Matrix Operations: Matrix Inversion Using Gauss-Jordan Elimination

Ziqi Shao, Hui Zheng, Weifan Li

CS5800 Seminar

Catalog

1. Introduction with example
2. Importance
3. Gauss-Jordan Elimination
4. Real-Life Examples

Introduction

- **Matrix inversion**
  - finding matrix A^{-1} such that AA^{-1} (identity matrix) $I_2 = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$
  - Inversion is only possible if the determinant of A is nonzero (matrix is invertible)

Introduction

- 2*2 Matrix Inversion:

- Ex.
- A = $\begin{pmatrix} 7 & 2 \\ 17 & 5 \end{pmatrix}$ -> $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- A $^{-1}$ = $\frac{1}{\text{ad} - \text{bc}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$= \frac{1}{7(5)-2(17)} \begin{pmatrix} 5 & -2 \\ -17 & 7 \end{pmatrix}$

$= \begin{pmatrix} 5 & -2 \\ -17 & 7 \end{pmatrix}$

- Matrix Identity:
  - An *identity* matrix, sometimes called a *unit matrix*, is a diagonal matrix with all its diagonal elements equal to 1, and zeroes everywhere else.

- A* A^{-1}
- $\begin{pmatrix} 7 & 2 \\ 17 & 5 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -17 & 7 \end{pmatrix}$

$= \begin{pmatrix} 7(5) + 2(-17) & 7(-2) + 2(7) \\ 17(5) + 5(-17) & 17(-2) + 5(7) \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Why is Matrix Inversion Important?

•Solves linear systems Ax=b

•A is a matrix representing coefficients.

•x is the vector of unknown variables you want to find.

•b is the vector of known outcomes.

Critical in fields such as

- Engineering (structural analysis, circuit theory).
- Computer graphics (transformations, rendering).
- Data Science (regression analysis, machine learning algorithms).

Gaussian Elimination (Gauss-Jordan Method)

Let's take the matrix:

$$A = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$$

Our goal is to find A^{-1} using **Gaussian Elimination (Gauss-Jordan Method)**.

Step 1: Augment A with the Identity Matrix

We create an **augmented matrix** by appending the identity matrix I:

$$\left[\begin{array}{cc|cc} 4 & 7 & 1 & 0 \\ 2 & 6 & 0 & 1 \end{array}\right]$$

We will now apply row operations to transform the **left side into the identity matrix** while the **right side becomes** A^{-1}.

Step 2: Make the First Pivot 1

$$\left[\begin{array}{cc|cc} 4 & 7 & 1 & 0 \\ 2 & 6 & 0 & 1 \end{array}\right]$$

We want the first pivot (top-left) to be **1**. We do this by **dividing the first row by 4**:

$$R_1 \leftarrow R_1/4$$

$$\left[\begin{array}{cc|cc} 1 & \frac{7}{4} & \frac{1}{4} & 0 \\ 2 & 6 & 0 & 1 \end{array}\right]$$

Step 3: Make the First Column's Second Entry 0

$$\left[\begin{array}{cc|cc} 1 & \frac{7}{4} & \frac{1}{4} & 0 \\ 2 & 6 & 0 & 1 \end{array}\right]$$

We want to make the **2 in the first column (row 2) into 0**. We do this by subtracting **2 times the first row** from the second row:

$$R_2 \leftarrow R_2 - 2R_1$$

$$\left[\begin{array}{cc|cc} 1 & \frac{7}{4} & \frac{1}{4} & 0 \\ 0 & \frac{10}{4} & -\frac{2}{4} & 1 \end{array}\right]$$

which simplifies to:

$$\left[\begin{array}{cc|cc} 1 & \frac{7}{4} & \frac{1}{4} & 0 \\ 0 & \frac{5}{2} & -\frac{1}{2} & 1 \end{array}\right]$$

Step 4: Make the Second Pivot 1

$$\left[\begin{array}{cc|cc} 1 & \frac{7}{4} & \frac{1}{4} & 0 \\ 0 & \frac{5}{2} & -\frac{1}{2} & 1 \end{array}\right]$$

We now **divide the second row by** $5/2$ to make the second pivot equal to 1:

$$R_2 \leftarrow R_2 \times \frac{2}{5}$$

$$\left[\begin{array}{cc|cc} 1 & \frac{7}{4} & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{array}\right]$$

