EE-559 – Deep learning

3.1. The perceptron

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The first mathematical model for a neuron was the Threshold Logic Unit, with Boolean inputs and outputs:

$$f(x) = \mathbf{1}_{\{w \sum_i x_i + b \geq 0\}}.$$

It can in particular implement

$$\begin{aligned} or(u, v) &= \mathbf{1}_{\{u+v-0.5 \geq 0\}} & (w = 1, b = -0.5) \\ and(u, v) &= \mathbf{1}_{\{u+v-1.5 \geq 0\}} & (w = 1, b = -1.5) \\ not(u) &= \mathbf{1}_{\{-u+0.5 \geq 0\}} & (w = -1, b = 0.5) \end{aligned}$$

Hence, **any Boolean function can be build with such units.**

(McCulloch and Pitts, 1943)

The perceptron is very similar

$$f(x) = \begin{cases} 1 & \text{if} \quad \sum_i w_i\, x_i + b \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

but the inputs are real values and the weights can be different.

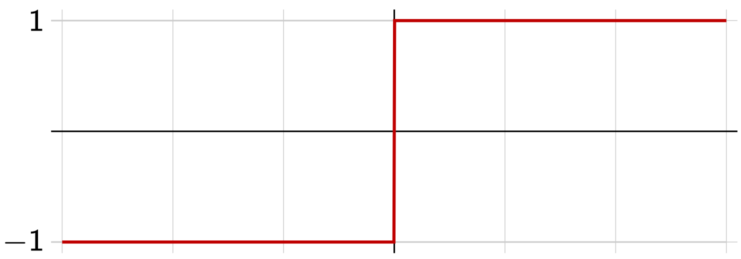
This model was originally motivated by biology, with w_i being the *synaptic weights*, and x_i and f firing rates.

It is a (very) crude biological model.

(Rosenblatt, 1957)

To make things simpler we take responses ± 1. Let

$$\sigma(x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ -1 & \text{otherwise.} \end{cases}$$

