

# (Ch.2) Terminology and Notation

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## 1 Terminologies and Notations

$X$ : non-negative random variable, failure time, survival time

The probability distribution of  $X$  can be specified in three ways (useful in survival applications):

- Survival function:  $S(x)$
- Probability density/mass function:  $f(x)/p(x)$
- Hazard function:  $h(x)$

## 2 $X$ continuous

$S(x)$ : survival function, non-increasing right-continuous function of  $x$ ,  $S(0) = 1$

$$S(x) = Pr(X > x), 0 < x < \infty$$

$f(x)$ : probability density function (pdf) of  $X$

$$f(x) = -dS(x)/dx$$

$$S(x) = \int_x^{\infty} f(s)ds$$

$h(x)$ : hazard function

$$\begin{aligned} h(x) &= \lim_{h \rightarrow 0+} \frac{Pr(x \leq X < x+h | X \geq x)}{h} \\ &= \frac{f(x)}{S(x)} \\ &= -\frac{d \log S(x)}{dx} \end{aligned}$$

<sup>1</sup>  $H(x)$ : cumulative hazard function

$$H(x) = \int_0^x h(s)ds$$

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$$1 \frac{Pr(x \leq X < x+h | X \geq x)}{h} = \frac{Pr(x \leq X < x+h, X \geq x)}{h Pr(X \geq x)} = \frac{Pr(x \leq X < x+h)}{h} \frac{1}{Pr(X \geq x)}$$

$$S(x) = \exp[-H(x)] = \exp\left[-\int_0^x h(s)ds\right]$$

$$f(x) = h(x) \exp[-H(x)]$$

Ex.1:  $X$  follows the exponential distribution with parameter  $\lambda$ .  $X \sim \text{Exp}(\lambda)$

$$f(x) = \lambda \exp(-\lambda x), \quad x \geq 0$$

$$\begin{aligned} S(x) &= \int_x^\infty \lambda \exp(-\lambda s) ds \\ &= -\exp(-\lambda s) \Big|_x^\infty \\ &= \exp(-\lambda x), \quad x \geq 0 \end{aligned}$$

$$h(x) = \frac{f(x)}{S(x)} = \lambda$$

### 3 X discrete

$X$ : discrete random variable taking values  $x_1, x_2, \dots$  with associated probability mass function

$$p(x_i) = \text{Pr}(X = x_i), \quad i = 1, 2, \dots,$$

$$S(x) = \text{Pr}(X \geq x) = \sum_{x_j > x} p(x_j)$$

The hazard at  $x_i$  is

$$\begin{aligned} h(x_i) &= \text{Pr}(X = x_i | X \geq x_i) \\ &= \frac{p(x_i)}{S(x_i^-)} = \frac{p(x_i)}{S(x_{i-1})} \\ &= \frac{S(x_{i-1}) - S(x_i)}{S(x_{i-1})} \\ &= 1 - \frac{S(x_i)}{S(x_{i-1})}, \quad i = 1, 2, \dots, \end{aligned}$$

$$S(x_0) = 1,$$

$$S(x) = \prod_{x_j \leq x} \frac{S(x_j)}{S(x_{j-1})} = \prod_{x_j \leq x} (1 - h(x_j))$$

$$\text{Ex.2: } S(x_3) = \frac{S(x_3)}{S(x_2)} \frac{S(x_2)}{S(x_1)} \frac{S(x_1)}{S(x_0)} = [1 - h(x_3)][1 - h(x_2)][1 - h(x_1)]$$

Ex.3: The lifetime  $X$  has the pmf  $p(x_j) = \text{Pr}(X = j) = 1/3, \quad j = 1, 2, 3.$

$$S(x) = \text{Pr}(X > x) = \sum_{x_j > x} p(x_j) = \begin{cases} 1, & \text{if } 0 \leq x < 1, \\ 2/3, & \text{if } 1 \leq x < 2, \\ 1/3, & \text{if } 2 \leq x < 3, \\ 0, & \text{if } x \geq 3. \end{cases}$$

$$h(x) = \begin{cases} \frac{1/3}{1} = 1/3, & \text{if } x = 1, \\ \frac{1/3}{2/3} = 1/2, & \text{if } x = 2, \\ \frac{1/3}{1/3} = 1, & \text{if } x = 3, \\ 0, & \text{otherwise.} \end{cases}$$

