EXERCISE 1

1.) Solve the Laplace-Poisson equation $L(u) = f$ inside a disk of radius a with Neumann boundary conditions—normal derivative is zero on the boundary. As the right-hand side, use the characteristic function of a disk of radius $b < a$; that is, f is 1 inside the disk of radius b and zero outside. Organize your exposition as follows:

- *(a) (30 Points) Present a clear derivation of the symbolic solution. Remember that clarity is not directly proportional to the length of exposition. In fact, suppress rote algebra and calculus so as not to distract the reader from the logic of the derivation*
- *(b) (20 Points) Plot the (truncated series) solution for $b = 1$ and $a = 2, 4, 8, 16$. Comment on the plots: do they conform with you intuition?*
- *(c) (10 Points) If $b = 1$ is held fixed and $a \to \infty$, what happens to the solution? Does the limit exist? If "Yes," try to find it.*

Solution (Part 1): In order to derive the solution to the Laplace-Poisson equation inside a disk with Neumann conditions,

$$\begin{aligned}
&\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = f, \quad \left.\frac{\partial u}{\partial r}\right|_{\partial\Omega} = 0 \\
&0 \le r < a, \quad 0 < \theta \le 2\pi, \\
&f = \begin{cases} 1, & r < b \\ 0, & r > b \end{cases},
\end{aligned} \tag{1}$$

we apply the Fundamental Theorem of Math 57, adapted for Math 157:

> **Fundamental Theorem of Math 157.** Let L be a partial differential operator. The solution to the nonhomogeneous equation
>
> $$L(u) = f,$$
>
> is given by $u = u_1 + u_2$, where u_1 satisfies the homogeneous equation $L(u) = 0$, with appropriate boundary conditions ($u_1|_{\partial\Omega} = g$), and u_2 satisfies the nonhomogeneous equation with homogeneous boundary conditions($u_2|_{\partial\Omega} = 0$). Thus
>
> $$\begin{aligned}
> &L(u) = L(u_1 + u_2) = L(u_1) + L(u_2) = 0 + f = f \\
> &u|_{\partial D} = u_1|_{\partial\Omega} + u_2|_{\partial\Omega} = g + 0 = g,
> \end{aligned}$$
>
> satisfying both.

Thus, we need to solve two PDEs:

$$\begin{aligned} &\Delta u_1 = 0 & \qquad & \Delta u_2 = f \\ &\partial_r u_1|_{\partial\Omega} = 0 & & \partial_r u_2|_{\partial\Omega} = 0 \end{aligned} \tag{2}$$

The solution to the Laplace Equation on the disk is given by

$$u_1(r,\theta) = \sum_{n=0} r^n (a_n \cos(n\theta) + b_n \sin(n\theta)),$$

which is solved by implementing separation of variables. In satisfying the boundary conditions, we find that $u_1(r,\theta) = 0$. Now, to solve the nonhomogeneous expression, we expand f into a Fourier series by the method of eigenfunction expansion. Thus, we must solve the Helmholtz equation, $\Delta u_2 + \lambda u_2 = 0$. In doing so, the solution to the Helmholtz equation is given by

$$u_2(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\big(\sqrt{\lambda_{mn}}r\big)(A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)),$$

where the eigenvalues are

$$\lambda_{mn} = \frac{k_{mn}^2}{a^2},$$

and k_{mn} are the zeros of the first derivative of the m-th order Bessel function of the first kind evaluated at the boundary, $J_m'\big(\sqrt{\lambda_{mn}}a\big) = 0$, to satisfy the Neumann boundary conditions. Now, in substituting our solution $u_2(r,\theta)$ into the nonhomogeneous equation, we obtain

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} -\lambda_{mn} J_m\big(\sqrt{\lambda_{mn}}r\big)(A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)) = f.$$

By projecting three times, once for each harmonic and once for the Bessel functions, we obtain the direct expression for the coefficients:

$$\begin{aligned} A_{mn} &= \frac{-1}{\pi\lambda_{mn}} \frac{\int_0^{2\pi}\int_0^b f(r',\theta') J_m\big(\sqrt{\lambda_{mn}}r'\big)\cos(m\theta')\ r'dr'd\theta'}{\int_0^a J_m^2\big(\sqrt{\lambda_{mn}}r'\big)\ r'dr'} \\ B_{mn} &= \frac{-1}{\pi\lambda_{mn}} \frac{\int_0^{2\pi}\int_0^b f(r',\theta') J_m\big(\sqrt{\lambda_{mn}}r'\big)\sin(m\theta')\ r'dr'd\theta'}{\int_0^a J_m^2\big(\sqrt{\lambda_{mn}}r'\big)\ r'dr'}, \end{aligned}$$

where the factor of π in each coefficient comes from explicitly computing the cosine and sine projections explicitly. Thus, the full solution is

$$
\begin{aligned}
u &= u_1 + u_2 \\
&= 0 + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\big(\sqrt{\lambda_{mn}}r\big)\big(A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)\big) \\
&= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[\frac{-1}{\pi\lambda_{mn}}\frac{\int_0^{2\pi}\int_0^b f(r',\theta')J_m\big(\sqrt{\lambda_{mn}}r'\big)\cos(m\theta')\ r'dr'd\theta'}{\int_0^a J_m^2\big(\sqrt{\lambda_{mn}}r'\big)\ r'dr'}\right] J_m\big(\sqrt{\lambda_{mn}}r\big)\cos(m\theta) \\
&+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[\frac{-1}{\pi\lambda_{mn}}\frac{\int_0^{2\pi}\int_0^b f(r',\theta')J_m\big(\sqrt{\lambda_{mn}}r'\big)\sin(m\theta')\ r'dr'd\theta'}{\int_0^a J_m^2\big(\sqrt{\lambda_{mn}}r'\big)\ r'dr'}\right] J_m\big(\sqrt{\lambda_{mn}}r\big)\sin(m\theta).
\end{aligned}
$$

The solution clearly satisfies the boundary condition because evaluating the solution for the Neumann problem produces the zeros of the first derivative of the m-th order Bessel function at the boundary. Further, the solution satisfies Poisson's equation. The method of eigenfunction expansion worked because we were able to use the Fundamental Theorem. Being able to apply nonhomogeneous boundary conditions, should they apply, to Laplace's equation and homogeneous boundary conditions to Poisson's equation, in conjunction with the properties of linearity, make it possible to successfully utilize the method of eigenfunction expansion for the Poisson equation. There is a slight problem with this solution though. The very first eigenvalue $\lambda_{01} = 0$. Thus, we encounter division by zero and the solution does not exist, or so it seems. In order to obtain a non-trivial solution that satisfies the proper boundary conditions, we turn to Linear Algebra. In essence, this problem is a lot like the standard $Ax = b$, where A is a singular linear operator because one of the eigenvalues is zero. For this exercise, we have $L(u) = f$, where the Neumann Laplacian is a singular operator. So, how do we go about solving this problem? Let's first examine an elementary example.

