theorem of Math 57 Part (a)

$$\frac{dx}{dt} + 2x = \cos(t), \;\; x(0) = 10.$$

Symbolic solution:

$$x = \frac{48}{5}e^{-2t} + \frac{2}{5}\cos(t) + \frac{1}{5}\sin(t).$$

Numerical validation:

```
%% Exercise 1 part a
t = linspace(0,6*pi,200);
dxdt = @(t,x) cos(t) - 2*x;
x0 = 10;
T = 6*pi;
[t,x] = ode45(dxdt,[0 T],x0);
x1 = (48/5)*exp(-2*t) + (2/5)*cos(t) + (1/5)*sin(t);

figure
hold on
plot(t,x,'ro');
plot(t,x1,'b-');
xlabel('time');
ylabel('x');
legend('Numerical Solution','Symbolic Solution');
title('Exercise 1 Part (a)');
```

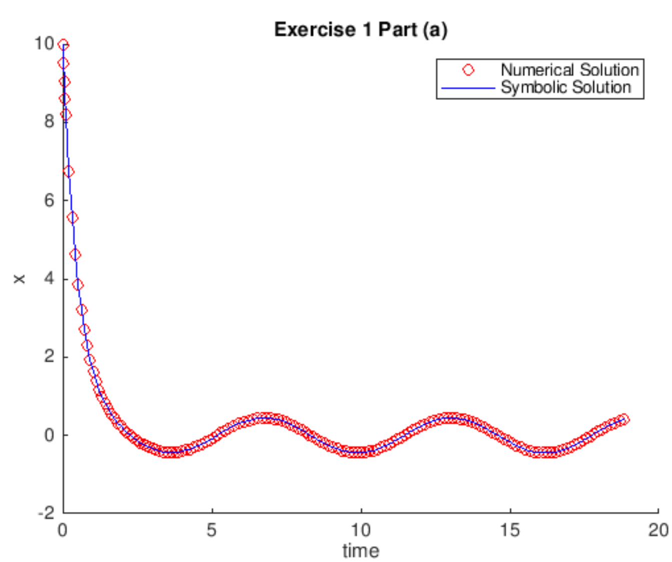


Figure 28: Part (a) solution.

Part (b)

$$\frac{dx}{dt} + 2x = e^{-t},\ x(0) = 10.$$

Symbolic solution:

$$x = 9e^{-2t} + e^{-t}.$$

Numerical validation:

```
%% Exercise 1 part b
t = linspace(0,5,200);
dxdt = @(t,x) exp(-t) - 2*x;
x0 = 10;
T = 5;
[t,x] = ode45(dxdt,[0 T],x0);
x1 = 9*exp(-2*t) + exp(-t);

figure
hold on
plot(t,x,'ro');
plot(t,x1,'b-');
xlabel('time');
ylabel('x');
legend('Numerical Solution','Symbolic Solution');
title('Exercise 1 Part (b)');
```

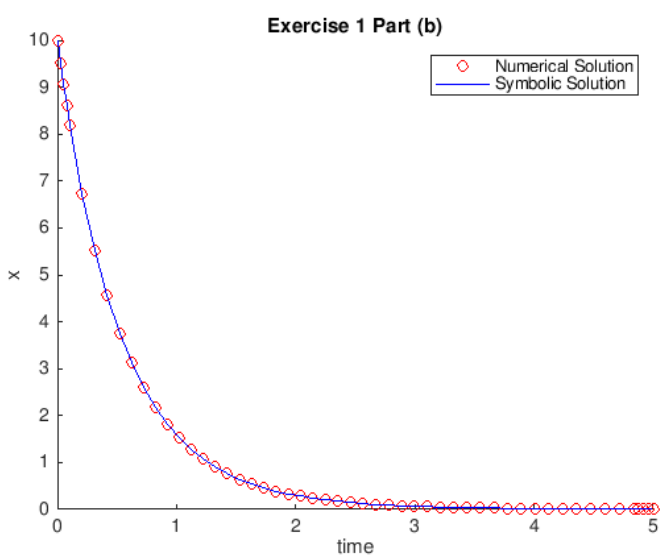


Figure 29: Part (b) solution.

Part (c)

$$\frac{dx}{dt} + 2x = 2\cos(t) + 3e^{-t},\ x(0) = 10.$$

Solution:

$$x = \frac{31}{5}e^{-2t} + 3e^{-t} + \frac{4}{5}\cos(t) + \frac{2}{5}\sin(t).$$

Numerical validation:

```
%% Exercise 1 part c
t = linspace(0,6*pi,200);
dxdt = @(t,x) 2*cos(t) + 3*exp(-t) - 2*x;
x0 = 10;
T = 6*pi;
[t,x] = ode45(dxdt,[0 T],x0);
x1 = (31/5)*exp(-2*t) + 3*exp(-t) + (4/5)*cos(t) + (2/5)*sin(t);

figure
hold on
plot(t,x,'ro');
plot(t,x1,'b-');
xlabel('time');
ylabel('x');
legend('Numerical Solution','Symbolic Solution');
title('Exercise 1 Part (c)');
```

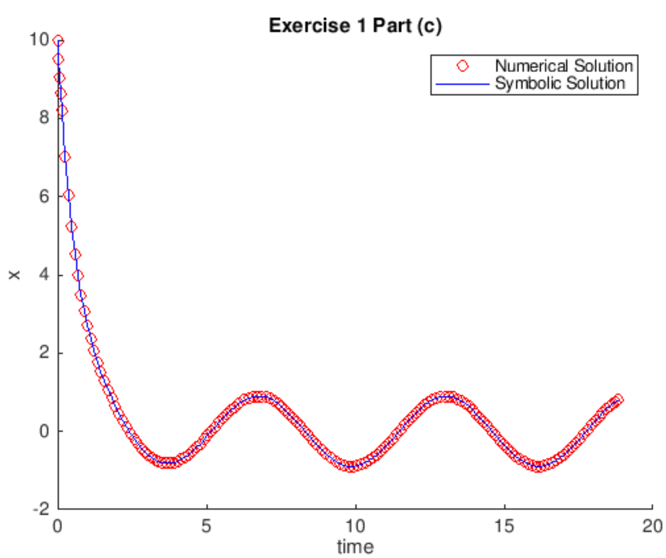


Figure 30: Part (c) solution.

