EE-559 – Deep learning

3.4. Multi-Layer Perceptrons

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So far we have seen linear classifiers of the form

$$\mathbb{R}^D \to \mathbb{R}$$
$$x \mapsto \sigma(w \cdot x + b),$$

with $w \in \mathbb{R}^D$, $b \in \mathbb{R}$, and $\sigma : \mathbb{R} \to \mathbb{R}$.

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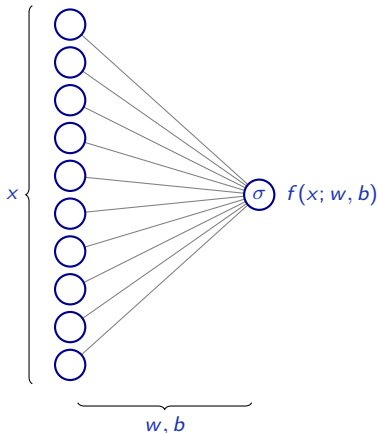$$\mathbb{R}^D \to \mathbb{R}$$
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with $w \in \mathbb{R}^D$, $b \in \mathbb{R}$, and $\sigma : \mathbb{R} \to \mathbb{R}$.

This can naturally be extended to a multi-dimension output by applying a similar transformation to every output, which leads to

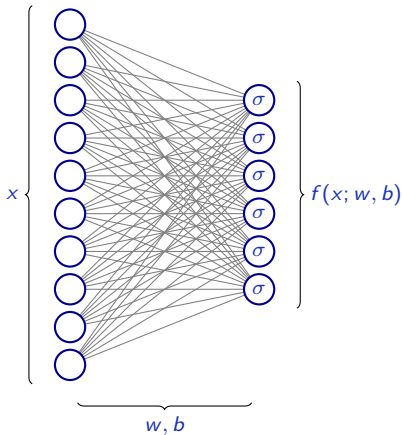$$\mathbb{R}^D \to \mathbb{R}^C$$
$$x \mapsto \sigma(wx + b),$$

with $w \in \mathbb{R}^{C \times D}$, $b \in \mathbb{R}^C$, and σ is applied component-wise.

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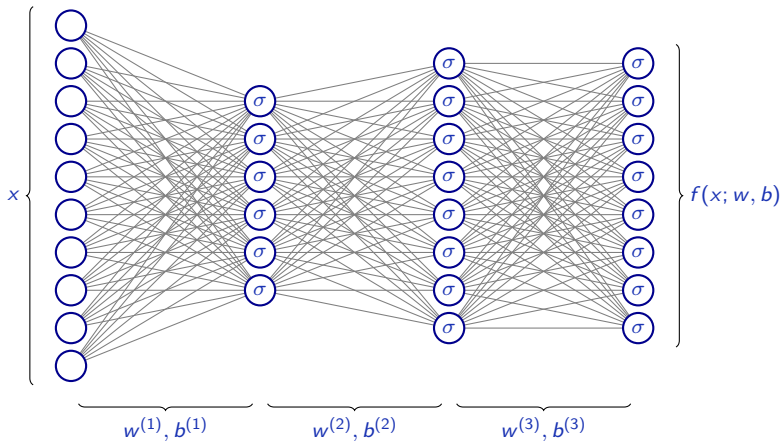
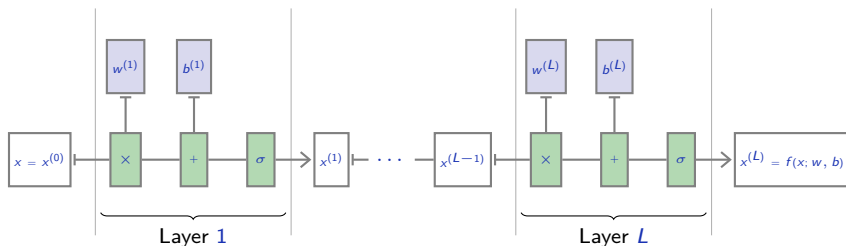This latter structure can be formally defined, with $x^{(0)} = x$,

$$\forall l = 1, \ldots, L, \ x^{(l)} = \sigma \left(w^{(l)} x^{(l-1)} + b^{(l)} \right)$$

