Realized Covariance Estimation in Dynamic Portfolio Optimization
Work in Progress

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Abstract

Mean-variance portfolio optimization requires both invertible and well-conditioned covariance matrices. As the number of dimensions increases relative to the number of sampled observations, the optimized portfolio weights become unstable as the covariance matrix becomes ill-conditioned. This paper compares the performance of covariance conditioning techniques applied to the realized covariance matrices of the portfolio of constituents of the Dow Jones Industrial Average. We use the volatility of portfolio returns derived from volatility-timing investment strategies employing different conditioning techniques as the criterion of assessment. In low dimensions realized covariance estimation offers precision gains over estimators constructed using daily level data. As the dimension of the portfolio increases, however, we see the increasing need for matrix conditioning to maintain the precision improvement offered by intraday data. We find that the relative performance of the single factor model provides a computationally tractable alternative to fully estimated realized covariance matrices in a global minimum variance dynamic portfolio setting.

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1 Introduction

Markowitz mean-variance (MV) optimization is the standard framework for optimal portfolio construction (See Chan, Karceski, and Lakonishok (1999), Jagannathan and Ma (2003), and references therein). MV optimization requires covariance matrices to be not only invertible, but also well-conditioned. Michaud (1989) points out that matrix inversion maximizes the effects of errors in the input assumptions and, as a result, practical implementation is problematic. Consistent with this, Britten-Jones (1999) find the sampling error of the weights of mean-variance efficient portfolios to be very large.

Realized covariance estimation has emerged as a viable candidate for covariance estimation. This class of estimators employs high-frequency data and provides more precise estimates. In a low dimensional setting, Fleming, Kirby, and Ostdiek (2003) have shown that realized covariance estimates provide utility gains over implied covariance estimates for risk averse investor following a MV optimization strategy. More recently, Liu (2008) determined that a manager tracking the S&P 500 index with the DJIA stocks will switch from from covariance estimates based on daily data to estimates using intraday data when there are less than 6 months of historical data available, or if she rebalances her portfolio daily.

The realized covariance literature has focused on improving these estimators by using techniques such as cross-market tick-matching, optimal sampling frequencies, and sub-sampling. Cross-market tick matching was introduced independently by Corsi (2006), and Hayashi and Yoshida (2005), and implementation concerns have been examined by Voev and Lunde (2007), Griffin and Oomen (2006), and Kyj, Ensor, and Ostdiek (2008). Optimal sampling is concerned with determining the optimal frequency that balances the amount of market microstructure effects introduced relative to the underlying covariance and has been discussed in Bandi and Russell (2006), Bandi, Russell, and Zhu (2008), Oomen (2006), and de Pooter, Martens, and van Dijk (2008). Sub-sampling, introduced by Zhang, Mykland, and Ait-Sahalia (2005), is a technique used to reduce the variance of realized covariance estimators. Realized kernel estimation using refresh time sampling introduced by Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008b) synthesizes all of these refinements into one estimator. For stocks which have frequent price updates, such as the DJIA constituents, Kyj, Ensor, and Ostdiek (2008) and Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008b) demonstrate that these realized covariance estimator refinements do not offer much improvement over the traditional calendar-time estimator introduced by Andersen, Bollerslev, Diebold, and Labys (2001).

In general, even without refinements, estimation of high dimensional realized covariance matrices is both computationally expensive and plagued by sampling error. The required computational time affects implementation feasibility and the sampling error can result in numerically ill-conditioned matrices, making inversion problematic. Under the assumption of normally distributed data, Ledoit and Wolf (2003) suggest that to minimize ill-conditioning the number of observations, n, needs to be at least ten times the number of dimensions, p. In the case of five-minute calendar time realized covariance estimation, for example, we have an effective sample size of $n = 78$ so this rule of thumb is exceeded when considering more than 7 assets. This is a paradox of high frequency data. What first appears to be a wealth of data is substantially diminished once asynchroneity and market microstructure effects are accounted for. Given the effective sample size available to estimate daily covariance matrices, we are once again confronted with sampling errors and ill-conditioned matrices.

Previous literature has addressed imprecise covariance matrix estimates by imposing more structure on the covariance matrix. Variants of shrinkage are employed to mitigate ill-conditioned matrices. Fleming, Kirby, and Ostdiek (2003) and de Pooter, Martens, and van Dijk (2008) use rolling estimators, Bandi, Russell, and Zhu (2008) use ARFIMA forecasting, Jagannathan and Ma (2003) use non-negative constraints on the portfolio weights, and Ledoit and Wolf (2003) use shrinkage toward the market estimate. Fan, Fan, and Lv (2007) find that the major advantage of factor models is in the estimation of the inverse of the covariance matrix and demonstrate that the factor model provides a better conditioned alternative to the fully estimated covariance matrix. To date, the realized covariance literature has focused on evaluating estimators for a very small number of assets. The exceptions to this are Voev (2008) and Liu (2008) who focus on forecasting the covariance matrix for portfolios of 15 assets and 30 assets, respectively. Both studies conclude that some form of smoothed realized covariance estimates provides improvements over daily level estimates.

Our paper contributes to the discussion by examining three features of interest to both academics and practitioners. We examine the performance of realized covariance estimators as the dimension increases from 3 to 30 and show the importance of conditioning the estimates at high dimensions. We show that in high dimensions, computationally sparse estimators offer portfolio volatilities that are similar to the best performing fully estimated covariance matrix. Finally, we examine in-sample performance to disentangle the precision of the estimator from the efficacy of the forecasting model.

This paper compares the performance of covariance conditioning techniques applied to the realized covariance matrices of the portfolio of constituents of the Dow Jones Industrial Average. We use the volatility of portfolio returns derived from volatility-timing investment strategies employing different conditioning techniques as the criterion of assessment. We assess the role that ill-conditioning plays on covariance estimation and portfolio optimization by examining subsets of increasing dimension. The relative performance of the single factor model is compared against a fully estimated realized covariance matrix in a dynamic global minimum variance setting. We conclude that the single factor model offers a compelling, computationally tractable alternative.