2. Second Order ODE

$$m\frac{d^2x}{dt^2}+r\frac{dx}{dt}=A\sin{(\omega t)},\ x(0)=x_0,\ x'(0)=x_1.$$

Symbolic solution:

$$x=-\frac{m}{r}\left(x_1+\frac{Am\omega}{m^2\omega^2+r^2}\right)e^{-\frac{r}{m}t}+\left[x_0+\frac{m}{r}\left(x_1+\frac{Am\omega}{m^2\omega^2+r^2}\right)+\frac{Ar}{\omega(m^2\omega^2+r^2)}\right]-\frac{Ar\cos{(\omega t)}}{m^2\omega^2+r^2}-\frac{Am\sin{(\omega t)}}{m^2\omega^2+r^2}.$$

Numerical Validation:

```
%% Exercise 2
w = 3;
m = 1;
r = .9;
A = 2;
t = linspace(0,4*pi,200);
T = 4*pi;
x10 = 1;
x20 = 2;
f = @(t,x) [x(2); (A*sin(w*t) - r*x(2))/m];
[t,x] = ode45(f,[0 T],[x10;x20]);
C = (x20 + (A*m*w)/(((m^2)*(w^2)) + (r^2)));
D = x10 + (m/r)*C + (A*r)/(((m^2)*(w^3)) + ((r^2)*(w)));
x1 = D + (-m/r)*C*exp((-r/m)*t) - (A*r*cos(w*t))/(((m^2)*(w^3)) + ((r^2)*(w))) -
    (A*m*sin(w*t))/(((m^2)*(w^2)) + (r^2));

figure
hold on
plot(t,x(:,1),'ro');
plot(t,x1,'b');
xlabel('time');
ylabel('x');
legend('Numerical Solution','Symbolic Solution');
title('Exercise 2');
```

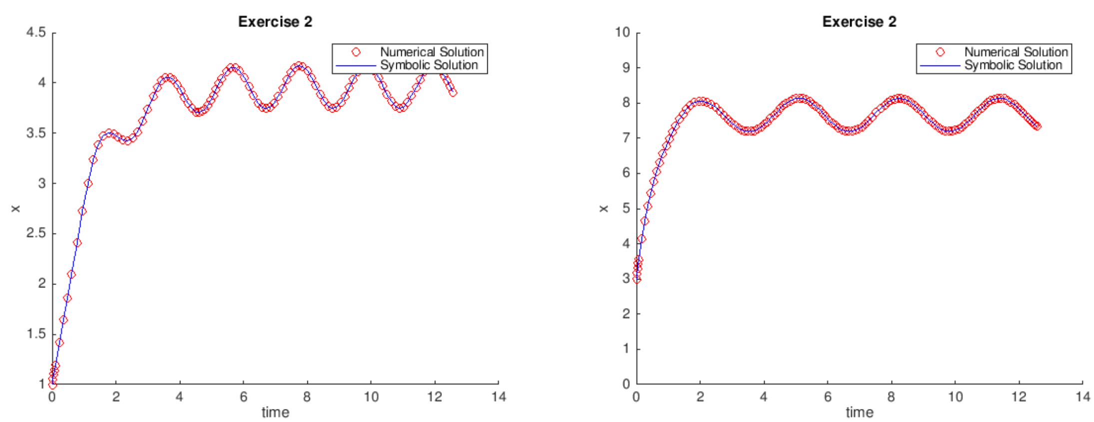


Figure 31: The solution on the left has the conditions of the code provided above. The solution on the right has the code provided below.

```
%% Solution represented by the rhs figure in Figure 31
w = 2;
m = 1.5;
r = 3;
A = 4;
t = linspace(0,4*pi,200);
T = 4*pi;
x10 = 3;
x20 = 8;
```

3. Linear Algebra and the Fundamental Theorem of Math 57

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

Solution:

$$x = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \;\middle|\; x_1 = 3 - 2x_2, \{x_1, x_2\} \in \mathbb{R} \right\}.$$

Some questions:

1. Is there a more elegant way to code my solution for exercise 2?
2. What would be a good resource to learn about correct mathematical notation?
3. Is there a symbolic way to solve equation (8)? I was able to solve it by examining different particular solutions, eventually realizing the relation between x_1 and x_2.
4. Why is it that solutions to some equations occur as a sum, as in the FTC of Math 57, whereas solutions to other equations occur as sets? Is there actually a difference?