Back to Basics: Let us look at the following singular system for inspiration, as provided by the wonderful Dr. Beltukov:

$$
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.
$$

This system is obviously singular: it's first eigenvalue is equal to zero and it's determinant is trivially zero. Further, there is only a solution provided that $a = 0$. So, how can be determine the general solution of this system, assuming that it exists? We project. The columns of the matrix span the yz-plane in $\mathbb{R}^3$, or equivalently, range$(A) \subset \mathbb{R}^3$. We want to represent the system such that the solution is in the range of A. Thus, we project the solution vector onto the yz-plane. This is what we obtain by doing so:

$$
\text{proj}_{[yz]}(b) = \langle \mathbf{j}, b\rangle \mathbf{j} + \langle \mathbf{k}, b\rangle \mathbf{k} = (0\ b\ c)^T.
$$

Now, we solve the system in the usual way and obtain

$$\mathbf{x} = \begin{bmatrix} t \\ b \\ c/2 \end{bmatrix},$$

where $t \in ker(A)$, and thus $\mathbf{x}$ is an affine space of solutions. We now apply this logic to the problem at hand.

Solution (Part 2): In order to solve equation (1), we need to project f onto a basis. However, we need to know the space in which we are working. Our domain is

$$\Omega = \{r, \theta \in \mathbb{R} \mid 0 < r \leq a,\ 0 < \theta \leq 2\pi\}.$$

Further, having already solved the Helmholtz equation, we can express f as an eigenfunction expansion of the eigenfunctions of the Neumann Laplacian by projecting onto them, where the coefficients are given by

$$a_{mn} = \frac{1}{\pi} \frac{\int_0^{2\pi} \int_0^b f(r', \theta') J_m\big(\sqrt{\lambda_{mn}} r'\big) \cos(m\theta')\ r' dr' d\theta'}{\int_0^a J_m^2\big(\sqrt{\lambda_{mn}} r'\big)\ r' dr'}$$
$$b_{mn} = \frac{1}{\pi} \frac{\int_0^{2\pi} \int_0^b f(r', \theta') J_m\big(\sqrt{\lambda_{mn}} r'\big) \sin(m\theta')\ r' dr' d\theta'}{\int_0^a J_m^2\big(\sqrt{\lambda_{mn}} r'\big)\ r' dr'}.$$

This means that the space in which we are working is $L^2(\Omega) \subset H$, because the eigenfunctions of the Neumann Laplacian serve as a basis for $L^2(\Omega)$. Now that we have projected f onto our desired basis, and we know the space in which we are operating, we can confidently solve the problem at hand. Our new PDE is given by

$$\begin{aligned}
&\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = f = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\big(\sqrt{\lambda_{mn}} r\big)\big(a_{mn}\cos(m\theta) + b_{mn}\sin(m\theta)\big), \\
&\frac{\partial u}{\partial r}\Big|_{\partial\Omega} = 0 \\
&0 < r < a, \quad 0 < \theta \leq 2\pi, \\
&f = \begin{cases} 1, & r < b \\ 0, & r > b \end{cases}.
\end{aligned} \tag{3}$$

Now, let's guess that our solution is an element in the space, namely

$$u_2(r, \theta) = c \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\big(\sqrt{\lambda_{mn}} r\big)\big(a_{mn}\cos(m\theta) + b_{mn}\sin(m\theta)\big),$$

a constant multiple of the right-hand side. In substituting our guess into the equation (3), we obtain

$$c\left[J_m''\left(\sqrt{\lambda_{mn}}r\right)+\frac{1}{r}J_m'\left(\sqrt{\lambda_{mn}}r\right)-\frac{m^2}{r^2}J_m\left(\sqrt{\lambda_{mn}}r\right)\right]=J_m\left(\sqrt{\lambda_{mn}}r\right),$$

where the harmonic terms vanish and we omit the summation signs to avoid cluttering. This may seem intimidating to solve, but Bessel functions have very nice recursion relations and derivative properties that make solving for c trivial, which are

$$\begin{aligned} J_m'(x) &= \frac{J_{m-1}(x)+J_{m+1}(x)}{2} \\ \frac{d^2}{dx^2}\big[J_m(x)\big] &= \frac{(m^2+m-x^2)J_m(x)-xJ_m(x)}{x^2}. \end{aligned}$$

Let's first make the substitution $s=\sqrt{\lambda_{mn}}r$. Now, in proceeding to solve for c, we obtain

$$\begin{aligned} c &= \frac{J_m(s)}{J_m''(s)+\frac{1}{r}J_m'(s)-\frac{m^2}{r^2}J_m(s)} \\ &= \frac{J_m(s)}{-\frac{m^2J_m(s)}{s^2}+\frac{(m^2+m-s^2)J_m(s)-sJ_{m-1}(s)}{s^2}+\frac{J_{m-1}(s)-J_{m-1}(s)}{2s}} \\ &= \frac{J_m(s)}{-J_m(s)} \\ &= -1. \end{aligned}$$

Thus, our full solution is the negative of f, given by

$$\boxed{u(r,\theta)=u_1+u_2=(-1)\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m\left(\sqrt{\lambda_{mn}}r\right)\left(a_{mn}\cos(m\theta)+b_{mn}\sin(m\theta)\right)},$$

where a_{mn} and b_{mn} are the same coefficients presented on the previous page. Here, our solution not only avoids division by zero, but also satisfies equation (3) and the boundary conditions. The resulting code and plots are shown below.

```
%% Exercise 1

N = 50;
m = 5;
n = 5;

%a = 2;
%a = 4;
%a = 8;
a = 16;