2 Methods

2.1 Realized Covariance Estimators

The discretely observed price process $p(t_i)$ is a function of both the latent price process $x(t_i)$ and the market microstructure effects $u(t_i)$, which are treated as "observation error" such that the price of asset A is observed as:

$$p_A(t_i) = x_A(t_i) + u_A(t_i), \quad i = 1, 2, \ldots, n. \tag{1}$$

Hence, returns are written as

$$r_A(t_i) = \Delta p_A(t_i). \tag{2}$$

Andersen, Bollerslev, Diebold, and Labys (2001) first proposed realized variance estimation using ad-hoc calendar-time sampling. Synchronous observations across markets are achieved by interpolating previous tick prices onto an ad-hoc common sampling grid (e.g., every 5 minutes) yielding m equally spaced intraday observations. These calendar time realized covariance estimators can be written as:

$$\widehat{\Sigma}_{AB}(m) = \sum_{i=1}^{m} r_A(i) \times r_B(i). \tag{3}$$

For variance reduction, Zhang, Mykland, and Ait-Sahalia (2005) advocate sub-sampling and averaging as a technique for exploiting the richness of high frequency data. They divide the time domain grid into K non-overlapping subgrids and average the estimates over the K subgrids to calculate the final estimate. As outlined in Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008a), sub-sampling is equivalent to using a Bartlett kernel.

Alternative methods for estimating realized covariance in the presence of market micro structure noise are numerous, but are beyond the scope of this paper. We focus instead on techniques to improve the condition number of the covariance matrix post estimation. Moreover, as shown in Kyj, Ensor, and Ostdiek (2008), for actively quoted stocks it is difficult to provide economic utility over the simple 5 minute previous tick estimator.

2.2 Ill-Conditioned Covariance Matrices

Many applied financial problems require a covariance matrix estimator that is not only invertible, but also well-conditioned. While the true covariance matrix is guaranteed to be well-conditioned, estimates may not be due to sampling error. The sample covariance matrix is a consistent estimate of the true covariance matrix as $\frac{p}{n} \to 0$, but when $\frac{p}{n} \to c$ it may be ill-conditioned. When the sample covariance matrix is not consistent it is due to the accumulation of a large number of small errors off the diagonal. Michaud (1989) points out that within the mean-variance context, ill-conditioned covariance estimates result in exaggerated estimation error. This problem has been identified as a barrier to practitioner adoption of the men-variance framework.

A positive definite matrix is necessary for matrix inversion, an necessary step in the mean-variance framework. The following three tests are necessary and sufficient conditions for a symmetric matrix A, with eigenvalues λ_i, to be positive definite:

1. $v^T Av > 0$ for all nonzero vectors v.
2. All the eigenvalues of A satisfy $\lambda_i > 0$.
3. All the upper left submatrices A_k have positive determinants.

The relationship between positive definiteness and invertibility is understood via the eigenvalues, where the determinant is defined as: $det(A) = \prod_{i=1}^{p} \lambda_i$. A matrix is invertible when the $det(A) \neq 0$, therefore the second test ensures that a positive definite matrix is invertible.

A well-conditioned operator is defined as having the property that all small perturbations of x lead to only small changes in $f(x)$. The condition number of a matrix A is defined as: $\kappa(A) = \|A\|\|A^{-1}\|$, where $\|\cdot\|$ is the Frobenius norm and can be expressed as: $\|A\| = \sqrt{tr(AA^T)}$. We note that $\|A\|^2 = \sum_{i=1}^{p}\sum_{j=1}^{p} A_{ij}^2$, and in our case A is symmetric so $\|A\|^2 = \sum_{i=1}^{p} \lambda_{Ai}^2$ where λ_{Ai} are the eigenvalues of A. In this setting, the condition number can be interpreted as the eccentricity of the ratio of eigenvalues. An ill-conditioned matrix is close to being non-invertible.

The definitions above state that the relative magnitude of the eigenvalues of the realized covariance matrices play a very prominent role in mean-variance asset allocation. Positive definiteness requires the eigenvalues be positive and the well-conditioned property imposes an additional requirement of non-vanishing eigenvalues. Intuitively, as an eigenvalue shrinks toward 0, this implies that a principal component vanishes and the rank is reduced, and we are now faced with collinearity in the system. As the smallest eigenvalue vanishes toward zero, the condition number explodes to infinity. This problem highlights the importance of preserving the smallest eigenvalue which is accomplished by imposing more structure to mitigate the imprecision of covariance estimates. In the following subsections we present two approaches, shrinkage and single-factor models, that have been shown to result in well-conditioned, consistent estimators of the true covariance matrix

2.3 Shrinkage Estimators

As the number of dimensions increases relative to the number of observations, the resulting covariance matrices become ill-conditioned and, in particular, are characterized by the smallest estimated eigenvalues being too small and the largest being too big relative to the true eigenvalues. Shrinking these estimated covariance matrices towards some idealized structure yields more stable estimates. The resulting eigenvalues are more compressed and the covariance estimates are better conditioned.

Generally, linear shrinkage can be written as:

$$\widetilde{\Sigma}_t(\alpha) = (1-\alpha)G + \alpha\widehat{\Sigma}_t, \quad \alpha \in [0,1] \tag{4}$$

where $\widehat{\Sigma}_t$ is the estimate of Σ_t, the covariance matrix of returns, and G is the idealized covariance structure. In the remainder of this section we consider two forms of linear shrinkage: the Ledoit-Wolf estimator, and the rolling estimator.