7. Homework 8

1. Linear, Homogeneous, Second Order ODE's with Constant Coefficients Part (a)

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = 0,\ x(0) = 0,\ \frac{dx}{dt}(0) = 1.$$

Symbolic solution:

$$x = -\frac{1}{2}e^{-3t} + \frac{1}{2}e^{-t}.$$

Numerical Solution:

```
%% Exercise 1 Part a
t = linspace(0,12,300);
dydt = @(t,y) [y(2); -4*y(2) - 3*y(1)];
y0 = [0;1];
[t,y] = ode45(dydt,[0 12],y0);
x = (-0.5)*exp(-3*t) + (0.5)*exp(-t);

figure
hold on
plot(t,y(:,1),'ro');
plot(t,x,'b-');
xlabel('time (sec)');
ylabel('x(t)');
title('Exercise 1 Part a')
legend('Numerical Solution','Symbolic Solution');
```

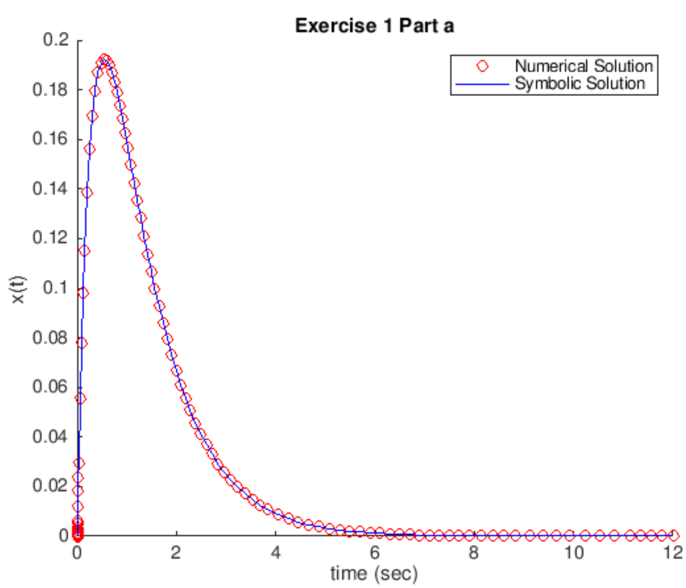


Figure 32: This solution describes a system whose motion is critically damped because the friction force dominates. This could also describe the critically damped flow of an RLC-circuit or any other circuit with a resistor and the equivalent of two energy storage elements.

Part (b)

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0,\ x(0) = 0,\ \frac{dx}{dt}(0) = 1.$$

Symbolic solution:

$$x = te^{-2t}.$$

Numerical solution:

```
%% Exercise 1 Part b

t = linspace(0,12,300);
dydt = @(t,y) [y(2); -4*y(2) - 4*y(1)];
y0 = [0;1];
[t,y] = ode45(dydt,[0 12],y0);
x = t.*exp(-2*t);

figure
hold on
plot(t,y(:,1),'ro');
plot(t,x,'b-');
xlabel('time (sec)');
ylabel('x(t)');
title('Exercise 1 Part b')
legend('Numerical Solution','Symbolic Solution');
```

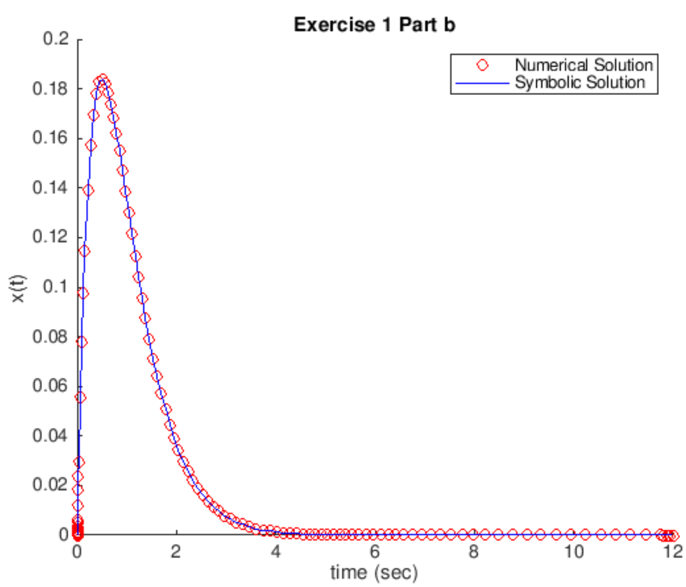


Figure 33: This solution also describes critically damped motion. However, because the spring constant has increased, the spring is stronger, so the spring force and the force of kinetic friction both dampen this motion more than in part (a), as can be seen in the absolute maximum of both graphs. Mathematically, as the coefficients of the first derivative and the displacement increase, the motion becomes more and more dampened.

Part (c)

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 10x = 0, \;\; x(0) = 0, \;\; \frac{dx}{dt}(0) = 1.$$

Symbolic solution:

$$x = \frac{\sqrt{6}}{6} e^{-2t} \sin(\sqrt{6}t).$$

Numerical solution:

```
%% Exercise 1 Part c

t = linspace(0,3*pi);
dydt = @(t,y) [y(2); -4*y(2) - 10*y(1)];
y0 = [0;1];
[t,y] = ode45(dydt,[0 3*pi],y0);
x = (sqrt(6)/6).*exp(-2*t).*sin(sqrt(6)*t);

figure
hold on
plot(t,y(:,1),'ro');
plot(t,x,'b-');
xlabel('time (sec)');
ylabel('x(t)');
title('Exercise 1 Part c')
legend('Numerical Solution','Symbolic Solution');
```

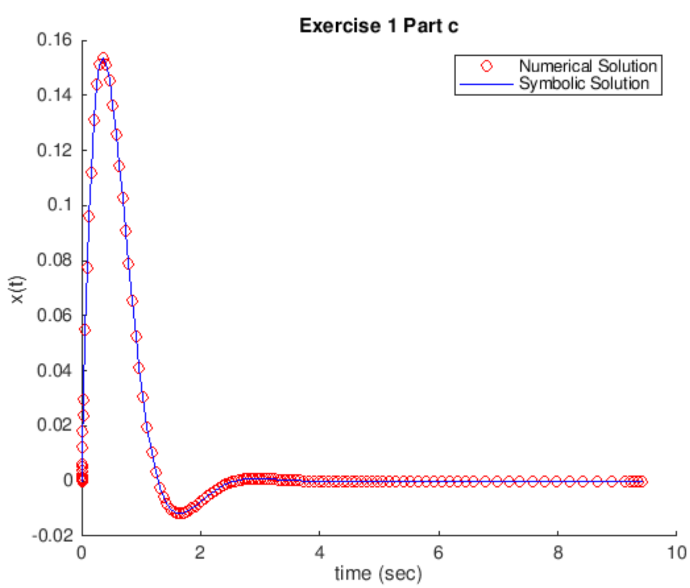


Figure 34: For this solution, both the spring force and the friction work in tandem to dampen the motion more than in parts (a) and (b), but because the spring force is greater than the friction force, the spring causes a slightly negative displacement. Then the spring pushes the cart with an even smaller positive displacement, where ultimately the friction force dominates and the system approaches its equilibrium position.