```

```
b = 1;

[r,theta] = meshgrid(linspace(0,a,N),linspace(0,2*pi,N));

% Right-hand side

f = @(r,t) 1.*(r<b) + 0.*(r>b);

% Computing first n zeros of derivatives of desired bessel eigenfunctions

z = zeros(n+1,n);

for j = 0:n
    for k = 1:n
        z(j+1,k) = BessDerivZerosBisect2(j,k,eps('single'));
    end
end

z(1,:) = circshift(z(1,:),1);
z(1,1) = 0;

% Solution for Neumann condition

u = zeros(size(r));

for ii = 1:n-1
    ll = (z(1,ii)/a).^2;
    aa0 = (integral2(@(rr,tt) ...
        f(rr,tt).*besselj(0,sqrt(ll).*rr).*rr,0,a,0,2*pi)...
            ./integral(@(rr) ...
                ((besselj(0,sqrt(ll).*rr)).^2).*rr,0,a))/(2*pi);
    u = u + besselj(0,sqrt(ll)*r).*aa0;
end

for mm = 1:m
    for nn = 1:n
        l = (z(mm+1,nn)/a).^2;
        aa = (integral2(@(rr,tt) ...
            f(rr,tt).*besselj(mm,sqrt(l).*rr).*cos(mm*tt).*rr,0,a,0,2*pi)...
             ./integral(@(rr) ((besselj(mm,sqrt(l).*rr)).^2).*rr,0,a))/(pi);
        bb = (integral2(@(rr,tt) ...
            f(rr,tt).*besselj(mm,sqrt(l).*rr).*sin(mm*tt).*rr,0,a,0,2*pi)...
             ./integral(@(rr) ((besselj(mm,sqrt(l).*rr)).^2).*rr,0,a))/(pi);
        u = u + besselj(mm,sqrt(l)*r).*(aa*cos(mm*theta) + ...
            bb*sin(mm*theta));
    end
end

u = -u;

Verification of Initial Condition
```

```

ff = zeros(size(r));

for ii = 1:n-1
    ll = (z(1,ii)/a).^2;
    aa0 = (integral2(@(rr,tt) ...
        f(rr,tt).*besselj(0,sqrt(ll).*rr).*rr,0,a,0,2*pi)...
            ./integral(@(rr) ...
                ((besselj(0,sqrt(ll).*rr)).^2).*rr,0,a))/(2*pi);
    ff = ff + besselj(0,sqrt(ll)*r).*aa0;
end

for mm = 1:m
    for nn = 1:n
        l = (z(mm+1,nn)/a).^2;
        aa = (integral2(@(rr,tt) ...
            f(rr,tt).*besselj(mm,sqrt(l).*rr).*cos(mm*tt).*rr,0,a,0,2*pi)...
            ./integral(@(rr) ((besselj(mm,sqrt(l).*rr)).^2).*rr,0,a))/(pi);
        bb = (integral2(@(rr,tt) ...
            f(rr,tt).*besselj(mm,sqrt(l).*rr).*sin(mm*tt).*rr,0,a,0,2*pi)...
            ./integral(@(rr) ((besselj(mm,sqrt(l).*rr)).^2).*rr,0,a))/(pi);
        ff = ff + besselj(mm,sqrt(l)*r).*(aa*cos(mm*theta) + ...
            bb*sin(mm*theta));
    end
end

% Plot Solutions

figure
mesh(r.*cos(theta),r.*sin(theta),u);
xlabel('$x$','Interpreter','latex');
ylabel('$y$','Interpreter','latex');
zlabel('$u(r,\theta)$','Interpreter','latex');
title(sprintf('Poisson Equation Solution: $a$ = ...
    %1.0f',a),'interpreter','latex');
set(gca,'fontsize',18);

figure
subplot(1,2,1)
mesh(r.*cos(theta),r.*sin(theta),f(r,theta));
xlabel('$x$','Interpreter','latex');
ylabel('$y$','Interpreter','latex');
zlabel('$f$','Interpreter','latex');
title('Actual Initial Condition');
set(gca,'fontsize',18);
zlim([-0.2 1.2]);

subplot(1,2,2)
mesh(r.*cos(theta),r.*sin(theta),ff);
xlabel('$x$','Interpreter','latex');
ylabel('$y$','Interpreter','latex');
```

```
zlabel('$f$','Interpreter','latex');
title('Eigenfunction Initial Condition');
set(gca,'fontsize',18);
zlim([−0.2 1.2]);
```

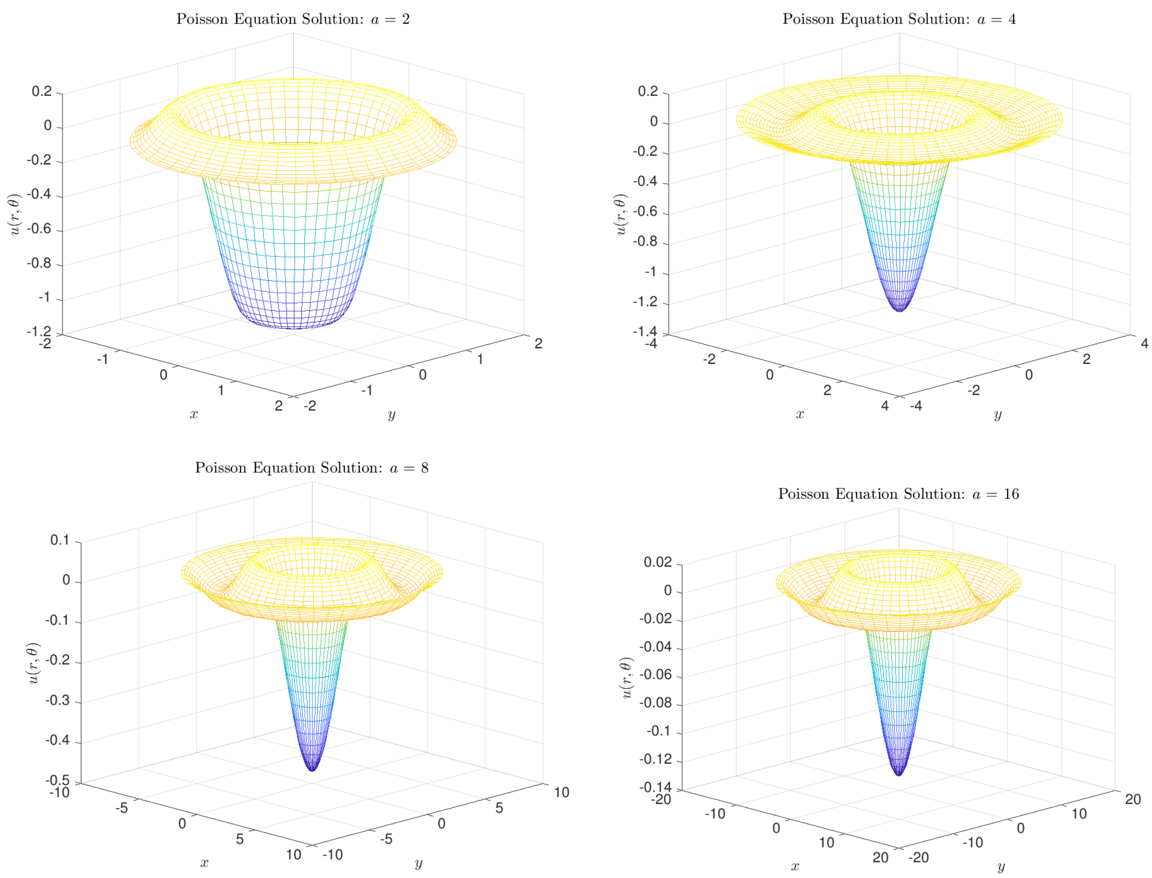


Figure 1: Plots of the Solutions to Poisson's Equation

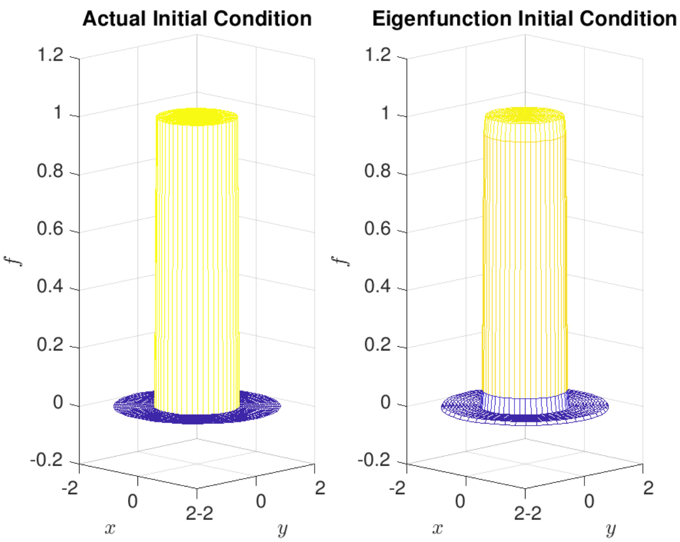


Figure 2: Verifying that the code is correct by plotting the initial condition.

Of course the truncated series does not appear to look like the initial condition because it was generated with the first five terms of each sum. Thus, I included the second figure to verify that for higher order sums—specifically the first five terms for the m-sum and the first 50 terms of the n-sum—the series indeed converges. Further, for the solution as $a \to \infty$, I think that the solution becomes a Fourier-Bessel Transform, also known as a Hankel transform.

