AN ILLUSTRATION OF A RIGIDITY THEOREM

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Latimer Galvan Harris-Ward

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AUTHOR: Latimer Galvan Harris-Ward

DATE SUBMITTED: Spring 2024

Department of Mathematics and Statistics

Dr. Arlo Caine
Thesis Committee Chair
Mathematics and Statistics

Dr. John Rock
Mathematics and Statistics

Dr. Ivan Ventura
Mathematics and Statistics

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ABSTRACT

This thesis explores rigidity theorems and their application to a specific case in differential geometry. Chapter 1 provides a foundational understanding of rigidity theorems by introducing four key examples: Mostow rigidity, the sphere theorem, Berger's hypersurface rigidity, and a rigidity theorem involving the Schwarzschild metric. Chapter 2 delves into the geometry of the n-sphere. We present stereographic and longitude-latitude coordinate systems for the unit circle and sphere. We also prove the compactness and connectedness of the n-sphere, explore visualization techniques for the 4-dimensional 3-sphere, and detail calculations for the Riemann curvature tensor, Ricci curvature, scalar curvature, and shape operator of the 2-sphere. Additionally, we provide examples of hypersurfaces in both Euclidean 3-space and the 3-sphere, alongside the metric tensors for the n-sphere in both stereographic and longitude-latitude coordinates (without proof). Chapter 3 focuses on Theorem 2 from the referenced paper, "Geometric Inequalities and Rigidity Theorems on Equatorial Spheres." We provide an in-depth explanation of the theorem's outline and its proof, focusing on the geometry. Finally, Chapter 4 outlines potential directions for future research related to the concepts explored in the thesis.

Contents

List of Figures

Chapter 1

Introduction

1.1 Rigidity in Differential Geometry: A Survey

Differential geometry is the branch of geometry that uses calculus and linear algebra to study the smooth features of curves and surfaces and higher dimensional shapes. These geometric objects, often called manifolds, can be one-dimensional (a curve), two-dimensional (a surface), or n-dimensional, where n is a positive integer greater than or equal to three. But differential geometry goes beyond these familiar shapes. It explores a broader universe of objects in different dimensions and investigates how these shapes can or cannot deform while preserving specific characteristics. One key concept in preserving characteristics under deformations is the *rigidity theorem*. While there is no consensus on a definition, we can think of rigidity theorems as statements that often show that simple geometric constraints can severely limit the possibilities of a manifold's shape.

Manifolds have basic shape characteristics, such as dimension, topology, orientation, and smoothness. The most sophisticated and fundamental feature of the shape of a manifold with a Riemannian metric is curvature. The curvature of a manifold describes how much it deviates from being "flat" like d-dimensional Euclidean space $\mathbb{E}^d$ in a way that depends only on its intrinsic properties, not how it is bent or stretched in a larger space. Because the curvature of a manifold is an intrinsic property, curvature arises in many rigidity theorems. For example, consider the following theorem on the rigidity of hypersurfaces [2].

Theorem 1.1.1. (Rigidity of Hypersurfaces) In Euclidean spaces $\mathbb{E}^d$ with dimension $d > 3$, generic hypersurfaces are determined by their inner geometry up to congruence.

There is much to unpack from this theorem. First, we examine Euclidean spaces $\mathbb{E}^d$. In three dimensions, where $d = 3$, we can think of Euclidean space as the geometrical representation of our three-dimensional reality. In Euclidean 3-space, we can measure lengths and angles like in real life. Euclidean d-space extends these ideas to higher dimensions. To understand hypersurfaces, we examine a sheet of paper in our three-dimensional reality. Technically, a sheet of paper is three-dimensional because it has some thickness. If we "ignore" the thickness of our paper, then our paper is a two-dimensional manifold in our three-dimensional reality. When a manifold has a dimension that is one fewer than the space in which it lives, we say the object is a *hypersurface*. This theorem is about hypersurfaces in higher-dimensional Euclidean spaces.

We need only to explore the last phrase of Theorem 1.1.1, namely "...generic hypersurfaces are determined by their inner geometry up to congruence." We can think of "inner geometry" as the intrinsic properties of the hypersurface, or features that can be determined from the metric itself. For example, Gauss's Theorema Egregium [5] shows that the Gaussian curvature of a surface in $\mathbb{E}^3$ can be computed from the metric itself. The final part of the theorem "up to congruence" is the "punch-line" of Theorem 1.1.1. Two geometric objects are congruent if they have the same shape and size, regardless of their different positions or orientations in space. So, when a statement is valid for a geometric object "up to congruence," this means that the statement is true regardless of the object's position and orientation in space. Therefore, our theorem states that most hypersurfaces in dimensions greater than three are uniquely defined by their internal characteristics, ignoring their positioning or orientation in space.

Theorem 1.1.1 is an example of a rigidity theorem because it connects a higher-dimensional hypersurface's curvature (an intrinsic geometric property) to its uniqueness under translations and rotations. If two "nice" hypersurfaces in $\mathbb{E}^6$ have the same curvature, but one is oriented differently and located elsewhere in the space, then they are both the same hypersurface. Another example of a rigidity theorem related to the central theme of this work is called the sphere theorem [2].

Theorem 1.1.2. (Sphere Theorem) If a complete, simply connected Riemannian manifold has sectional curvature K satisfying

$$\frac{1}{4} < K \leq 1,$$

then it is homeomorphic to a sphere.

A manifold is simply connected if it has no "holes," meaning if we draw a loop on the manifold, we can shrink it to a point without ever leaving the manifold. For example, a paper without holes is a simply connected manifold because we can shrink every loop we draw on the paper down to a point. The sectional curvature of a manifold measures how much the manifold deviates from being flat in a specific direction at a given point. Lastly, two objects are homeomorphic if we can continuously map the points of one object to the other and vice versa—no breaking, tearing, or gluing. So, Theorem 1.1.2 states the sectional curvature of a complete manifold with no "holes" forces the manifold to be topologically equivalent to a sphere as long as the sectional curvatures lie between 1/4 and 1, inclusive.

Frequently, rigidity theorems are not so friendly. For example, consider the Mostow Rigidity Theorem [2].

Theorem 1.1.3. (Mostow Ridigity) Two space forms of constant negative curvature and dimension larger than two whose fundamental groups are isomorphic (as groups) must be isometric.

This theorem is significantly more complicated to unpack. First, imagine you are sitting inside a basketball. No matter where you are in the ball, the walls always curve toward you—–the inside of the basketball has constant negative curvature. The fundamental group of a topological space captures the “loopiness” of the space by considering all possible loops within it and classifying them based on whether we can shrink them to a point without leaving the space. To understand the theorem, let us examine an analogy. Imagine two different, crumpled pieces of paper that represent the spaces. Suppose both have constant negative curvature and the same fundamental group. In this case, the theorem states you can flatten one crumpled piece into the exact shape of the other, highlighting their identical form.

Yet another rigidity theorem [3], and the last one before we discuss the theorem in this paper, is more recent and explores spaces with applications in general relativity, Einstein's theory on the gravitation of the universe.

Theorem 1.1.4. Let (M, g) be an asymptotically Schwarzschildean 3-manifold of non-negative scalar curvature. If it contains a complete, properly embedded stable minimal surface Σ, then (M, g) is isometric to the Euclidean space $\mathbb{R}^3$, and Σ is an affine plane.

We look to an example in two dimensions to understand the idea of this theorem. Imagine you place balls of different sizes on the ground—some close to each other and others far away—making sure they do not roll away. Now, place a large rectangular bed sheet over these balls, ensuring no wrinkles, kinks, cusps, sharp corners, or pits. Ignoring the balls underneath, we see this sheet as a landscape with hills and valleys. This landscape represents a two-dimensional curved space. Because we made sure the landscape is resting smoothly on the balls, the landscape has non-negative curvature—in some areas, the landscape has hills, and in others, the landscape is perfectly flat. Now, if we place a small round object on the hills, the round object will roll away until it reaches a flat section of the sheet and stops moving. However, if we placed a round object on the perfectly flat sheet section far from the hills, the object would not roll. We can think of this phenomenon as the sheet being asymptotically Schwarzchildean. So, let us act as though as we approach any of the four edges of our sheet, we are moving infinitely far in the direction of the edge without ever reaching the edge.

The Schwarzchild metric describes the warped geometry of spacetime around a massive spherically symmetric object like a star or black hole [4]. The metric tells us how the object's gravity affects distances, lengths, and directions, making spacetime appear curved. In an asymptotically Schwarzchildean space, we feel the effects of its gravitational pull when we are close to a massive, spherical object. When we are far away from the massive, spherical object, we do not experience any gravitational effects [4]. Back to our analogy, let us place an extraordinary kind of flexible rubber sheet (the minimal surface) on our landscape. This rubber sheet minimizes its area like a soap film but also has a specific stability property. The theorem states if we can smoothly place this rubber sheet on the landscape, without tearing or folding, and extend it infinitely in all directions (to the ends of the landscape), then the entire landscape must be flat! The unique rubber sheet would be a flat plane resting on the flat landscape.

Remark 1.1.5. Our explanation for Theorem 1.1.4 is limited because it takes place in two dimensions instead of three. Furthermore, we could not characterize the rubber sheet's completeness. That is because a manifold is complete if you can travel along the straightest possible path (called a geodesic) without reaching an edge or any other obstacle.

In this section, we have seen four rigidity theorems and examined curvature, hypersurfaces, and other objects and their properties. In the next section, we introduce the rigidity theorem that we are interested in this paper, which examines conditions for which a manifold becomes part of the upper half of the n-dimensional sphere.

1.2 The Rigidity Theorem of Interest

In this section, we introduce the main theorem from [6] in which we are interested and discuss the main conceptual ideas. It concerns hypersurfaces M within the unit $(n+1)$-sphere $\mathbb{S}^{n+1}$

Theorem: Let $n \geq 2$. Let M be a connected, embedded, two-sided hypersurface in $\mathbb{S}^{n+1}$ with boundary ∂M. Suppose $\text{int}(M)$ is C^{n+1} and M is C^1 up to boundary. Suppose M and ∂M satisfy the following conditions:

(1) M satisfies $R \geq n(n-1)$ (R is the scalar curvature);

(2) ∂M is contained in $\mathbb{S}^{n+1}$;

(3) M is tangent to $\mathbb{S}^n_+$ at ∂M from the region enclosed by ∂M.

Then M is a portion of the hemisphere $\mathbb{S}^n_+$.

What does this theorem mean? First, we need a special surface M. This surface M, a manifold, must be a single, two-sided object, like a sheet you would place on your bed between the fitted sheet and the comforter. It cannot have any breaks, tears, creases, or holes. If we ripped the sheet in two, we would not label those two pieces of linen as a single sheet. Additionally, this sheet cannot self-intersect and must be smooth everywhere except at the edges, which may be rough.

Now, we discuss the three conditions. Condition (1) states that our sheet M must be highly curved, like an inflated parachute. Condition (2) states that the edges of our sheet ∂M must be completely intact, without sharp corners and jagged edges. So, the sheet would have rounded edges with no corners or jagged sections (the edge does not have to be

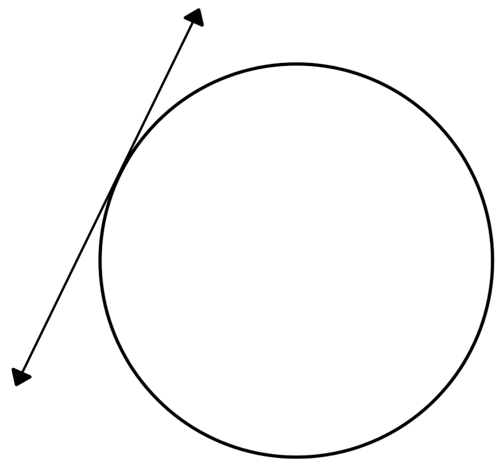


Figure 1.1: This is an example of a line tangent to a circle.

perfectly round). The last condition is perhaps the most interesting condition of the three. Condition (3) states that our sheet must be tangent to the upper half of the unit equatorial n-sphere from the region enclosed by its boundary ∂M. To understand this condition, first, we conceptualize what it means for two objects to be tangent to each other. If we have a circle and a line, the line would be tangent to the circle if the line touches or intersects the circle exactly at one point. We can see this behavior in Figure 1.1. Now, imagine a swimmer with a swim cap. Suppose the swim cap is inflated while on the swimmer's head so that only the edge of the swim cap contacts the swimmer's head. In that case, the swim cap will not be tangent to the swimmer's head at the boundary of the swim cap from the inside of the swim cap (the region the swim cap encloses). However, if we deflated the swimmer's cap to mold and hug the swimmer's head, the cap would be tangent to the swimmer's head from inside the swimmer's cap. This is the idea behind condition (3).

Let us explore why this theorem is a rigidity theorem. To recall, a rigidity theorem is a statement that shows how simple geometric constraints can severely limit the possibilities of a manifold's shape. For this theorem, we take a special manifold M with boundary ∂M, both satisfying certain topological conditions. Geometrically, if our manifold has a scalar curvature greater than or equal to the product of the dimension and one less the dimension, then the manifold is topologically part of the upper half of the unit equatorial n-sphere. We can see the relationship between the geometry in the theorem (the scalar curvature) and the resulting topology (the manifold being part of the upper hemisphere of the unit n-sphere). In the next section, we provide an outline of the paper.

1.3 Outline

Our goal in this paper is to expound the proof of the theorem in Section 1.2. In this chapter, we provided some background on rigidity theorems in differential geometry and explored four rigidity theorems. We also explored the theorem of interest and why it is a rigidity theorem.

In Chapter 2, we examine the geometry of spheres. In Section 2.1, we introduce the longitude-latitude and stereographic coordinates for the 1- and 2-spheres, along with illustrations to help us visualize the sphere in both coordinate systems. In Section 2.2, we look at the topology of the unit n-sphere, proving that it is compact and connected for $n = 3$ and then for general positive integers n. After proving the compactness and connectedness of $\mathbb{S}^n$, we discuss how to visualize $\mathbb{S}^3$ using stereographic coordinates. The stereographic coordinates allow us to project the 4-dimensional 3-sphere onto an extension of $\mathbb{R}^3$ with an extra point at infinity. Section 2.3 covers Riemannian metrics, shape, and curvature. Metrics allow us to define the concepts of length and angle measures between tangent vectors at each point on manifolds, giving manifolds intrinsic shape. We explore some examples of how Riemannian metrics give abstract manifolds their shape when embedded into larger spaces. Then, we further examine the concept of curvature and compute the Riemann curvature tensor, the Ricci curvature, and the Scalar curvature for the 2-sphere in Section 2.4. To finish the chapter, we define the shape operator and look at examples of hypersurfaces and in $\mathbb{R}^3$ and $\mathbb{S}^3$ and the spherical metric for $\mathbb{S}^n$ in stereographic and longitude-latitude coordinates in Section 2.5.

In Chapter 3 we list and describe the tools we need to understand for the proof of the theorem of interest. These results come from the paper where the theorem of interest was published, [6], and include geometry, topology, and analysis. Because we are interested in the geometric aspects of the theorem of interest, we relegate the details of the more topological and analytical aspects to the appendices. In Section 3.2, we explore the geometrical details of the proof, expounding on most of the tools we explored in Section 3.1. Lastly, in Chapter 4, we discuss future work.