2. Particle Under a Net 2π-Periodic Force, with Mass of Zero

$$\frac{d^2x}{dt^2} = a\cos(t) + b\sin(t),\ \ x(0) = \frac{dx}{dt} = 0. \tag{51}$$

Symbolic solution:

$$x = a + bt - a\cos(t) - b\sin(t).$$

Numerical solution:

```
%% Exercise 2

t = linspace(0,6*pi);
a = 1;
b = 1;
dydt = @(t,y) [y(2);a*cos(t) + b*sin(t)];
[t,y] = ode45(dydt,[0 6*pi],[0;0]);
x = a + b.*t - a*cos(t) - b*sin(t);
x1 = (a + b*t);

figure
hold on
plot(t,y(:,1),'ro');
plot(t,x,'b-');
xlabel('time (sec)');
legend('Numerical Solution','Symbolic Solution');
ylabel('x(t)');
ylim([0 .5]);
title('Exercise 2')
```

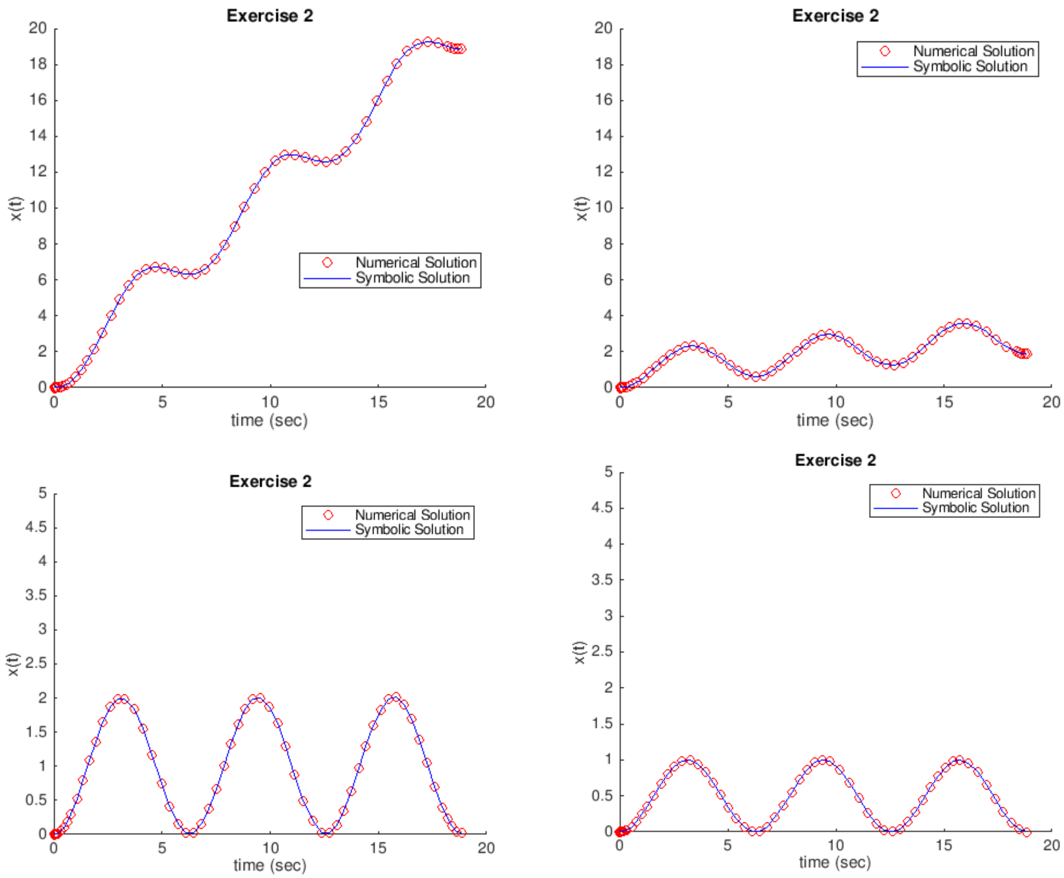


Figure 35: Various solutions to equation (4) with varying coefficients a and b. Graph (1,1) has $a = b = 1$; graph (1,2) has $a = 1$ and $b = 0.1$; graph (2,1) has $a = 1$ and $b = 0.01$; and graph (2,2) has $a = 0.5$ and $b = 0$.

Upon finding the solution to IVP (51), I thought it was quite interesting: the harmonic component oscillates about the linear component of the solution. As a consequence, whether the particle oscillates near the origin or not depends on (1) the slope of the linear component of the solution, the coefficient b, and (2) how we define "near the origin," for the coefficient a, which is the intercept of the the linear component and ultimately, the displacement. For the coefficient b, as long as $b > 0$, the solution will oscillate away from the origin. Of course, as $b \to 0$ the solution will take longer and longer to move away from the origin. For the coefficient a, if we define "around the origin" as the interval bound by $x > 0$ and $x \leq 1$, then a must lie on the interval of $(0, 0.5]$. This is a direct result of the linear component of the solution. When $b = 0$, the solution has the form $x = a - a\cos(t)$, where a represents both the amplitude of the harmonic and the amount by which the harmonic is shifted upward. So, for the values of a and b such that $b = 0$ and $0 < a \leq 1$, the particle will oscillate around the origin.

