UCRL-ID-125393

Sampling a Random Variable Distributed According to Planck's Law

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June, 1970

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Work performed under the auspices of the U.S. Department of Energy by the Lawrence Livermore National Laboratory under Contract W-7405-ENG-48.

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SAMPLING A RANDOM VARIABLE DISTRIBUTED ACCORDING TO PLANCK'S LAW

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In some Monte Carlo radiation transport calculations one must sample a random variable which is described by Planck's Radiation Law. That is, one must sample a random variable X where X is described by the density function

$$f_X(x) = \begin{cases} \dfrac{15}{\pi^4} \dfrac{x^3}{e^x - 1} & \text{if } x > 0 \\ 0 \text{ if } x \leq 0 \end{cases} \qquad (*)$$

We discuss a series expansion method and a rejection method in this note.

Series Expansion Method

Rewrite (*) as

$$f_X(x) = \frac{15}{\pi^4} \frac{x^3 e^{-x}}{1 - e^{-x}} \text{ if } x > 0 \qquad (**)$$

What follows is valid only for $x > 0$. Employ the expansion $\frac{1}{1-\theta} = \sum_{k=0}^{\infty} \theta^k$ if $0 < \theta < 1$ and (**) may be written as

$$f_X(x) = \frac{15}{\pi^4} x^3 e^{-x} \sum_{k=0}^{\infty} e^{-kx}$$

With a little manipulation this may be cast into the form

$$f_X(x) = \sum_{n=1}^{\infty} \left(\frac{90}{\pi^4} \cdot \frac{1}{n^4}\right)\left(\frac{n^4}{6} x^3 e^{-nx}\right)$$

If we define

$$\pi_n = \frac{90}{\pi^4} \frac{1}{n^4} \qquad 1 \leq n < \infty$$

$$f_n(x) = \frac{1}{6} n^4 x^3 e^{-nx} \qquad 1 \leq n < \infty$$

then

$$f_X(x) = \sum_{n=1}^{\infty} \pi_n f_n(x) \qquad x > 0, \quad 1 \leq n < \infty \tag{+}$$

where $\sum_{n=1}^{\infty} \pi_n = 1$ and each f_n is a density function. Let the random variable X_n be represented by f_n. Then (+) suggests the following algorithm for sampling X:

(i) Sample a random variable which is uniformly distributed in (0,1). Call the result u_1.

(ii) If $\mu_1 \leq \pi_1$ sample X_1. Call the result x_1. Then x_1 is a sample of X.

(iii) If the test in (ii) fails, check to see if $u_1 \leq \pi_1 + \pi_2$. If the answer is yes, sample X_2. Call the result x_2. Then x_2 is a sample of X.

(iv) Continue this procedure until

$$u_1 \leq \sum_{k=1}^{n} \pi_n$$

Then sample X_n. Call the result x_n.

Then x_n is a sample of X.

To paraphrase: Sample X_n with probability π_n.

This process is potentially open ended and could go on forever. In practice, however, it is terminated after some maximum number of tests by rounding off a number in the "if" test. And in fact, except for a miracle, the sample is acquired long before this maximum number of tests is made. These assertions are supported in what follows.

Figure 1 contains a listing of the code for the series expansion method. Notice the number 1.Ø8232 in the IF TEST. This is a six place representation of $\pi^4/90$. The seventh digit is 3. By rounding down in this manner we have assured that the test will pass by the 47th cycle irrespective of what R1 is. In effect we are using a 47 term series expansion for the Planck Law density function.

We claimed that the IF TEST passes long before the 47th cycle. To show this proceed as follows:

Let U be a random variable which is uniformly distributed in (0,1). Define a random variable C by

$$C = 1 \quad \text{if } U \leq \pi_1$$

$$C = 2 \quad \text{if } \pi_1 < U \leq \pi_1 + \pi_2$$

$$\vdots$$

$$C = k \quad \text{if } \sum_{i=1}^{k-1} \pi_i < U \leq \sum_{i=1}^{k} \pi_i$$

Then C counts the number of times required for the IF TEST to pass. It is clear that C is described by the probability mass function p_C where

$$p_C(x) = \begin{cases} \frac{90}{\pi^4} \cdot \frac{1}{x^4} & \text{if } x \text{ is a counting number} \\ 0 & \text{otherwise} \end{cases}$$

The expectation of C is

$$E(C) = \sum_{k=1}^{\infty} \frac{90}{\pi^4} k \frac{1}{k^4} \cong 1.1$$

So, "on the average", 1.1 cycles are required. This makes sense when we notice that $\pi_1 \cong .92$ which implies that only one test is required 92% of the time. Figure 1 contains a flow chart of the infinite series method. When compiled on the CHIP compiler, each sample requires about 21 microseconds of 7600 time.