2.3.1 Ledoit-Wolf Estimators

Ledoit and Wolf (2003, 2004b) introduced an estimator that is an optimal linear combination of a target matrix and the sample covariance matrix under squared error loss:

$$E[\|\widetilde{\Sigma}_t - \Sigma_t\|^2], \tag{5}$$

where $\widetilde{\Sigma}_t(\alpha) = (1-\alpha)G + \alpha\widehat{\Sigma}_t, \quad \alpha \in [0,1]$, and G is the target matrix. Intuitively, this estimator seeks to minimize the quadratic distance between the true and the estimated covariance matrices. We use the equicorrelated matrix suggested by Ledoit and Wolf (2004a) as our target matrix. The equicorrelated matrix is defined by all the off-diagonal elements of the covariance matrix having the average sample correlation. Φ, with elements ϕ_{ij}, denotes the unobserved true equicorrelated covariance matrix and F, with elements f_{ij}, is the corresponding estimate. The diagonal elements are the variance elements of the sample covariance matrix. The optimal shrinkage parameter α is estimated according to the method outlined in Ledoit and Wolf (2003), where $a_{i,j}$ corresponds to elements of matrix A:

$$\alpha = 1 - \max\{0, \min\{\theta^*, 1\}\}, \tag{6}$$

where

$$\theta^* = \frac{\sum_{i=1}^{p}\sum_{j=1}^{p} Var(s_{ij}) - Cov(f_{ij}, s_{ij})}{\sum_{i=1}^{p}\sum Var(f_{ij} - s_{ij}) + (\phi_{ij} - \sigma_{ij})^2}. \tag{7}$$

Our formulation of shrinkage stated in Equation 4 is different from the one used in Ledoit and Wolf. We define θ^*, the Ledoit Wolf shrinkage parameter in Equation 7 and in Equation 6 we show how the Ledoit Wolf shrinkage parameter relates to our shrinkage parameterization. Voev (2008) outlines the adaptation of the Ledoit-Wolf estimator for realized covariance estimators, drawing upon the the asymptotic variance and asymptotic covariance results derived in Barndorff-Nielsen and Shephard (2004). We avoid estimating the covariance of the covariance matrix due to the rapid explosion in parameters to $(p \times p) \times (p \times p)$, which makes estimation in high frequency even more onerous. Instead we assume that the covariance process is locally stationary, hence $s_{ij,t-l} \sim \sigma_{ij,t}$ for all t and a window of l values. We assume that $E[s_{ij,t-l}] = \sigma_{ij,t}$ and we use a lagged window of covariance values to estimate θ^*. In this formulation S and $s_{i,j}$ represent the realized covariance matrix estimate and its individual elements. The quantities $Var(s_{ij}), Cov(f_{ij}, s_{ij}), Var(f_{ij} - s_{ij})$, and $(\phi_{ij} - \sigma_{ij})^2$ are calculated using the daily time series of realized covariance estimates.

The results of the robustness analysis in Appendix A validate our approach of using local stationarity of the time series of realized covariance estimates. Specifically we see that the performance of the Ledoit Wolf shrinkage technique is not overly sensitive to the estimation of the α parameter. While our method may result in less precise α estimates, the sensitivity analysis suggests that in this application misspecification does not have a impact on the shrinkage estimator.

The positive definiteness of the Ledoit-Wolf estimator can be considered by examining

$$v^T\widetilde{\Sigma}_t v = (1-\alpha)v^T G v + \alpha v^T \widehat{\Sigma}_t v,$$

where positive definiteness requires $v^T\widetilde{\Sigma}_t v > 0$ for all nonzero vectors v. To ensure positive definiteness, the target matrix G is chosen to be positive definite and the shrinkage factor α is chosen to maintain positive definiteness of the resulting estimator. As stated in Ledoit and Wolf (2004b) this procedure shrinks the estimated eigenvalues towards the eigenvalues of the target matrix. The resulting eigenvalues have a smaller maximum and a larger minimum, resulting in a better conditioned estimator.

2.3.2 Rolling Estimators

Rolling estimation is a common feature in covariance applications (See Fleming, Kirby, and Ostdiek (2003), Bandi, Russell, and Zhu (2008), de Pooter, Martens, and van Dijk (2008), Bandi and Russell (2006) for esmpales in the realized covariance literature). This approach is motivated by conditional heteroskedasticity, a well know property of financial time series. In the presence of conditional heteroskedasticity, estimation of the covariance matrix involves a trade-off between considering a sufficiently large number of observations to obtain an unbiased and consistent estimate and considering a short enough history to accommodate changes in the covariance structure. Rolling realized covariance estimation attempts to balance the statistical power obtained using a large sample against potential problems caused by heteroskedasticity. It is easy to see that

$$\widetilde{\Sigma}_t = (1-\alpha)\widetilde{\Sigma}_{t-1} + \alpha\widehat{\Sigma}_t, , \quad \alpha \in [0,1] \tag{8}$$

is a variant of shrinkage estimation where the current realized covariance estimate is shrunk toward a function of past estimates. $\widetilde{\Sigma}_t$ is in fact an exponentially weighted rolling covariance estimator for Σ_t. Based on the work of Foster and Nelson (1996) and Andreou and Ghysels (2002) we use an exponentially weighted rolling scheme which provides MSE efficiency gains for realized covariance estimators.

We estimate the shrinkage parameters, or optimal decay parameters, by noting that Equation 8 resembles a multivariate GARCH(1,1) and adopting the GARCH estimation technique. As noted by, Kawakatsu (2006) and Tse and Tsui (2002), multivariate GARCH models suffer from the "curse of dimensionality" in that the number of parameter estimates grows quadratically the models do not reliably deliver positive definite covariance matrices. Restricting α to being a scalar value eliminates the "curse of dimensionality" problem and helps maintain positive definiteness. We determine α using maximum likelihood estimation as outlined in Tse and Tsui (2002):

$$\ell = \sum_{t=1}^{N} \left(-\frac{1}{2} \log |2\pi \widetilde{\Sigma}_t| - \frac{1}{2} tr(\widetilde{\Sigma}_t^{-1} \widehat{\Sigma}_t) \right). \tag{9}$$

Determining positive definiteness of the rolling estimator draws upon Foster and Nelson (1996) and assumes that $\widetilde{\Sigma}_{t-1}$ is a consistent estimator of Σ_{t-1}. The idealized target matrix, therefore, possesses the desirable properties of the true covariance matrix: positive definite and well-conditioned. Similar to the Ledoit-Wolf estimator, rolling estimation shrinks the eigenvalues of the realized covariance estimator towards the more consistent eigenvalues of the rolling estimator resulting in better conditioned covariance matrices.