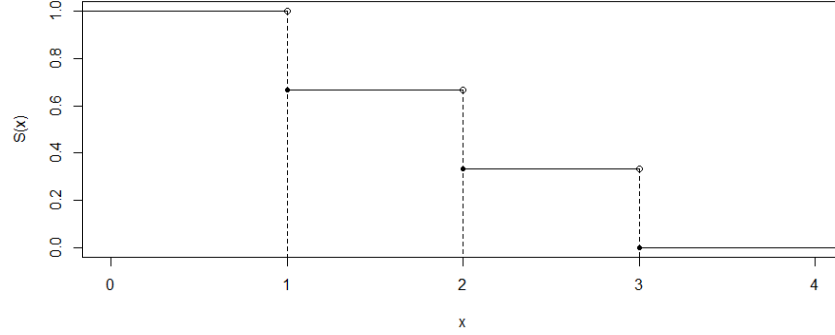


Figure 1: Survival curve(right continuous)

## 4 The Mean Residual Life Function and Median Life

### Mean residual life

The mean residual life (expected residual life) at time  $x$ :  $\text{mrl}(x) = E(X - x | X > x)$

**For a continuous random variable:**

$$\text{mrl}(x) = \frac{\int_x^\infty (t - x)f(t)dt}{S(x)} = \frac{\int_x^\infty S(t)dt}{S(x)}.$$

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When  $x = 0$ ,  $\text{mrl}(0) = E(X)$

$$\mu = E(X) = \int_0^\infty tf(t)dt = \int_0^\infty S(t)dt.$$

$$\text{Var}(X) = 2 \int_0^\infty tS(t)dt - \left[ \int_0^\infty S(t)dt \right]^2$$

3

**For a discrete random variable:**

$$\text{mrl}(x) = \frac{(x_{i+1} - x)S(x_i) + \sum_{j \geq i+1} (x_{j+1} - x_j)S(x_j)}{S(x)}, \text{ for } x_i \leq x < x_{i+1}$$

<sup>2</sup>Use the integration by part:  $\int_x^\infty (t - x)f(t)dt = \int_x^\infty S(t)dt - (t - x)S(t)|_x^\infty$ . Note that  $\lim_{t \rightarrow \infty} tS(t) = 0$  since  $E(X) < \infty$ .

<sup>3</sup>Use the integration by part:  $\int_x^\infty 2tS(t)dt = \int_x^\infty t^2f(t)dt - t^2S(t)|_x^\infty$

## Median life

The 0.5th quantile (median life) is found by solving the equation  $S(x_{0.5}) = 0.5$ .

$$S(x_{0.5}) \leq 0.5, \text{ i.e. } x_{0.5} = \inf \{t : S(t) \leq 0.5\}$$

## Homework 1.(Ch.2)

1. (# 2.13) Suppose that  $X$  has a geometric distribution with probability mass function

$$p(x) = p(1-p)^{x-1}, \quad x = 1, 2, \dots$$

- (a) Find the survival function of  $X$ . (Hint: Recall that for  $0 < \theta < 1$ ,  $\sum_{j=k}^{\infty} \theta^j = \theta^k / (1 - \theta)$  )
- (b) Find the hazard rate of  $X$ . Compare this rate to the hazard rate of an exponential distribution.

2. (# 2.12) Let  $X$  have a uniform distribution on the interval 0 to  $\theta$  with density function

$$f(x) = \begin{cases} \frac{1}{\theta}, & \text{for } 0 \leq x \leq \theta \\ 0, & \text{otherwise} \end{cases}$$

- (a) Find the survival function of  $X$ .
- (b) Find the hazard rate of  $X$ .
- (c) Find the mean residual-life function.