Solution (Part 3): Now, after revisiting this after a couple of years and re-doing the calculation, I have been able to, without a doubt, complete this exercise in its entirety. I was on the right track, but my second solution is incorrect, more incorrect than the first solution. But the idea was correct. When we project f onto the appropriate space, we obtain the following graph in Figure 3, where Figure 3a show f (left) and Figure 3b shows f expressed as an element of $L^2(\Omega)$ (right). To understand how we arrive at the correct solution, we re-examine the derivation of the solution from the first solution. First, we understand that there are two PDEs that we need to solved: the first being the general solution to the homogeneous PDE, which we previously believed was equal to zero, and the particular solution to the nonhomogeneous PDE, which in general is

$$u_2(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\big(\sqrt{\lambda_{mn}}r\big)\big(A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)\big), \tag{4}$$

both found in equation (2) with the homogeneous equation on the left and the nonhomogeneous equation on the right. Now, my error arose in the second solution by thinking the general solution in equation (4) was the projection of f onto $L^2(\Omega)$, which is incorrect. The above is, as we know from the first attempt of the solution, the solution to the Helmholtz equation. Since we know from the first solution that the first eigenvalue $\lambda_{01} = 0$, we know that the right-hand side of equation (1) is not in the image of $L^2(\Omega)$, that is $f \notin im(L^2(\Omega))$.

So, in order to solve equation (1), we must, as I correctly described in solution 2, project f onto $L^2(\Omega)$. To project f onto this particular space, we know that, through the Helmholtz equation and its solution,

$$\Delta u = -\lambda u = f.$$

meaning that

$$f = -\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\lambda_{mn}J_m\left(\sqrt{\lambda_{mn}}r\right)\left(A_{mn}\cos(m\theta)+B_{mn}\sin(m\theta)\right).$$

However, before continuing with the projections, if we compute the first few terms of the summation in m for equation (4), we find that the only valid value for m is $m=0$ and that the other values of m result in zeros. Thus, we obtain

$$f = -\sum_{n=1}^{\infty}\lambda_{0n}A_nJ_0\left(\sqrt{\lambda_{0n}}r\right), \tag{5}$$

which means

$$u_2(r,\theta) = \sum_{n=1}^{\infty}A_nJ_0\left(\sqrt{\lambda_{0n}}r\right). \tag{6}$$

Further, since we know the space is defined by linear combinations of zero-order Bessel functions, we can write f as a linear combination of them,

$$f = \sum_{n=1}^{\infty}C_nJ_0\left(\sqrt{\lambda_{0n}}r\right), \tag{7}$$

Now, we may project onto the space in equation (7) in order to express f as an element of the space. To project f onto the space, we compute the inner product of both sides of equation (7) with $J_0\left(\sqrt{\lambda_{0n}}r\right)$ in order to compute the coefficients C_n,

$$\left\langle f, J_0\left(\sqrt{\lambda_{0n}}r\right)\right\rangle = C_n\left\langle J_0\left(\sqrt{\lambda_{0n}}r\right), J_0\left(\sqrt{\lambda_{0n}}r\right)\right\rangle,$$

where the inner product $\langle\cdot,\cdot\rangle$ is the L^2 inner product

$$\langle h, g\rangle = \int_0^{2\pi}\int_0^a h(r)g(\theta)\ r\,dr\,d\theta,$$

and the sum is not actually omitted, but rather implied. In solving for C_n, we find that

$$C_n = \frac{\left\langle f, J_0\left(\sqrt{\lambda_{0n}}r\right)\right\rangle}{\left\langle J_0\left(\sqrt{\lambda_{0n}}r\right), J_0\left(\sqrt{\lambda_{0n}}r\right)\right\rangle}.$$

Now that we have an expression for the right-hand side that resides in the space $L^2(\Omega)$, we can equate equations (5) and (7),

$$-\sum_{n=1}^{\infty} \lambda_{0n} A_n J_0\left(\sqrt{\lambda_{0n}} r\right) = f = \sum_{n=1}^{\infty} C_n J_0\left(\sqrt{\lambda_{0n}} r\right).$$

By taking the inner product of both sides again with $J_0\left(\sqrt{\lambda_{0n}} r\right)$, we find that

$$-\lambda_{0n} A_n \left\langle J_0\left(\sqrt{\lambda_{0n}} r\right), J_0\left(\sqrt{\lambda_{0n}} r\right)\right\rangle = C_n \left\langle J_0\left(\sqrt{\lambda_{0n}} r\right), J_0\left(\sqrt{\lambda_{0n}} r\right)\right\rangle.$$

Since inner products return scalars, we can simplify the inner products on each side and we find that

$$A_n = -\frac{C_n}{\lambda_{0n}}.$$

Thus the solution for the nonhomogeneous problem is

$$u_2(r,\theta) = -\sum_{n=1}^{\infty} \frac{C_n}{\lambda_{0n}} J_0\left(\sqrt{\lambda_{0n}} r\right).$$

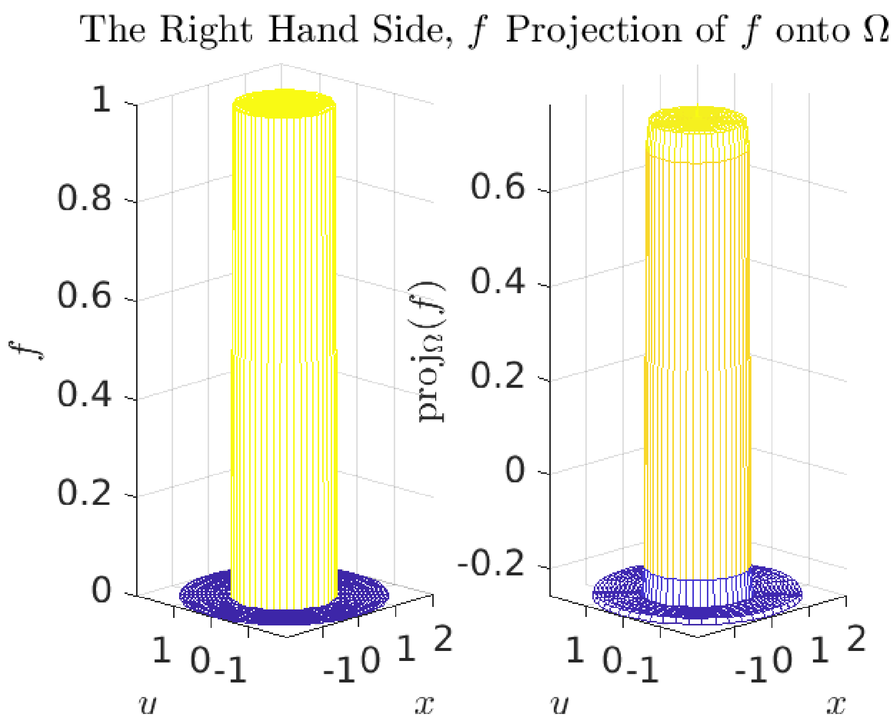
But what about the solution to the homogeneous PDE? Is it equal to zero? Certainly not. The reason being that the first eigenvalue is still zero, that is $\lambda_{01} = 0$. What does this mean? This means that the homogeneous PDE has a non-trivial solution. And the only solution for the homogeneous problem is $u_1(r,\theta) = K$, where K is a constant. We can check by substituting $u_1 = K$ into the PDE and we find that indeed the boundary value problem is satisfied. This also means that the sum goes from 2 to ∞. And we expect this result due to the spherical symmetry of the system. Thus the full solution is

$$\boxed{u(r,\theta) = u_1 + u_2 = K - \sum_{n=2}^{\infty} \frac{C_n}{\lambda_{0n}} J_0\left(\sqrt{\lambda_{0n}} r\right).}$$

