1. DH Convention on Fig. 3

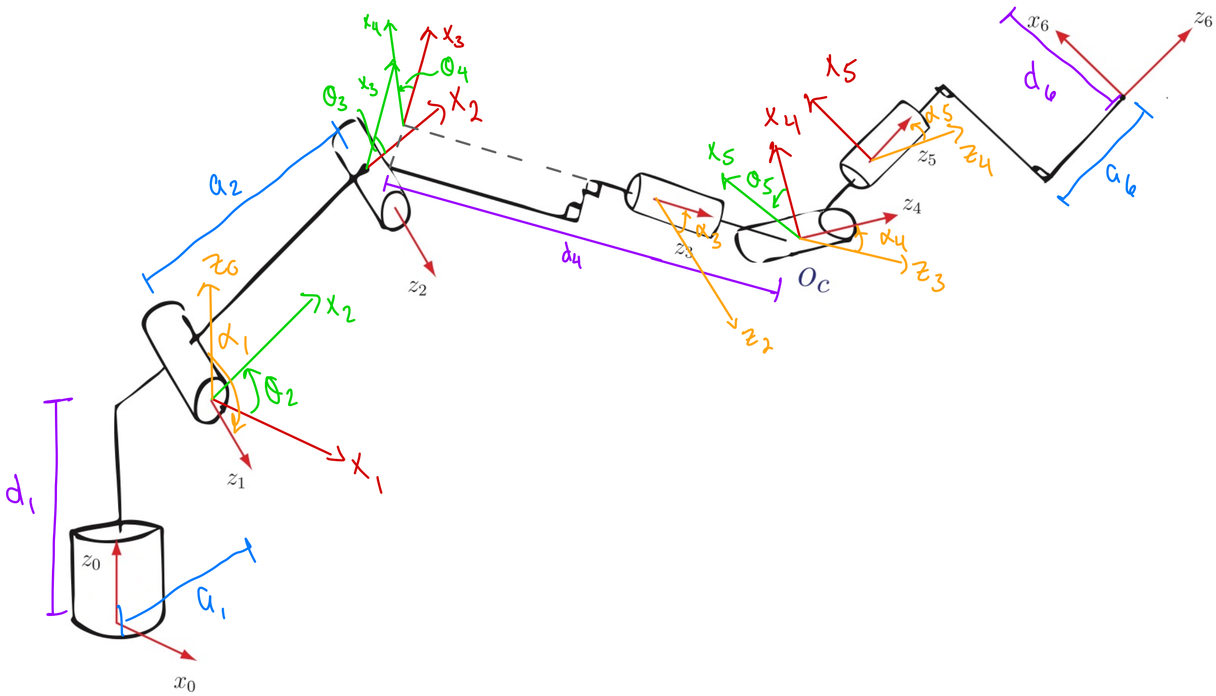
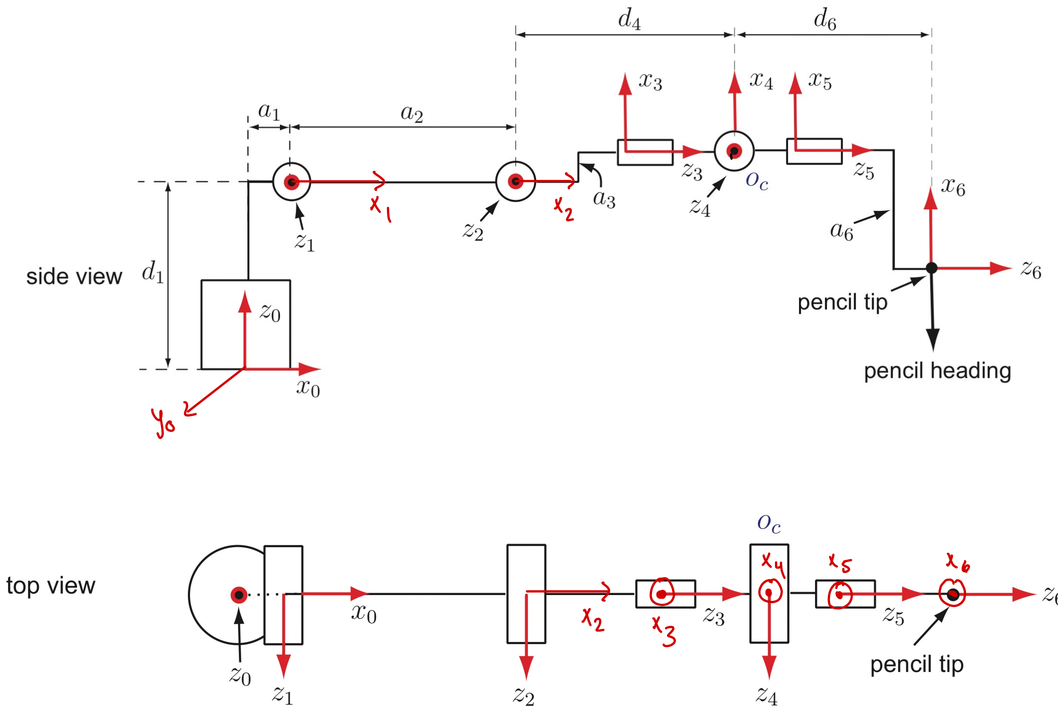


Figure 3: Schematic diagram of KUKA robot arm not at rest.