Chapter 2

The Geometry of Spheres

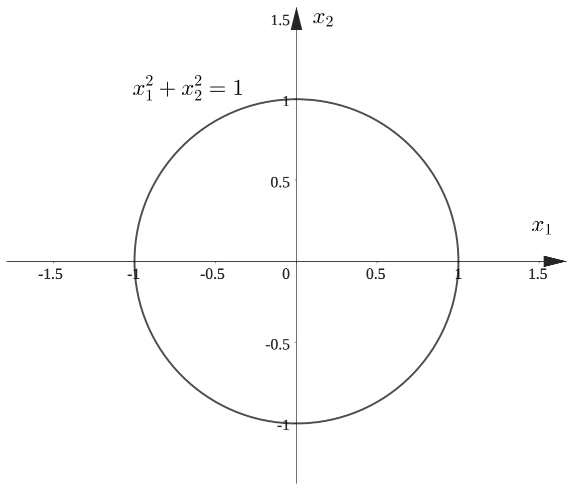
2.1 The n-Sphere and Local Coordinates

The unit n-sphere, denoted $\mathbb{S}^n$, is the boundary of the $(n+1)$-dimensional unit ball. When centered at the origin, $\mathbb{S}^n$ consists of all points $\mathbf{x} \in \mathbb{R}^{n+1}$ at distance 1 from the origin, or in set-builder notation,

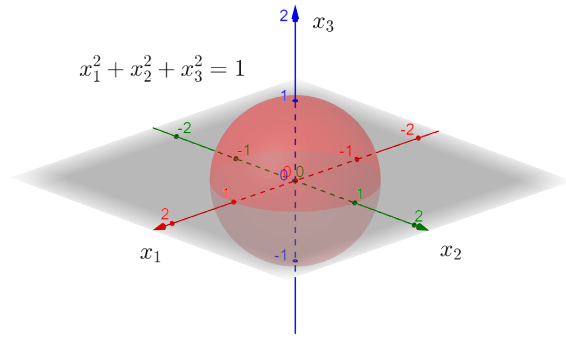
$$\mathbb{S}^n = \left\{ \mathbf{x} = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1 \right\}. \tag{2.1}$$

We can also write this set using the Euclidean norm $\|\cdot\| : \mathbb{R}^{n+1} \to \mathbb{R}$, where for $\mathbf{x} \in \mathbb{R}^{n+1}$, we have $\|\mathbf{x}\| = \sqrt{\sum_{i=1}^{n+1} x_i^2}$ and $\mathbb{S}^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : \|\mathbf{x}\| = 1\}$. When $n \geq 3$, $\mathbb{S}^n$ is impossible to visualize from a birds-eye perspective because the plot would be in dimensions four and higher, but we can look to the cases where $n = 1$ and $n = 2$ to give us a good idea of how to think of $\mathbb{S}^n$ when $n \geq 3$.

The one-dimensional unit sphere $\mathbb{S}^1$ is a subset of $\mathbb{R}^2$, the boundary of the unit disk, i.e., the unit circle centered at the origin. The two-dimensional unit sphere $\mathbb{S}^2$ is a subset of $\mathbb{R}^3$, the boundary of the unit ball centered at the origin in $\mathbb{R}^3$. See Figure 2.1. We use local parameterizations to precisely locate points on these manifolds. One way we can achieve this is with longitude/latitude coordinates. For the surface $\mathbb{S}^2$, this parameterization helps



(a) The 1-sphere, or $\mathbb{S}^1$, is the set of all points in the plane whose distance from the origin is equal to 1.



(b) The 2-sphere, $\mathbb{S}^2$, is the set of all points in space whose distance from the origin is equal to 1.

Figure 2.1: Graphical representations of the 1-sphere and the 2-sphere.

us locate points on the surface of the Earth and can be generalized to help us locate points on spheres of any dimension. We can describe $\mathbb{S}^1$ through the mapping

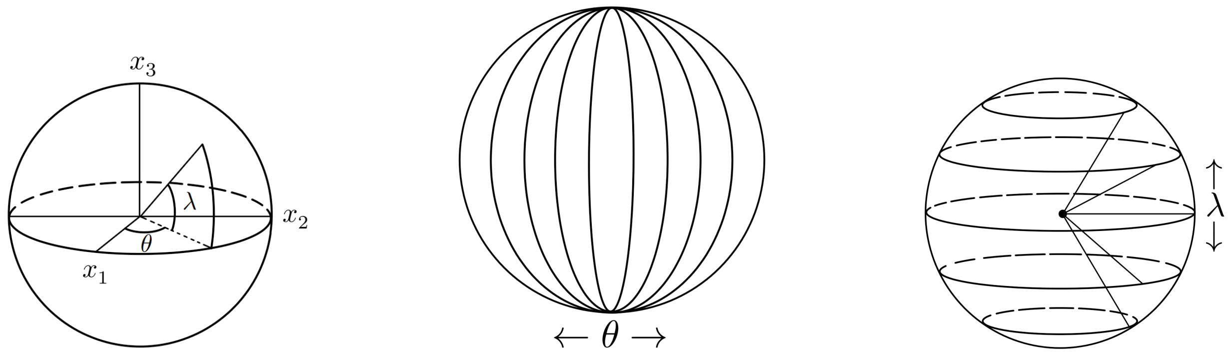
$$f_1 : [0, 2\pi] \subseteq \mathbb{R} \to \mathbb{R}^2, \quad f_1(\theta) = (\cos(\theta), \sin(\theta)),$$

and we can describe $\mathbb{S}^2$ through the mapping

$$f_2 : [0, 2\pi] \times [-\pi/2, \pi/2], \quad f_2(\theta, \lambda) = (\cos(\theta)\cos(\lambda), \sin(\theta)\cos(\lambda), \sin(\lambda)).$$

Then we interpret $\theta \in [0, 2\pi]$ as *longitude*, and $\lambda \in [-\pi/2, \pi/2]$, as *latitude*. To make these mappings one-to-one, we must restrict the domains slightly. Otherwise, we can live with the fact that the points $(1, 0)$ on $\mathbb{S}^1$ and $(0, 0, \pm 1)$ (and the points on a semi-circle connecting them) are described by multiple parameters.

Another way in which we can describe locations on $\mathbb{S}^1$ and $\mathbb{S}^2$ is by using stereographic coordinates. The stereographic projection of $\mathbb{S}^1$ is in Figure 2.3a. For $t \in \mathbb{R}$, the intersection of the ray from $(1, 0)$ through $(0, t)$ and the unit circle occurs at the point



(a) θ measures longitude and λ measures latitude.

(b) Longitude semi-circles indexed by θ and parametrized by λ.

(c) Latitude circles indexed by λ and parameterized by θ.

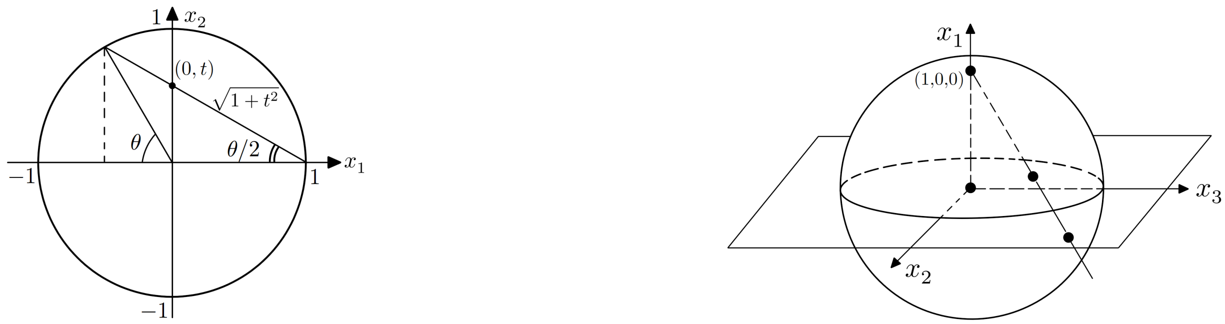
Figure 2.2: Longitude and latitude coordinates for $\mathbb{S}^2$.

$(x_1, x_2) = \left(\dfrac{t^2 - 1}{1 + t^2}, \dfrac{2t}{1 + t^2} \right)$. We can similarly describe $\mathbb{S}^2$ using stereographic projection as in Figure 2.3b. The ray from $(1, 0, 0)$ through $(0, t_1, t_2)$ intersects the unit sphere at the point

$$(x_1, x_2, x_3) = \left(\frac{t_1^2 + t_2^2 - 1}{1 + t_1^2 + t_2^2}, \frac{2t_1}{1 + t_1^2 + t_2^2}, \frac{2t_2}{1 + t_1^2 + t_2^2} \right). \tag{2.2}$$

Stereographic coordinates are useful because they describe almost every point of the curved sphere in a higher dimension using every point of a linear space in a lower dimension. Every point on $\mathbb{S}^1 \setminus \{(1, 0)\}$ in $\mathbb{R}^2$ corresponds to a point on the real line $\mathbb{R}^1$ and every point on $\mathbb{S}^2 \setminus \{(1, 0, 0)\}$ in $\mathbb{R}^3$ corresponds to a point in the real plane $\mathbb{R}^2$. To capture all of $\mathbb{S}^1$ and $\mathbb{S}^2$, we must include a point at infinity for $\mathbb{R}^1$ and $\mathbb{R}^2$. More precisely, $\mathbb{S}^1$ is homeomorphic to the one-point compactification $\mathbb{R}^1 \cup \{\infty\}$ of $\mathbb{R}^1$ and $\mathbb{S}^2$ is homeomorphic to the one-point compactification $\mathbb{R}^2 \cup \{\infty\}$.

The set of all points of $\mathbb{S}^2$ with fixed x_3 coordinate between -1 and 1 form a circle in space surrounding the x_3-axis. With latitude $\lambda = \sin^{-1}(x_3)$ the function $\theta \mapsto f_2(\theta, \lambda)$ parameterizes such a circle, called a *latitude circle*. Fixing θ, the function $\lambda \mapsto f_2(\theta, \lambda)$ parameterizes a semi-circle from $(0, 0, -1)$ to $(0, 0, 1)$, called a *semi-circle of longitude*, or



(a) Stereographic projection of S^1.

(b) Stereographic projection of S^2.

Figure 2.3: Stereographic projections.

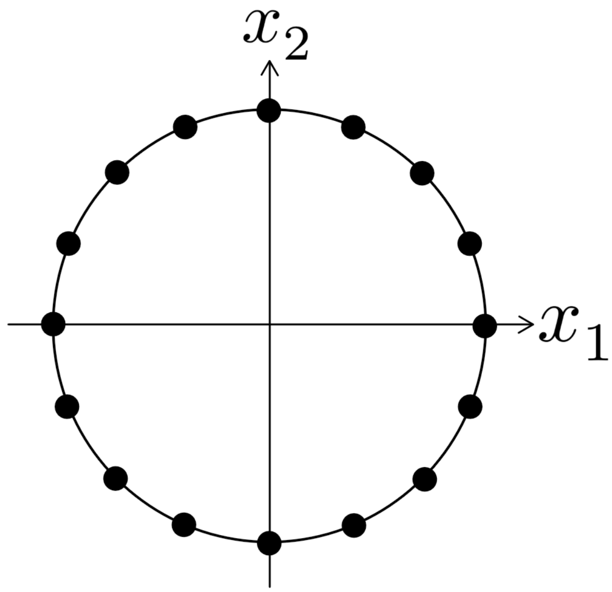
meridian. It is instructive to consider how such curves appear in stereographic coordinates (t_1, t_2). Since

$$t_1 = \frac{x_2}{1 - x_1} \text{ and } t_2 = \frac{x_3}{1 - x_1} \tag{2.3}$$

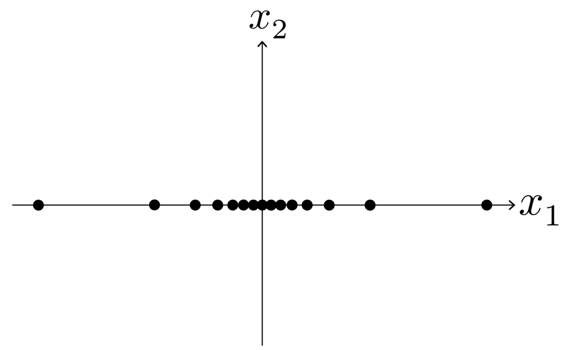
we can parameterize the circle at latitude λ via

$$(t_1, t_2) = \left(\frac{\sin(\theta)\cos(\lambda)}{1 - \cos(\theta)\cos(\lambda)}, \frac{\sin(\lambda)}{1 - \cos(\theta)\cos(\lambda)} \right) \tag{2.4}$$

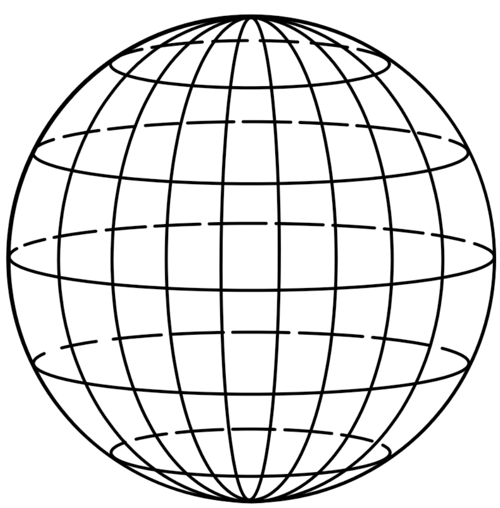
by varying θ over $[0, 2\pi]$. Likewise, the semi-circle at longitude θ is parameterized by the same formula by varying λ over $[-\pi/2, \pi/2]$. This is the result of composing the longitude/latitude parametrization f_2 with stereographic projection (2.3) (i.e., the inverse of the stereographic parametrization (2.2)). The images are shown in Figure 2.4. Just as the circles of latitude and semi-circles of longitude meet at right angles on the surface of $\mathbb{S}^2$ so do their images in stereographic coordinates. However, the spacing between evenly spaced circles of latitude or semi-circles of longitude, is not preserved. This is because the stereographic projection preserves angles between vectors but distorts lengths in a manner that varies from point to point.



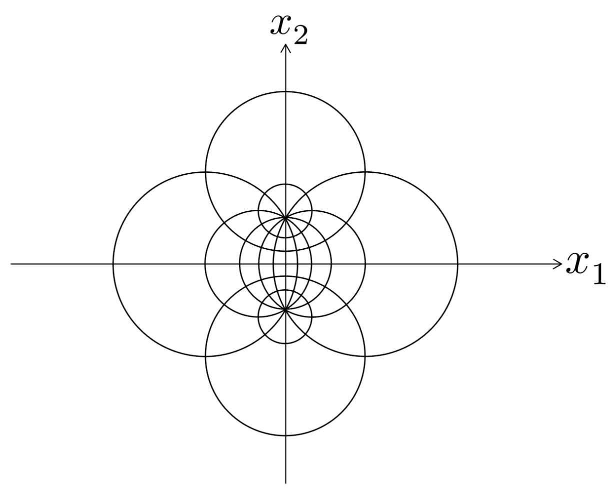
(a) Points on $\mathbb{S}^1$ evenly spaced by longitude.



(b) Image of evenly spaced points under stereographic projection to $\mathbb{R}^1$.



(c) Evenly spaced curves of latitude and longitude on $\mathbb{S}^1$.



(d) Image under stereographic projection to $\mathbb{R}^2$.

Figure 2.4: Images of latitude/longitude coordinate sets under stereographic projections from the point $(1, 0, 0)$.

2.2 Topology of S^n

In the previous section we discussed $\mathbb{S}^1$ and $\mathbb{S}^2$ and two ways we can visualize them locally. We cannot, however, observe $\mathbb{S}^3$ from a birds-eye perspective like we did for $\mathbb{S}^1$ and $\mathbb{S}^2$ because it lives in $\mathbb{R}^4$. Our three-dimensional perception limits our ability to visualize objects in dimensions greater than three. Despite this challenge, we can use more abstract concepts to understand the nature of $\mathbb{S}^3$. We first present a proof that $\mathbb{S}^3$ is compact as a topological subspace of $\mathbb{R}^4$ using the Heine-Borel theorem. We can think of compactness as the "next best thing to finiteness."[1] In other words, while compact sets may not be finite in size, they exhibit many desirable properties that finite sets have, like sequences and subsequences that never escape to infinity and extrema for continuous functions defined on compact sets.