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Such a model is a **Multi-Layer Perceptron (MLP).**

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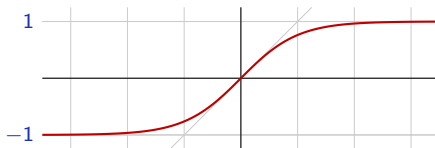
Consequently:

⚠ **The activation function σ should be non-linear**, or the resulting MLP is an affine mapping with a peculiar parametrization.
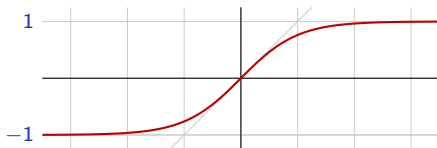
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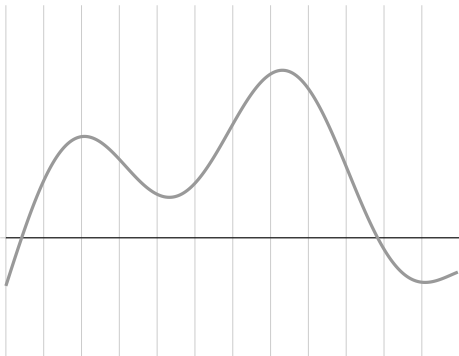
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and the rectified linear unit (ReLU)
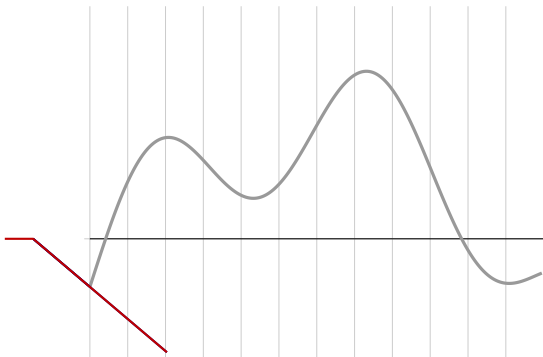
$$x \mapsto \max(0, x)$$

Universal approximation

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$$f(x) = \sigma(w_1 x + b_1)$$
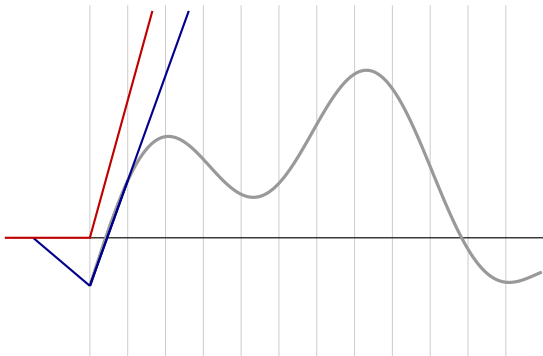
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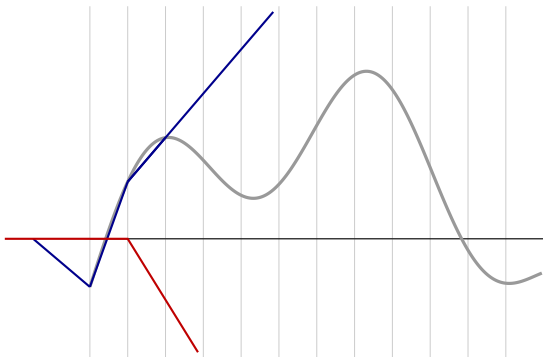
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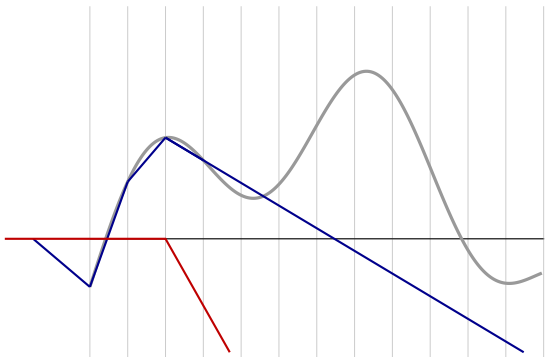$$f(x) = \sigma(w_1 x + b_1) + \sigma(w_2 x + b_2) + \sigma(w_3 x + b_3) + \ldots$$