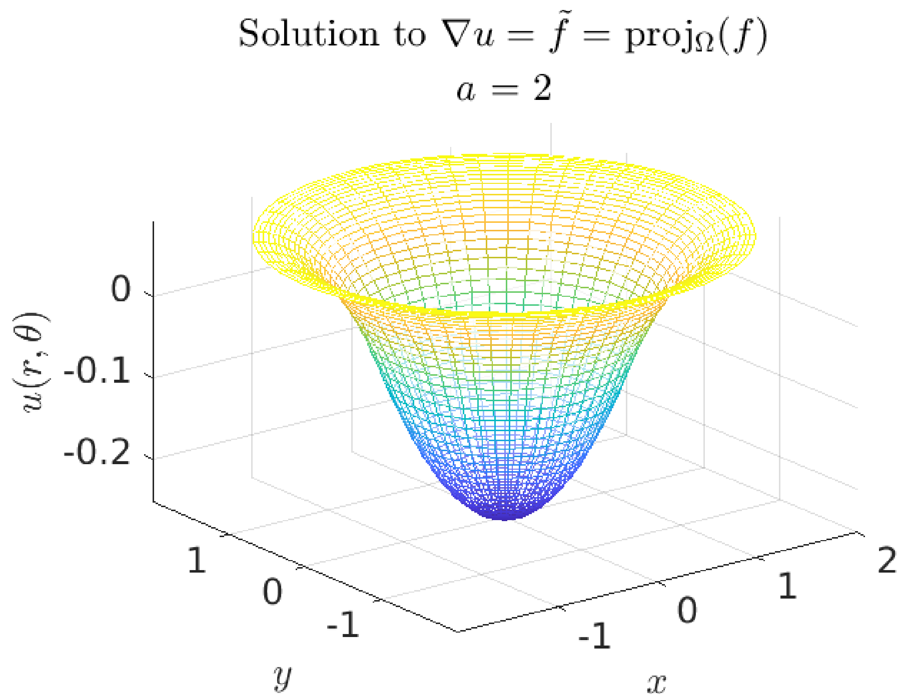
Further, in the limit as $a \to \infty$ this result means that the average of the right-hand side over the region is equal to zero

$$\bar{f} = \frac{1}{\mathrm{vol}(\Omega)} \int_{\Omega} f \, d\Omega = \frac{b^2}{a^2} = 0.$$

Thus $C_1 = \bar{f} = \lambda_{01} \cdot K = 0$. This is true for the Poisson kernel. Physically, we can visualize the solution as the electric potential of a point charge in the limiting case.

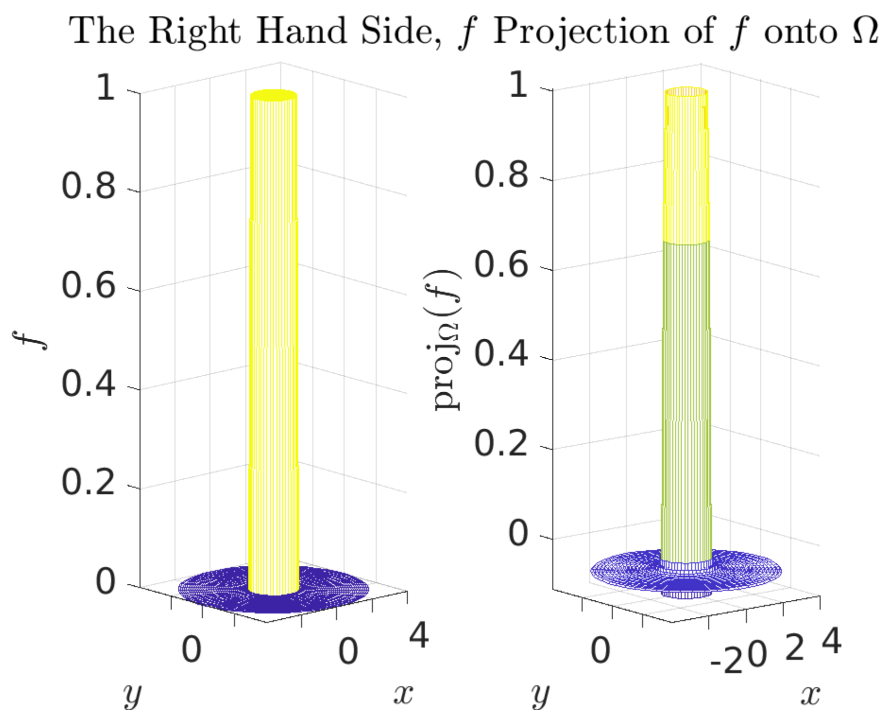


(a)

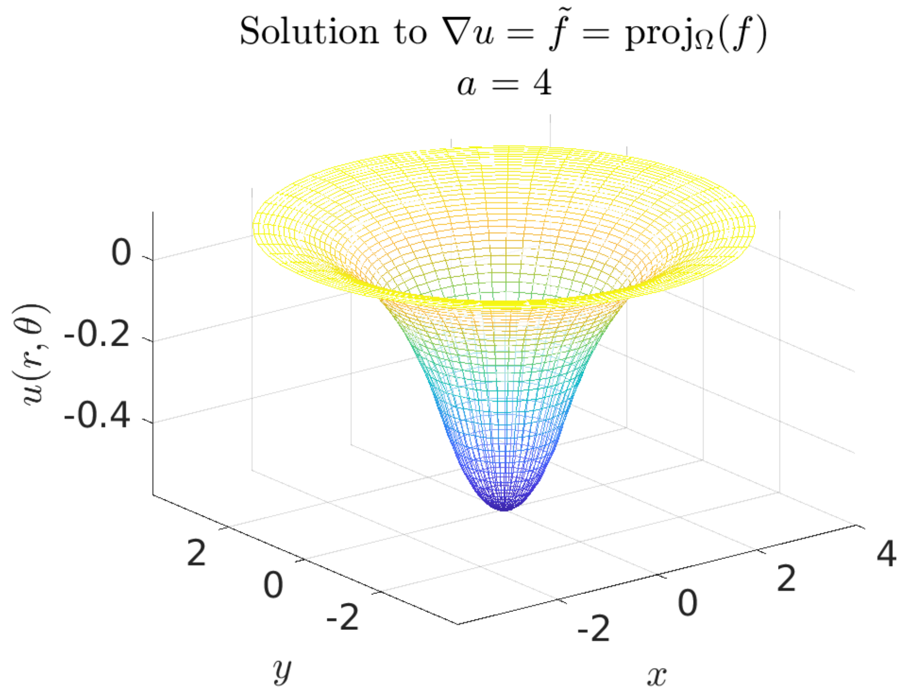


(b)

Figure 3: The correct projection of f onto $L^2(\Omega)$ and the correct solution for $a = 2$.