DH Table:

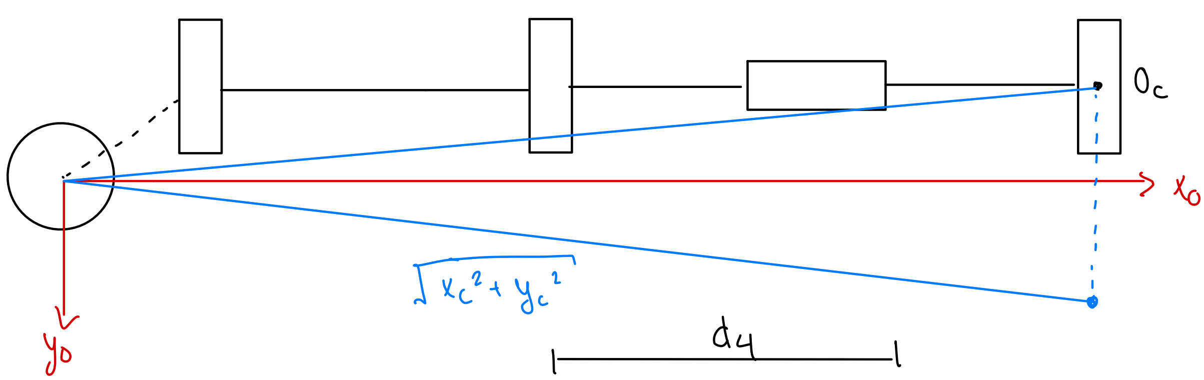
link	d_i [mm]	a_i [mm]	θ_i	α_i
1	400	25	$\theta_1 = 0$	$\frac{\pi}{2}$
2	0	315	θ_2	0
3	0	35	θ_3	$\frac{\pi}{2}$
4	365	0	θ_4	$-\frac{\pi}{2}$
5	0	0	θ_5	$\frac{\pi}{2}$
6	161.44	296.23	$\theta_6 = 0$	0