Theorem 2.2.1. The unit 3-sphere, $\mathbb{S}^3$, is compact.

Proof. The Heine-Borel Theorem states that a closed and bounded subset of $\mathbb{R}^n$ is compact. Observe that $\mathbb{S}^3$ is bounded because all points of $\mathbb{S}^3$ are one unit from the origin.

To demonstrate that $\mathbb{S}^3$ is closed, we will show that its complement is open, or for every point $\mathbf{x} \in \mathbb{R}^4 \setminus S^3$, there exists a real number $\varepsilon > 0$ such that the ε-ball $B(\mathbf{x}, \varepsilon)$ is contained in $\mathbb{R}^4 \setminus \mathbb{S}^3$. First, notice that $\mathbb{R}^4 \setminus \mathbb{S}^3 = \mathrm{Int}(B^4) \cup (\mathbb{R}^4 \setminus B^4)$, where B^4 is the closed unit 4-ball. By definition, $\mathrm{Int}(B^4)$ is an open subset of $\mathbb{R}^4$ because it is the largest open subset of $\mathbb{R}^4$ contained in B^4. Thus, it remains to show $\mathbb{R}^4 \setminus B^4$ is open.

We have that $\mathbb{R}^4 \setminus B^4 = \{\mathbf{x} \in \mathbb{R}^4 : \|\mathbf{x}\| > 1\}$. Let $\mathbf{a} \in \mathbb{R}^4 \setminus B^4$ be an arbitrary point. Then $\|\mathbf{a}\| > 1$. We must find an $\varepsilon > 0$ such that $B(\mathbf{a}, \varepsilon) \subseteq \mathbb{R}^4 \setminus B^4$. Let $\varepsilon = (\|\mathbf{a}\| - 1)/2$, and suppose that $\mathbf{b} \in B(\mathbf{a}, \varepsilon)$ is arbitrary. Then $\|\mathbf{a} - \mathbf{b}\| < \varepsilon$. By the Reverse Triangle Inequality, we have

$$|\|\mathbf{a}\| - \|\mathbf{b}\|| \leq \|\mathbf{a} - \mathbf{b}\| < \varepsilon = (\|\mathbf{a}\| - 1)/2,$$

[1]Credit: Ryan Aniceto.

which implies $\|\mathbf{a}\| - \|\mathbf{b}\| < (\|\mathbf{a}\| - 1)/2$. Notice that $2\|\mathbf{b}\| - 1 > \|\mathbf{a}\| > 1$, further implying $\|\mathbf{b}\| > 1$, so $\mathbf{b} \in \mathbb{R}^4 \setminus B^4$. Because $\mathbf{b}$ is an arbitrary point of $B(\mathbf{a}, \varepsilon_{\mathbf{a}})$ we conclude that $B(\mathbf{a}, \varepsilon) \subseteq \mathbb{R}^4 \setminus B^4$. Further, because $\mathbf{a}$ was an arbitrary point of $\mathbb{R}^4 \setminus B^4$, we conclude that $\mathbb{R}^4 \setminus B^4$ is an open subset of $\mathbb{R}^4$. The union of two open sets is open, so $\mathrm{Int}(B^4) \cup (\mathbb{R}^4 \setminus B^4) = \mathbb{R}^4 \setminus \mathbb{S}^3$ is open, which means $\mathbb{S}^3$ is closed. Therefore, by the Heine-Borel Theorem, $\mathbb{S}^3$ is compact in $\mathbb{R}^4$. ■

The same argument used in the proof of Theorem 2.2.1 with minor changes can be made to prove that the n-sphere $\mathbb{S}^n$ is compact. A birds-eye view of the spaces $\mathbb{S}^1$ and $\mathbb{S}^2$ also suggests that they are connected topological spaces. Neither one can be separated by two or more disjoint open sets. Is the same true for $\mathbb{S}^n$? By generalizing the longitude/latitude parametrization of $\mathbb{S}^2$ to a longitude/latitudes parametrization of $\mathbb{S}^n$ we can prove that $\mathbb{S}^n$ is connected for every dimension n because it is the continuous image of a connected set.

We again define the longitude as θ, where $\theta \in [0, 2\pi]$. However, instead of having one latitude, $\mathbb{S}^n$ has $n-1$ latitudes, $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$, with $\lambda_k \in [-\pi/2, \pi/2]$, where $k = 1, \ldots, n-1$. We describe $\mathbb{S}^n$ through the mapping $f_n : D \subseteq \mathbb{R}^n \to \mathbb{S}^n \subseteq \mathbb{R}^n$, where $D = [0, 2\pi] \times [-\pi/2, \pi/2]^{n-1}$ and $f_n(\theta, \lambda_1, \ldots, \lambda_{n-1}) = (x_1, \ldots, x_{n+1})$ is defined by

$$x_1 = \cos(\theta) \prod_{k=1}^{n-1} \cos(\lambda_k) \tag{2.5}$$

$$x_2 = \sin(\theta) \prod_{k=1}^{n-1} \cos(\lambda_k) \tag{2.6}$$

$$x_3 = \sin(\lambda_1) \prod_{k=2}^{n-1} \cos(\lambda_k) \tag{2.7}$$

$$\vdots$$

$$x_{n+1} = \sin(\lambda_{n-1}). \tag{2.8}$$

Theorem 2.2.2. The unit n-dimensional sphere, $\mathbb{S}^n$, is compact and connected.

Proof. To prove that $\mathbb{S}^n$ is connected and compact, we demonstrate that $\mathbb{S}^n$ equals the forward image of a compact and connected set under a continuous mapping. That is, for our mapping $f_n : D \to \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$, we will show that $f(D) = \mathbb{S}^n$. Notice that D is both connected and compact because closed bounded intervals are connected and are also compact in $\mathbb{R}$. Furthermore, finite Cartesian products of compact and connected sets are compact and connected. Next, f is continuous because its components are products of basic, continuous functions.

For the first containment, namely $f(D) \subseteq \mathbb{S}^n$, we induct on the dimension n. For the base case, let $n = 1$. Given $\theta \in D = [0, 2\pi]$, we have $f(\theta) = (x_1, x_2) = (\cos(\theta), \sin(\theta)) \in f(D)$. Since $x_1^2 + x_2^2 = \cos^2(\theta) + \sin^2(\theta) = 1$ we conclude that $f(\theta) = (x_1, x_2) \in \mathbb{S}^1$. Thus, every element of $f(D)$ is in $\mathbb{S}^1$ because $\theta \in D$ was arbitrary, and so $f(D) \subseteq \mathbb{S}^1$.

Now, given $(\theta, \lambda_1, \ldots, \lambda_{n-1}) \in D = [0, 2\pi] \times [-\pi/2, \pi/2]^{n-1}$, assume that $y_1^2 + y_2^2 + y_3^2 + \cdots + y_n^2 = 1$, where

$$y_1 = \cos(\theta) \prod_{k=1}^{n-2} \cos(\lambda_k) \tag{2.9}$$

$$y_2 = \sin(\theta) \prod_{k=1}^{n-2} \cos(\lambda_k) \tag{2.10}$$

$$y_3 = \sin(\lambda_1) \prod_{k=2}^{n-2} \cos(\lambda_k) \tag{2.11}$$

$$\vdots$$

$$y_n = \sin(\lambda_{n-2}) \tag{2.12}$$

for the case $n - 1$. Notice that $x_k = y_k \cos(\lambda_{n-1})$ for $k = 1, \ldots, n$ and $x_{n+1} = \sin(\lambda_{n-1})$. It follows that

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 + x_{n+1}^2 &= \cos^2(\lambda_{n-1})(y_1^2 + y_2^2 + y_3^2 + \cdots + y_n^2) + x_{n+1}^2 \\ &= \cos^2(\lambda_{n-1}) \cdot 1 + \sin^2(\lambda_{n-1}) \\ &= 1. \end{aligned}$$

Therefore $f(\theta, \lambda_1, \ldots, \lambda_{n-1}) = (x_1, \ldots, x_{n+1}) \in \mathbb{S}^n$, and so $f(D) \subseteq \mathbb{S}^n$ because $(\theta, \lambda_1, \ldots, \lambda_{n-1}) \in D$ was arbitrary.

For the other containment, we similarly induct on the dimension n. For the base case, let $n = 1$. Given $(x_1, x_2) \in D = [0, 2\pi]$, let $\theta = \operatorname{Arg}(x_1 + ix_2)$. Then $f(\theta) = (\cos(\theta), \sin(\theta)) = (x_1, x_2)$. Hence $(x_1, x_2) \in f(D)$. Now, for induction, let us assume that for $D' = [0, 2\pi] \times [-\pi/2, \pi/2]^{n-2}$, we have $f(D') \supseteq \mathbb{S}^{n-1}$, i.e., given $(y_1, \ldots, y_n) \in \mathbb{S}^{n-1}$ we can find $(\theta, \lambda_1, \ldots, \lambda_{n-2}) \in D'$ such that equations (2.9)-(2.12) are satisfied. Let $(x_1, \ldots, x_{n+1}) \in S^n$ be arbitrary. We must find $(\theta, \lambda_1, \ldots, \lambda_{n-1})$ in D solving the equations (2.5)-(2.8).

Case 1: Suppose that $x_{n+1} = \pm 1$ which forces $x_1 = x_2 = \cdots = x_n = 0$ because $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1$. If $\lambda_{n-1} = \pm\pi/2$, then $\sin(\lambda_{n-1}) = \pm 1$ while $\cos(\lambda_{n-1}) = 0$. Thus, $x_{n+1} = \pm 1$ while $x_1 = x_2 = \cdots = x_n = 0$ in (2.5)-(2.8). In this case, $(\theta, \lambda_1, \ldots, \lambda_{n-2})$ are unrestricted. Therefore, by choosing $(\theta, \lambda_1, \ldots, \lambda_{n-2})$ in D', we have found a $(\theta, \lambda_1, \ldots, \lambda_{n-1}) \in D$ where $f(\theta, \lambda_1, \ldots, \lambda_{n-1}) = (x_1, \ldots, x_{n-1})$.

Case 2: Suppose that $x_{n+1} \neq \pm 1$ and let $\lambda_{n-1} = \sin^{-1}(x_{n+1})$. Then $\lambda_{n-1} \neq \pm\pi/2$ and $\cos(\lambda_{n-1}) = \sqrt{1 - x_{n+1}^2} \neq 0$. Substituting for $\cos(\lambda_{n-1})$ in the first n equations of (2.5)-(2.8) and dividing both sides by $\cos(\lambda_{n-1}) = \sqrt{1 - x_{n+1}^2}$ gives equations (2.9)-(2.12) where

$$y_1 = \frac{x_1}{\sqrt{1 - x_{n+1}^2}}, \; y_2 = \frac{x_2}{\sqrt{1 - x_{n+1}^2}}, \ldots, y_n = \frac{x_n}{\sqrt{1 - x_{n+1}^2}}. \tag{2.13}$$

Observe that

$$y_1^2 + y_2^2 + \cdots + y_n^2 = \frac{x_1^2 + \cdots + x_n^2}{1 - x_{n+1}^2} = 1 \tag{2.14}$$

because $x_1^2 + \cdots + x_n^2 = 1 - x_{n+1}^2$ due to the fact that $(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n$. Hence $(y_1, \ldots, y_n) \in \mathbb{S}^{n-1}$. By assumption, there exists a $(\theta, \lambda_1, \ldots, \lambda_{n-2}) \in D'$ such that equations (2.9)-(2.12) are satisfied. It follows that the equations (2.5)-(2.8) are satisfied using $(\theta, \lambda_1, \ldots, \lambda_{n-1})$. Therefore we have found $(\theta, \lambda_1, \ldots, \lambda_{n-1}) \in D$ such that $f(\theta, \lambda_1, \ldots, \lambda_{n-1}) = (x_1, \ldots, x_{n+1})$.

In either case, we have found a $(\theta, \lambda_1, \ldots, \lambda_{n-1}) \in D$ such that $f(\theta, \lambda_1, \ldots, \lambda_{n-1}) = (x_1, \ldots, x_{n+1}) \in \mathbb{S}^n$. So $(x_1, \ldots, x_{n+1}) \in f(D)$. Because $(x_1, \ldots, x_{n+1}) \in S^n$ was arbitrary, shows that $f(D) \supseteq \mathbb{S}^n$. Because we have $f(D) \subseteq \mathbb{S}^n$ and $f(D) \supseteq \mathbb{S}^n$, then $f(D) = \mathbb{S}^n$. Since D is compact and connected, $f(D)$ is compact and connected because f is continuous. Therefore, $\mathbb{S}^n$ is compact and connected. ■

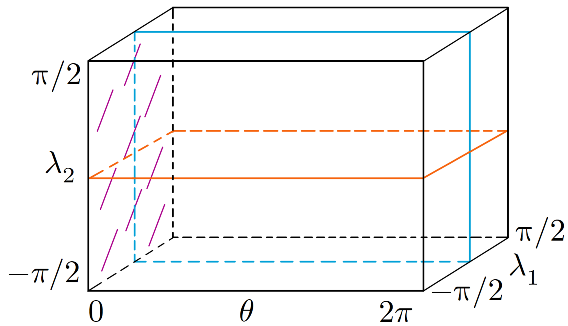
Now, we discuss how to visualize $\mathbb{S}^3$. Recall that earlier in the section, we learned that $\mathbb{S}^1 \cong \mathbb{R} \cup \{\infty\}$ and $\mathbb{S}^2 \cong \mathbb{R}^2 \cup \{\infty\}$ using stereographic coordinates. Naturally, this extends to n-dimensions, where $\mathbb{S}^n \cong \mathbb{R}^n \cup \{\infty\}$. So, when $n = 3$, $\mathbb{S}^3$ is homeomorphic to the one point compactification $\mathbb{R}^3 \cup \{\infty\}$. First, we extend the stereographic coordinates for $\mathbb{S}^2$ from equation (2.2) to $\mathbb{R}^n$, giving

$$\left(\frac{\|\mathbf{t}\|^2 - 1}{1 + \|\mathbf{t}\|^2}, \frac{2\mathbf{t}}{1 + \|\mathbf{t}\|^2} \right), \tag{2.15}$$

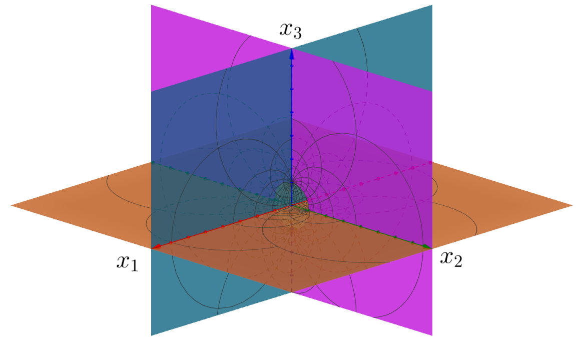
where $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$. So when $n = 3$, we get

$$\left(\frac{t_1^2 + t_2^2 + t_3^2 - 1}{1 + t_1^2 + t_2^2 + t_3^2}, \frac{2t_1}{1 + t_1^2 + t_2^2 + t_3^2}, \frac{2t_2}{1 + t_1^2 + t_2^2 + t_3^2}, \frac{2t_3}{1 + t_1^2 + t_2^2 + t_3^2} \right). \tag{2.16}$$

How do we visualize $\mathbb{S}^3$ using stereographic projections? We look to Figure 2.5. In Figure 2.5a, we see the domain $[0, 2\pi] \times [-\pi/2, \pi/2]^2$, a rectangular prism, and in Figure 2.5*b*, we see the image of the purple, blue, and orange planes from Figure 2.5a under stereographic projection. The purple, blue, and orange planes correspond to $\theta = 0$, $\lambda_1 = 0$, and $\lambda_2 = 0$, respectively. We can see what happens to our projection as we vary θ, λ_1, and λ_2 in Figures 2.6a - 2.6f. Figure 2.6a shows what happens when $0 < \theta < \pi$ and Figure 2.6b shows what happens when $\theta = \pi$. Notice that as θ begins at 0 in Figure 2.5b and progresses to π in Figure 2.6b, the purple plane wraps itself towards the positive x_1 axis until it collapses in on itself, forming a unit circle in the x_2, x_3-plane. Once $\theta > \pi$, the purple plane expands outwardly in the negative x_1 direction as a balloon expands until it looks like the plane from Figure 2.5b. For Figures 2.6c and 2.6d, we see what happens when $0 < \lambda_1 < \pi/2$ and



(a) Longitudes and latitudes of $\mathbb{S}^3$.

