Matrix Operations: Inverting Matrices Efficiently

Team Members: Ziqi Shao, Hui Zheng, Weifan Li

What is Matrix Inversion?

Given a square matrix A, its inverse A^{-1} is the unique matrix that satisfies:

$$A * A^{-1} = I$$

Where I is the identity matrix.

Ex.

2*2 Matrix Inversion:

$$A = \begin{pmatrix} 7 & 2 \\ 17 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{7(5)-2(17)} \begin{pmatrix} 5 & -2 \\ -17 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -2 \\ -17 & 7 \end{pmatrix}$$

Test for $A* A^{-1}$

$$\begin{pmatrix} 7 & 2 \\ 17 & 5 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -17 & 7 \end{pmatrix} =$$

$$\begin{pmatrix} 7(5)+2(-17) & 7(-2)+2(7) \\ 17(5)+5(-17) & 17(-2)+5(7) \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

Importance of Matrix Inversion

Matrix inversion is fundamentally about solving equations involving matrices.

Ex. Ax=b

To find the unknown vector x, you can multiply both sides of the equation:

$x=A^{-1}b$

Matrix inversion is crucial for solving systems of linear equations, and is widely applied in:

- **Engineering:** Structural analysis, circuit theory, **Computer Graphics:** Geometric transformations, **Machine Learning:** Linear regression, neural networks

Gauss-Jordan Elimination

Gauss-Jordan elimination systematically transforms to find the inverse :

Steps:

1. Write augmented matrix [A] [I]
2. Use row operations to convert A into identity matrix.
3. Transformed identity matrix I becomes A^{-1}.

Example Problem:

Gaussian Elimination (Gauss-Jordan Method)

Let's take the matrix:

$$A = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$$

Our goal is to find A^{-1} using **Gaussian Elimination (Gauss-Jordan Method)**.

Step 1: Augment A with the Identity Matrix

We create an **augmented matrix** by appending the identity matrix I:

$$\left[\begin{array}{cc|cc} 4 & 7 & 1 & 0 \\ 2 & 6 & 0 & 1 \end{array}\right]$$

We will now apply row operations to transform the **left side into the identity matrix** while the **right side** becomes A^{-1}.

Step 2: Make the First Pivot 1

$$\left[\begin{array}{cc|cc} 4 & 7 & 1 & 0 \\ 2 & 6 & 0 & 1 \end{array}\right]$$

We want the first pivot (top-left) to be **1**. We do this by **dividing the first row by 4**:

$$R_1 \leftarrow R_1/4$$

$$\left[\begin{array}{cc|cc} 1 & \frac{7}{4} & \frac{1}{4} & 0 \\ 2 & 6 & 0 & 1 \end{array}\right]$$

Step 3: Make the First Column's Second Entry 0

$$\left[\begin{array}{cc|cc} 1 & \frac{7}{4} & \frac{1}{4} & 0 \\ 2 & 6 & 0 & 1 \end{array}\right]$$

We want to make the **2 in the first column (row 2) into 0**. We do this by subtracting **2 times the first row** from the second row:

$$R_2 \leftarrow R_2 - 2R_1$$

$$\left[\begin{array}{cc|cc} 1 & \frac{7}{4} & \frac{1}{4} & 0 \\ 0 & \frac{10}{4} & -\frac{2}{4} & 1 \end{array}\right]$$

which simplifies to:

$$\left[\begin{array}{cc|cc} 1 & \frac{7}{4} & \frac{1}{4} & 0 \\ 0 & \frac{5}{2} & -\frac{1}{2} & 1 \end{array}\right]$$

Step 4: Make the Second Pivot 1

$$\left[\begin{array}{cc|cc} 1 & \frac{7}{4} & \frac{1}{4} & 0 \\ 0 & \frac{5}{2} & -\frac{1}{2} & 1 \end{array}\right]$$

We now **divide the second row by** $5/2$ to make the second pivot equal to 1:

$$R_2 \leftarrow R_2 \times \frac{2}{5}$$

$$\left[\begin{array}{cc|cc} 1 & \frac{7}{4} & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{array}\right]$$

Step 5: Make the Second Column's First Entry 0

$$\left[\begin{array}{cc|cc} 1 & \frac{7}{4} & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{array}\right]$$

We now want to eliminate the $\frac{7}{4}$ **above the second pivot**. We do this by subtracting $\frac{7}{4}$ **times the second row from the first row**:

$$R_1 \leftarrow R_1 - \frac{7}{4}R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{5} & -\frac{7}{10} \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{array}\right]$$

Final Result:

Thus, the inverse matrix A^{-1} is:

$$A^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{7}{10} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$$