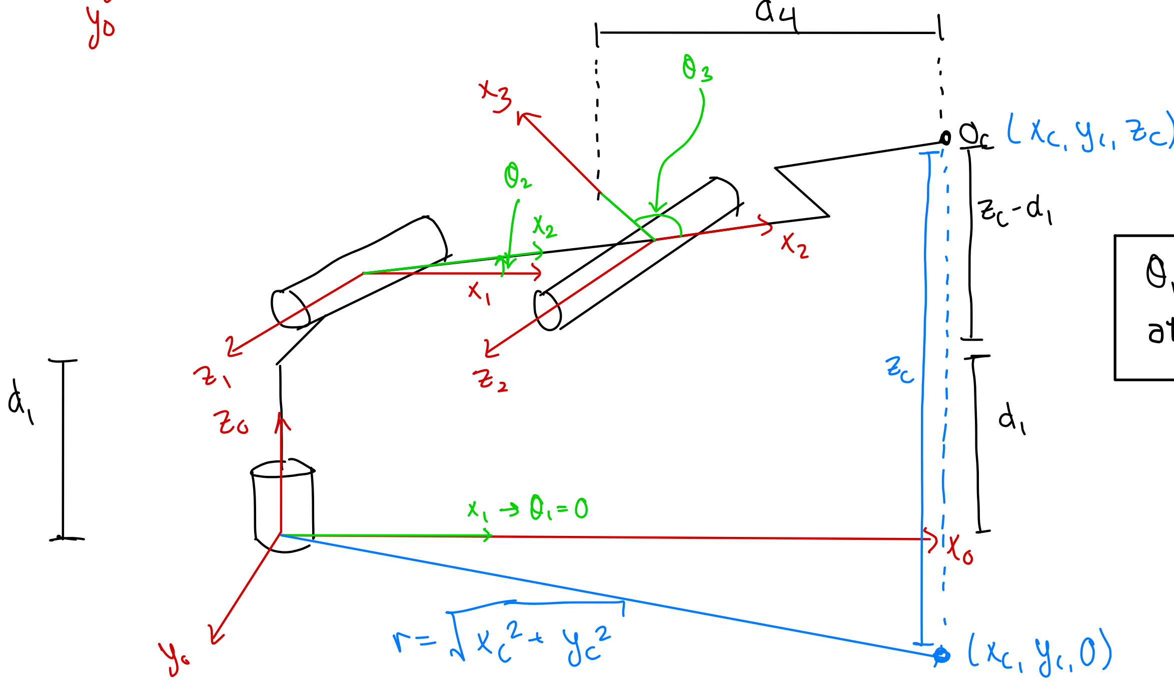
2. Inverse Kinematics



Part 1: inverse position kinematics

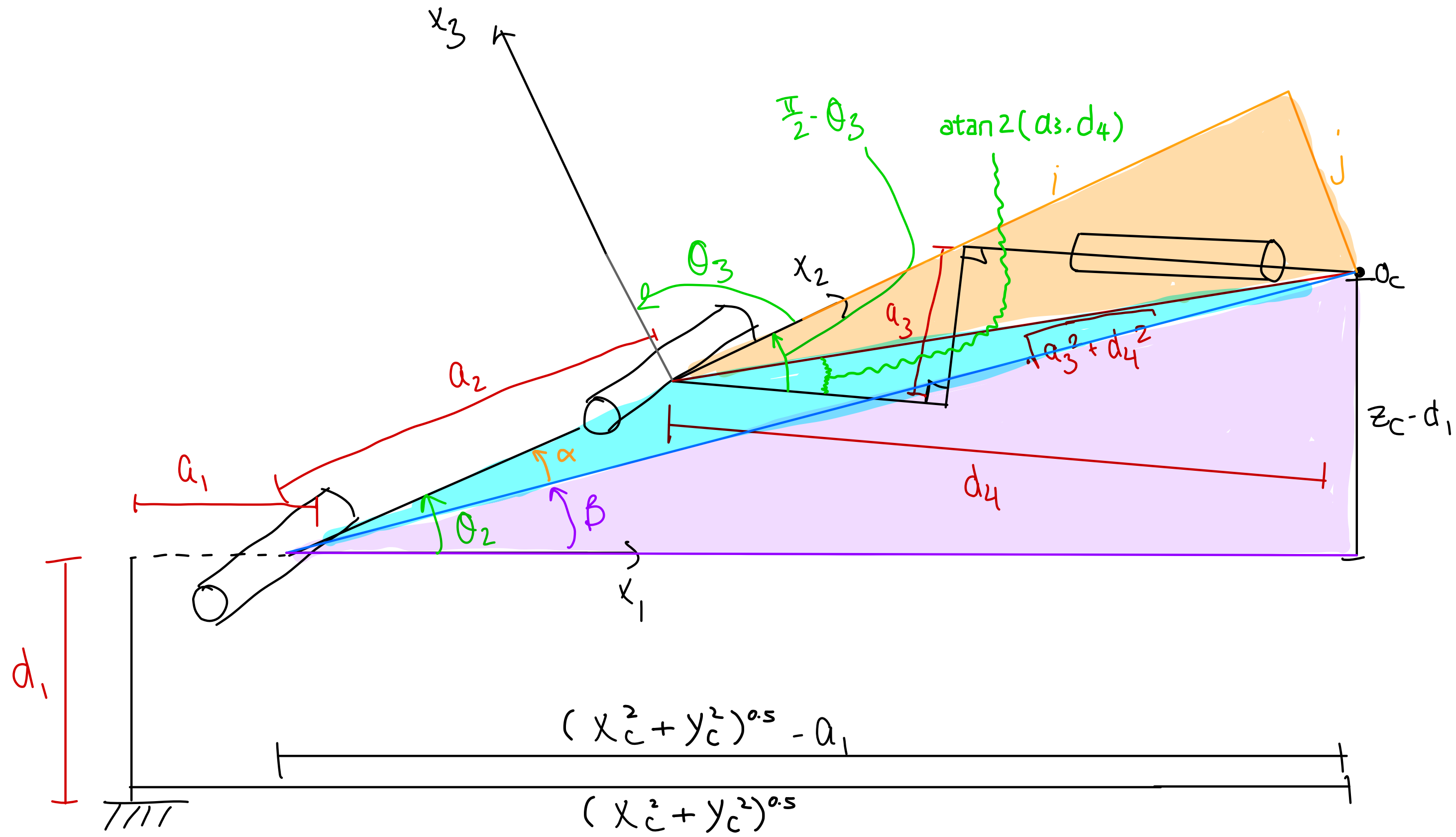
from the top view:



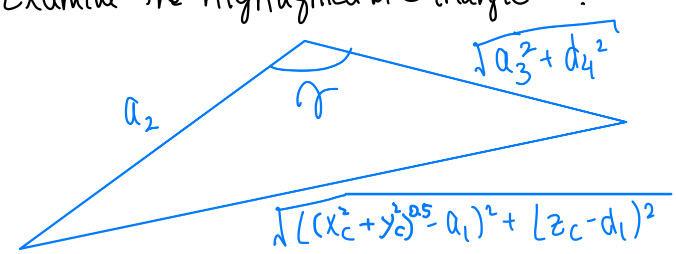


$$\boxed{\theta_1 = \operatorname{atan2}(y_c, x_c)}$$

examine joints 2 and 3 in the $x_0 z_0$ plane:



Examine the highlighted blue triangle.



from the cosine law: $((x_c^2+y_c^2)^{0.5}-a_1)^2+(z_c-d_1)^2 = a_2^2+a_3^2+d_4^2-2a_2\sqrt{a_3^2+d_4^2}\cos\gamma$

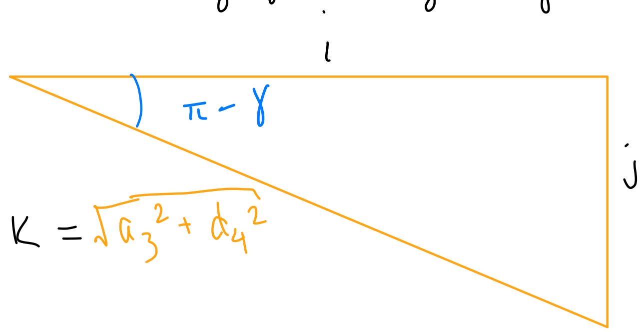
$$\cos\gamma = \frac{a_2^2+a_3^2+d_4^2-((x_c^2+y_c^2)^{0.5}-a_1)^2-(z_c-d_1)^2}{2a_2\sqrt{a_3^2+d_4^2}} = D$$

$\gamma = \text{atan2}(\sqrt{1-D^2}, D)$ b/c we want the elbow-up (ie: positive) solution

$$\gamma = \pi - \left(\frac{\pi}{2}-\theta_3-\text{atan2}(a_3,d_4)\right) = \frac{\pi}{2}+\theta_3+\text{atan2}(a_3,d_4)$$

$$\boxed{\begin{aligned}\theta_3 &= \gamma - \frac{\pi}{2} - \text{atan2}(a_3,d_4)\\ &= \text{atan2}(\sqrt{1-D^2}, D) - \frac{\pi}{2} - \text{atan2}(a_3,d_4)\end{aligned}}$$

Examine the highlighted orange triangle:



$$i = K\cos(\pi - \gamma) = -K\cos\gamma$$

$$j = K\sin(\pi - \gamma) = K\sin\gamma$$

Examine all 3 highlighted triangles: $\theta_2 = \alpha + \beta$

$$\alpha = \operatorname{atan2}(j,\ a_2 + i)$$

$$= \operatorname{atan2}\left(\sqrt{a_3^2 + d_4^2}\,\sin\gamma,\ a_2 - \sqrt{a_3^2 + d_4^2}\,\cos\gamma\right)$$

$$\beta = \operatorname{atan2}\left(z_c - d_1,\ (x_c^2 + y_c^2)^{0.5} - a_1\right)$$

$$\boxed{\theta_2 = \operatorname{atan2}\left(\sqrt{a_3^2 + d_4^2}\,\sin\gamma,\ a_2 - \sqrt{a_3^2 + d_4^2}\,\cos\gamma\right) + \operatorname{atan2}\left(z_c - d_1,\ (x_c^2 + y_c^2)^{0.5} - a_1\right)}$$

where $\cos\gamma = D$ (as defined above)

Part 2: Inverse orientation kinematics

$R_6^0 = R_3^0 R_6^3 = R_d \iff R_6^3 (R_3^0)^T R_d.$

Given $(R_3^0)^T R_d$, find (q_4, q_5, q_6) s.t. $R_6^3(q_4, q_5, q_6) = M$ where $M = (R_3^0)^T R_d$

$$R_6^3 = \begin{bmatrix} * & * & c_4 s_5 \\ * & * & s_4 s_5 \\ -s_5 c_6 & s_5 c_6 & c_5 \end{bmatrix} = M$$

where $c_i = c_{q_i}$ and $s_i = s_{q_i}$

corresponding to the ZYZ Euler angles

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}.$$

If $(m_{13})^2 + (m_{23})^2 \neq 0$ and in the elbow up (i.e.: positive atan2 convention), we have:

$$\boxed{\begin{aligned} q_4 &= \theta_4 = \operatorname{atan2}(m_{23}, m_{13}) \\ q_5 &= \theta_5 = \operatorname{atan2}\left(\sqrt{1-(m_{33})^2}, m_{33}\right) \\ q_6 &= \theta_6 = \operatorname{atan2}(m_{32}, -m_{31}) \end{aligned}}$$

Note: if $(m_{13})^2 + (m_{23})^2 = 0$, then a kinematic singularity exists