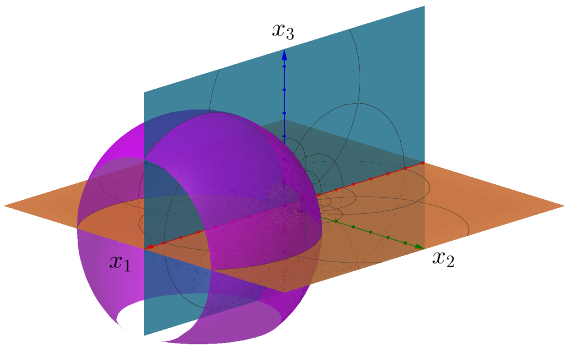


(b) This is a GeoGebra rendering of the stereographic of $\mathbb{S}^3$ in $\mathbb{R}^3$ when $\theta = 0$ (purple plane), $\lambda_1 = 0$ (blue plane), and $\lambda_2 = 0$ (orange plane).

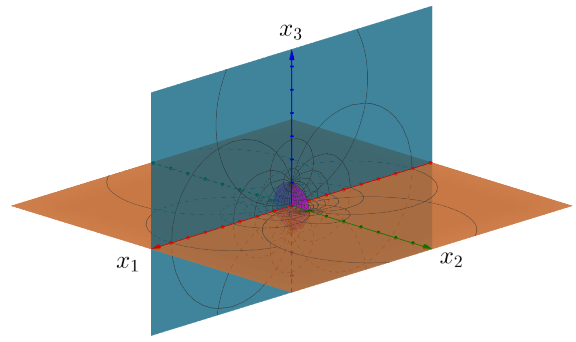
Figure 2.5: The stereographic projection of $\mathbb{S}^3$.

$-\pi/2 < \lambda_1 < 0$, respectively. As $\lambda_1 \to \pi/2$, the blue plane wraps itself towards the positive x_2 axis until it rolls itself into a smooth, 3-dimensional crescent, eventually collapsing onto the positive hemisphere of the purple unit circle in Figure 2.6b. The same happens in the negative direction as $\lambda_1 \to -\pi/2$. Lastly, as $\lambda_2 \to \pi/2$, the orange plane wraps itself towards the positive x_3 axis until it collapses to a point at the top of the upper hemisphere of the purple unit circle, and the same happens in the negative x_3 direction as $\lambda_2 \to -\pi/2$. Now, we have a comprehensive view of the stereographic projection of $\mathbb{S}^3$ into $\mathbb{R}^3$.

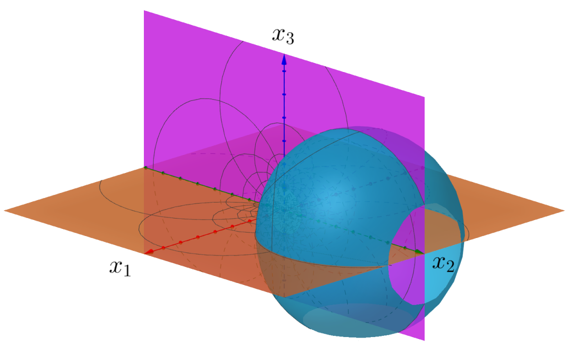
This section covered the latitude/longitude and stereographic coordinates for $\mathbb{S}^1$, $\mathbb{S}^2$, $\mathbb{S}^3$, and $\mathbb{S}^n$. We then proved that $\mathbb{S}^3$ is compact (closed and bounded) and that $\mathbb{S}^n$ is compact and connected. Afterward, we surveyed the stereographic projection of $\mathbb{S}^3$ and how each parameter changed under the stereographic projection, helping us understand how alterations affect the visual representation of the projection. Looking ahead, we will explore how Riemannian metrics define the shapes of spheres. These metrics provide insight into the intrinsic curvature properties that shape spheres across dimensions.



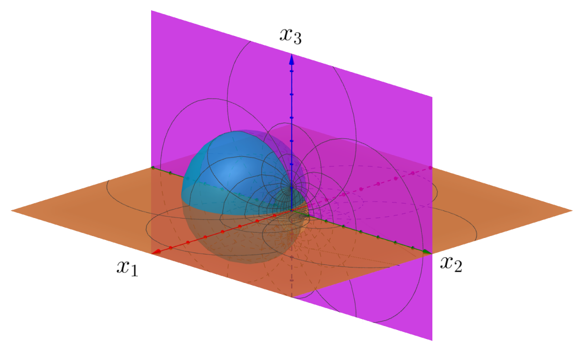
(a) $0 < \theta < \pi$



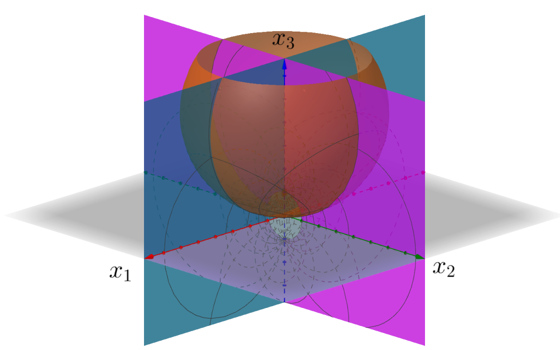
(b) $\theta = \pi$



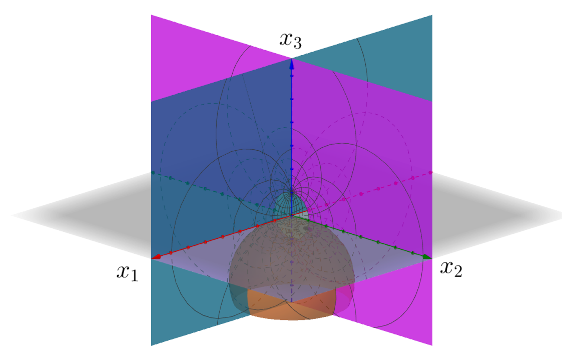
(c) $0 < \lambda_1 < \pi/2$



(d) $-\pi/2 < \lambda_1 < 0$



(e) $0 < \lambda_2 < \pi/2$



(f) $-\pi/2 < \lambda_2 < 0$

Figure 2.6: Varying θ, λ_1, and λ_2 under the stereographic projection.

2.3 Riemannian Metrics and Shape

In exploring spheres and their properties, we encountered two coordinate systems. We learned how to visualize $\mathbb{S}^3$ using stereographic coordinates and how we can think of $\mathbb{S}^n$ using longitude/latitude coordinates. Now, we investigate the framework underlying these shapes and their intrinsic properties. First, we develop the tools to define smooth manifolds, which describe spaces that locally resemble Euclidean space. Smooth manifolds extend our intuition from familiar geometric objects to more complex, abstract spaces. After defining smooth manifolds, we look at inner products and define Riemannian metrics on smooth manifolds, which endow smooth manifolds with notions of length and angles, giving us the ability to compute distances and curvature. Then, using Riemannian geometry, we will examine how metrics give spheres their shape, gaining insight into the geometry and topology of n-dimensional spheres.

We begin with the definition of a topology. A *topology* τ on a non-empty set X satisfies: (1) $\varnothing$ and X are in τ; (2) If V_α is in τ for all α in some index set Λ, then their union is in τ; (3) If V_i is in τ for $i = 1, ..., k$, then their intersection is in τ. The pair (X, τ) forms a *topological space* denoted as X, when there is no confusion. Elements of τ are *open sets*. A *neighborhood* of a point $p \in X$ is an open set containing p. A topological space has a *basis*, the smallest collection of open sets needed to define the topology. If this basis is countable, the topology is *second countable*. Two topological spaces are *homeomorphic* if there exists a bi-continuous bijection between them, and those spaces are *homeomorphic*. A space is *Hausdorff* if any two points in X have disjoint neighborhoods. An open cover for X is a collection of open sets whose union contains X.

Topology is concerned with the idea of stretching, shrinking, and deforming geometric objects without breaking them [1]. We need the ideas we just covered to understand the definition of a locally Euclidean topological space [8].

Definition 2.3.1. (Locally Euclidean of Dimension n) A topological space X is *locally Euclidean of dimension* n if every point p in X has a neighborhood U such that there is a homeomorphism ϕ from U onto an open subset of $\mathbb{R}^n$. We call the pair $(U, \phi : U \to \mathbb{R}^n)$ a *chart*, U a *coordinate neighborhood or open set*, and ϕ a *coordinate map or system* on U.

Now, we can define topological manifolds [8].

Definition 2.3.2. (Topological Manifold) A *topological manifold* is a Hausdorff, second countable, locally Euclidean space. It has dimension n if it is locally Euclidean of dimension n.

To define smooth manifolds, we need to know what compatible charts and atlases are. Two charts $(U, \phi : U \to \mathbb{R}^n)$ and $(V, \psi : V \to \mathbb{R}^n)$ of a topological space are C^∞-*compatible* if the two maps

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V), \quad \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

are C^∞ (smooth) [8]. A C^∞ *atlas* on a locally Euclidean space X is a collection $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}$ of pairwise C^∞-compatible charts that cover X, i.e., such that $M = \cup_\alpha U_\alpha$. An atlas $\mathcal{M}$ on a locally Euclidean space is *maximal* if it is not contained in a larger atlas. Now, we may define smooth manifolds.

Definition 2.3.3. (Smooth Manifold) A *smooth* or C^∞ *manifold* is a topological manifold X together with a maximal atlas. The maximal atlas is also called a *differentiable structure* on X. A manifold has dimension n if all its connected components have dimension n.

For example, a 1-dimensional manifold is a *curve*, a 2-dimensional manifold is a *surface* (like $\mathbb{S}^2$), and an n-dimensional manifold is an n-*manifold*.

Now, we define Riemannian metrics, Riemannian manifolds, and general metrics. Then, we explore some examples. To properly define Riemannian metrics and manifolds, we need to know what inner products and inner product spaces are, and for inner products we need vector spaces. Herein, we give brief descriptions of the key definitions and encourage the reader to explore [8, 9] for the detailed definitions. A *vector space* V over a field $\mathbb{F}$ is a collection of objects called vectors with operations called vector addition and scalar multiplication. The set V satisfies specific properties such as closure under addition and scalar multiplication, associativity, commutativity, distributivity, and the existence of additive and multiplicative identities. When $\mathbb{F} = \mathbb{R}$, then V is a *real vector space*. An *inner product* on a real vector space V is a function $\langle \cdot, \cdot \rangle$ from $V \times V$ to $\mathbb{R}$ that is bilinear, symmetric, and positive definite, and an *inner product space* is the ordered pair $(V, \langle \cdot, \cdot \rangle)$. Now, we define a Riemannian metric.

Definition 2.3.4. (Metric, Riemannian Metric, Riemannian Manifold) Suppose M is a smooth manifold. When we say g is a *metric* on M, we mean to say that g, with respect to some fixed point $p \in M$, is a function from the the Cartesian product of the tangent space to M at p with itself to $\mathbb{R}$, that is $g_p : T_pM \times T_pM \to \mathbb{R}$ satisfying the following properties:

1. g_p is *bilinear*; for all $u, v, w \in T_pM$ and scalars $a, b, c \in \mathbb{R}$, we have $g_p(au + bv, cw) = acg_p(u, w) + bcg_p(v, w)$ and $g_p(au, bv + cw) = abg_p(u, v) + acg_p(u, w)$,
2. g_p is *symmetric*; for all $u, v \in T_pM$, we have $g_p(u, v) = g_p(v, u)$, and
3. g_p is *positive definite*; for all $u \in T_pM$, we have $g_p(u, u) \geq 0$, with equality if and only if $u = 0$.

When the assignment $p \to g_p(X_p, Y_p)$ is smooth, or C^∞ for any two vector fields X and Y on M, then g_p is a *Riemannian metric*. A *Riemannian manifold* is the pair (M, g_p) consisting of a smooth manifold M and a Riemannian metric g_p [9].

Metrics allow us to define the notions of length and angle measure for abstract tangent vectors in T_pM, and the tangent vectors $u \in T_pM$ are equivalence classes of curves passing through p. Using g_p, we can define the length of $u \in T_pM$ as $\sqrt{g_p(u,u)}$ and the angle between arbitrary non-zero vectors $u, v \in T_pM$ as the value $\theta \in [0, \pi]$ such that

$$\cos(\theta) = \frac{g_p(u,v)}{\sqrt{g_p(u,u)}\sqrt{g_p(v,v)}}.$$

Metrics are helpful because they give manifolds their shape, among other features. Since manifolds are connected, non-compact topological spaces that are smooth, without boundary, and contractible, manifolds can potentially have the appearance of any object with these properties. Therefore, a manifold does not inherently have a "shape" until it has a corresponding Riemannian metric. For now, we continue exploring the geometry of spheres by looking at different types of curvature in the next section.

2.4 The Riemann Curvature Tensor, Ricci Curvature, and Scalar Curvature

In the last section, we developed the machinery to define Riemannian metrics and Riemannian manifolds. Now, we use that machinery to explore curvature. The first object that tells us about curvature is the Riemann curvature tensor. To define the Riemann curvature tensor, we briefly describe Einstein summation notation, affine connections, and the curvature operator. Einstein summation notation is a concise way to express mathematical expressions involving summation over repeated indices. In differential geometry, where computations often involve vectors, tensors (like the the metric), and their components, Einstein notation simplifies notation by implying summation over repeated indices. This notation conventionally omits explicit summation symbols, assuming summation over any index that appears twice, once as a subscript and once as a superscript. For example, in $\mathbb{R}^n$ with a vector $v = \sum_{i=1}^n v^i e_i$ with components $v^1, \ldots, v^n$ and basis vectors $e_1, \ldots, e_n$, we write $v = v^i e_i$ with Einstein notation. Furthermore, we can use the Einstein summation when writing inner products of vectors $x = x^i e_i$ and $y = y^i e_i$ in $\mathbb{R}^n$. Taking the inner product of x and y using our metric gives us $g(x, y) = g(x^i e_i, y^i e_i) = g(e^i, e^j)x^i y^j = g_{ij} x^i y^j$, where $g_{ij} = g(e_i, e_j)$ are the components of an $n \times n$ matrix A with entries $g_{ij} = g(e_i, e_j)$. For each index, when we sum, we add up n terms, or the dimension.

Turning to affine connections, we often study smooth functions defined on open subsets U of $\mathbb{R}^n$. Directional derivatives allow us to analyze how such functions change along vector fields, which are smooth assignments of tangent vectors to each point in U. However, more than a simple directional derivative is required when dealing with more general curved spaces. This generality demands we use more general operators called affine connections. An affine connection denoted ∇, can differentiate vector fields on a smooth manifold M. The covariant derivative, $\nabla_X(Y)$, acting on a vector field Y along a vector field X, encodes how Y changes in the direction of X along the manifold. We can use the covariant derivative to define the curvature operator

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

for vector fields X and Y. When we apply the curvature operator on a third vector field Z, we get another vector field $R(X, Y)Z$. The curvature operator is then used with the metric to define the Riemann curvature tensor [9].