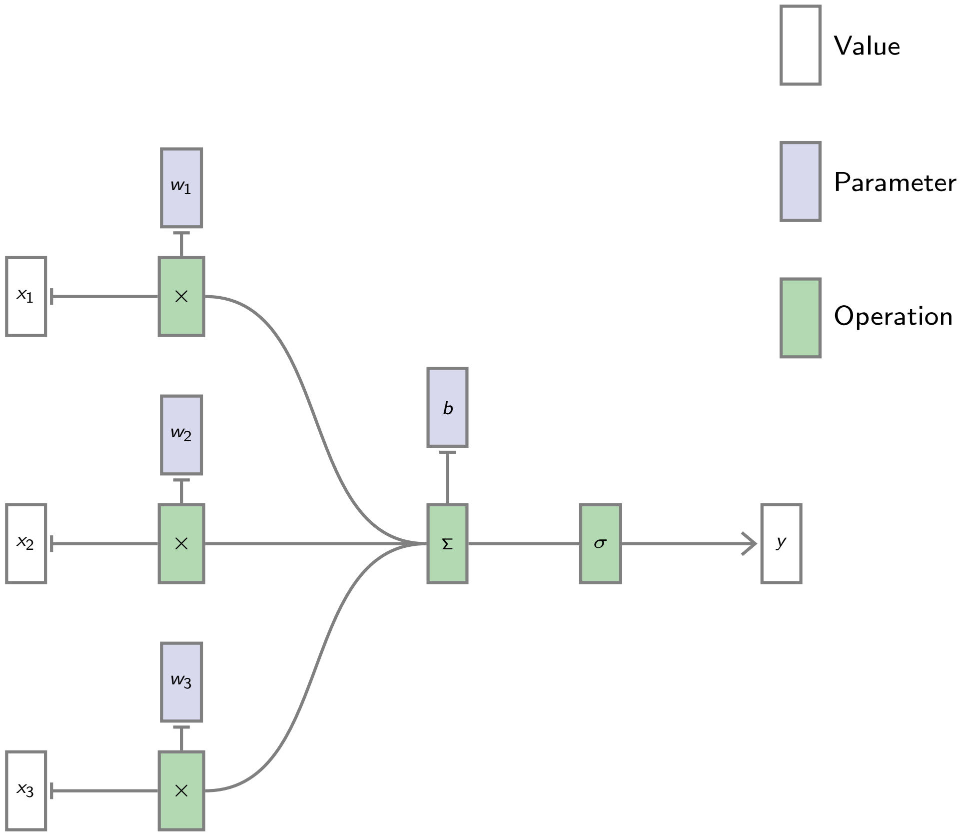


The perceptron classification rule boils down to

$$f(x) = \sigma(w \cdot x + b).$$

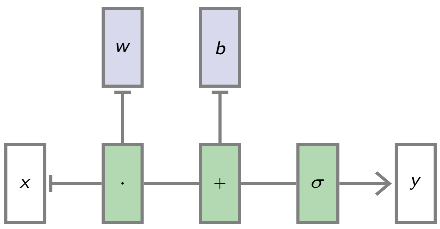
For neural networks, the function σ that follows a linear operator is called the **activation function**.

We can represent this “neuron” as follows:



We can also use tensor operations, as in

$$f(x) = \sigma(w \cdot x + b).$$



Given a training set

$$(x_n, y_n) \in \mathbb{R}^D \times \{-1, 1\}, \quad n = 1, \dots, N,$$

a very simple scheme to train such a linear operator for classification is the **perceptron algorithm:**

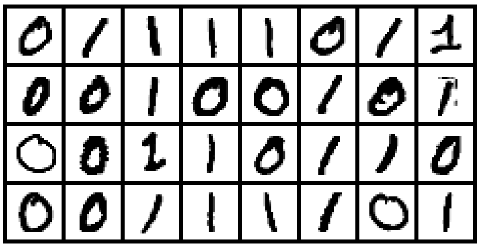
1. Start with $w^0 = 0$,
2. while $\exists n_k$ s.t. $y_{n_k} \left(w^k \cdot x_{n_k} \right) \leq 0$, update $w^{k+1} = w^k + y_{n_k} x_{n_k}$.

The bias b can be introduced as one of the ws by adding a constant component to x equal to 1.

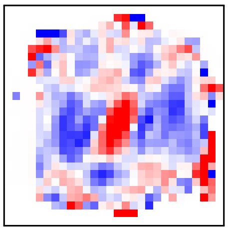
(Rosenblatt, 1957)


```
def train_perceptron(x, y, nb_epochs_max):
    w = torch.zeros(x.size(1))

    for e in range(nb_epochs_max):
        nb_changes = 0
        for i in range(x.size(0)):
            if x[i].dot(w) * y[i] <= 0:
                w = w + y[i] * x[i]
                nb_changes = nb_changes + 1
        if nb_changes == 0: break;

    return w
```

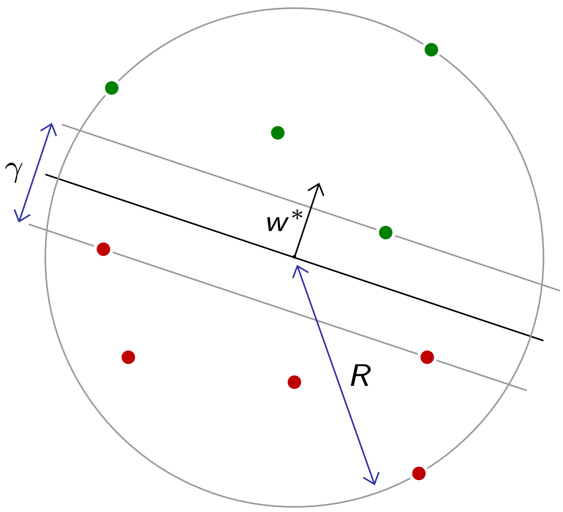
This crude algorithm works often surprisingly well. With MNIST's "0"s as negative class, and "1"s as positive one.



```
epoch 0 nb_changes 64 train_error 0.23% test_error 0.19%
epoch 1 nb_changes 24 train_error 0.07% test_error 0.00%
epoch 2 nb_changes 10 train_error 0.06% test_error 0.05%
epoch 3 nb_changes 6 train_error 0.03% test_error 0.14%
epoch 4 nb_changes 5 train_error 0.03% test_error 0.09%
epoch 5 nb_changes 4 train_error 0.02% test_error 0.14%
epoch 6 nb_changes 3 train_error 0.01% test_error 0.14%
epoch 7 nb_changes 2 train_error 0.00% test_error 0.14%
epoch 8 nb_changes 0 train_error 0.00% test_error 0.14%
```



We can get a convergence result under two assumptions:



1. The x_n are in a sphere of radius R:
$$\exists R > 0,\ \forall n,\ \|x_n\| \leq R.$$
2. The two populations can be separated with a margin $\gamma > 0$.
$$\exists w^*,\ \|w^*\| = 1,\ \exists \gamma > 0,\ \forall n,\ y_n\,(x_n \cdot w^*) \geq \gamma/2.$$

To prove the convergence, let us make the assumption that there still is a misclassified sample at iteration k, and w^{k+1} is the weight vector updated with it. We have

$$\begin{aligned} w^{k+1} \cdot w^* &= \left(w^k + y_{n_k} x_{n_k}\right) \cdot w^* \\ &= w^k \cdot w^* + y_{n_k} \left(x_{n_k} \cdot w^*\right) \\ &\geq w^k \cdot w^* + \gamma/2 \\ &\geq (k+1)\,\gamma/2. \end{aligned}$$

Since

$$\|w^k\| \|w^*\| \geq w^k \cdot w^*,$$

we get

$$\begin{aligned} \|w^k\|^2 &\geq \left(w^k \cdot w^*\right)^2 / \|w^*\|^2 \\ &\geq k^2 \gamma^2 / 4. \end{aligned}$$

And

$$\begin{aligned} \|w^{k+1}\|^2 &= w^{k+1} \cdot w^{k+1} \\ &= \left(w^k + y_{n_k} x_{n_k}\right) \cdot \left(w^k + y_{n_k} x_{n_k}\right) \\ &= w^k \cdot w^k + 2\,\underbrace{y_{n_k}\, w^k \cdot x_{n_k}}_{\leq 0} + \underbrace{\|x_{n_k}\|^2}_{\leq R^2} \\ &\leq \|w^k\|^2 + R^2 \\ &\leq (k+1)\,R^2. \end{aligned}$$

Putting these two results together, we get

$$k^2\gamma^2/4 \leq \|w^k\|^2 \leq \; k\,R^2$$

hence

$$k \leq 4R^2/\gamma^2,$$

hence no misclassified sample can remain after $\left\lfloor 4R^2/\gamma^2 \right\rfloor$ iterations.

This result makes sense:

- The bound does not change if the population is scaled, and
- the larger the margin, the more quickly the algorithm classifies all the samples correctly.

The perceptron stops as soon as it finds a separating boundary.

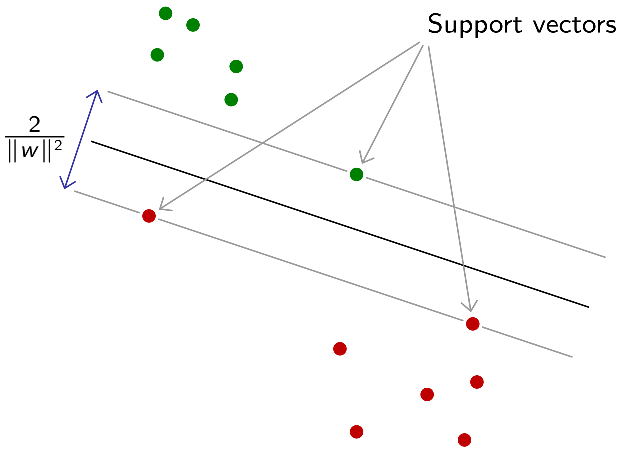
Other algorithms maximize the distance of samples to the decision boundary, which improves robustness to noise.

Support Vector Machines (SVM) achieve this by minimizing

$$\mathscr{L}(w, b) = \lambda\|w\|^2 + \frac{1}{N}\sum_n \max(0, 1 - y_n(w \cdot x_n + b)),$$

which is convex and has a global optimum.

$$\mathscr{L}(w, b) = \lambda \|w\|^2 + \frac{1}{N} \sum_n \max(0, 1 - y_n(w \cdot x_n + b))$$



Minimizing $\max(0, 1 - y_n(w \cdot x_n + b))$ pushes the nth sample beyond the plane $w \cdot x + b = y_n$, and minimizing $\|w\|^2$ increases the distance between the $w \cdot x + b = \pm 1$.

At convergence, only a small number of samples matter, the "support vectors".

The term

$$\max(0, 1 - \alpha)$$

is the so called "hinge loss"

References

W. S. McCulloch and W. Pitts. A logical calculus of the ideas immanent in nervous activity. *The bulletin of mathematical biophysics*, 5(4):115–133, 1943.

F. Rosenblatt. The perceptron–A perceiving and recognizing automaton. Technical Report 85-460-1, Cornell Aeronautical Laboratory, 1957.