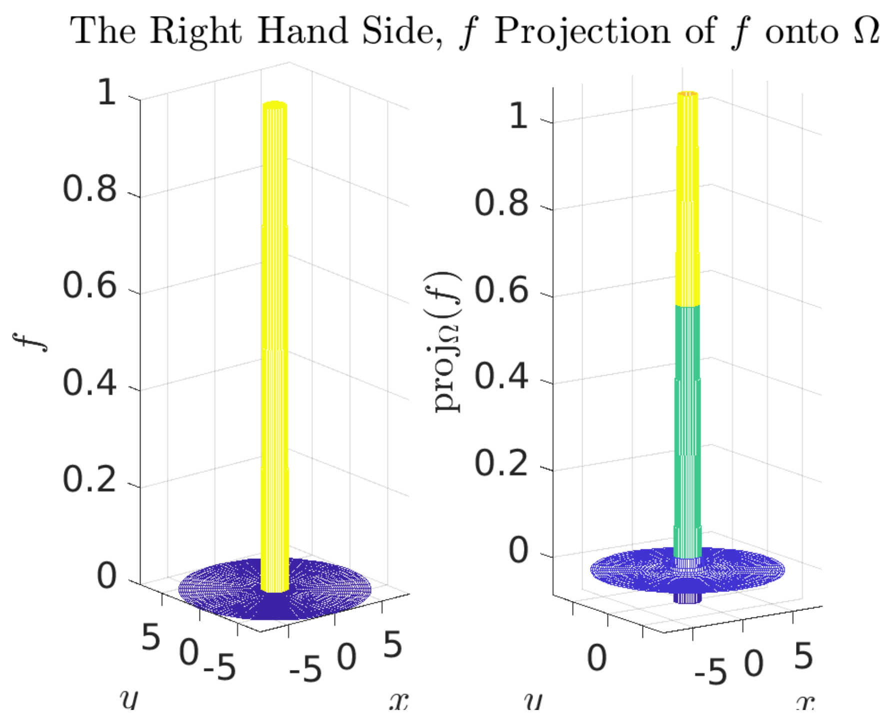


(a)

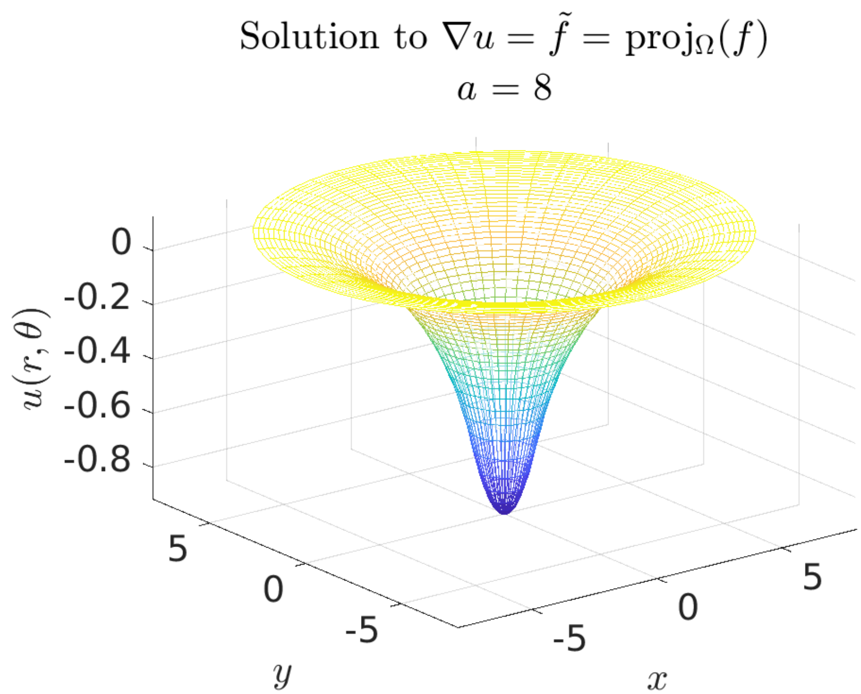


(b)

Figure 4: The right-hand side for $a = 4$.

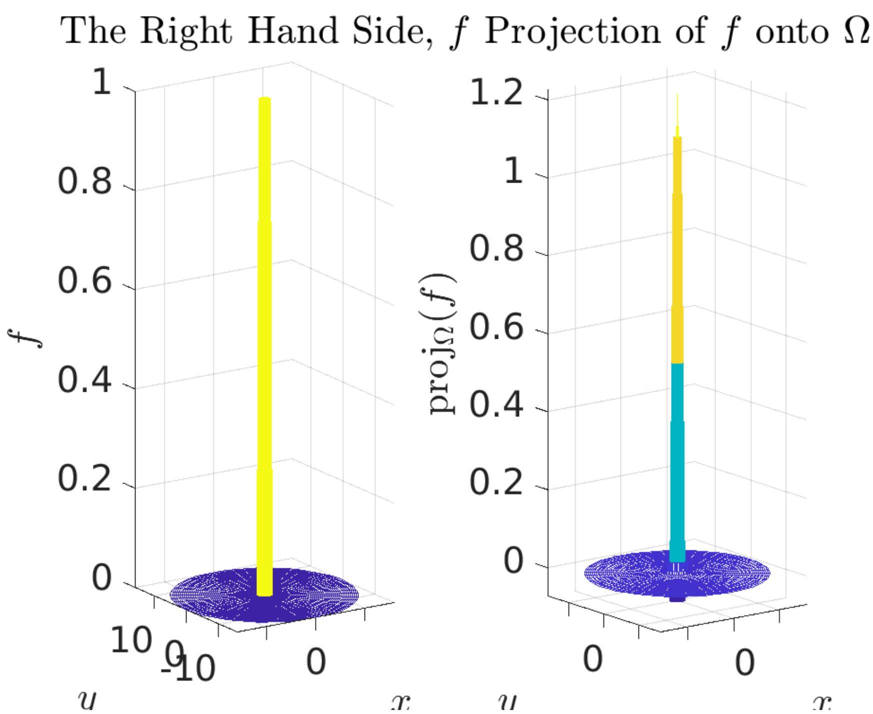


(a)

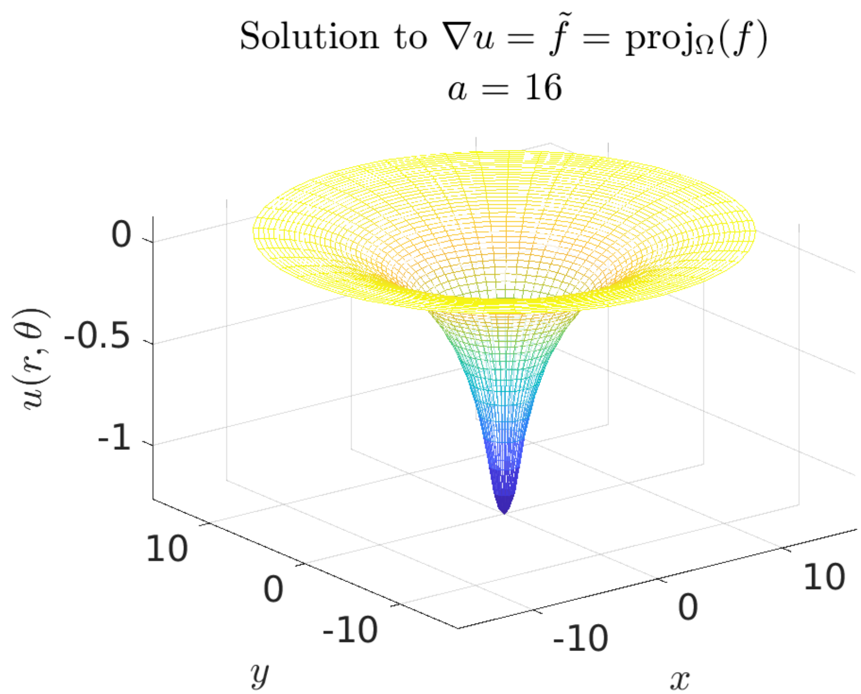


(b)

Figure 5: The right-hand side for $a = 8$.