Definition 2.4.1. (Riemann Curvature Tensor) If ∇ is the Levi-Cevita connection in TM, the tangent bundle of M, then $R(X, Y)$ is its curvature operator and the *Riemann curvature tensor* is $R(X, Y, Z, W) = g(W, R(X, Y)Z)$, where X, Y, Z, and W are vector fields on M and g is the Riemannian metric on M.

The Riemann curvature tensor tracks how much the order in which we take covariant derivatives matter. In flat space (like $\mathbb{R}^n$), the order commutes—$\nabla_X \nabla_Y Z = \nabla_Y \nabla_X Z$ for any vector fields X, Y, and Z. However, on a curved manifold, this commutativity may not hold. This non-commutativity is what the Riemann curvature tensor, denoted R, captures.

The Riemann curvature tensor measures the "curvature" of the manifold by quantifying the difference between commuting covariant derivatives and taking the covariant derivative of the Lie bracket of vector fields. The quantity $g(W, R(X,Y)Z)$ is the inner product of W with $R(X,Y)Z$. If W is a unit vector field, then $g(W, R(X,Y)Z)$ returns the scalar component of $R(X,Y)Z$ in the direction of W. If we compute the Riemann curvature tensor for coordinate vector fields e_i, e_j, e_k, and e_l, which we can think of as basis vectors for some finite dimensional vector space, we get $R(e_i, e_j, e_k, e_l) = R_{ijk}{}^{l}$.

The Riemann curvature tensor stores all the information necessary to obtain the other two curvature measures we are interested in, the Ricci curvature and the scalar curvature. Again, we leave the precise definitions to chapter 5 of [9]. The *Ricci curvature* is given by $R_{ik} = R_{ijk}{}^{j}$, and the *scalar curvature* is $R = g^{ik}R_{ik}$, where the upper indices for the metric refer to the entries of the inverse matrix A^{-1}. Notice that for the Ricci curvature, we are summing over the j index, and for the scalar curvature, we sum over both the i and k indices. The Ricci curvature is the contraction of the Riemann curvature tensor and the scalar curvature is the metric trace of the Ricci curvature, both at some point p in the manifold [9]. One last notion of curvature that we need for later is called the mean curvature. The mean curvature is an extrinsic measure of curvature for a manifold embedded in some ambient space. This means that the mean curvature is affected by how a manifold, which some Riemannian metric, sits in space. To understand the mean curvature of a surface, imagine you shrink wrap a bumpy surface. The mean curvature tells us, on average, how much we have to push or pull the wrap to conform to the bumps, where it would be positive for hills and negative for valleys.

As an example, to compute the Riemann curvature tensor, the Ricci curvature, and scalar curvature of a 2-sphere with radius r, we use the local parametrization

$$\mathbf{x}(\theta, \lambda) = (r\cos(\theta)\cos(\lambda), r\sin(\theta), \cos(\lambda), r\sin(\lambda))$$

by longitude $\theta \in [0, 2\pi]$ and latitude $\lambda \in [-\pi/2, \pi/2]$ of the sphere of radius $r > 0$ centered at the origin. Because we are dealing with the 2-sphere, our dimension is $n = 2$. So, there are $2^4 = 16$ components for the curvature tensor. This means the Ricci curvature has $s^2 = 4$ components and the scalar curvature has 2 components. To find the components of the curvature tensor, we use the coordinate equation of the Riemann curvature tensor

$$R_{ijk}{}^{l} = \frac{\partial \Gamma^l_{ik}}{\partial x^j} - \frac{\partial \Gamma^l_{jk}}{\partial x^i} + \left(\Gamma^p_{ik} \Gamma^l_{jp} - \Gamma^p_{jk} \Gamma^l_{ip} \right), \tag{2.17}$$

where the Γ terms are *Christoffel Symbols*. Christoffel symbols give us a way to describe affine connections in a way depending on our local coordinates [9]. So, the non-zero Christoffel symbols for the 2-sphere are $\Gamma^2_{11} = \frac{1}{2}\sin(2\lambda)$ and $\Gamma^1_{21} = \Gamma^1_{12} = -\tan(\lambda)$ in longitude/latitude coordinates. While there are 16 components of the Riemann curvature tensor, the curvature tensor is skew-symmetric, where $R_{ijk}{}^{l} = -R_{ijk}{}^{l}$. This skew-symmetry causes 8 of the components to be zero. Using equation (2.17), we find that 4 more of the components of 0, and using the skew-symmetry of the tensor, we find that $R_{121}{}^{2} = \cos^2(\lambda) = -R_{211}{}^{2}$ and $R_{122}{}^{1} = -1 = R_{212}{}^{1}$. The components of the Ricci curvature are given by $R_{ik} = R_{ijk}{}^{j} = R_{i1k}{}^{1} + R_{i2k}{}^{2}$, which is symmetric. We get $R_{11} = \cos^2(\lambda)$, $R_{12} = R_{21} = 0$, and $R_{22} = 1$. To calculate the scalar curvature, we need the components of the inverse matrix representing the metric, which we can find from the partial derivatives of the local parametrization. We find that

$$[g_{ij}] = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_\theta \cdot \mathbf{x}_\theta & \mathbf{x}_\theta \cdot \mathbf{x}_\lambda \\ \mathbf{x}_\lambda \cdot \mathbf{x}_\theta & \mathbf{x}_\lambda \cdot \mathbf{x}_\lambda \end{bmatrix} = \begin{bmatrix} r^2\cos^2(\lambda) & 0 \\ 0 & r^2 \end{bmatrix}, \tag{2.18}$$

and

$$[g^{ij}] = [g_{ij}]^{-1} = \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{r^2\cos^2(\lambda)} & 0 \\ 0 & \frac{1}{r^2} \end{bmatrix}. \tag{2.19}$$

The scalar curvature is $R = g^{ik}R_{ik} = g^{11}R_{11} + g^{12}R_{12} + g^{21}R_{21} + g^{22}R_{22} = 2/r^2$. The scalar curvature is the weakest measure of curvature for a manifold. For the case of the 2-sphere of radius r centered at the origin, the scalar curvature is equal to twice the Gaussian curvature $K = 1/r^2$. Now that we have calculated the Riemann curvature tensor, the Ricci curvature, and scalar curvature, we end the chapter with a discussion of the shape of hypersurfaces using the shape operator and some examples of hypersurfaces in $\mathbb{R}^3$ and $\mathbb{S}^3$.

2.5 The Shape Operator and Example Hypersurfaces

In the last section, we briefly explored Riemannian geometry, curvature, and how to calculate the Riemann curvature tensor, the Ricci curvature, and the scalar curvature. In this section, we define the shape operator. To do so requires a discussion of regular points, regular values, normal vector fields, and hypersurfaces.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and $p \in \mathbb{R}^n$ be an arbitrary point. Then p is a *regular point* of f if at least one partial derivative $\partial f/\partial x^i(p)$ is nonzero. A point $q \in \mathbb{R}^n$ is a *regular value* if its preimage $f^{-1}(q)$ consists entirely of regular points; otherwise, it is a *singular value* [9]. A *hypersurface* in $\mathbb{R}^n$ is the zero set $Z(f)$ of a C^∞ (smooth) function f on $\mathbb{R}^n$. For example, consider a hypersurface $M \subseteq \mathbb{R}^3$ defined as the zero set of the smooth function $f(x, y, z)$. Assuming its partial derivatives are not zero simultaneously on M, 0 would be a regular of f, meaning $M = f^{-1}(0)$ is a regular level set, and thus a regular submanifold of $\mathbb{R}^3$. This means that M inherits the Euclidean metric on $\mathbb{R}^3$ [9]. If we let $N = \nabla f = \langle f_x, f_y, f_z\rangle$ be the gradient vector field of f on M and 0 a regular value of f, then N is a nowhere-vanishing normal vector field along the smooth hypersurface $M = Z(f)$ [9]. The *normal vector field* N assigns to every point on M a vector perpendicular or normal to the surface. If we divide N by its length, we get a unit normal vector field. Now, we may define the shape operator.

Definition 2.5.1. (Shape Operator) Let p be a point on a hypersurface M in $\mathbb{R}^n$ and $X_p \in T_pM$ be a tangent vector. Then the *shape operator*, or *Weingarten map*, $s : T_pM \to T_pM$ is defined to be $s(X_p) = -D_{X_p}N$, where D_{X_p} is a special directional derivative operator in the direction of X_p called the Riemannian connection on $\mathbb{R}^n$ satisfying

(i) $D_XY - D_YX - (XY - YX) = 0$,

(ii) $D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z = 0$,

(iii) $Xg(Y, Z) = g(D_X Y, Z) + g(Y, D_X Z)$,

where X, Y, Z are C^∞ vector fields on $\mathbb{R}^n$.

The shape operator depends on the unit normal vector field and the point p in M. It encodes in it information about the hypersurface's curvature at p in a local manner whereas the Riemann curvature tensor encodes curvature information in a global manner. More specifically, the unit eigenvectors of the the shape operator are the principal directions of the hypersurface M at p and the eigenvalues of the shape operator are the principal curvatures of the hypersurface M at p. Because the shape operator is self-adjoint, the eigenvalues are real.

As an example computation, let us calculate the shape operator for the manifold M being the 2-sphere of radius R centered at the origin. Using the longitude and latitude coordinates, we have local parametrization

$$\mathbf{x}(\theta, \lambda) = (R\cos(\theta)\cos(\lambda), R\sin(\theta)\cos(\lambda), R\sin(\lambda))$$

for $\theta \in [0, 2\pi]$ and $\lambda \in [-\pi/2, \pi/2]$. To calculate the shape operator, we first need a basis for the tangent space T_pM for a point $p = \mathbf{x}(\theta, \lambda)$, which we choose to be the partial derivatives of $\mathbf{x}$ with respect to θ and λ

$$\begin{aligned}
\mathbf{x}_\theta(\theta, \lambda) &= (-R\sin(\theta)\cos(\lambda), \quad R\cos(\theta)\cos(\lambda), 0), \\
\mathbf{x}_\lambda(\theta, \lambda) &= (-R\cos(\theta)\sin(\lambda), -R\sin(\theta)\sin(\lambda), R\cos(\lambda)).
\end{aligned}$$

Now, to get the unit normal vector field N, we take the cross product of $\mathbf{x}_\theta$ and $\mathbf{x}_\lambda$. Because the shape operator s is a linear operator on T_pM, then we can completely characterize s by evaluating it on the basis vectors of T_pM. So, we must compute $s(\mathbf{x}_\theta) = -D_{\mathbf{x}_\theta} N$ and

$s(\mathbf{x}_\lambda) = -D_{\mathbf{x}_\lambda} N$, which gives us

$$s(\mathbf{x}_\theta) = -D_{\mathbf{x}_\theta} N = -\frac{\partial N}{\partial \theta} = (\sin(\theta)\cos(\lambda), -\cos(\theta)\cos(\theta), 0)$$
$$s(\mathbf{x}_\lambda) = -D_{\mathbf{x}_\lambda} N = -\frac{\partial N}{\partial \lambda} = (\cos(\theta)\sin(\lambda), \quad \sin(\theta)\sin(\lambda), -\cos(\lambda))$$

Notice that $s(\mathbf{x}_\theta) = -\frac{1}{R}\mathbf{x}_\theta$ and $s(\mathbf{x}_\lambda) = -\frac{1}{R}\mathbf{x}_\lambda$. Furthermore, because s is a linear operator on T_pM, there are two consequences: (1) for any tangent vector $X \in T_pM$, we can write $s(X)$ as a linear combination of the basis vectors $\mathbf{x}_\theta$ and $\mathbf{x}_\lambda$, and (2) we can completely characterize s by a matrix A. Using these two consequences, we have $s(\mathbf{x}_\theta) = -\frac{1}{R}\mathbf{x}_\theta = a\mathbf{x}_\theta + b\mathbf{x}_\lambda$ and $s(\mathbf{x}_\lambda) = -\frac{1}{R}\mathbf{x}_\lambda = b\mathbf{x}_\theta + c\mathbf{x}_\lambda$, which means $a = -1/R$, $b = 0$, and $c = -1/R$. So, the matrix representation of the shape operator relative to the basis $\mathbf{x}_\theta, \mathbf{x}_\lambda$ is

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} -\frac{1}{R} & 0 \\ 0 & -\frac{1}{R} \end{bmatrix}.$$

If we take the determinant of this matrix, we get the Gaussian curvature for a sphere $K = \det(A) = 1/R^2$. Having defined and computed the shape operator for the 2-sphere in longitude/latitude coordinates, we conclude the section, and thus the chapter, with some examples of hypersurfaces in $\mathbb{R}^3$ and $\mathbb{S}^3$.

Let M be the smooth manifold given by $\mathbb{R}^2 = \{(u,v) : u, v \in \mathbb{R}\}$. Consider the metrics

$$g_1 = (du)^2 + (dv)^2$$

and

$$g_2 = \left(1 + 4u^2\right)(du)^2 + 2(4uv)du\ dv + \left(1 + 4v^2\right)(dv)^2.$$

We demonstrate the local parametrization $\phi(u,v) = (u,v,c)$, where c is a constant, is an isometric embedding of (M, g_1) as a plane in $\mathbb{R}^3$ and $\psi(u,v) = (u,v,u^2+v^2+k)$, for another constant k, is an isometric embedding of (M, g_2) as a paraboloid in $\mathbb{R}^3$. An *isometry* is a function between metric spaces that preserves the notion of length.

Example 1: $\phi : M \to \mathbb{R}^3$, defined by $\phi(u,v) = (u,v,c)$, is an isometric embedding of (M, g_1) as a plane in $\mathbb{R}^3$.

Proof. To demonstrate that $\phi : N \to \mathbb{R}^3$ is an isometric embedding of (N, g_1) as a plane in $\mathbb{R}^3$, we must demonstrate

(i) ϕ is differentiable

(ii) ϕ is a homeomorphism onto $\phi(N)$

(iii) the differential of ϕ, namely $[d\phi_p]$, has full column rank at all points $p \in N$

(iv) the standard metric of $(\mathbb{R}^3, g_{\mathbb{R}^3})$, given by $g_{\mathbb{R}^3} = (dx)^2 + (dy)^2 + (dz)^2$, returns g_1 when we make the substitutions $x = u$, $y = v$, and $z = c$.

First, notice that ϕ is differentiable on N because its component functions are differentiable on N. This implies ϕ is continuous on N. The image of ϕ is given by $\phi(N) = \{p \in \mathbb{R}^3 \mid \exists (u, v) \in N \ni \phi(u, v) = p\} \subseteq \mathbb{R}^3$. Let $(x, y, z) = \phi(u, v) \in \mathbb{R}^3$ be an arbitrary point. If $x = u$, $y = v$, and $z = c$ for $(u, v) \in N$, then $(u, v, c) \in \phi(N)$. Therefore, ϕ restricts to $\phi : N \to \phi(N)$, which is surjective. Now, suppose that $\phi(a_1, b_1) = \phi(a_2, b_2) \in \phi(N)$, where (a_1, b_1) and (a_2, b_2) are arbitrary elements of N. Then $(a_1, b_1, c) = (a_2, b_2, c)$, implying $a_1 = a_2$ and $b_1 = b_2$. Therefore, $\phi_1 : N \to \phi(N)$ is one-to-one. So, ϕ is bijective and continuous. Notice that $\phi^{-1}(x, y, z)$ is continuous from $\phi(N)$ to N because it is a projection given by $\phi^{-1}(x, y, z) = (x, y)$ for arbitrary $(x, y, z) \in \phi(N)$. Because $\phi : N \to \phi(N)$ is bijective and continuous with a continuous inverse, then ϕ is a homomorphism onto $\phi(N)$. The differential of ϕ at $p \in N$ is

$$[d\phi_p] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

which is linearly independent for all $p \in N$, with full column rank. Lastly, let $x = u$, $y = v$, and $z = c$. Substituting these into the metric $g_{\mathbb{R}^3}$ gives $g_{\mathbb{R}^3} = (dx)^2 + (dy)^2 + (dz)^2 = (du)^2 + (dv)^2 + 0 = (du)^2 + (dv)^2 = g_1$. Therefore, in satisfying conditions (i) through (iv), we have that ϕ is an isometric embedding of (N, g_1) as a plane in $\mathbb{R}^3$. ■

Example 2: $\psi : N \to \mathbb{R}^3$, defined by $\psi(u,v) = (u, v, u^2 + v^2 + k)$, is an isometric embedding of (N, g_2) as a paraboloid in $\mathbb{R}^3$.