2.4 Factor Model

Factor models offer an extreme variation of shrinkage models that greatly greatly simplifying the estimation of the covariance matrix. Assuming conditional independence of contemporaneous returns of a large number (p) of assets given a small number (K) of factors, dramatically reduces the number of parameters needed to estimate the cross-sectional dependence among returns. Chamberlain and Rothschild (1983) discuss the relationship between factor structure and asset pricing. Chan, Karceski, and Lakonishok (1999) and Jagannathan and Ma (2003) both show that factor models can reduce the variance of optimal mean-variance portfolios and offer utility gains over strategies employing full sample covariance matrices. Specifically, Jagannathan and Ma (2003) compare the performance of portfolios determined using sample covariance matrix estimates (based on daily data), but constrained to have non-negative portfolio weights, to the performance of portfolios based on factor and shrinkage covariance estimators. They show that the single factor model performs very well when the number of observation is not much greater than the number of dimensions. Han (2006) discusses the importance of factor models in high dimensional settings. Bollerslev and Zhang (2003) employ high-frequency data in a multi-factor model and find improved asset pricing predictions when compared with conventional monthly rolling estimates. In this paper, we assess realized covariance estimation using a single factor model.

The single factor model introduced by Sharpe (1963) states that:

$$r_{A,t} = \alpha_A + \beta_A r_{M,t} + \epsilon_{A,t}. \tag{10}$$

where r_A is the return of the individual stock, r_M is the return of the market as represented by the index, and β represents the systemic risk. We assume that $\epsilon_t \sim N(0, \sigma_\epsilon^2)$ and that residuals ϵ_A are uncorrelated to market returns. The resulting covariance matrix is:

$$\Phi = \sigma_{M,M}^2 \beta\beta^T + D. \tag{11}$$

From ordinary least squares regression we know that $\beta_A = \frac{\sigma_{A,M}}{\sigma_M^2}$. We can estimate b, representing the vector of estimates for β, by computing the realized covariances with the index and the realized variance of the index. The matrix D represents the diagonal matrix of residual variances $VAR(\epsilon)$. It is natural to set $\phi_{i,i} = s_{i,i}$. The estimated Φ can be written as:

$$\widehat{\phi}_{i,j} = \begin{cases} s_i^2 & \text{if i=j} \\ s_{M,M}^2 bb^T & \text{if i} \neq \text{j}. \end{cases} \tag{12}$$

The assumption implicit in Equation 11 is that all the co-variation between stocks is captured by the co-variation estimated by the market index, that is, the single factor captures all the systemic risk across all the assets. AS can be seen by examining $v^T \Phi v$, the covariance matrix of the single-factor model is always positive definite:

$$v^T \Phi v = v^T \left(\sigma_M^2 \beta \beta^T + D \right) v = \sigma_M^2 \underbrace{v^T \beta \beta^T v}_{>0} + \underbrace{v^T D v}_{>0}. \tag{13}$$

For all nonzero vectors v, $v^T D v > 0$ as D is a diagonal matrix of positive values. Likewise, $\beta^T v = z$, is a scalar, and as a result $v^T \beta \beta^T v = z^2 > 0$.

In terms of conditioning the matrix, Fan, Fan, and Lv (2007) show that when $K = o(p)$, where K is the number of factors, the inverse of the factor model covariance matrix converges to the true inverse covariance faster than the inverse of the sample covariance matrix, implying that the factor model is a better conditioned alternative to the fully estimated covariance matrix.

3 Assessment Criteria

3.1 Volatility Timing

We consider portfolio allocation as our criterion for assessing estimator performance. Chan, Karceski, and Lakonishok (1999) and Jagannathan and Ma (2003) advocate the use of the Global Minimum Variance (GMV) portfolio to avoid the problematic estimation of μ_t, the vector of expected returns. The GMV portfolio is independent of μ_t and is the solution to the following optimization problem:

$$\begin{aligned} min_{w_t} \quad & w_t' \Sigma_t w_t \\ s.t. \quad & w_t' \iota = 1 \end{aligned} \tag{14}$$

where Σ is the covariance matrix and ι is a unitary vector of length p. The GMV weights are given as:

$$w_{t,GMV} = \frac{\Sigma_t^{-1} \iota}{\iota' \Sigma_t^{-1} \iota}. \tag{15}$$

At the end of day t we estimate the covariance either using a long horizon of historical open-to-close returns or that day's intraday returns. We then use the covariances to determine the GMV portfolio weights and compare the covariance estimators on the basis of the volatility of the resulting time series of portfolio returns.

3.2 Forecasting

Forecasting emerges as a nontrivial issue in the assessment of the estimators. We control for this by presenting both in-sample and out-of sample forecast results. For the in-sample experiments, the time series of realized portfolio returns are defined based on the log difference of the open price to close price of day t. As such, the returns are known and only the covariance structure is left to estimation. This allows us to focus on the incremental benefits of covariance conditioning methods, independent of forecast models. In the out-of-sample experiments, one step ahead returns are defined as the log difference between the closing price of day $t+1$ and the opening price of day $t+1$. We distinguish between these two settings because the out-of-sample results rely on both the precision of the estimate and the robustness of the forecast model, an issue beyond the scope of this paper.

Nonetheless, to get an idea of out-of-sample peformance we consider the classic Markovian forecast, the simplest forecasting model which relies upon the long memory property of realized covariance. The underlying assumption of this forecasting model is that $E[S_{t+1}|\Im_t] = S_t$, where $\Im_t$ is the information set up to time t. Forecasting realized covariance matrices is challenging due to the constraint that the resulting forecasts must to be positive definite. This in part motivates the choice of this simple method. Voev (2008) and Bauer and Vorkink (2007) consider more sophisticated multivariate forecasting via Cholesky decomposition and matrix exponentiation respectively. Unfortunately, Cholesky decomposition is of limited applicability for high dimensional problems as it requires that the initial estimate be positive definite. In this study all of our forecasted estimators are positive definite and better conditioned than the original realized covariance estimator. We can attribute this to the dimension of the problem and the simplicity of the forecasting methods.

4 Empirical Analysis

We present the mean, median, standard deviation, and minimum and maximum weights of the respective volatility-timing portfolios generated using different covariance estimators. We consider four classes of estimators: naive estimators used as benchmark strategies, fully estimated realized covariance matrices, shrinkage estimators, and single factor estimators. For each of the realized covariance estimators, we also consider a sub-sampled version. We assess the performance of the estimators by considering the volatility of the time series of returns of the dynamic Global Minimum Variance (GMV) portfolios based on the estimated covariances. To facilitate comparison with previous literature, we also consider strategies with nonnegativity constraints imposed on the portfolio weights. While we report the mean and median return for each strategy as a indicator of the nature of the portfolio, the weights in each strategy are independent of expected return estimates.