3. Higher Order Differential Equations

$$\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} = 0,\ x(0) = x_0,\ \dot{x}(0) = x_1,\ \ddot{x}(0) = x_2,\ \dddot{x}(0) = x_3. \tag{52}$$

Symbolic solution:

$$x = -x_2\cos(t) - x_3\sin(t) + (x_1 + x_3)t + (x_0 + x_2).$$

Numerical solution:

```
%% Exercise 3

t = linspace(0,10*pi,200);
x0 = 1;
x1 = .5;
x2 = 3;
x3 = .1;
dydt = @(t,y) [y(2);y(3);y(4);-y(3)];
y0 = [x0;x1;x2;x3];
[t,y] = ode45(dydt,[0 10*pi],y0);
x = -x2*cos(t) + -x3*sin(t) + (x1 + x3)*t + (x0 + x2);

figure
hold on
plot(t,y(:,1),'ro');
plot(t,x,'b-');
ylim([0 20]);
legend('Numerical Solution','Symbolic Solution');
title('Exercise 3')
```

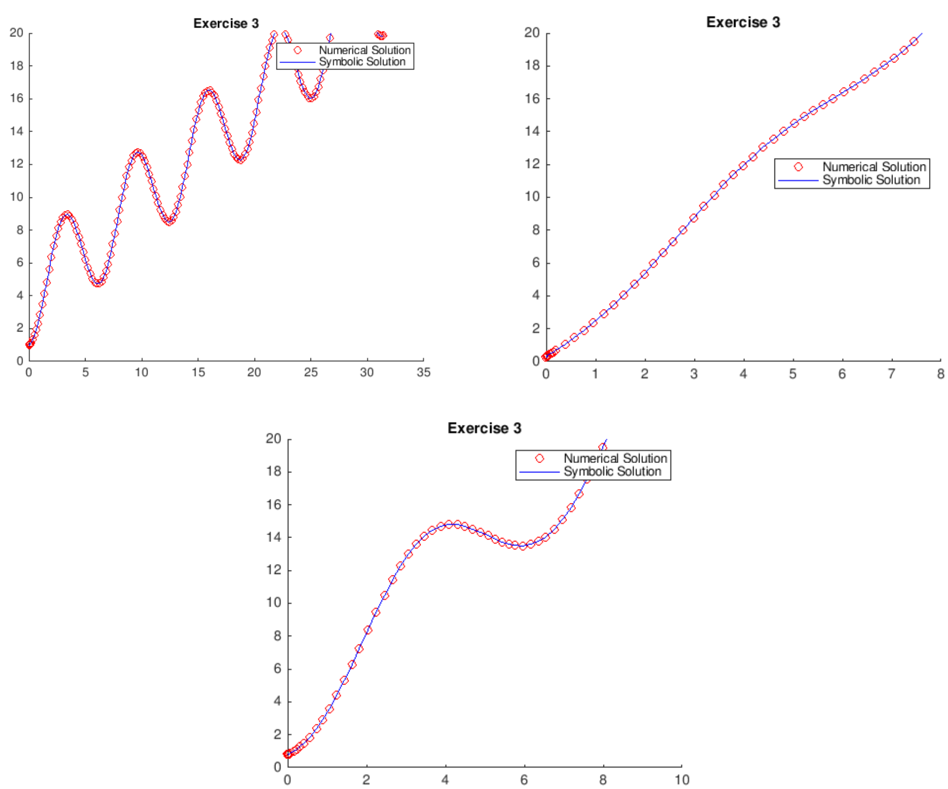


Figure 36: The solution to IVP (5). The upper left solution has initial conditions that are provided in the code above. The upper right solution has initial conditions $x_0 = 0.3$, $x_1 = 1.9$, $x_2 = 0.34$ and $x_3 = 0.74$. The bottom solution has initial conditions $x_0 = 0.8$, $x_1 = 1$, $x_2 = 3$ and $x_3 = 1.05$.

What is interesting about the solution to both this IVP and the IVP provided in exercise two is that they share the same form of the solution: a linear combination of a line and a simple harmonic with frequency 1. This makes sense because when the order of the ODE is reduced for IVP (52), what is obtained is a linear combination of complex exponentials, which, when Euler's formula is applied, become simple harmonics whose terms can be collected, providing part of the form of the solution for IVP (52),

$$\frac{d^2x}{dt^2} = C_1 \cos(t) + C_2 \sin(t),$$

which becomes

$$x = -C_1 \cos(t) - C_2 \sin(t) + C_3 t + C_4,$$

when integrated twice with respect to time, which is exactly the same for as the solution to IVP (51).

4. Questions

1. Why do indistinguishable solutions to the second order characteristic equation yield a second general solution $Cte^{\lambda t}$? Why is the t term there? Is there a way to derive this for a general case of n indistinguishable solutions to an nth order characteristic equation?

2. Why do the solutions to IVP (51) and (52) have the same form even though IVP (52) is a fourth order ODE?

8. Homework 9

1. Amplitude Gain

Show that $G(\omega) = 1/|p(i\omega)|$ has a unique global maximum and find the corresponding resonant frequency, given that $p(\lambda) = a\lambda^2 + b\lambda + c$.