Proof. Notice that ψ is differentiable on N because its component functions are differentiable on N. It follows that ψ is continuous. Let $(x, y, z) = \psi(u, v) \in \mathbb{R}^3$ be an arbitrary point. If $x = u$, $y = v$, and $z = u^2 + v^2 + k$ for $(u, v) \in N$, then $(u, v, u^2 + v^2 + k) \in \psi(N) = \{p \in \mathbb{R}^3 \mid \exists (u, v) \in N \ni \psi(u, v) = p\}$. Therefore, $\psi : N \to \psi(N)$ is surjective onto its image. Now, let (a_1, b_1) and (a_2, b_2) be arbitrary points in N and suppose that $\psi(a_1, b_1) = \psi(a_2, b_2)$. Then $(a_1, b_1, a_1^2 + b_1^2 + k) = (a_2, b_2, a_2^2 + b_2^2 + k)$. This implies that $a_1 = a_2$ and $b_1 = b_2$, further implying the equality of $a_1^2 + b_1^2 + k$ and $a_2^2 + b_2^2 + k$. So, $\psi : N \to \mathbb{R}^3$ is injective. Therefore, ψ is bijective and continuous. Now, notice that $\psi^{-1}(x, y, z) = (x, y)$, which is continuous because it is a projection, so $\psi^{-1} : \psi(N) \to N$ is continuous. Therefore, ψ is a homeomorphism onto $\psi(N)$. Considering the differential $[d\psi_p]$, we have

$$[d\psi_p] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2u & 2v \end{bmatrix},$$

which is linearly independent for all $p = (u, v) \in N$. Lastly, because $x = u$, $y = v$, and $z = u^2 + v^2 + k$, then $dx = du$, $dy = dv$, and $dz = 2udu + 2vdv$. Therefore, substituting these values in the metric $g_{\mathbb{R}^3}$, we get

$$\begin{aligned} g_{\mathbb{R}^3} &= (dx)^2 + (dy)^2 + (dz)^2 \\ &= (du)^2 + (dv)^2 + (2udu + 2vdv)^2 \\ &= (du)^2 + (dv)^2 + 4u^2(du)^2 + 2(4uv)du\ dv + 4v^2(dv)^2 \\ &= \left(1 + 4u^2\right)(du)^2 + 2(4uv)du\ dv + \left(1 + 4v^2\right)(dv)^2 \\ &= g_2. \end{aligned}$$

Therefore, in satisfying conditions (i) through (iv), we have that ψ is an isometric embedding of (N, g_2) as a paraboloid in $\mathbb{R}^3$. ∎

Now that we have seen some examples of hypersurfaces in $\mathbb{R}^3$, we derive the spherical metric for the 3-sphere in stereographic coordinates and longitude-latitude coordinates. Let $U = \mathbb{R}^3$ be our hypersurface in $\mathbb{S}^3 \subseteq \mathbb{R}^4$. Denote by $\mathbf{x} : U \to \mathbb{S}^3$, with $\mathbf{x}(U) = \mathbb{S}^3 \backslash \{(0,0,0,1)\}$, be the stereographic projection $\mathbf{x}(x, y, z) = (a, b, c, h)$ from equation (2.16) where $t_1 = x$, $t_2 = y$, $t_3 = z$, and

$$a = \frac{2x}{1+\sigma^2}, \quad b = \frac{2y}{1+\sigma^2}, \quad c = \frac{2z}{1+\sigma^2}, \quad h = \frac{\sigma^2 - 1}{1+\sigma^2}, \quad \sigma^2 = x^2 + y^2 + z^2. \tag{2.20}$$

The spherical metric on $\mathbb{S}^3$ is given by the equation $g_{\mathbb{S}^3} = (da)^2 + (db)^2 + (dc)^2 + (dh)^2$. To compute $g_{\mathbb{S}^3}$ in terms of the flat metric $g_{\mathbb{R}^3} = (dx)^2 + (dy)^2 + (dz)^2$ on $\mathbb{R}^3$, we compute the differentials of a, b, c, and d, giving us

$$\begin{aligned}
da &= \frac{2}{(1+\sigma^2)^2} \left(dx \left(1+\sigma^2\right) - xd\left(\sigma^2\right)\right), \\
db &= \frac{2}{(1+\sigma^2)^2} \left(dy \left(1+\sigma^2\right) - yd\left(\sigma^2\right)\right), \\
dc &= \frac{2}{(1+\sigma^2)^2} \left(dz \left(1+\sigma^2\right) - zd\left(\sigma^2\right)\right), \\
d(\sigma^2) &= \frac{2d\left(\sigma^2\right)}{(1+\sigma^2)^2}.
\end{aligned}$$

Squaring each differential and adding them together gives

$$\begin{aligned}
g_{\mathbb{S}^3} &= (da)^2 + (db)^2 + (dc)^2 + (dh)^2 \\
&= \frac{4}{(1+\sigma^2)} \Big[\left(1+\sigma^2\right)^2 \left((dx)^2 + (dy)^2 + (dz)^2\right) - \\
&\quad \left(1+\sigma^2\right) \left[d\left(\sigma^2\right)\right]^2 + \sigma^2 \left[d\left(\sigma^2\right)\right] + \left[d\left(\sigma^2\right)\right] \Big] \\
&= \frac{4}{(1+\sigma^2)} \Big[\left(1+\sigma^2\right)^2 \left((dx)^2 + (dy)^2 + (dz)^2\right) - \left(1+\sigma^2\right) \left[d\left(\sigma^2\right)\right]^2 + \left(1+\sigma^2\right) \left[d\left(\sigma^2\right)\right] \Big] \\
&= \frac{4}{\left(1+\sigma^2\right)^4} \Big[\left(1+\sigma^2\right)^2 \left((dx)^2 + (dy)^{+}(dz)^2\right) \Big] \\
&= \frac{4}{\left(1+\sigma^2\right)^2} \left((dx)^2 + (dy)^2 + (dz)^2\right) \\
&= \frac{4 g_{\mathbb{R}^3}}{\left(1+\sigma^2\right)^2}.
\end{aligned}$$

If we were to perform this computation for $\mathbb{S}^n$, we would find that the spherical metric has the same form but instead with $\sigma^2 = \sum_{i=1}^n x_i^2$ and our flat metric on $\mathbb{R}^n$ as $g_{\mathbb{R}^n} = \sum_{i=1}^n (dx_i)^2$, where $x_1, \ldots, x_n$ are the coordinates on $\mathbb{R}^n$.

To obtain the spherical metric in longitude/latitude coordinates, we let $D = [0, 2\pi] \times [-\pi/2, \pi/2]^2 \subseteq \mathbb{R}^3$ and let $f_3 : D \to \mathbb{S}^3$ be the parametrization $f_3(\theta, \lambda_1, \lambda_2) = (x_1, x_2, x_3, x_4)$ from Section 2.2, where

$$\begin{aligned}
x_1 &= \cos(\theta)\cos(\lambda_1)\cos(\lambda_2) \\
x_2 &= \sin(\theta)\cos(\lambda_1)\cos(\lambda_2) \\
x_3 &= \sin(\lambda_1)\cos(\lambda_2) \\
x_4 &= \sin(\lambda_2).
\end{aligned}$$

Now, we take the differential of the coordinates to get

$$\begin{aligned}
dx_1 &= -\sin(\theta)\cos(\lambda_1)\cos(\lambda_2)d\theta - \cos(\theta)\sin(\lambda_1)\cos(\lambda_2)d\lambda_1 - \cos(\theta)\cos(\lambda_1)\sin(\lambda_2)d\lambda_2 \\
dx_2 &= \quad \cos(\theta)\cos(\lambda_1)\cos(\lambda_2)d\theta - \sin(\theta)\sin(\lambda_1)\cos(\lambda_2)d\lambda_1 - \sin(\theta)\cos(\lambda_1)\sin(\lambda_2)d\lambda_2 \\
dx_3 &= \quad \cos(\lambda_1)\cos(\lambda_2)d\lambda_1 - \sin(\lambda_1)\sin(\lambda_2)d\lambda_2 \\
dx_4 &= \quad \cos(\lambda_2)d\lambda_2.
\end{aligned}$$

Squaring all four differentials and adding them together gives us the spherical metric

$$\begin{aligned}
g'_{\mathbb{S}^3} &= (dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_4)^2 \\
&= \left(\cos^2(\lambda_1)\cos^2(\lambda_2)\right)(d\theta)^2 + \left(\cos^2(\lambda_2)\right)(d\lambda_1)^2 + (d\lambda_2)^2 .
\end{aligned}$$

If we let $\lambda_2 = 0$ for x_4, we find that $g'_{\mathbb{S}^3}$ reduces to the metric for the 2-sphere $g'_{\mathbb{S}^2} = (dx_1)^2 + (dx_2)^2 + (dx_3)^2 = (\cos^2(\lambda_1))\,(d\theta))^2 + (d\lambda_1)^2$ and for the unit circle, we have $g'_{\mathbb{S}^1} = (dx_1)^2 + (dx_2)^2 = (d\theta)^2$. While it may be possible to demonstrate with induction that for $\mathbb{S}^n$, the metric is given by

$$g'_{\mathbb{S}^n} = \left(\prod_{i=1}^{n-1}\cos^2(\lambda_i)\right)(d\theta)^2 + \sum_{i=1}^{n-1}\left(\prod_{k=i+1}^{n-1}\cos^2(\lambda_k)\right)(d\lambda_i)^2 , \tag{2.21}$$

which works for $n = 1$, $n = 2$, and $n = 3$, the detailed mathematical proof of this result is deferred to future endeavors, allowing for a more comprehensive examination of the proof in the next chapter, where we explore the proof of the theorem of interest in this paper.

Chapter 3

A Rigidity Theorem

3.1 The Precise Rigidity Theorem

In Chapter 1, we discussed rigidity theorems and why they are important to differential geometry, and in Chapter 2, we learned about the n-sphere $\mathbb{S}^n$ and how we can think about it. Now, we restate Theorem 1.2 from [6] here with the additional machinery we will need to examine the proof of the theorem.

Theorem 3.1.1. Let $n \geq 2$. Let M be a connected, embedded, two-sided hypersurface in $\mathbb{S}^{n+1}$ with boundary ∂M and scalar curvature function R. Suppose $\mathrm{int}(M)$ is C^{n+1} and M is C^1 up to boundary. Suppose M and ∂M satisfy the following conditions:

(1) M satisfies $R \geq n(n-1)$;

(2) ∂M is contained in $\mathbb{S}^{n+1}$;

(3) M is tangent to the equatorial hemisphere $\mathbb{S}^n_+$ at ∂M from the region enclosed by ∂M.

Then M is a portion of the hemisphere $\mathbb{S}^n_+$.

As a reminder, the theorem states that if a hypersurface M and its boundary ∂M in $\mathbb{S}^{n+1}$ satisfy the curvature condition $R \geq n(n-1)$ and the boundary condition that ∂M is tangent to $\mathbb{S}^n_+$ in a special way, respectively, then M must be a portion of the upper half of the equatorial sphere $\mathbb{S}^n_+$. To prove Theorem 3.1.1, we need several lemmas and propositions, a new definition, and a corollary. We state them here and explore what they mean. The first important result is Lemma 3.1.2.

Lemma 3.1.2. Let M be a C^2 hypersurface in $(\mathbb{R}^{n+1}, g_S)$. Let $\Sigma = M \cap \{x^{n+1} = \epsilon\}$ be a regular level set. Let ν and η be unit normal vectors to $M \subset (\mathbb{R}^{n+1}, g_0)$ and $\Sigma \subset \left(\mathbb{R}^n \times \{x^{n+1} = \epsilon\}, g_0|_{\{x^{n+1}=\epsilon\}}\right)$, respectively. Let H and H_Σ be the mean curvature scalars with respect to $\phi\nu$ and $\phi\eta$, respectively. Then

$$\begin{aligned}
&H\left[g_0(\nu,\eta)H_\Sigma + (n-1)g_0\left(\nu, \partial_{n+1}\right)x^{n+1}\right] \\
&\geq \frac{1}{2}[R - n(n-1)] + \frac{n}{2(n-1)}\left[g_0(\nu,\eta)H_\Sigma + (n-1)g_0\left(\nu, \partial_{n+1}\right)x^{n+1}\right]^2,
\end{aligned}$$

where the equality holds at $p \in \Sigma$ if and only if the following conditions hold at $p \in \Sigma$:

(i) Σ is umbilic in $(\mathbb{R}^n \times \{\epsilon\}, g_S)$. We denote the principal curvature of Σ by κ.

(ii) M has a principal curvature $g_0(\nu,\eta)\kappa + g_0\left(\nu, \partial_{n+1}\right)\phi_{n+1}$ in $(\mathbb{R}^{n+1}, g_S)$ with multiplicity at least $n-1$.

The above lemma connects differential geometry with analysis. However, because we are looking at the geometric aspects of rigidity theorems, we reserve the details involved in lemma 3.1.2 to the [6]. Instead, we provide an overview. First, to set the scene, we begin with a C^2 hypersurface M in $(\mathbb{R}^{n+1}, g_0)$. This means that the dimension of M is n and it is twice continuously differentiable in $\mathbb{R}^{n+1}$. The space $(\mathbb{R}^{n+1})$ is imbued with a metric g_S, the spherical metric on $\mathbb{S}^{n+1}$ in stereographic coordinates, given by

$$g_S = \phi^{-2} g_0 \quad \text{where} \quad \phi = \frac{1 + \sum_{i=1}^{n+1}(x_i)^2}{2}, \tag{3.1}$$

where $g_0 = \sum_{i=1}^{n+1}(dx_i)^2$ is the standard metric on $\mathbb{R}^{n+1}$. Then, we take a "slice" of M by intersecting with the hyperplane $\varepsilon = x_{n+1}$. This gives us the new manifold $\Sigma = M \cap \{\varepsilon\}$, which has dimension n and lives in $\mathbb{R}^n \times \{\varepsilon\}$ with the induced metric from $g_{\mathbb{S}}$. This means Σ inherits the tools that allow one to measure distances and angles from the larger space $\mathbb{R}^{n+1}$. The unit normal vector to M in $(\mathbb{R}^{n+1}, g_S)$ and the unit normal vector to Σ in $\mathbb{R}^n \times \{\varepsilon\}$ with the induced metric are denotes ν and η, respectively. The quantities H and H_Σ represent the mean curvature scalars of M and Σ, respectively.

Now, the inequality involves the mean curvatures H and H_Σ, the metric terms, the $(n+1)^{\text{th}}$ term, and the dimension n. The inequality relates the curvature of the "slice" Σ to the curvature of the overall hypersurface M and the geometry determined by the metric. We can achieve equality at $p \in \Sigma$ if and only if Σ satisfies conditions (i) and (ii). Condition (i) states Σ must be umbilic, meaning all principal curvatures (the measures of curvature in all directions at p) are equal. Condition (ii) states the hypersurface M must have a principal curvature that matches a specific combination of the curvature of Σ and the geometry determined by the metric, with a multiplicity of at least $n-1$. We continue our survey of the important results with Lemma 3.1.3.