4.1 Data: Dow Jones Industrial Average

We consider the the covariance structure of the quotes of the stocks in the Dow Jones Industrial Average (DJIA) over the period from January 1, 2002 to December 31, 2006. We use the first year to generate estimates and implement dynamic volatility timing for the remaining four years. We consider a single factor model with the DJIA as the sole factor. We use the DIAMONDS Trust, Series 1 (DIA) as a proxy for the DJIA. This exchange traded fund (ETF) holds a portfolio of the equity securities that comprise the Dow Jones Industrial Average and is traded on the American Stock Exchange. The stock price data is from the TAQ database and is filtered according to the trade-quote matching technique suggested in Lee and Ready (1991) and Henker and Wang (2006).

Table 1 presents the stocks in the DJIA over our sample period with ticker symbol, average annual mean, and average annual median. Variance estimates are summarized by average annualized realized volatility estimates using high frequency data (HFvol), average annualized volatility of realized variance estimates using high frequency data multiplied by 100 (HFvv100), lag one autocorrelation of realized variance using high frequency data (HFauto), average annualized open-to-close returns squared (LFvol), average annualized volatility of open-to-close returns squared (LFvv), lag one autocorrelation of open-to-close returns squared (LFauto). Consistent with expectations, volatility estimates using intraday data are much lower than open-to-close squared returns and they display a greater degree of autocorrelation as well.

4.2 Results

To implement the Ledoit Wolf and the rolling estimators, we must first estimate the shrinkage parameter, α. Table 2 presents these parameter estimates based on a rolling window of historical data. The estimates are consistent with our intuition that sub-sampled estimators, denoted by "Sub", result in more precise estimates and require less smoothing, hence a larger α. Recall that we have reparameterized the Ledoit Wolf formula to correspond with the rolling formula, so our $1 - \alpha$ should be compared to the Ledoit Wolf parameter estimates reported in the literature.

To implement the factor realized covariance estimator, we must first fit the factor model to the data each day. Table 3 presents the average of the daily estimate of the β coefficients, the average of the daily estimated correlation ρ, and the average of the daily estimated R^2 for each stock. Note that the average coefficient estimate is 0.73 and the average R^2 is only 0.18. Estimated β ranges from 0.53 for JNJ to 1.16 for INTC and the R^2 ranges from 0.05 for HP to 0.27 for GE.

Table 4 presents the in-sample performance of the global minimum variance portfolios using open-to-close returns. We present the mean, median and standard deviation of the returns to the dynamic portfolio. We also present the minimum portfolio weight, the median minimum, the median maximum, and the maximum portfolio weights across all days in the sample. These final four statistics allow us to monitor the presence of ill-conditioned estimates that result in extreme portfolio weights.

In panel (a) we present the benchmark strategies, which use open-to-close data, representing conventional covariance estimation. DIA presents the results of a static investment in the ETF and Equal provides the results of an investment strategy with equal weight in each asset. Note, that the means and medians of DIA and Equal, the benchmark portfolios, are very different. This suggests that the means are in fact misleading and are being driven by a few outlier portfolio returns. (The finding that mean open-to-close returns are near zero is consistent with the results presented in Cliff, Cooper, and Gulen (2007).) Both of these strategies represent static portfolios, i.e. the portfolio weights are independent of the covariance matrix, so the outlier portfolio returns are not artifacts of either our estimation methodologies or our trading strategy.

We also report the results for dynamic GMV strategies based on covariance estimates that use open-to-close returns. The covariance matrix for $\bar{S}$ Year is determined by using one year of historical open-to-close returns. The performance of this estimator provides the benchmark for comparison for the remaining high frequency estimators. The $\bar{S}$Year-W_+ estimator uses the same covariance estimates as $\bar{S}$ Year, but constrains the portfolio weights to be non-negative. The $\bar{S}$ Year SI estimator is a single factor model applied to open-to-close returns. The $\bar{S}$ 78days estimator, which uses only 78 days of historical returns, provides an indication of performance when the number of observations equals the number of observations used for the realized covariance estimators. Finally, we include the $\bar{S}$ 30days estimator, which uses only 30 days of historical returns, to provide an indication of performance when the number of observations equals the number of assets.

Note, 30 days provides us with a positive definite covariance matrix, but the resulting minimum and maximum weights indicate that this estimator is ill-conditioned. Indeed these extreme portfolio weights lead, in turn, to very high portfolio return volatility.

In panel (b) we present the results of the fully estimated Realized Covariance (RC), Realized Covariance ($RC - W_+$) with the non-negativity constraint on portfolio weights, Realized Covariance Subsampled (RC-Sub), and the Realized Covariance averaged over one year ($\overline{RC}$ Year) estimators. The Realized Covariance Sub-sampled estimator averaged over one year is not presented as it provides the same results as the $\overline{RC}$ Year estimator. The RC estimator generates portfolios with higher volatility than the $\bar{S}$ Year benchmark and examination of the portfolio weight characteristics indicate that this estimator is vulnerable to being ill-conditioned. The sub-sampled estimator provides better performance in terms of portfolio volatility than both the RC and benchmark estimators. We attribute the lower portfolio volatility of the sub-sampled estimator to a substantial reduction in the extreme weights. This suggests that when spurious extreme weights are eliminated there are gains in precision from using sub-sampled realized covariance in place of a long time series of daily level data. The $\overline{RC}$ Year estimator provides lower volatility than the RC, but not lower than the RC-Sub estimator. We also see that the weights of the $\overline{RC}$ Year do not have extreme positions. It appears to be a better conditioned estimator than the RC and RC-Sub. However, the higher volatility of returns for RC than for RC-Sub suggests that averaging a year of RC estimates results in a loss of precision. Evidently, important short horizon information is lost by averaging over this long horizon.