Symbolic solution: to find the symbolic solution, first take the modulus of the characteristic equation evaluated at $i\omega$; then, apply the first derivative test and solve for ω where $G(\omega) = 0$; this will give the resonant frequency, ω_{res}

$$\omega_{res} = \frac{1}{a}\sqrt{\frac{2ac - b^2}{2}}.$$

Numerical solution:

```
%% Exercise 1

w = linspace(0,2);
a = 2.5;
b = 1.1;
c = .75;
G = 1./sqrt((c - a*(w.^2)).^2 + (b*w).^2);
dG = diff(G);
dw = diff(w);
dGdw = dG./dw;
[yvalue,xvalue] = max(G);
dGdw1 = -((2*(a^2)*(w.^3) + (b^2)*(w) - 2*a*c*w)./(((a^2)*(w.^4) + (b^2)*(w.^2) -
    2*a*c*(w.^2) + (c^2)).^(3/2)));
wres = (1/a)*sqrt((2*a*c - b^2)/2);

figure
hold on
plot(w,G,'b-');
plot(w(1:end-1),dGdw,'mo');
plot(w,dGdw1,'r-');
plot(wres,yvalue,'ko');
xlabel('\omega');
ylabel('G(\omega)');
legend('G(\omega)', 'Numerical Derivative of G', 'Symbolic Derivative of G','(\omega_{res},
    G(\omega_{res}))');
```

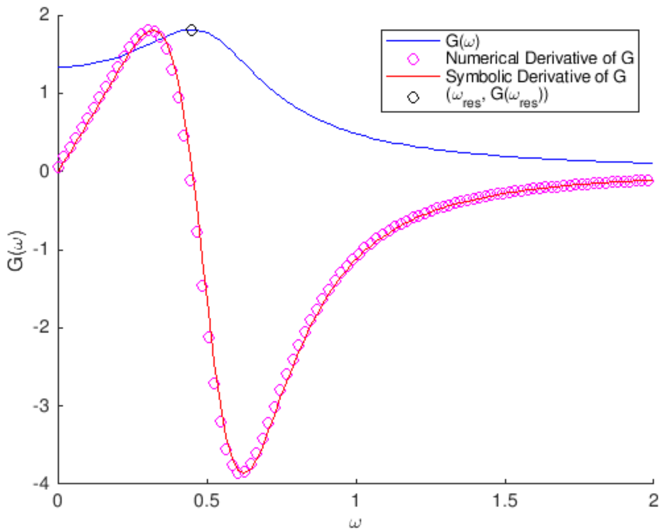


Figure 37: Plot for Exercise 1, where $a = 2.5$, $b = 1.1$, and $c = .75$.

2. Phase Shift

Symbolic solution:

$$\Phi - \phi = \cos^{-1}\left(\frac{A\cos(\omega t) + B\sin(\omega t)}{\sqrt{A^2+B^2}}\right) - \cos^{-1}\left(\frac{[(\frac{1}{C} - L\omega^2)A - R\omega B]\cos(\omega t) + [(\frac{1}{C} - L\omega^2)B + R\omega A]\sin(\omega t)}{\sqrt{A^2+B^2}\sqrt{(\frac{1}{C} - L\omega^2)^2 + R^2\omega^2}}\right).$$

Numerical solution: I was unable to code this exercise. I couldn't find my error.

3. RLC Circuit

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = \sin^{-1}(\sin(t)) = \sum_{n=0}^{\infty}\frac{1}{i\pi n^2}\left[\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{3n\pi}{2}\right)\right](e^{int} - e^{-int}),$$
$$x(0) = \dot{x}(0) = 0;$$

Symbolic solution:

$$x = x_c + x_p = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t} + \sum_{n=0}^{\infty}\frac{1}{i\pi n^2}\left[\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{3n\pi}{2}\right)\right]\left(\frac{e^{int}}{p(in)} - \frac{e^{-int}}{p(-in)}\right),$$

where $p(\lambda) = L\lambda^2 + R\lambda + \frac{1}{C}$. The constants C_1 and C_2 can be found as such:

$$\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} -x_p(0) \\ -\dot{x_p}(0) \end{bmatrix}.$$

Numerical solution: I couldn't successfully code this exercise either, but this is what I have so far.

```
%% Exercise 3

t = linspace(0,2*pi);
```

```
n = 1:2:(2*size(t'));
R = 0.1;
L = 1;
C = 10;
y = asin(sin(t));
b_n = 2*sin(n*pi*.5)./(pi*n.^2) - 2*sin(3*n*pi*.5)./(pi*n.^2);
f = b_n*sin(n'*t);
%p1 = -L*n.^2 + R*i*n + (1/C);
%p2 = -L*n.^2 - R*i*n + (1/C);
%xp = (b_n./(2*i))'*(exp(i*n'*t)/p1 - exp(-i*n'*t)/p2);
dxdt = @(t,x) [x(2); (f - R*x(2) - (1/C)*x(1)/L)];
x0 = [0;0];
T = 2*pi;
[t,x] = ode45(dxdt,[0 T],x0);



figure
hold on
plot(t,x(:,1),'b-');
%plot(t,y,'r-');
%plot(t,f,'b-');
```

I don't understand MATLAB and I need to continue learning how to use it, especially in an efficient and effective manner. That was my only difficulty; otherwise, I feel I have an okay grasp of linear homogeneous and non-homogeneous ordinary differential equations, with constant coefficients.

9. Homework 10

1. Homogeneous Matrix-Vector ODE

$$\frac{d}{dt} = \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} -1 & 4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{53}$$

Symbolic solution:
Part a

$$\frac{1}{4}\frac{d^2x}{dt^2} + \frac{1}{2}\frac{dx}{dt} + \frac{5}{4}x = 0, \ x(0) = 0, \ \dot{x}(0) = 4.$$

$$x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = e^{-t}(C_1 e^{2it} + C_2 e^{-2it}) = 2e^{-t}\sin(2t). \tag{54}$$

Part b

$$\begin{bmatrix} x \\ y \end{bmatrix} = -i \begin{bmatrix} 1 \\ \frac{i}{2} \end{bmatrix} e^{\lambda_1 t} + i \begin{bmatrix} 1 \\ -\frac{i}{2} \end{bmatrix} e^{\lambda_2 t}, \ \lambda_{1,2} = -\frac{1}{2} \pm 2i. \tag{55}$$

Applying Euler's Formula to the first component of equation (55) the vector provides the solution in equation (54).