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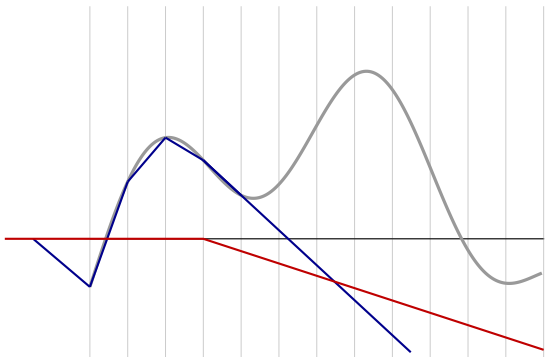$$f(x) = \sigma(w_1 x + b_1) + \sigma(w_2 x + b_2) + \sigma(w_3 x + b_3) + \dots$$

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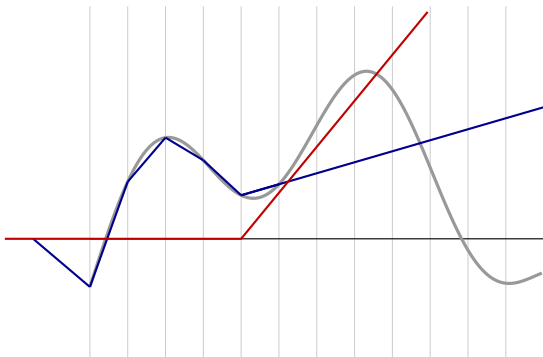$$f(x) = \sigma(w_1 x + b_1) + \sigma(w_2 x + b_2) + \sigma(w_3 x + b_3) + \dots$$

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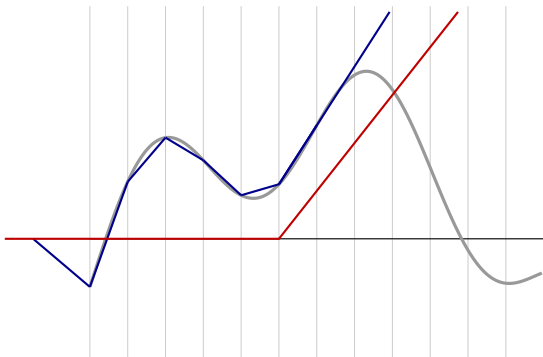$$f(x) = \sigma(w_1 x + b_1) + \sigma(w_2 x + b_2) + \sigma(w_3 x + b_3) + \ldots$$

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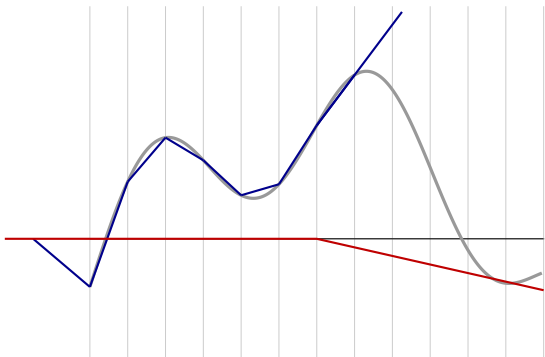$$f(x) = \sigma(w_1 x + b_1) + \sigma(w_2 x + b_2) + \sigma(w_3 x + b_3) + \dots$$

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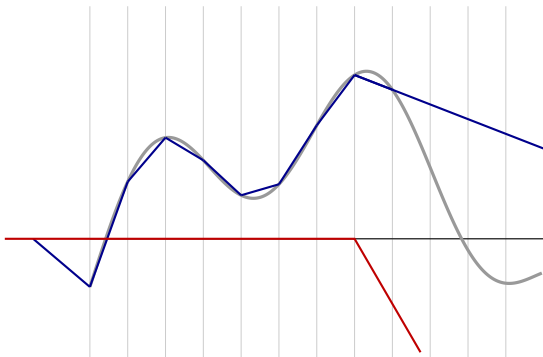$$f(x) = \sigma(w_1 x + b_1) + \sigma(w_2 x + b_2) + \sigma(w_3 x + b_3) + \dots$$