The algorithm for sampling a random variable which is described by a density of the form $x^3 e^{-nx}$ is discussed in UCIR # 474 .

The Rejection Method

The infinite series method described above is not theoretically exact. Practically speaking its degree of inexactness is comparable to that which results from representing real numbers by terminated decimals. But, for those who prefer theoretical exactness, we present a rejection technique.

Our density function is

$$f_X(x) = \frac{15}{\pi^4} \frac{x^3 e^{-x}}{(1-e^{-x})} \qquad x \geq 0 \qquad (*)$$

Multiply and divide (*) by $1 + e^{-x}/x$ and manipulate the result somewhat to obtain

$$f(x) = \left[\frac{6}{6.25}\left(\frac{1}{6}x^3e^{-x}\right) + \frac{.25}{6.25}\left(4\ x^2e^{-2x}\right)\right]\frac{(6.25)\left(\frac{15}{\pi^4}\right)x}{(1-e^{-x})(x+e^{-x})} \qquad (**)$$

Identify as follows

$$\pi_1 \equiv 6/6.25 \quad , \quad \pi_2 \equiv .25/6.25\ .$$

$$f_1(x) \equiv \frac{1}{6}x^3e^{-x} \qquad x \geq 0$$

$$f_2(x) \equiv 4\ x^2e^{-2x} \qquad x \geq 0$$

$$h(x) \equiv \frac{(6.25)\left(\frac{15}{\pi^4}\right)x}{(1-e^{-x})(x+e^{-x})}$$

and (**) becomes

$$f(x) = \left[\pi_1 f_1(x) + \pi_2 f_2(x)\right] h(x) \qquad x \geq 0 \qquad (+)$$

where the f_i are densities and $h(x)$ is a rejection function. $\underset{0 \leq x < \infty}{\text{Max}}\ h(x) \equiv \hat{h}$ is approximately 1.16 which implies an acceptance efficiency of about 86%$(1/\hat{h})$.

Let X_1, X_2 be represented by f_1 and f_2. Then (+) suggests the following sampling procedure:

(i) Sample a random variable which is uniformly distributed in (0,1). Call the result u_1.

(ii) If $u_1 < \pi_1$, sample X_1. Call the result x_1. If $u_1 > \pi_1$ sample X_2. Call the result x_2.

(iii) Sample a random variable which is uniformly distributed in (0,1). Call the result u_2.

(iv) Let x_i, $i = 1,2$, be the result of (ii). If $u_2 \leq h(x_i)/\hat{h}$,

x_i is a sample of $\tilde{X}$. If $u_2 > h(x_i)/\hat{h}$, start again at (i).

Figure 2 shows a listing and flow chart of the code. The algorithm used for sampling f_1 and f_2 are discussed in UCIR # 474 . When compiled on the CHIP compiler, each sample requires about 30 microseconds of 7600 time.

Comparison of the Two Methods

Table 1 shows the results of a test of both the series and rejection calculations. The results are compared with each other and with a numerical integration of the density function. 100,000 samples were taken and the number which fell into 100 intervals of length .1 was determined. The columns labeled "interval number" give k where $I_k = (.1(k-1), 1k]$ is the k-th interval. The "numerical integration" column gives the number of samples which should have fallen into the k-th interval according to the density function. The other two columns give the results of the two sampling techniques.

We have not performed any statistical analysis on the data. A glance at the table indicates nothing to make us prefer one method over the other.

```
      SUBROUTINE PLANK(X)
      R1=RNFL(0)
      R2=RNFL(0)
      R3=RNFL(0)
      R4=RNFL(0)
      X=-ALOG(R1*R2*R3*R4)
      A=Y=Z=1.
      R1=RNFL(0)
    5 IF(1.08232*R1.LE.A) GO TO 10
      Y=Y+1.
      Z=1./Y
      A=A+Z*Z*Z*Z
      GO TO 5
   10 X=X*Z
      RETURN
      END
```