(a)



(b)

Figure 6: The right-hand side for $a = 16$.

Exercise 2

2.) Consider the following PDE:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = f(x) = e^{-2x^2}. \tag{3}$$

(a) *(10 Points) Find a formula for the Fourier transform of $u(x,t)$ (taken with respect to x).*

(b) *(20 Points) Plot the solution of (1) for $t = 1, 2, 33, 4$. You can do that either by finding an explicit formula for $u(x,t)$ or by inverting the Fourier transform numerically.*

Solution: To begin, apply the Fourier transform to equation (3), known as the Transport equation, and the initial condition to obtain

$$\frac{d\widehat{u}}{dt} = i\xi\widehat{u}, \quad \widehat{u}_0 = \widehat{f(x)} = \widehat{f}(\xi).$$

The solution is thus

$$\widehat{u} = e^{i\xi t}\widehat{f}(\xi) = \widehat{f(x+t)}.$$

In using the addition identity of the Fourier transform we can obtain the explicit solution of the Transport equation, which is

$$\boxed{u(x,t) = \left[\widehat{f(x+t)}\right]^{\vee} = f(x+t) = e^{-2(x+t)^2}}.$$

Below is code that compared the explicit solution with the numerical Fourier inversion. The plots of the results are below the code.

```
%% Exercise 2

t = [1 2 3 4]; % Desired times

x = linspace(-5,5,100); % Length of finite domain

u = zeros(length(x),1); % Solution vectors
uu = u;

% Initial condition

f = @(x) exp(-2*x.^2);

% Transform Solution

figure
```

```
for tt = 1:length(t)
    for j = 1:length(x)
        u(j,:) = integral2(@(w,y) ...
            exp(1i*w.*(x(j)-y+t(tt))).*f(y),-10,10,-10,10)/(2*pi);
    end
    hold on
    subplot(2,2,tt)
    plot(x,u,'b-');
    xlabel('Rod Position','interpreter','latex');
    ylabel('$u(x,t)$','interpreter','latex');
    legend(sprintf('$t$ = %1.0f',t(tt)),'interpreter','latex');
    title('Transform Solution','interpreter','latex');
    set(gca,'fontsize',18);
    ylim([-0.05 1.05]);
end

% Explicit Solution

figure
for j = 1:length(t)
    uu = exp(-2*(x+t(j)).^2);
    hold on
    subplot(2,2,j);
    plot(x,uu,'r-');
    xlabel('Rod Position','interpreter','latex');
    ylabel('$u(x,t)$','interpreter','latex');
    legend(sprintf('$t$ = %1.0f',t(j)),'interpreter','latex');
    title('Explicit Solution','interpreter','latex');
    set(gca,'fontsize',18);
    ylim([-0.05 1.05]);
end
```

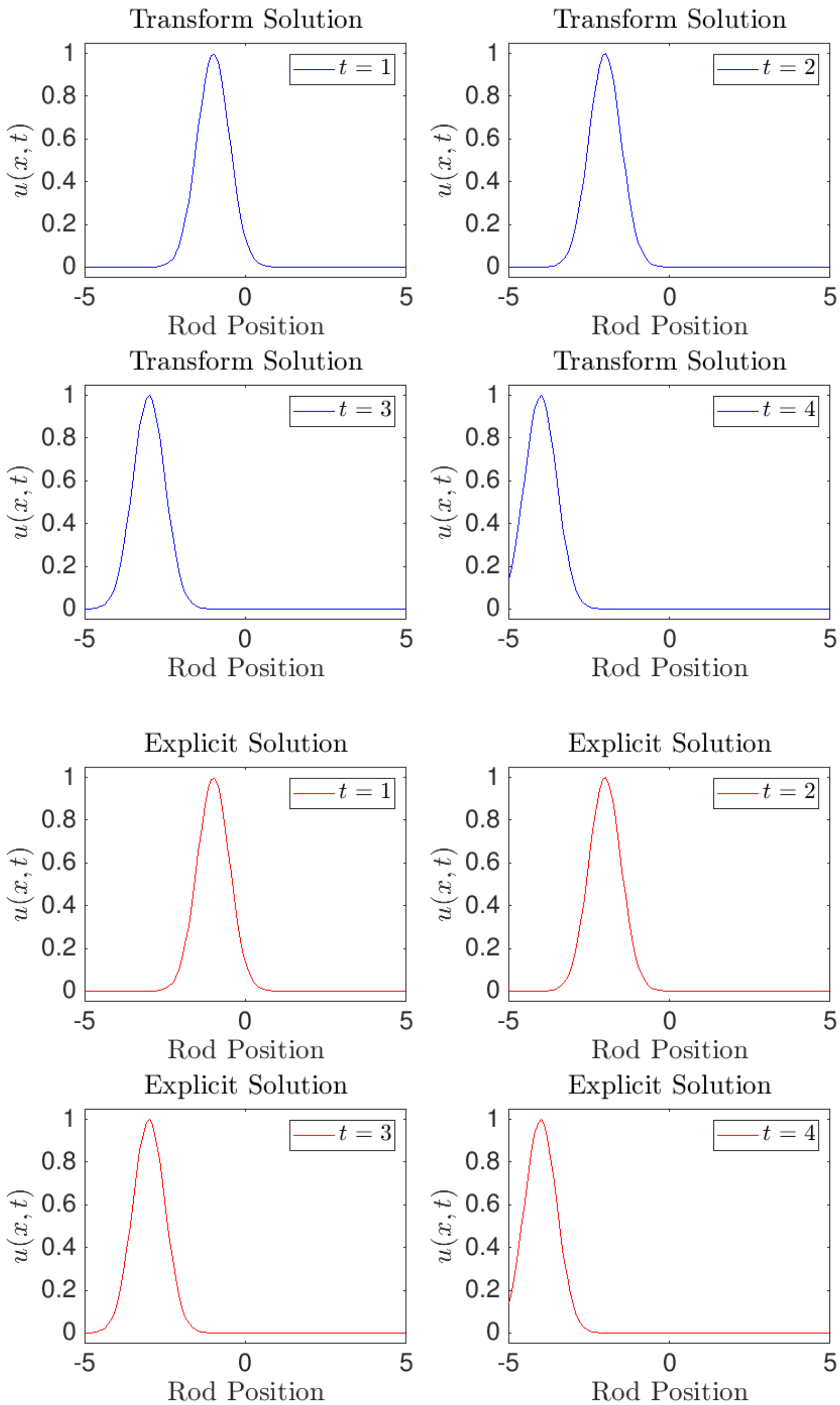


Figure 7: The plots of the numerically transformed solution and the explicit solution. They produce the exact same plots, as expected.