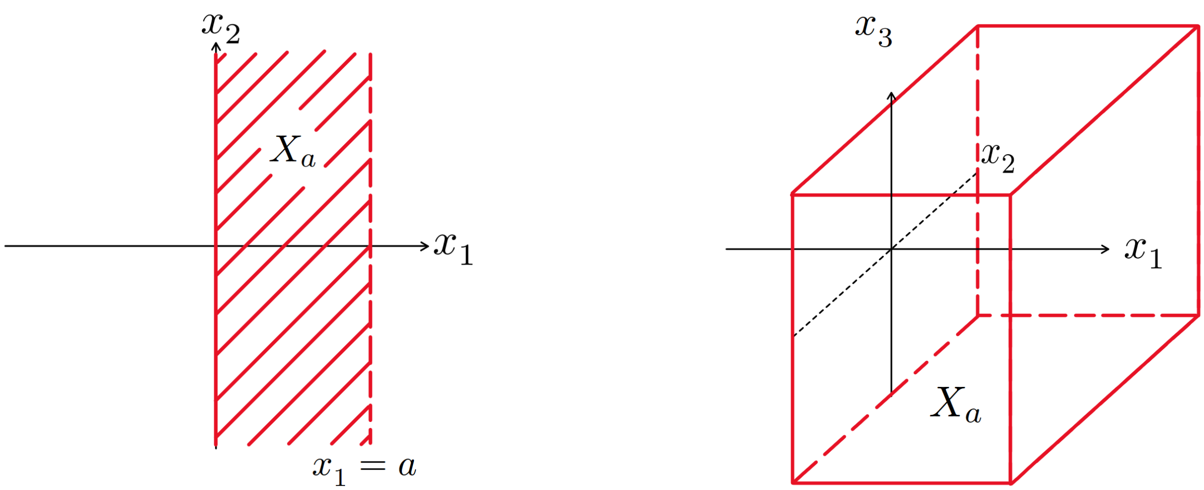
Lemma 3.1.3. Let W be an open subset in $X_a = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_1 < a\}$ for some $a > 0$, and let ∂W be the boundary of W in X_a. Let $u \in C^2(W) \cap C^1(\bar{W})$ satisfy that, for a constant vector $v \in \mathbb{R}^n, u(x) = v \cdot x$ and $Du = v$ on ∂W. If $H(u) \geq 0$ in W, where

$$\begin{aligned} H(u) =& \frac{1+|x|^2+u^2}{2} \sum_{i,j=1}^{n} \left(\delta_{ij} - \frac{u_i u_j}{1+|Du|^2}\right) \frac{u_{ij}}{\sqrt{1+|Du|^2}} \\ &+ \frac{n}{\sqrt{1+|Du|^2}} \left(u - \sum_{i=1}^{n} x_i u_i\right) \end{aligned}$$

is the mean curvature of the graph of $x_{n+1} = u\left(x_1, \ldots, x_n\right)$ in $(\mathbb{R}^{n+1}, g_S)$ with respect to the upward unit normal vector $\phi\nu$, where $\nu = \frac{(-Du,1)}{\sqrt{1+|Du|^2}}$, and

$$\phi = \frac{1+\sum_{i=1}^{n+1}\left(x_i\right)^2}{2},$$

then $u(x) > v \cdot x$ somewhere in W, unless $u \equiv v \cdot x$ in W.



(a) The set X_a in $\mathbb{R}^2$.

(b) The set X_a in $\mathbb{R}^3$

Figure 3.1: Visualizations of the set X_a from Lemma 3.1.3.

For this lemma, we start with a subset W of $X_a \subseteq \mathbb{R}^n$, which is the set of all points in $\mathbb{R}^n$ where the first coordinate x_1 is between 0 and some positive real number a. In $\mathbb{R}^2$, X_a is the portion of the right half-plane between 0 and a, and in $\mathbb{R}^3$, X_a is the portion of 3-space consisting of quadrants I, II, V, and VI, between 0 and a. We can see this in Figure 3.1. We denote by ∂W the boundary of W in X_a in the subspace topology of X_a, meaning $\partial W = X_a \cap \partial_{\mathbb{R}^n} W$, where $\partial_{\mathbb{R}^n} W$ is the boundary of W in $\mathbb{R}^n$. Next, the function we let $u \in C^2(W) \cap C(\bar{W})$, which means that u is twice continuously differentiable on W and u is once continuously differentiable on the closure of W, which includes being differentiable on the boundary ∂W. This particular function u satisfies the condition that for some constant vector $v \in \mathbb{R}^n$, we have $u(x) = v \cdot x$ for all $x \in W$ and $Du = v$ on ∂W. Now, for the result of the lemma, if the mean curvature of u is non-positive, $H(u) \geq 0$, throughout the region W, then there must exist a point inside W where the value of u is strictly greater than the dot product of v and x at that point. So, if the graph bends outwards everywhere, or not at all, the function u must take on a value greater than the linear function defined by v at some point within the domain. The idea of "bending outwards" means that our graph balloons outwards like a parachute. The exception to this ballooning condition would be if $u(x)$ is identically equal to $v \cdot x$ throughout the region W, in which case the graph would be a flat plane and the condition on $H(u)$ being non-positive would be satisfied. Now, the next proposition is an extension of the lemma we just met, and imposes stronger conditions on the curvature of the graph of u.

Proposition 3.1.4. Let W be an open subset in X_a for some $a > 0$. Denote by ∂W the boundary of W in X_a. Let $u \in C^2(W) \cap C^1(\bar{W})$ be a bounded function that satisfies $u(x) = v \cdot x$ and $Du = v$ on ∂W for some $v \in \mathbb{R}^n$. Suppose that the graph of u in $(\mathbb{R}^{n+1}, g_S)$ satisfies $R \geq n(n-1)$. If either $H(u) \geq 0$ or $H(u) \leq 0$ everywhere in W, then $u(x) = v \cdot x$ in W, and in particular $H(u) \equiv 0$.

Notice that most of the conditions are the same except we require the scalar curvature R to satisfy $R \geq n(n-1)$, which indicates significant curvature since as n increases, the scalar curvature must be on the order of n^2. Now, if the mean curvature is non-positive or non-positive everywhere in W, then the function $u(x)$ is the constant linear function $v \cdot x$ in W and the mean curvature is identically zero. This means that the graph of u is a plane through the origin with normal v, which does not curve or bend anywhere in W. So, the graph of u is flat. However, this proposition does not guarantee that we achieve flatness on the boundary of the graph of u in W because of the condition that $u(x) = v \cdot x$ and $Du = v$ on ∂W for some $v \in \mathbb{R}^n$. The next tool we need is a definition that will help us with Remark 3.1.6. Although the definition has two parts in [6], we are primarily interested in part (2), which is related to condition (3) in Theorem 3.1.1.

Definition 3.1.5. Let M be a hypersurface in $\mathbb{S}^{n+1}$. Suppose that ∂M is contained in a great sphere $\mathbb{S}^n \subset \mathbb{S}^{n+1}$. Suppose that ∂M is contained in the hemisphere $\mathbb{S}^n_+$, then M is said to be tangent to $\mathbb{S}^n_+$ at ∂M from the the region enclosed by ∂M if ∂M encloses an open subset $V \subset \mathbb{S}^n_+$ so that M is locally the graph of a bounded function u in a collar neighborhood of ∂V in V with $u = 0, |Du| = 0$ on ∂V.

In this definition, we start with an n-dimensional surface M contained in $\mathbb{S}^{n+1}$, a hypersurface, with boundary ∂M contained in a great sphere $\mathbb{S}^n \subseteq \mathbb{S}^{n+1}$. We can think of the n-dimensional sphere as a "slice" of $\mathbb{S}^{n+1}$. The important condition that must be satisfied is that the boundary of M must be contained in the upper half of the great sphere $\mathbb{S}^n_+$. Now, we are defining "M is tangent to $\mathbb{S}^{n+1}$ at ∂M from the region enclosed by ∂M." In this definition, we are interested in how M touches the upper half of the great sphere, specifically from the "inside" of the boundary loop ∂M. For M to be tangent to the upper half of the great sphere, its boundary ∂M must enclose an open subset V of the the upper hemisphere, creating an open "pocket" inside the upper hemisphere. To understand this definition, we return to our illustration from Section 1.2. For this illustration, we have $n = 2$. We let the hypersurface M be a swim cap with essentially no thickness, the surface of the swimmer's head representative of the great sphere $\mathbb{S}^2$, and three-dimensional space with an extra point out at infinity, $\mathbb{R}^3 \cup \{\infty\}$, representative of $\mathbb{S}^3$. The upper "half" of the swimmer's head (above the nose) will represent $\mathbb{S}^n_+$. Now, as it stands, the swim cap is not necessarily on the upper half of the swimmer's head. However, to satisfy the condition that the boundary of the cap is part of the upper half of the swimmer's head, we need the edge of the cap to contact the upper half of the swimmer's head. With this, imagine a small pocket of air trapped between the cap and the swimmer's head. We can think of this region as the open set V. Taking a closer look at the cap's behavior near its boundary, we can approximate the cap in this local area by the graph of a function of two variables, like a height function depending on the latitude and longitude above the swimmer's nose. This function would be u. The function's value and gradient would be zero on the edge, where the cap meets the swimmer's forehead, corresponding to ∂V. Because the height function is zero, then the cap at the edge sits perfectly on the swimmer's forehead in this local area. This would be like the edge of the cap touching the swimmer's forehead without fitting snuggly. However, because the gradient, the first derivative of a real-valued function $u(x, y)$, is zero in this local region, this condition forces the cap to fit snuggly on the swimmer's forehead. In this state, the swimmer's cap is tangent to the swimmer's head at the boundary (edge) of the cap from the region enclosed (the air trapped between the swimmer's head and the cap) by the boundary. If you remember, this definition is related to condition (3) of Theorem 3.1.1. The next statement we address is Remark 3.1.6, which uses the definition we just explored.

Remark 3.1.6. If M satisfies (2) in Definition 3.1.5, then there exists a stereographic projection Φ such that $\mathbb{S}^n_+$ is mapped via Φ onto the closed half-space $\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \geq 0\}$, and there is an open subset $U \subset \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \geq 0\}$ such that $\Phi(\partial M) = \partial U$, and $\Phi(M)$ is locally a graph of a bounded function u in a collar neighborhood of ∂U in U such that $u = 0$ and $|Du| = 0$ on ∂U.

We need a hypersurface M satisfying the previous definition for this remark. Once we have this hypersurface, there exists a specific stereographic projection Φ such that Φ maps the upper hemisphere $\mathbb{S}^n_+$ onto the closed half-space $\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \geq 0\}$. This half-space is like a flat space with a boundary at $x_1 = 0$. Furthermore, Φ maps the boundary ∂M of the hypersurface onto the boundary ∂U of a subset of the half-space. Under Φ, the set U becomes the projected version of the open region V enclosed by ∂M. Most importantly, Φ preserves the local behavior of the hypersurface near the boundary. This means that the projected hypersurface $\Phi(M)$ in the half-space can also be locally represented by the graph of a function u whose properties are the same as those in Definition 3.1.5. This remark is used directly in the proof of Theorem 3.1.1. The next two statements are used to prove the last statement. The first is Corollary 3.1.7.

Corollary 3.1.7. Let M be a connected, embedded, two-sided hypersurface in $\mathbb{S}^{n+1}$ with boundary ∂M. Suppose $\mathrm{int}(M)$ is C^2 and M is C^1 to the boundary. Suppose M is tangent to the great sphere $\mathbb{S}^n$ along an open subset $\Gamma \subset \partial M$. If $p \in \Gamma$ is a strictly convex boundary point and, in a neighborhood of p in M, the scalar curvature $R \geq n(n-1)$ and the mean curvature is weakly convex, then a neighborhood of p in M is contained in $\mathbb{S}^n$.

For this corollary, we imagine a tightly stretched rubber sheet, which represents the hypersurface M, draped over a large ball, representing $\mathbb{S}^{n+1}$. The sheet needs to satisfy several conditions. The sheet must be whole with no holes (connected), smoothly integrated within the ball (embedded), have two distinct sides (two-sided), and smooth on the interior of the sheet ($\mathrm{int}(M)$ is C^2). The interior smoothly transitions to the edge (C^1 to the boundary). Also, the sheet must be tangent to $\mathbb{S}^n$ along an open subset γ of the boundary. Now, consider a point p on the edge where the sheet bulges outwards from the ball, representing the strictly convex boundary point. If the sheet's scalar curvature is greater than $n(n-1)$ and the mean curvature is weakly convex (a sort of quasi non-negativity), then in a small area around p the sheet cannot "bulge" out any further. The sheet must stay entirely within the surface of the ball. Now, we review the penultimate result Lemma 3.1.8.

Lemma 3.1.8. Let M be a C^{n+1} hypersurface in $\mathbb{S}^{n+1}$ (possibly with boundary). Let $M_0 = \{p \in \mathrm{int}(M) : A = 0 \text{ at } p\}$ be set of (interior) geodesic points, where A is the shape operator of M. Let M_0' be a nonempty connected component of M_0. Then M_0' is contained in a great n-sphere $\mathbb{S}^n$ of $\mathbb{S}^{n+1}$ so that M is tangent to the sphere $\mathbb{S}^n$ at every point of M_0'.

Here our hypersurface M, living in $\mathbb{S}^{n+1}$ is C^{n+1}, which suggests a high degree of smoothness. This means our hypersurface can be described by well-behaved functions up to the order of $n+1$ derivatives. Also, M possibly could have a boundary or edge. Now, we take a brief segue to explore the shape operator. The shape operator A is an operator that stores information about the intrinsic curvature of the hypersurface at a point. It describes how the hypersurface "bends" in different directions at that point, independent of how it is embedded into the higher-dimensional space. The set M_0 is the set of all points in the interior of M where the shape operator evaluated at those points is zero. At these points, the hypersurface locally resembles a flat plane regarding its intrinsic curvature. Now we look at a specific piece M_0' of M_0. This piece is a nonempty connected component of M_0. This means that M_0' is a single, continuous, non-empty chunk of the hypersurface that is not connected to other pieces through the sphere's interior. Under these conditions, the set M_0' must be completely contained within $\mathbb{S}^n$ so that M is tangent to $\mathbb{S}^n$ at every point of M_0'. Ultimately, Lemma 3.1.8 states that a smooth hypersurface in a unit sphere can have regions where the curvature vanishes. Any continuous chunk of these points must lie completely within a sphere $\mathbb{S}^n \subseteq \mathbb{S}^{n+1}$, and the hypersurface is tangent to $\mathbb{S}^n$ at all points within that chunk. Now, we examine our last result before understanding the proof of Theorem 3.1.1.

Proposition 3.1.9. Let W be an open subset in X_a for some $a > 0$. Denote by ∂W the boundary of W in X_a. Let $u \in C^{n+1}(W) \cap C^1(\bar{W})$ satisfy $u(x) = v \cdot x$ and $Du = v$ on ∂W for some $v \in \mathbb{R}^n$. If the graph of u in $(\mathbb{R}^{n+1}, g_S)$ satisfies $R \geq n(n-1)$, then $u(x) \equiv v \cdot x$ in W.

We start with identical conditions described in Lemma 3.1.3 and the same metric described after Lemma 3.1.2. The difference between this proposition and Lemma 3.1.3 is the conclusion. For this Proposition, if the graph of u in $(\mathbb{R}^{n+1}, g_S)$ has a scalar curvature R greater than or equal to $n(n-1)$, then $u(x)$ is identically equal to $v \cdot x$. This means that if the scalar curvature R is greater than or equal to $n(n-1)$, then u is a linear function, and its graph is a plane in $(\mathbb{R}^{n+1}, g_S)$ under the spherical metric. Now that we have surveyed all of our tools, we can tackle the proof of Theorem 3.1.1.