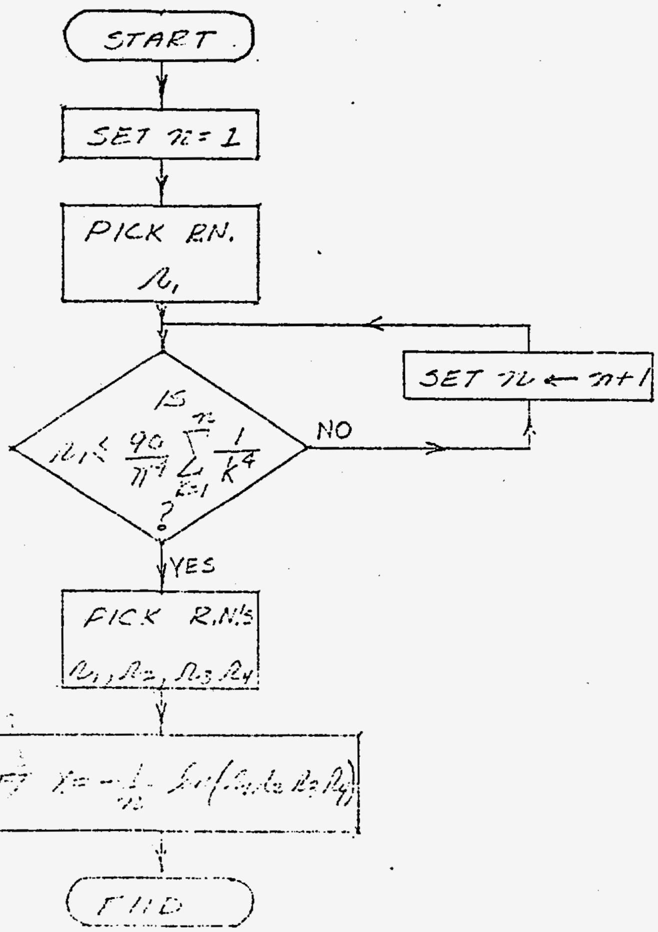

FIGURE 1. SERIES METHOD.

FORTRAN LISTING AND FLOWCHART

```
      SUBROUTINE PLANKER(X)
    4 R1=RNFL(Ø)
      R2=RNFL(Ø)
      R3=RNFL(Ø)
      R4=RNFL(Ø)
      IF(R4.LE..96) GO TO 1Ø
      TT=R1*R2*R3
      X=-.5*ALOG(TT)
      T=SORT(TT)
      GO TO 3Ø
   1Ø R4=RNFL(Ø)
      T=R1*R2*R3*R4
      X=-ALOG(T)
   3Ø R1=RNFL(Ø)
      RL=1.1642*R1*(1.-T)*(X+T)
      IF(RL.GT.X) GO TO 4
      RETURN
      END
```

