Homework 1.(Ch.2) Solution

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1. (# 2.13) Suppose that X has a geometric distribution with probability mass function

$$p(x) = p(1-p)^{x-1}, \quad x = 1, 2, \cdots$$

(a) Find the survival function of X. (Hint: Recall that for $0 < \theta < 1, \sum_{j=k}^{\infty} \theta^j = \theta^k/(1-\theta)$)
(b) Find the hazard rate of X. Compare this rate to the hazard rate of an exponential distribution.

Sol:
(a)
Let $S(x)$ be the survival function, and $[x]$ is the greatest integer function, where $[x] = \max\{y : y \leq x, y \text{ is integer }\}$.

$$\begin{aligned} S(x) &= \sum_{j>x} p(1-p)^{j-1} \\ &= \sum_{j=[x]+1}^{\infty} p(1-p)^{j-1} \\ &= p(1-p)^{[x]} \sum_{j=0}^{\infty} (1-p)^j \\ &= (1-p)^{[x]} \end{aligned}$$

$$S(x) = \begin{cases} 1, & \text{if } 0 \leq x < 1, \\ 1-p, & \text{if } 1 \leq x < 2, \\ (1-p)^2, & \text{if } 2 \leq x < 3, \\ \vdots & \\ (1-p)^k, & \text{if } k \leq x < k+1, \\ \vdots & \end{cases}$$

(b)
Let $h(x)$ be the hazard function

$$h(x) = \begin{cases} \frac{p(x)}{S(x^-)} = p, & \text{if } x = 1, 2, 3, \cdots \\ 0, & \text{otherwise.} \end{cases}$$

$$
\begin{aligned}
E(X) &= \sum_{j=1}^{\infty} j \cdot p(1-p)^{j-1} \\
&= p\left[\sum_{j=1}^{\infty}(1-p)^{j-1} + \sum_{j=2}^{\infty}(1-p)^{j-1} + \cdots\right] \\
&= p\left[\frac{1}{1-(1-p)} + \frac{1-p}{1-(1-p)} + \cdots\right] \\
&= \sum_{j=0}^{\infty}(1-p)^{j} = \frac{1}{p}
\end{aligned}
$$

For both geometric distribution and exponential distribution, the hazard rate is constant and is equal to the reciprocal of the mean.

2. (# 2.12) Let X have a uniform distribution on the interval 0 to θ with density function

$$
f(x) = \begin{cases} \frac{1}{\theta}, & \text{for } 0 \le x \le \theta \\ 0, & \text{otherwise} \end{cases}
$$

(a) Find the survival function of X.
(b) Find the hazard rate of X.
(c) Find the mean residual-life function.
Sol:
(a)
Let $S(x)$ be the survival function,

$$
\begin{aligned}
S(x) &= \int_{x}^{\theta} \frac{1}{\theta} dt \\
&= 1 - \frac{x}{\theta}, \quad 0 \le x < \theta
\end{aligned}
$$

$$
S(x) = \begin{cases} 1 - \frac{x}{\theta}, & \text{if } 0 \le x < \theta, \\ 0 & \text{if } x \ge \theta \end{cases}
$$

(b)
Let $h(x)$ be the hazard function

$$
h(x) = \frac{f(x)}{S(x)} = \frac{1}{\theta - x}, \quad 0 \le x \le \theta
$$

$$
h(x) = \begin{cases} \frac{1}{\theta - x}, & \text{if } 0 \le x \le \theta, \\ 0 & \text{otherwise} \end{cases}
$$

(c)
Let $mrl(x)$ be the mean residual-life function

$$
\begin{aligned}
mrl(x) &= \frac{\int_x^\theta (t-x)\frac{1}{\theta}dt}{\frac{\theta-x}{\theta}} \\
&= \frac{\theta}{\theta-x}\left[\frac{1}{2\theta}t^2 - \frac{x}{\theta}t|_x^\theta\right] \\
&= \frac{1}{2(\theta-x)}\left[\theta^2 - 2x\theta - x^2 + 2x^2\right] \\
&= \frac{(\theta-x)^2}{2(\theta-x)} = \frac{\theta-x}{2}, \quad 0 \le x \le \theta
\end{aligned}
$$