Panel (c) presents the results for the shrinkage estimators. As with the $\overline{RC}$ Year estimator, the rolling estimator yields portfolio volatilities that are lower than the RC estimator, but not lower than the RC-Sub estimator. As expected, the rolling estimators result in lower portfolio volatility than the $\overline{RC}$ Year estimator. The rolling estimators are better conditioned, with less extreme weights, and the volatility of the dynamic portfolio is similar to the $\bar{S}$ Year benchmark. The Ledoit Wolf estimators offer comparatively low portfolio return volatility. The extreme weights are similar to the $\bar{S}$ Year benchmark. This suggests that this estimator is able to capture the day-to-day covariance changes, while smoothing out some of the estimation error.

Panel (d) presents the results using the Factor model. A striking feature of these results is that, for this set of stocks, a single-factor model is capable of generating GMV portfolios with lower volatility than a fully estimated covariance matrix. The sub-sampled volatility results of the parsimonious Factor model and the Full RC-Sub are similar. On the other hand, the weight characteristics are quite different, with the Factor model displaying much less extreme weights. Considered on balance, the Factor model offers the smallest medians of extreme weights across all the estimators considered. This suggests that the Factor models are better conditioned than the alternative estimators. Compared to the performance of $\bar{S}$ Year Factor, the Factor model based on open-to-close returns, we see lower volatility of returns. It is clear that the conditioning strategies substantially reduce the magnitude of the minimum and maximum weights. Conditioning allows us to derive sensible portfolio optimization weights without directly constraining those weights.

To furhter investigate the effects of ill-conditioning on our covariance estimations, we consider our GMV strategies across a range of dimensions. We generate portfolios by averaging a number of randomly sampled subsets, thereby avoiding stock selection bias in the lower dimensional portfolios. The results, presented in Table 5, highlight the importance that dimensionality plays in the covariance estimation problem, and confirms that by moving from 3 assets to 30, we find ourselves in a new setting.

Panel (a) presents the benchmark estimators. We see that as the number of assets increases, the volatility of the equally weighted portfolio decreases, as expected with greater diversification. Likewise, for the $\bar{S}$ Year estimator, we see a steady decline in volatility of portfolio returns as the number of assets increases. The $\bar{S}$ 30days estimator outperforms the $\bar{S}$ Year and $\bar{S}$ 78days estimators for dimensions 3 through 20. This highlights the importance of using a short horizon to capture relevant information that can be lost when using an excessively long horizon. At 30 assets, we see that the $\bar{S}$ 30days estimator performs very poorly because it is close to singular at this dimension.

Panel (b) provides results for the fully estimated realized covariance estimators. Recall that theses estimators use only 78 intraday observations. As demonstrated by previous literature, at low dimensions the realized covariance estimator outperforms the equally weighted and $\bar{S}$ Year estimators. Only at dimensions 20 and 30 do we see that the realized covariance estimator fails to outperform the $\bar{S}$ Year estimator. The subsampled variant, RC-Sub, provides lower volatility of portfolio returns than the RC estimator for the entire set of dimensions considered. This suggests the RC estimator becomes ill-conditioned at dimensions 20 and 30, and that sub-sampling offers a better conditioned estimator by averaging across a number of covariance estimates each day. The Factor estimator is constructed using elements of the RC estimator and it shows how conditioning strategies play a greater role as the dimension increases. For low dimensions, 3 to 10, the Factor estimator produces similar results to the RC estimator, and from 15 to 30 the Factor estimator provides much lower variance than the RC estimator.

4.3 One-step Ahead Forecasts

Table 6 presents the out-of-sample performance characteristics of dynamic global minimum variance portfolios using open-to-close returns. Incorporating close-to-open returns into the high-frequency estimators is a non-trivial matter and at this stage we present results which do not include close-to-open returns. We consider a series of covariance matrix forecasts and evaluate one-step ahead performance of the optimal portfolios. We present the mean, median and volatility of the resulting portfolio returns. The DIA and Equal portfolios show much larger means and medians than in the in-sample open-to-close results in Table 4. This suggests that overnight returns are greater than returns during the day and is consistent with the results presented in Cliff, Cooper, and Gulen

(2007), who find this phenomenon consistently present across exchanges, days of the week, days of the month, and months of the year.

Neither the RC nor RC-Sub estimations outperform the benchmark. We see that the precision gains realized by the RC-Sub in Table 4 are lost, perhaps due to the simplicity of the forecasting technique. On the other hand, the conditioned estimators derived from the RC all outperform the benchmark. The rolling estimators and $\overline{RC}$ Year estimators perform best. These estimators take advantage of a longer horizon of historical data. The Ledoit Wolf and the parsimonious Single Factor estimators continue to provide portfolios with some of the smallest volatilities. These results suggest that the Markovian forecast is perhaps too simple and underline the importance of developing more sophisticated forecasting models.

5 Future Work and Conclusion

Our paper contributes to the realized covariance literature by examining three features of interest to both academics and practitioners. We examine the performance of realized covariance estimators as the dimension increases from 3 to 30 and show the importance of conditioning the estimates at higher dimensions. We show that at high dimensions, computationally simple estimators offer portfolio volatilities that are similar to the best performing fully estimated covariance matrix. Finally, we examine in-sample performance to disentangle precision from forecasting. Both estimation and forecasting present sources of error, and we argue that by only looking at the final forecast results, the precision of an estimator can be overshadowed by the errors of a naive forecast.

As future work we will consider more sophisticated one-step ahead forecasts using the single factor estimators. The current in-sample results using the single-factor model are encouraging and suggest that this is a sensible starting point for high dimensional covariance forecasting. An attractive feature of the single-factor model is that it is straightforward to apply, as there is no intermediate step requiring estimation of a smoothing factor. Furthermore, the single-factor approach circumvents the problem of ensuring positive definite forecasts. Our empirical analysis has focused primarily on open-to-close returns. In future work we will include overnight returns according to techniques suggested by Gallo (2001) and Hansen and Lunde (2005). We will then consider the robustness of the resulting volatilities of returns using different covariance estimators within a global minimum variance portfolio allocation strategy. This will allow for the replication of more practitioner oriented trading strategies and will allow us to examine allocation strategies in the presence of transaction costs.

We have compared a number of conditioning techniques for realized covariance and the characteristics of the Markowitz mean-variance optimized portfolios they generate. We argue that in the presence of linear versus quadratic growth in computational complexity and ill-conditioned matrices, factor models offer a tractable solution to portfolio optimization problems.