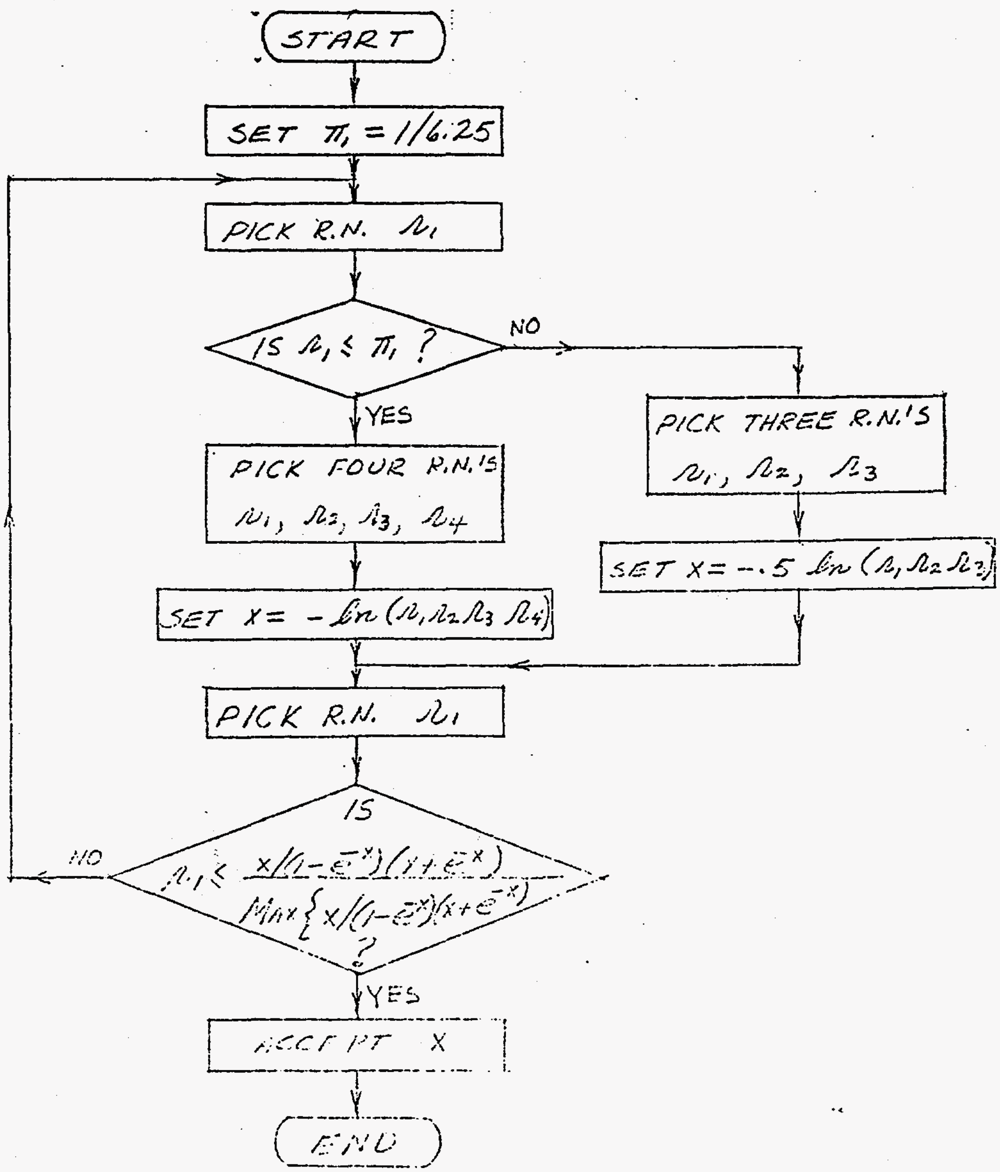

FIGURE 2. REJECTION METHOD.
FORTRAN LISTING AND FLOWCHART

INTERVAL NUMBER	NUMERICAL INTEGRATION	SERIES METHOD	REJECTION METHOD
1	0	1	4
2	33	36	31
3	86	98	80
4	158	142	167
5	247	257	257
6	350	347	326
7	462	476	460
8	582	572	561
9	706	740	707
10	833	836	820
11	960	943	951
12	1085	1114	1090
13	1208	1193	1224
14	1326	1273	1353
15	1438	1475	1488
16	1545	1521	1554
17	1644	1617	1599
18	1735	1727	1762
19	1819	1784	1748
20	1894	1948	1899
21	1969	1966	1921
22	2017	2049	2002
23	2066	2052	2127
24	2106	2101	2098
25	2138	2223	2191
26	2162	2107	2129
27	2178	2122	2212
28	2187	2148	2280
29	2188	2177	2165
30	2183	2210	2179
31	2172	2132	2212
32	2155	2120	2165
33	2132	2126	2161
34	2105	2067	2061
35	2073	2105	2068
36	2037	2055	2038
37	1998	2029	2004
38	1956	1950	2025
39	1911	1939	1919
40	1863	1845	1893
41	1814	1804	1811
42	1763	1831	1796
43	1711	1644	1733
44	1657	1679	1629
45	1603	1635	1611
46	1549	1481	1540
47	1495	1439	1472
48	1440	1484	1401
49	1386	1405	1375
50	1330	1333	1304
51	1279	1258	1275
52	1227	1187	1202
53	1176	1195	1173
54	1125	1102	1116
55	1076	1077	1134
56	1027	1050	1006
57	983	968	953
58	935	908	925
59	890	914	892
60	848	842	851
61	806	810	826
62	766	824	753
63	727	730	687
64	690	692	692
65	654	653	638
66	620	601	595
67	587	576	581
68	555	560	557
69	525	539	523
70	496	469	510
71	469	444	458
72	442	469	456
73	417	448	438
74	393	387	361
75	370	363	354
76	349	368	354
77	328	339	334
78	309	307	283
79	290	265	278
80	273	296	280
81	256	264	238
82	241	279	247
83	226	229	200
84	212	223	225
85	199	215	218
86	186	184	189
87	175	188	170
88	164	182	157
89	153	155	139
90	143	146	143
91	134	131	145
92	125	120	129
93	117	122	112
94	109	99	119
95	102	93	92
96	96	88	82
97	89	76	82
98	83	80	74
99	78	77	79
100	72	64	71

TABLE 1. A COMPARISON OF THE SERIES AND REJECTION RESULTS WITH A NUMERICAL INTEGRATION OF THE DENSITY FUNCTION.

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