3.2 The Proof

We restate the theorem in question once more.

Theorem 3.1.1. Let $n \geq 2$. Let M be a connected, embedded, two-sided hypersurface in $\mathbb{S}^{n+1}$ with boundary ∂M. Suppose $\text{int}(M)$ is C^{n+1} and M is C^1 up to boundary. Suppose

M and ∂M satisfy the following conditions:

(1) M satisfies $R \geq n(n-1)$;

(2) ∂M is contained in $\mathbb{S}^{n+1}$;

(3) M is tangent to $\mathbb{S}^n_+$ at ∂M from the region enclosed by ∂M.

Then M is a portion of the hemisphere $\mathbb{S}^n_+$.

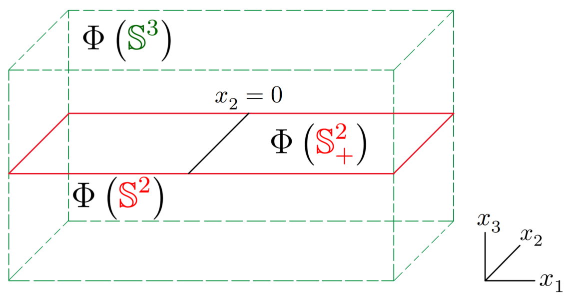
The proof in the paper is as follows.

Proof. The key is to choose a stereographic projection of Φ from $\mathbb{S}^n$ minus a point to $\mathbb{R}^n$ as in Remark 3.5 such that there is an open subset $U \subseteq \{(x_1, \ldots, x_n) : x_1 \geq 0\}$ where $\Phi(\partial M) = \partial U$ and $\Phi(M)$ is locally the graph of a function u on a collar neighborhood of U, $u = 0$, and $|Du| = 0$ on ∂U. The authors then consider the set

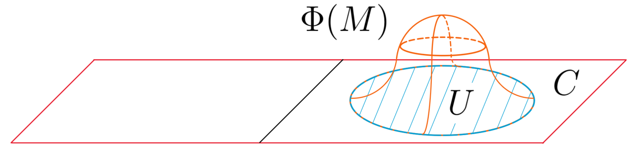
$$I = \big\{a \in \mathbb{R}^+ : u \equiv 0 \text{ on } U \cap X_a\big\} \tag{3.2}$$

The set I is closed by continuity of u. By Proposition 3.8, I is non-empty and open. This implies that $I = \mathbb{R}^+$ and hence $u \equiv 0$ on U; that is, M is contained in the great n-hemisphere. ■

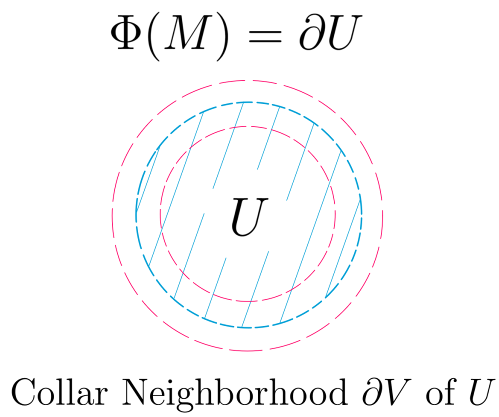
Our goal in this section is to understand the proof using the tools we developed in the previous section through the lens of what we learned in Chapter 2. The proof begins with using the stereographic projection from Remark 3.1.6. Recall that in Remark 3.1.6, if our hypersurface M satisfies Definition 3.1.5, then we can choose a particular stereographic projection Φ and the projection point such that Φ maps the upper half of the equatorial n-sphere $\mathbb{S}^n_+$ onto the closed half-space $C = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq 0\}$. Furthermore, there is an open subset $U \subset C$ such that the image of the boundary of the hypersurface $\Phi(\partial M)$ is equal to the boundary of U, ∂U. Further still, the image of the hypersurface $\Phi(M)$



(a) $\mathbb{S}^3$, $\mathbb{S}^2$, and $\mathbb{S}^2_+$ under stereographic projection Φ.



(b) The sets U, C, and $\Phi(M)$.



(c) The set U and the collar neighborhood around ∂U.

Figure 3.2: This is an illustration of Remark 3.1.6 for the proof.

is locally a graph of a bounded function u in a collar neighborhood of ∂U in U such that $u = 0$ and $|Du| = 0$ on ∂U.

To better understand what is happening in this remark, let us look at the case when $n = 2$. We are looking at a hypersurface M satisfying condition (2) in Definition 3.1.5. So, M is a 3-dimensional hypersurface living in $\mathbb{S}^3 \subseteq \mathbb{R}^4$, where its boundary ∂M lives in the upper equatorial hemisphere $\mathbb{S}^2_+$ and M is tangent to the upper equatorial 2-hemisphere $\mathbb{S}^2_+$ from the region enclosed by ∂M. We can see an illustration of $\mathbb{S}^3$, $\mathbb{S}^2$, and $\mathbb{S}^2_+$ under the stereographic projection from the point $(1, 0, 0, 0)$ in Figure 3.2a. The box's interior represents $\mathbb{R}^3$, and the surrounding dashed green box represents the additional point at infinity. So, we have $\mathbb{R}^3 \cup \{\infty\} \cong \mathbb{S}^3$. The red plane passing through the center of the box is the equatorial 2-sphere $\mathbb{S}^2$ under stereographic projection. Letting x_1 point to the right, x_2 into the page, and x_3 upwards, the right half of the equatorial sphere (the red plane) corresponds to the upper equatorial hemisphere $\mathbb{S}^2_+$. Notice that the right half of the red plane, including the x_2-axis, corresponds to the closed half-space C. Figure 3.2b shows a zoomed-in view of $\mathbb{S}^2$, $\mathbb{S}^2_+$, the open set $U \subset C$, and one possibility of the image of the hypersurface $\Phi(M)$. As a reminder, because ∂M is contained in $\mathbb{S}^2_+$, we have chosen a stereographic projection Φ and the projection point that maps $\mathbb{S}^2_+$ onto C, and there exists an open subset $U \subset C$ where the following results hold. The open subset U is contained in C such that the boundary of our hypersurface ∂M under our choice of stereographic projection Φ is equal to the boundary ∂U of our open set U. Furthermore, the image $\Phi(M)$ of our hypersurface M is locally a graph of a bounded function u in a collar neighborhood of ∂U in U where $u = 0$ and $|Du| = 0$ on ∂U. This means that $\Phi(M) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = u(x_1, x_2)\}$, at least for the points (x_1, x_2) in the collar neighborhood of ∂U. The collar neighborhood of U is an annular neighborhood that surrounds the boundary ∂U of U. We can see this Figure 3.2c. So, for points in the collar neighborhood of ∂U, we have $u(x_1, x_2) = 0$, which means the image $\Phi(M)$ of our hypersurface under our stereographic projection and the magnitude of the derivative $|Du|$

are equal to zero on ∂U. Not only does the image $\Phi(\partial M)$ of the boundary of M perfectly coincide with ∂U, since $u = 0$, but also the image $\Phi(M)$ of M seamlessly and smoothly transitions into the boundary ∂U. We can think of rotating a Gaussian curve about the x_3-axis to create a 2-dimensional bell curve, which tapers to zero in the radial direction. We can see this behavior in $\phi(M)$ in Figure 3.2b.

Now that we have set the stage let us progress to the second sentence of the proof. We choose the set $I = \{a \in \mathbb{R}^+ : u \equiv 0 \text{ on } U \cap X_a\}$, where $X_a = \{(x_1, \ldots, x_n) : 0 \leq x_1 < a\}$ and $a > 0$ is a positive real number. In our example, with $n = 2$, X_a corresponds to a strip in the right half of the red plane (the equatorial 2-hemisphere) starting at $x_1 = 0$ and ending at a. We chose this particular set I because we can use the properties of our special stereographic projection from Remark 3.1.6 and use Proposition 3.1.9. As a reminder, we are trying to show that if M, a connected, embedded, and two-sided smooth surface with a somewhat smooth boundary, has scalar curvature $R \geq n(n-1)$ and its boundary contained in the n-sphere $\mathbb{S}^n$, while being tangent to the upper half of the equatorial n-sphere $\mathbb{S}^n_+$, then M is a portion of the upper half of $\mathbb{S}^n_+$. So far, our hypersurface M satisfies conditions (2) and (3). We use the set I to demonstrate that M satisfies condition (1). First, we show that I is a closed set. Let (a_n) be a sequence in I that converges to a point a_0. We want to show that $a_0 \in I$ and that for some $x_0 \in U \cap X_{a_0}$ we have $u(x_0) = 0$. Let us choose a sequence (x_n) such that $x_n \in U \cap X_{a_n}$ converges to x_0. Notice that u is a continuous function because it returns closed sets to closed sets, namely, $u^{-1}(\{0\}) = \{(x_1, \ldots, x_n) \in U : u(x_1, \ldots, x_n) = 0\} = U \cap K$ for some closed set $K \subseteq \mathbb{R}^+ \times \mathbb{R}^{n-1}$. Then, $u(x_0) = u\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} u(x_n) = \lim_{n\to\infty} 0 = 0$. The first equality follows from our sequence choice, the second equality follows from the continuity of u, and the third equality follows from the fact that $a_n \in I$. Because x_0 is arbitrary, then $u \equiv 0$ on $U \cap X_{a_0}$ implies that $a_0 \in I$, and because a_0 is an arbitrary limit point of I, then I is closed.

Next, we demonstrate that I is open. By assumption, we have that for some real number

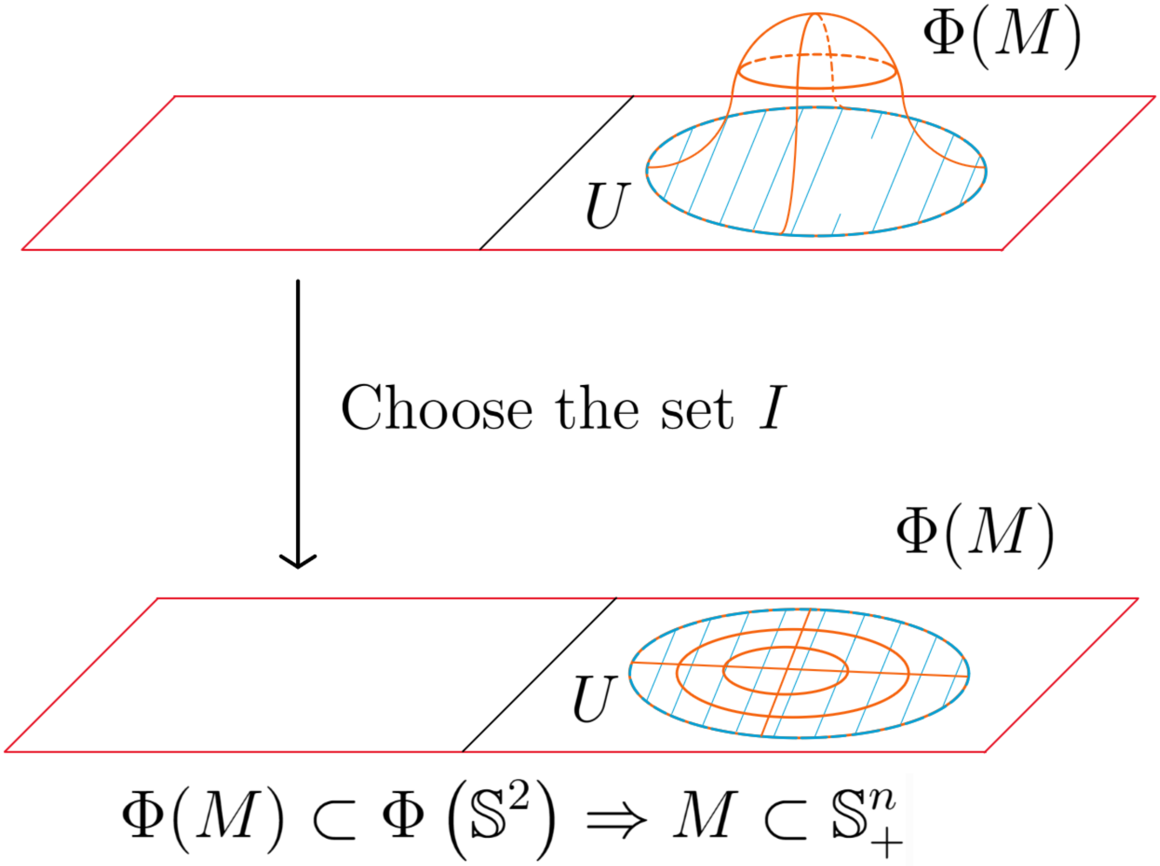


Figure 3.3: Visualization of how I flattens $\Phi(M)$.

$a > 0$ where $U \cap X_a \neq \varnothing$, $u = 0$ and $|Du| = 0$ on $\partial U \cap X_a$ and $R \geq n(n-1)$ on the graph of u in $(\mathbb{R}^{n+1}, g_S)$. By Proposition 3.1.9 with $v = 0$ and $W = U \cap X_a$, we have $u \equiv 0$ on all of $W = U \cap X_a$, which means that $a \in I$, so $I \neq \varnothing$. It follows that $b \in I$ for all $0 < b < a$ because either the same is true for b or $U \cap X_b = \varnothing$, so $b \in I$ vacuously. Therefore, $I = \bigcup_{a \in \mathbb{R}^+} (0, a)$, so I is open in $\mathbb{R}^+$. Since $\mathbb{R}^+$ is connected, the only subsets of $\mathbb{R}^+$ that are open and closed are $\mathbb{R}^+$ and $\varnothing$. Because $I \neq \varnothing$, it must be the case that $I = \mathbb{R}^+$, and so $u \equiv 0$ on all of U since $u \equiv 0$ on $U \cap X_a$ for all $a > 0$. This implies that the image $\Phi(M)$ of our hypersurface M is locally a graph of a bounded function u not only in the collar neighborhood of ∂U but on all of U where $u = 0$ and $|Du| = 0$. In our illustration, where $n = 2$, this means $\Phi(M)$ is completely contained in the right half of the red plane, which corresponds to the image of the upper half of the equatorial 2-hemisphere $\mathbb{S}^2_+$ under Φ. We can see this in Figure 3.3. So, $\Phi(M) \subset \Phi(\mathbb{S}^n_+)$, which implies that M is contained in the upper half of the equatorial n-sphere, and we have concluded the proof.

We must note that in using Proposition 3.1.9 in the proof, we are not only using the result of the proposition, but also the tools used to prove it. To prove Proposition 3.1.9, the authors in [6] used Proposition 3.1.4, Corollary 3.1.7, and Lemma 3.1.8, among other results not mentioned in this paper.

Chapter 4

Future Work

Having looked at the proof of Theorem 3.1.1, there are several directions in which we could extend and deepen the investigation initiated in this thesis. Firstly, an immediate avenue for future work involves an even more detailed exploration of the proof outlined in Chapter 3. While we discussed the tools used for the proof and examined the critical ideas, examining the proofs of the lemmas, propositions, and corollaries in Section 3.1 and seeing how the authors in [6] used them would be good direction. This endeavor presents an opportunity to delve into the intricacies of the theorem.

Building upon the examples of hypersurfaces in $\mathbb{R}^3$ and $\mathbb{S}^3$ presented in Chapter 2, we can explore a broader range of hypersurfaces and their geometric properties. This exploration can include investigating intrinsic and extrinsic curvature and classifying hypersurfaces based on their geometric characteristics. Furthermore, we could explore the proof for the spherical metric in terms of stereographic coordinates and longitude-latitude coordinates. In Chapter 1, we also introduced several rigidity theorems, including Mostow rigidity and the sphere theorem, which have profound implications in geometry and topology. It would be fascinating to delineate general characteristics to define rigidity theorems in differential geometry.

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