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APPENDIX

A Robustness to Shrinkage Parameter Estimation

We suspect that the sampling errors with respect to the estimated shrinkage parameter are rather large, and warranting a robustness analysis. In this section, we show the robustness of the shrinkage methods for different values of of the shrinkage parameter α. Table 7 presents the resulting volatility of portfolio returns using GMV as the investment strategy. We conclude that the variation in results is quite modest with respect to a large range of shrinkage parameters. Recall that our target matrix is the the constant correlation matrix which is somewhat similar to the single-factor estimator. In particular, both of these estimators have the same diagonal elements. We can easily see that the size and frequency of extreme weights are reduced as we smooth. Again, we see that out of sample forecasts call for more smoothing than do in sample estimates.

Table 1: Components of DJIA. Symbols denote the following time periods of stocks included in Dow 30 (DAYMONTHYEAR), $\dagger$: $08042004 - 31122006$, § : $01012002 - 07042004 and 22112005 - 31122006$, $*$: $01012002 - 07042004$, and $\ddagger$: $01012002 - 21122005$.

Ticker	mean	median	HFvol	HFvv100	HFauto	LFvol	LFvv	LFauto
AA	−0.2464	−0.3674	0.0314	0.0055	0.7223	0.2618	0.0080	0.1459
$AIG^{\dagger}$	−0.0937	−0.1371	0.0198	0.0038	0.4640	0.1707	0.0077	0.1124
AXP	0.1726	0.0947	0.0253	0.0066	0.7726	0.2252	0.0092	0.1555
BA	0.1325	0.0965	0.0284	0.0054	0.7856	0.2371	0.0076	0.1316
C	0.0089	−0.0274	0.0264	0.0095	0.7876	0.2236	0.0158	0.4667
CAT	0.0410	0.0000	0.0263	0.0035	0.6383	0.2387	0.0074	0.0163
DD	−0.0075	−0.0602	0.0243	0.0043	0.7719	0.1980	0.0069	0.0696
DIS	0.2263	0.2074	0.0297	0.0068	0.7740	0.2463	0.0080	0.2084
EK^{*}	0.0028	−0.0914	0.0305	0.0062	0.4320	0.2629	0.0116	0.0394
GE	−0.0662	−0.1024	0.0245	0.0057	0.7622	0.2149	0.0078	0.2470
GM	−0.3653	−0.4409	0.0318	0.0071	0.4776	0.3044	0.0124	0.1951
HD	−0.0633	−0.0879	0.0281	0.0063	0.8200	0.2388	0.0087	0.1965
HON	−0.1248	−0.0342	0.0310	0.0076	0.4500	0.2564	0.0105	0.1096
HP	−0.0780	−0.1490	0.0343	0.0049	0.5528	0.3126	0.0097	0.0780
IBM	0.0776	0.1176	0.0219	0.0035	0.8145	0.1980	0.0058	0.1192
INTC	−0.2160	−0.2499	0.0345	0.0085	0.8006	0.3197	0.0130	0.2686
IP^{*}	−0.1229	−0.2022	0.0250	0.0033	0.6857	0.2175	0.0060	0.1003
JNJ	0.0916	0.0234	0.0208	0.0042	0.7908	0.1647	0.0042	0.1474
JPM	0.0728	0.0686	0.0304	0.0142	0.6278	0.2785	0.0285	0.4086
KO	0.1579	0.1221	0.0203	0.0032	0.8292	0.1607	0.0038	0.1565
MCD	0.2326	0.2685	0.0272	0.0056	0.5839	0.2271	0.0081	0.1783
MMM	−0.0034	−0.1164	0.0205	0.0028	0.6948	0.1745	0.0046	0.0247
MO	0.0282	−0.0584	0.0244	0.0078	0.2257	0.2175	0.0130	0.0827
MRK	0.1240	0.0802	0.0261	0.0069	0.3822	0.2247	0.0101	0.1198
MSFT	−0.0701	−0.0479	0.0253	0.0051	0.8092	0.2299	0.0086	0.4602
$PFE^{\dagger}$	−0.2124	−0.1928	0.0207	0.0043	0.1208	0.1723	0.0043	0.0583
PG	0.2939	0.2822	0.0182	0.0020	0.7682	0.1469	0.0031	0.1533
$SBC^{\ddagger}$	−0.0334	−0.0502	0.0332	0.0097	0.7717	0.2617	0.0100	0.1725
$T^{\S}$	−0.0955	−0.0938	0.0283	0.0056	0.5980	0.2581	0.0090	0.2568
UTX	0.0375	−0.0133	0.0240	0.0039	0.6611	0.2094	0.0064	0.1225
$VZ^{\dagger}$	−0.1619	−0.1374	0.0183	0.0015	0.2982	0.1365	0.0018	0.0396
WMT	−0.0614	−0.1353	0.0226	0.0042	0.8232	0.1864	0.0050	0.1681
XOM	0.1011	0.2437	0.0229	0.0037	0.8012	0.1958	0.0064	0.0776
DIA	0.0053	0.0887	0.0158	0.0024	0.8398	0.1394	0.0039	0.0610

Table 2: Optimal weight parameters for 4 different realized covariance estimators.

	Rolling		Ledoit Wolf	
α	RC	RC-Sub	LW	LW-Sub
min	.0100	.0100	.0591	.1662
mean	.0187	.0212	.4637	.5364
max	.0621	.0684	.6738	.7479