2. Non-homogeneous Matrix-Vector ODE

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(t), \quad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{56}$$

Symbolic solution:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{\lambda_2 i\sqrt{3}}{3} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} e^{\lambda_1 t} - \frac{\lambda_1 i\sqrt{3}}{3} \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} e^{\lambda_2 t} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cos(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(t), \ \lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i. \tag{57}$$

Numerical solution:

```
%% Non-Homogeneous Matrix-Vector ODE

t = linspace(0,2*pi);
dxdt = @(t,x) [x(2);-x(1) - x(2) - sin(t)];
T = 2*pi;
x0 = [0;0];
[t,x] = ode45(dxdt,[0 T],x0);
A = [0 1;-1 -1];
I = [1 0;0 1];
[V,D] = eig(A);
lambda1 = D(1);
lambda2 = D(4);
a1b1 = (A^-1 + I)^-1 *[0;-1];
a2b2 = A*a1b1;
xp = a1b1*cos(t)' + a2b2*sin(t)';
B = ones(2,2);
B(2,:) = [lambda1 lambda2];
C = V^-1 * [1;0];
xc = C(1)*V(:,1)*exp(lambda1*t)' + C(2)*V(:,2)*exp(lambda2*t)';
x1 = xc + xp;
```

```
figure
hold on
plot(t,x(:,1),'ro');
plot(t,x(:,1),'b-');
xlabel('t');
ylabel('x');
legend('Numerical Solution','Matrix-Vector Solution');
title('Non-homogeneous Matrix-Vector ODE');
```

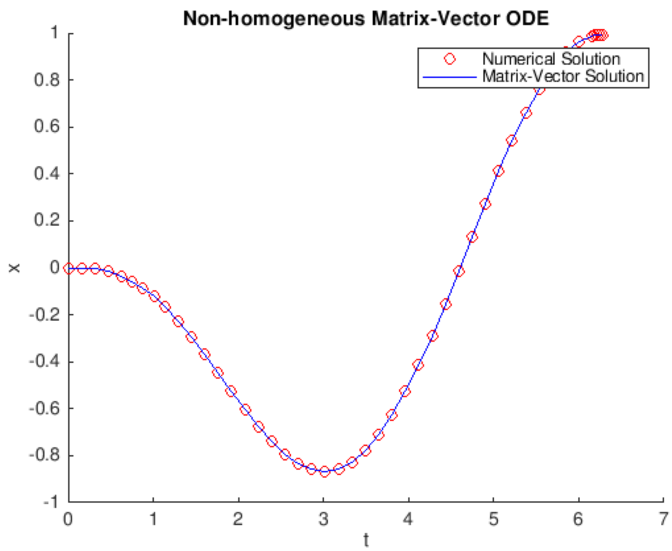


Figure 38: Solution to equation (56).

3. Questions

1. In exercise 2, the matrix A has a property such that $A^2 = A^{-1}$. Why is that and is this a special type of matrix? If so, what matrices have this property? I wasn't able to find anything online.

2. Is it common that some eigenvectors contain eigenvalues as components?

10. Exam 2

Exercise 1, Part C

$$\ddot{x} + 4x = \sum_{\omega=1,\ \omega is odd}^{5} \cos(\omega t),\ x(0) = \dot{x}(0) = 0$$

Symbolic solution:

$$x = x_c + x_p = C_1 e^{2it} + C_2 e^{-2it} + \sum_{\omega=1,\ \omega is odd}^{5} \frac{\left(e^{i\omega t} + e^{-i\omega t}\right)}{2p(i\omega)},$$

where $p(\lambda) = \lambda^2 + 4$ and C_1 and C_2 can be found from

$$\begin{bmatrix} 0 & 0 \\ 2i & -2i \end{bmatrix} \begin{bmatrix} C_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x(0) - x_p(0) \\ \dot{x}(0) - \dot{x}_p(0) \end{bmatrix}.$$

Numerical Solution:

```
%% Exercise 1
t = linspace(0,4*pi);
dxdt = @(t,x) [x(2); cos(t) + cos(3*t) + cos(5*t) - 4*x(1)];
x0 = [0;0];
T = 4*pi;
[t,x] = ode45(dxdt,[0 T],x0);
xp = 0.1667*(exp(i*1*t) + exp(-i*1*t)) - 0.1000*(exp(i*3*t) + exp(-i*3*t)) - 0.0238*(exp(i*5*t) + exp(-
A = ones(2,2);
A(2,:) = [2*i -2*i];
b = [-0.0429; 0];
C = linsolve(A,b);
xc = C(1)*exp(2*i*t) + C(2)*exp(-2*i*t);
x1 = xc + xp;

figure
hold on
plot(t,x(:,1),'ro');
plot(t,x1,'b-');
xlabel('t');
ylabel('x');
legend('Numerical Solution','Symbolic Solution');
title('Exercise 1 Part c');
```