EXERCISE 3

3.) Use Finite Differences to solve the following one-dimensional boundary value problem:

$$\frac{d^2u}{dx^2} - \frac{1}{1+x}\frac{du}{dx} + 16u = -\frac{4\cos(4x)}{1+x}, \quad u(0) = u(\pi) = 0. \tag{4}$$

Present your solution in the following format:

1. *Explain how you discretize the problem into a linear system; in particular, explain the entries in the matrix of that system.*
2. *Produce several plots of the solution with increasing number of subdivisions; these plots should show convergence.*
3. *Conclude with well-documented code.*

Solution: To begin, let us define

$$u(x_n) = u_n, \quad a_n = \frac{1}{1+x_n}, \quad f_n = -a_n 4\cos(4x_n)h^2$$

In rewriting equation (4) in terms of the central differences formula for the first and second derivatives, we obtain

$$\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} - a_n\left(\frac{u_{n+1} - u_{n-1}}{h}\right) + 16u_n = -a_n 4\cos(4x_n),$$

where h is the step-size of the grid. By multiplying both sides by h^2 and simplifying by collecting terms, we obtain the following:

$$(1 + a_n h)u_{n-1} + (16h^2 - 2)u_n + (1 - a_n h)u_{n+1} = f_n.$$

Lastly, to satisfy the initial condition, we have for n = 1 and for n = N,

$$(1 + a_n h)u_0 = 0, \quad (1 - a_n h)u_{N+1} = 0.$$

Thus, the matrix vector system that we need to solve is

$$\begin{bmatrix} (16h^2-2) & (1-a_nh) & & & \\ (1+a_nh) & (16h^2-2) & (1-a_nh) & & \\ & & \ddots & & \\ & & (1+a_nh) & (16h^2-2) & (1-a_nh) \\ & & & (1+a_nh) & (16h^2-2) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N \end{bmatrix}.$$

The code is below, in addition to the plots. The code is structured as follows: the matrix system is formed with an additional row and column so as to contain the a_n's. Then, afterward, the additional row and column are removed so that the remaining system is $N \times N$ as intended and is solved. The solution is plotted with grid sizes ranging from $N = 50 - 1600$, to show convergence.

```matlab
%% Exercise 3

N = [50 100 200 400 800 1600]; % Number of subdivisions

for j = 1:length(N)

    % Semi-discretization

    L = pi; % length
    h = L/(N(j)+1); % step-size
    x = (1:N(j))*h; % 1D grid

    % Generating the matrix

    f = (-4*cos(4*x)./(1+x)); % function evaluated on grid
    a = 1./(1+x);
    d = eye(N(j)+1).*((16*h^2) - 2); % main diagonal
    da = diag(1-a*h,1); % subdiagonal above
    db = diag(1+a*h,-1); % subdiagonal below
    A = d + da + db; % Full matrix with Dirichlet conditions
    A(:,N(j)+1) = []; % Delete extra column
    A(N(j)+1,:) = []; % Delete extra row

    % Solving the linear system

    u = A\(f*h^2)';

    % Plotting to show convergence

    hold on
    plot(x,u);
    xlabel('$x$','interpreter','latex');
    ylabel('$u(x)$','interpreter','latex');
    title('Solution');
    set(gca,'fontsize',18);
end

legend('N = 50','N = 100','N = 200','N = 400','N = 800','N = 1600');
```

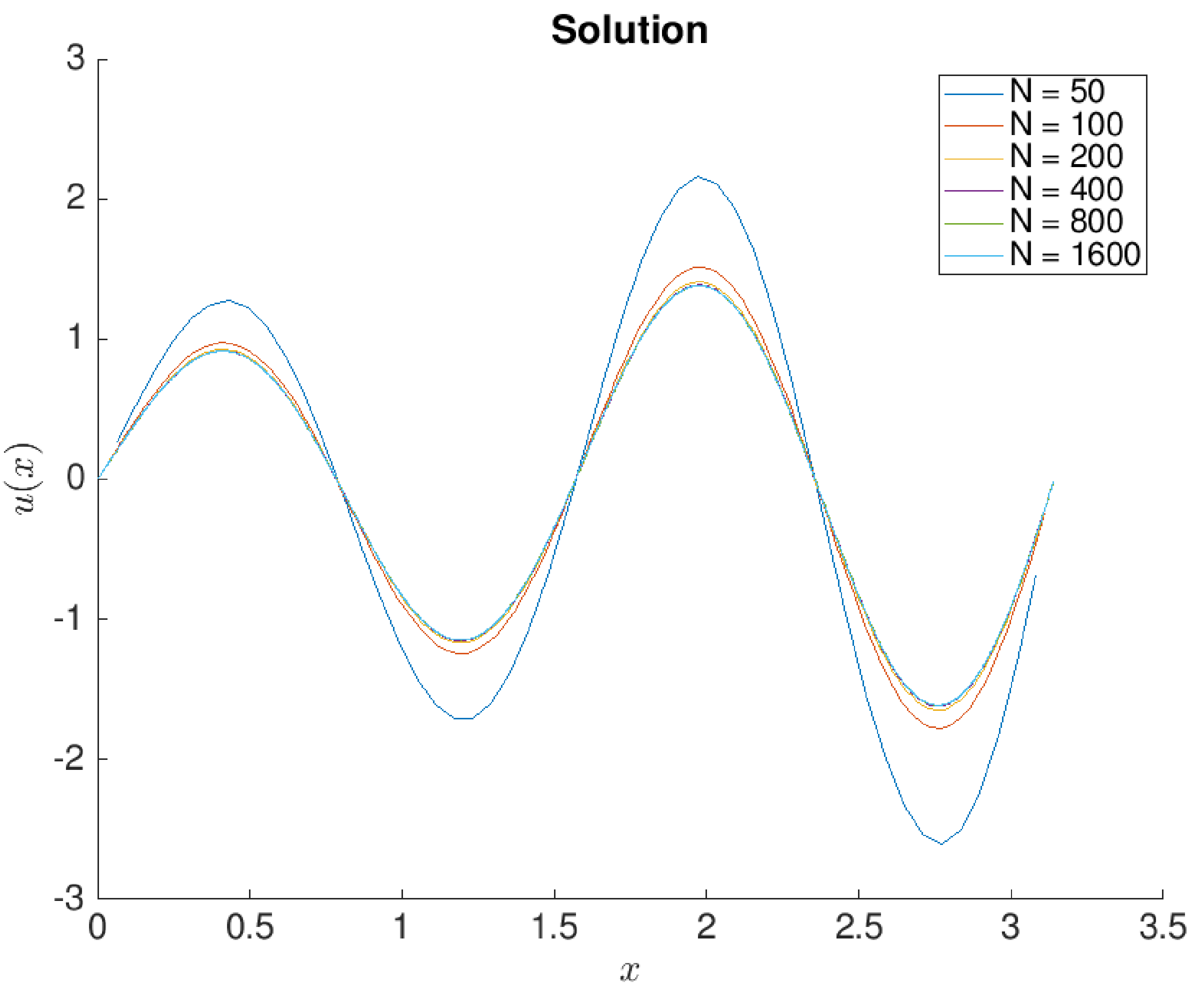


Figure 8: Plot of the solution to equation (4) with increasing N, which indeed does converge.