Table 3: Factor Model Estimates

Ticker	β	ρ	R^2
AA	0.87	0.39	0.15
AIG	0.70	0.39	0.15
AXP	0.73	0.47	0.23
BA	0.82	0.43	0.18
C	0.82	0.52	0.27
CAT	0.91	0.48	0.23
DD	0.78	0.47	0.22
DIS	0.76	0.39	0.16
EK	0.67	0.32	0.10
GE	0.79	0.52	0.27
GM	0.76	0.37	0.14
HD	0.84	0.45	0.20
HON	0.94	0.45	0.21
HP	0.57	0.22	0.05
IBM	0.74	0.50	0.25
INTC	1.16	0.50	0.25
IP	0.74	0.42	0.17
JNJ	0.53	0.40	0.16
JPM	0.85	0.48	0.23
KO	0.58	0.43	0.18
MCD	0.71	0.38	0.15
MMM	0.72	0.50	0.25
MO	0.57	0.37	0.14
MRK	0.67	0.39	0.15
MSFT	0.81	0.49	0.24
PFE	0.63	0.34	0.11
PG	0.57	0.44	0.19
SBC	0.81	0.43	0.18
T	0.63	0.34	0.11
UTX	0.78	0.47	0.22
VZ	0.61	0.36	0.13
WMT	0.74	0.48	0.23
XOM	0.68	0.44	0.19
Mean	0.74	0.43	0.18
Median	0.74	0.43	0.18

Table 4: Performance Characteristics of Global Minimum Variance (GMV) portfolios using open-to-close returns from 2003-2006, in sample

	Portfolio Returns			min *weights*		max *weights*	
Model	Mean	Median	σ	min	median	max	median
(a) Benchmark: Daily data							
DIA	0.0115	0.0961	0.1083				
Equal	0.0135	0.1108	0.1102	0.0333	0.0333	0.0333	0.0333
$\bar{S}$ Year	0.1204	0.1084	0.0859	-0.1709	-0.0982	0.5282	0.2341
$\bar{S}$ Year-W_+	0.0781	0.0687	0.0932	0	0	0.3226	0.1589
$\bar{S}$ Year SI	0.1837	0.1804	0.0914	-0.9603	-0.0772	0.4891	0.2299
$\bar{S}$ 78days	0.1030	0.1455	0.0702	-0.4025	-0.1723	0.7330	0.3385
$\bar{S}$ 30days	0.5641	0.1762	0.4085	-123.6657	-1.8889	139.1857	1.3922
(b) Full							
RC	0.0822	0.0818	0.0984	-4.4785	-0.1581	3.6658	0.3281
RC-W_+	0.0592	0.0715	0.0927	0	0	3.6658	0.2539
RC-Sub	0.0605	0.0763	0.0802	-2.7305	-0.1146	2.8784	0.2892
$\overline{RC}$ Year	0.0870	0.0594	0.0898	-0.2056	-0.0807	0.3683	0.1855
(c) Shrinkage							
Rolling	0.0991	0.0686	0.0858	-0.2657	-0.0774	0.4310	0.2167
Rolling-Sub	0.0956	0.0726	0.0856	-0.5212	-0.0797	0.6122	0.2152
Ledoit Wolf	0.0769	0.0617	0.0778	-0.2022	-0.0603	0.7743	0.2751
Ledoit Wolf-Sub	0.0794	0.0984	0.0761	-0.2987	-0.0621	0.7994	0.2719
(d) Single-Factor							
Factor	0.0745	0.1108	0.0804	-0.3507	-0.0514	0.7117	0.1981
Factor-Sub	0.0795	0.0888	0.0793	-0.4399	-0.0455	0.6925	0.2028

Table 5: Volatility of Global Minimum Variance (GMV) portfolios of different dimension, open-to-close returns, in-sample

	Number of Assets						
Model	3	5	10	15	20	25	30
(a) Benchmark: Daily data							
Equal	0.1405	0.1282	0.1176	0.1148	0.1128	0.1110	0.1102
$\bar{S}$ Year	0.1300	0.1180	0.1012	0.0963	0.0921	0.0880	0.0859
$\bar{S}$ 78 days	0.1263	0.1140	0.0963	0.0868	0.0802	0.0747	0.0702
$\bar{S}$ 30days	0.1229	0.1087	0.0863	0.0772	0.0737	0.1085	0.4085
(b) Full							
RC	0.1271	0.1147	0.1007	0.0944	0.1072	0.0896	0.0984
RC-Sub	0.1259	0.1137	0.0981	0.0907	0.0886	0.0894	0.0802
Factor	0.1270	0.1142	0.1009	0.0925	0.0888	0.0828	0.0804

Table 6: Performance Characteristics of Global Minimum Variance (GMV) portfolios using open-to-close returns from 2003-2006, one step ahead Forecasts

Model	Mean	Median	σ
(a) Benchmark: Daily data			
DIA	0.0115	0.0961	0.1083
Equal	0.0078	0.1097	0.1102
$\bar{S}$ Year	0.1245	0.1069	0.0952
$\bar{S}$ Year-W_+	0.0828	0.0798	0.0966
$\bar{S}$ SI Year	0.0935	0.0689	0.0959
$\bar{S}$ 78days	0.1065	0.1139	0.1047
$\bar{S}$ 30days	0.8274	0.2657	0.9846
(b) Full			
RC	0.1533	0.1243	0.1173
RC-W_+	0.1106	0.1171	0.1139
RC-Sub	0.1662	0.1297	0.1042
$\overline{RC}$ Year	0.0856	0.0619	0.0900
(c) Shrinkage			
Rolling	0.1137	0.1119	0.0989
Rolling-Sub	0.1110	0.1021	0.1064
Ledoit Wolf	0.0976	0.1023	0.0962
Ledoit Wolf Sub	0.1119	0.1276	0.0947
(d) Single-Factor			
Factor	0.0906	0.0683	0.0957
Factor-Sub	0.0976	0.0939	0.0937

Table 7: Robustness analysis of smoothing parameters; Ledoit Wolf

α	σ		$\min(w)$		$\max(w)$	
	in	out	min	median	max	median
.1	0.0807	0.0964	−0.1087	−0.0394	0.7261	0.2561
.2	0.0791	0.0958	−0.1113	−0.0423	0.7340	0.2600
.3	0.0781	0.0955	−0.1150	−0.0472	0.7398	0.2653
.4	0.0774	0.0956	−0.1277	−0.0541	0.7533	0.2689
.5	0.0772	0.0960	−0.1456	−0.0615	0.7662	0.2748
.6	0.0773	0.0968	−0.1672	−0.0705	0.7783	0.2772
.7	0.0779	0.0983	−0.2882	−0.0817	0.7893	0.2836
.8	0.0805	0.1020	−2.2709	−0.0967	3.3067	0.2924
.9	0.0849	0.1441	−9.3478	−0.1190	6.2236	0.3034