Step 5: Make the Second Column's First Entry 0

$$\left[\begin{array}{cc|cc} 1 & \frac{7}{4} & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{array}\right]$$

We now want to eliminate the $\frac{7}{4}$ **above the second pivot**. We do this by subtracting $\frac{7}{4}$ **times the second row from the first row**:

$$R_1 \leftarrow R_1 - \frac{7}{4} R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{5} & -\frac{7}{10} \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{array}\right]$$

Time complexity: O(n3)

Main Elimination Steps in Gauss–Jordan:

1. Scale the Pivot Row so the Pivot Becomes 1
2. Zero Out Entries in Column j Above and Below the Pivot.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

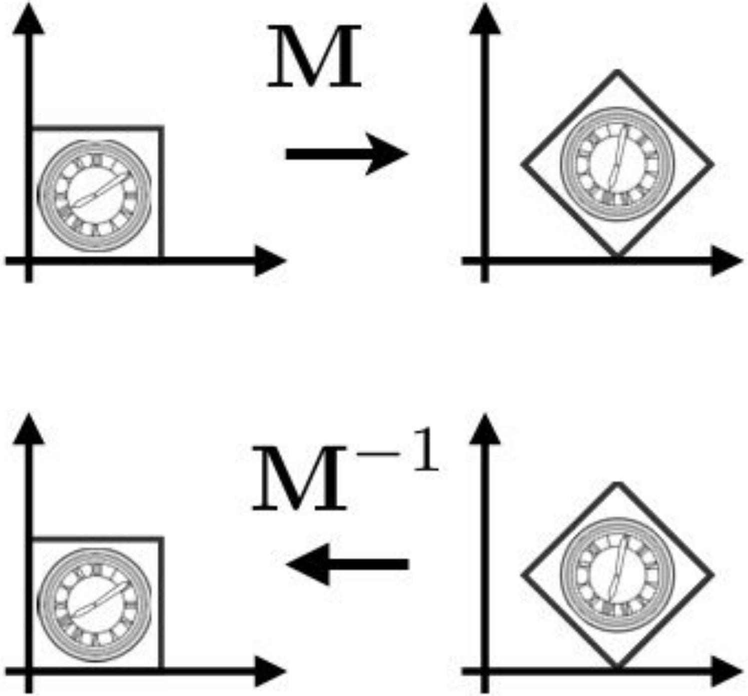
Application: Inverse Transform

translation

rotation

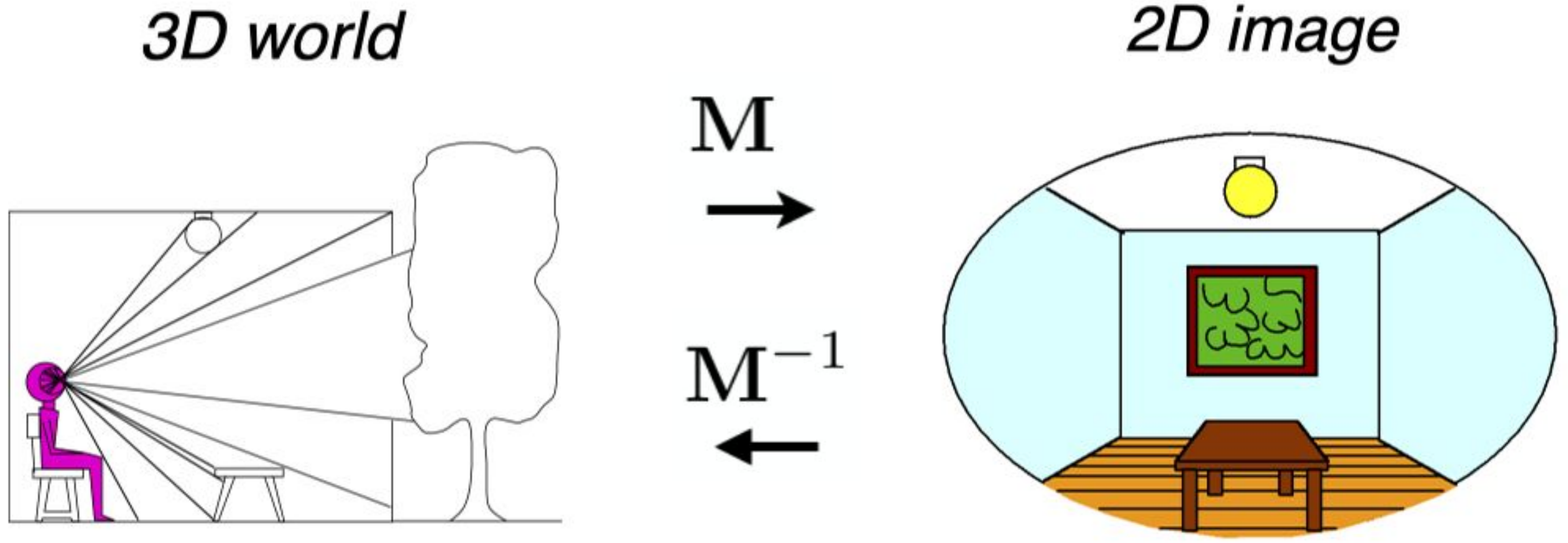
affine

perspective



Inverse Transform

Human Visual Perception: inverting 2d images to 3d space

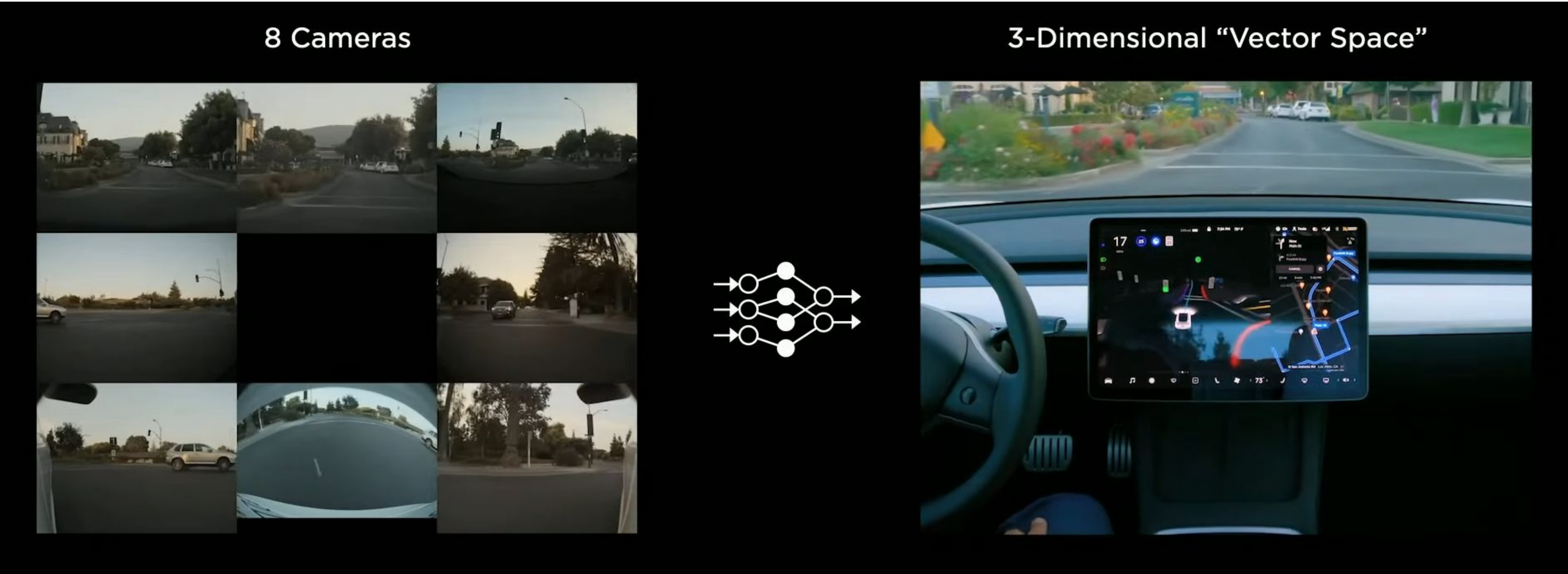


Point of observation

Figures © Stephen E. Palmer, 2002

Inverse Transform

Tesla Self-Driving: inverting more 2d images to 3d space



Inverse Transform

Bird Eyes View: inverting 2d images to one 2d image