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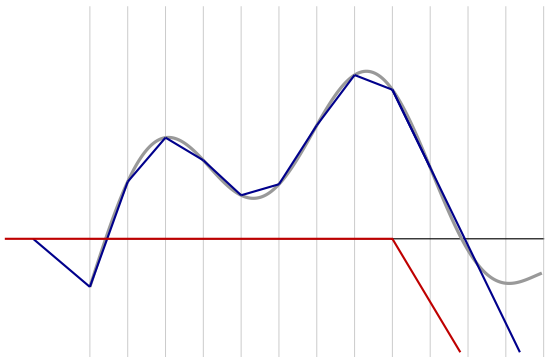$$f(x) = \sigma(w_1 x + b_1) + \sigma(w_2 x + b_2) + \sigma(w_3 x + b_3) + \dots$$

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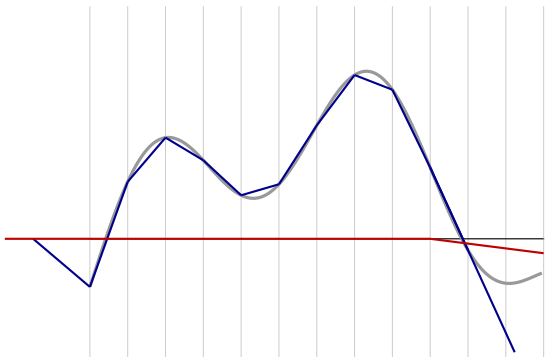
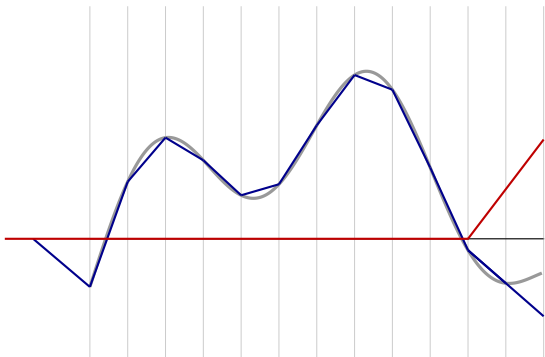$$f(x) = \sigma(w_1 x + b_1) + \sigma(w_2 x + b_2) + \sigma(w_3 x + b_3) + \ldots$$

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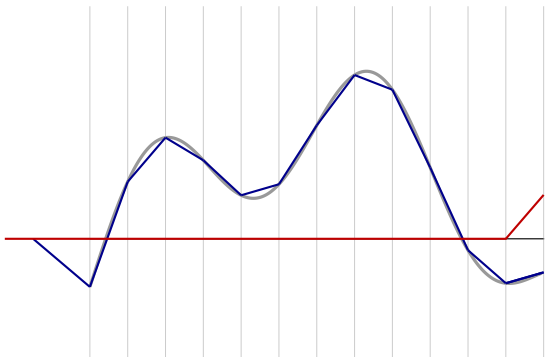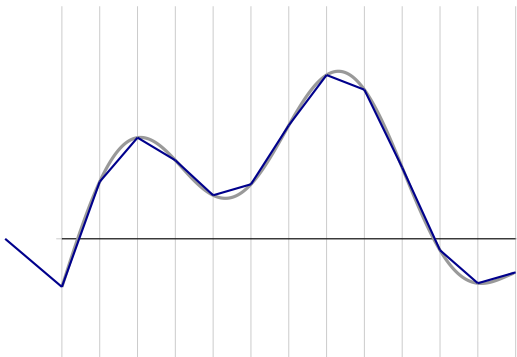$$f(x) = \sigma(w_1 x + b_1) + \sigma(w_2 x + b_2) + \sigma(w_3 x + b_3) + \dots$$
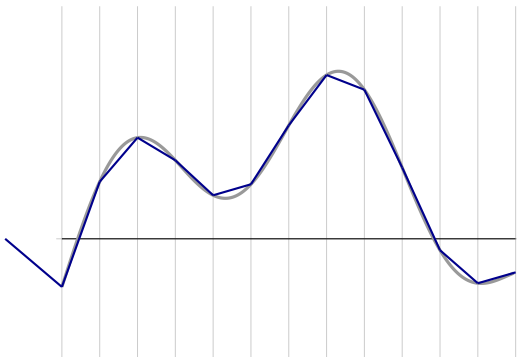
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$$f(x) = \sigma(w_1 x + b_1) + \sigma(w_2 x + b_2) + \sigma(w_3 x + b_3) + \ldots$$

This is true for other activation functions under mild assumptions.