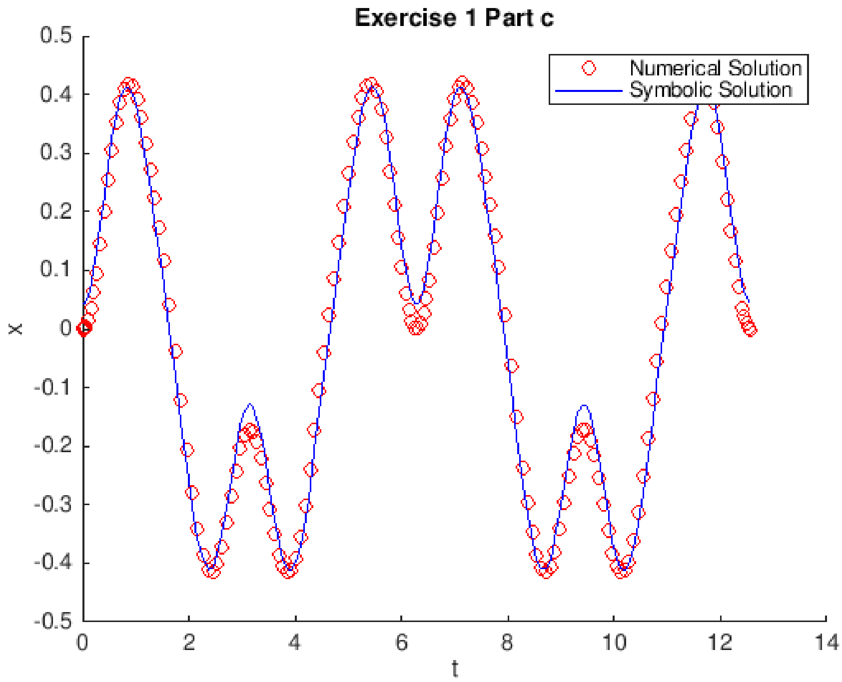


Figure 39: Exercise 1 part c.

Exercise 4

$$\ddot{x} + 4x = \sin(\omega t), \;\; x(0) = \dot{x}(0) = 0.$$

Symbolic Solution when $\omega \neq 2$:

$$x = x_c + x_p = \frac{\sin(\omega t)}{p(i\omega)} - \frac{\omega \sin(2t)}{2p(i\omega)},$$

where $p(\lambda) = \lambda^2 + 4$. The solution when $\omega = 2$ is

$$\lim_{\omega \to 2} x = \lim_{\omega \to 2} \frac{\sin(\omega t)}{p(i\omega)} - \frac{\omega \sin(2t)}{2p(i\omega)} \overset{\text{(H)}}{=} \lim_{\omega \to 2} \frac{t\cos(\omega t)}{p'(i\omega)} - \frac{\sin(2t)}{2p'(i\omega)} = \frac{t\cos(2t)}{4} - \frac{\sin(2t)}{8}$$

Thus when $\omega = 2$,

$$x = \frac{t\cos(2t)}{4} - \frac{\sin(2t)}{8}.$$

Numerical Solution:

```
%% Exercise 4

w = 2;
t = linspace(0,5*pi);
dxdt = @(t,x) [x(2);sin(w*t) - 4*x(1)];
x0 = [0;0];
T = 4*pi;
[t,x] = ode45(dxdt,[0 T],x0); % numerical solution
p = (i*w)^2 + 4;
```

```
x1 = sin(w*t)/p - w*sin(2*t)/(2*p); % for omega = 1
x2 = -0.5*(0.5*t.*cos(2*t) - 0.25*sin(2*t)); % for omega = 2

figure
hold on
plot(t,x(:,1),'ro'); % for omega = 1
plot(t,x1,'b-');
xlabel('t');
ylabel('x');
legend('Numerical Solution','Symbolic Solution');

figure
hold on
plot(t,x(:,1),'ro'); % for omega = 2
plot(t,x2,'b');
xlabel('t');
ylabel('x');
legend('Numerical Solution','Symbolic Solution');
```

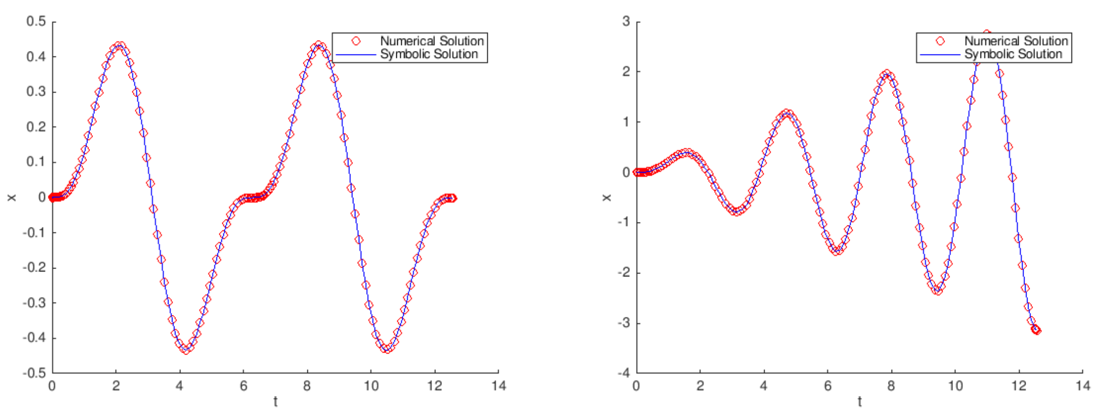


Figure 40: The solution to exercise 4.

In doing exercise 1 on the exam, I realized I needed the initial condition but because I knew the slow maths would be more time consuming, I left the solution as is. However, given this opportunity, I decided to look at the solution in MATLAB. In doing so, I validated the symbolic solution that I obtained on my exam. Even though there is a good agreement between the numerical and symbolic solutions, the agreement is not as close for other plots because the constants that were determined do not include all of the significant figures. For exercise 4, I did not realize that I had made a mistake with my solution. Further, I did not realize that I needed to utilize l'Hospital's rule because the form of the solution at $\omega = 2$ is in the form of 0/0. Having corrected my solution and performing l'Hospital's rule correctly, I was able to obtain the solution.