

EE-559 – Deep learning

3.2. Probabilistic view of a linear classifier

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Consider the following class populations

$$\forall y \in \{0, 1\}, x \in \mathbb{R}^D,$$

$$\mu_{X|Y=y}(x) = \frac{1}{\sqrt{(2\pi)^D |\Sigma|}} \exp \left(-\frac{1}{2} (x - m_y) \Sigma^{-1} (x - m_y)^T \right).$$

That is, they are Gaussian with **the same covariance matrix Σ** . This is the **homoscedasticity** assumption.

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$$P(Y = 1 \mid X = x) = \sigma\left(\log \frac{\mu_{X|Y=1}(x)}{\mu_{X|Y=0}(x)} + \log \frac{P(Y = 1)}{P(Y = 0)}\right).$$

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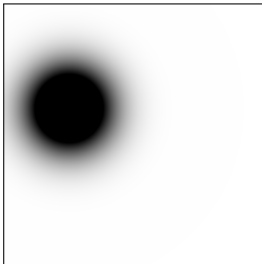
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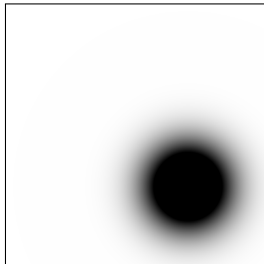
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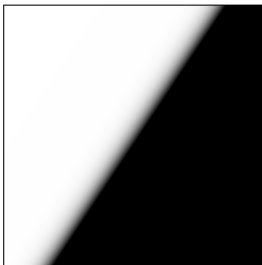
The homoscedasticity makes the second-order terms vanish.



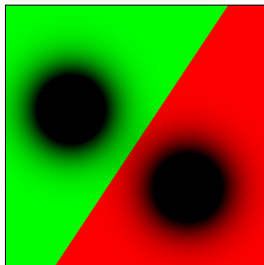
$\mu_{X|Y=0}$

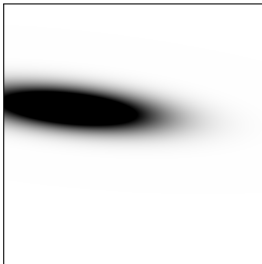


$\mu_{X|Y=1}$

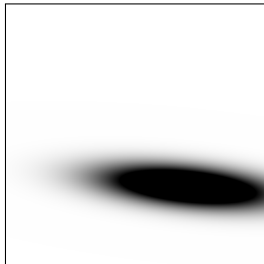


$P(Y = 1 \mid X = x)$

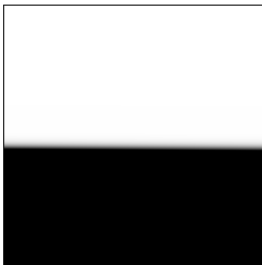




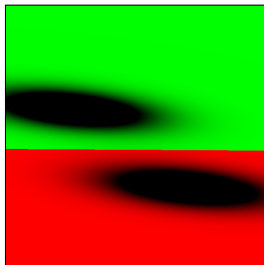
$\mu_{X|Y=0}$

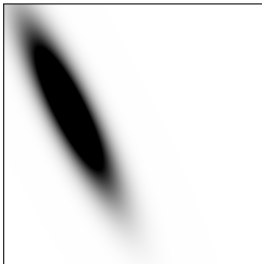


$\mu_{X|Y=1}$

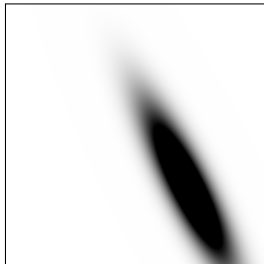


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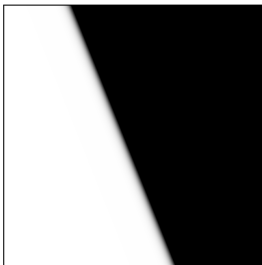




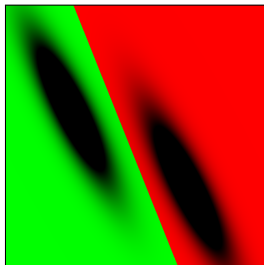
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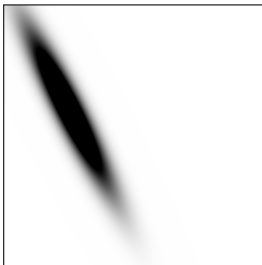


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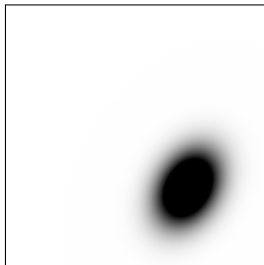


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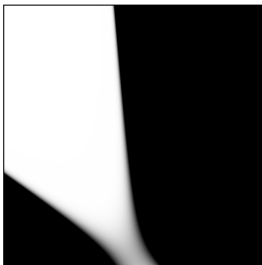




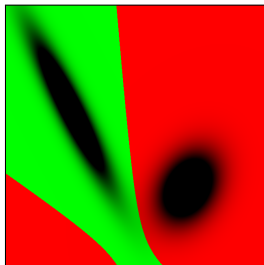
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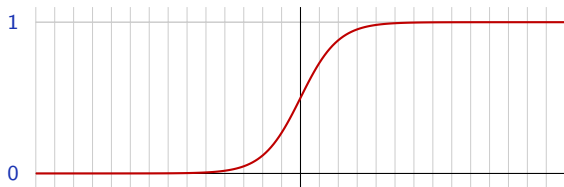
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Note that the (logistic) sigmoid function

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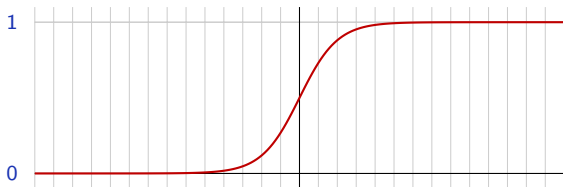
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So the overall model

$$f(x; w, b) = \sigma(w \cdot x + b)$$

looks very similar to the perceptron.

We can use the model from LDA

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First, to simplify the next slide, note that we have

$$1 - \sigma(x) = 1 - \frac{1}{1 + e^{-x}} = \sigma(-x),$$

hence if Y takes value in $\{-1, 1\}$ then

$$\forall y \in \{-1, 1\}, \quad P(Y = y \mid X = x) = \sigma(y(w \cdot x + b)).$$

We have

$$\begin{aligned}\log \mu_{W,B}(w, b \mid \mathcal{D} = \mathbf{d}) \\&= \log \frac{\mu_{\mathcal{D}}(\mathbf{d} \mid W = w, B = b) \mu_{W,B}(w, b)}{\mu_{\mathcal{D}}(\mathbf{d})} \\&= \log \mu_{\mathcal{D}}(\mathbf{d} \mid W = w, B = b) + \log \mu_{W,B}(w, b) - \log Z \\&= \sum_n \log \sigma(y_n(w \cdot x_n + b)) + \log \mu_{W,B}(w, b) - \log Z'\end{aligned}$$

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This is the **logistic regression**, whose loss aims at minimizing

$$-\log \sigma(y_n f(x_n)).$$



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We will come back sometime to a probabilistic interpretation, but most of the methods will be envisioned from the signal-processing and optimization angles.

The end