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First, we can use the previous result for the sin function

$$\forall A > 0, \epsilon > 0, \ \exists N, \ (\alpha_n, a_n) \in \mathbb{R} \times \mathbb{R}, n = 1, \ldots, N,$$

$$\text{s.t.} \max_{x \in [-A,A]} \left| \sin(x) - \sum_{n=1}^{N} \alpha_n \sigma(x - a_n) \right| \leq \epsilon.$$

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And the density of Fourier series provides

$$\forall \psi \in \mathscr{C}([0,1]^D, \mathbb{R}), \delta > 0, \exists M, (v_m, \gamma_m, c_m) \in \mathbb{R}^D \times \mathbb{R} \times \mathbb{R}, m = 1, \ldots, M,$$

$$\text{s.t.} \ \max_{x \in [0,1]^D} \left| \psi(x) - \sum_{m=1}^{M} \gamma_m \sin(v_m \cdot x + c_m) \right| \leq \delta.$$

So, $\forall \xi > 0$, with

$$\delta = \frac{\xi}{2}, A = \max_{1 \leq m \leq M} \max_{x \in [0,1]^D} |v_m \cdot x + c_m|, \text{ and } \epsilon = \frac{\xi}{2 \sum_m |\gamma_m|}$$

we get, $\forall x \in [0,1]^D$,

$$\left| \psi(x) - \sum_{m=1}^{M} \gamma_m \left(\sum_{n=1}^{N} \alpha_n \sigma(v_m \cdot x + c_m - a_n) \right) \right|$$

$$\leq \underbrace{\left| \psi(x) - \sum_{m=1}^{M} \gamma_m \sin(v_m \cdot x + c_m) \right|}_{\leq \frac{\xi}{2}}$$

$$+ \underbrace{\sum_{m=1}^{M} |\gamma_m| \underbrace{\left| \sin(v_m \cdot x + c_m) - \sum_{n=1}^{N} \alpha_n \sigma(v_m \cdot x + c_m - a_n) \right|}_{\leq \frac{\xi}{2 \sum_m |\gamma_m|}}}_{\leq \frac{\xi}{2}}$$
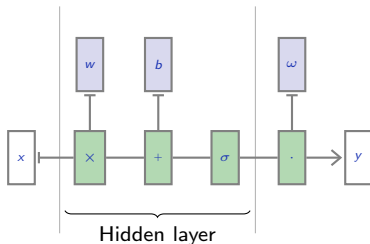
So we can approximate any continuous function

$$\psi : [0, 1]^D \to \mathbb{R}$$

with a one hidden layer perceptron

$$x \mapsto \omega \cdot \sigma(w\, x + b),$$

where $b \in \mathbb{R}^K$, $w \in \mathbb{R}^{K \times D}$, and $\omega \in \mathbb{R}^K$.

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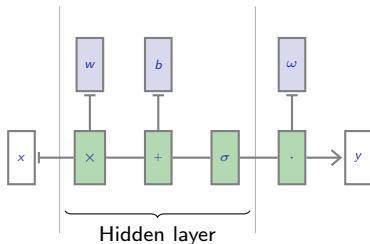where $b \in \mathbb{R}^K$, $w \in \mathbb{R}^{K \times D}$, and $\omega \in \mathbb{R}^K$.

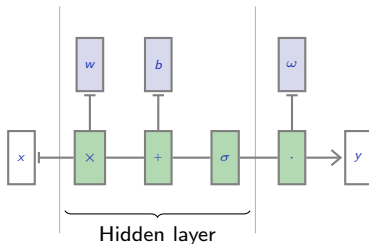Hidden layer

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⚠ A better approximation requires a larger hidden layer (larger K), and this theorem says nothing about the relation between the two.

The end