MATH1023 Final Summary

Fall 2022

- In the Final Exam, you are only allowed to use all the results in this note and those from the **Midterm Summary**. Remember to verify **ALL the conditions** needed to apply them.
- You are **not allowed** to use results from HWs or Worksheets not stated here.
- For the results in **red** below, you need to quote the **name of the Theorems** in order to use them. Otherwise there will be at least **1 mark penalty**.

1 Basic functions and properties

- **Basic derivatives:**

$$(x^p)' = px^{p-1}, \qquad p \in \mathbb{R}$$
$$(a^x)' = a^x \log a, \qquad a > 0$$
$$(\log x)' = \frac{1}{x}$$
$$(\sin x)' = \cos x$$
$$(\cos x)' = -\sin x$$
$$(\arctan x)' = \frac{1}{1+x^2}$$

- **Taylor Series of functions at $x = 0$:**

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n} + o(x^n)$$
$$(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + \cdots + \binom{p}{n} x^n + o(x^n), \qquad p \in \mathbb{R}$$
$$\binom{p}{n} = \prod_{k=1}^{n} \frac{p-k+1}{k}$$
$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + o(x^n)$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$$
$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + o(x^8)$$
$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + o(x^{2n+2})$$

- **Important examples:**
  - **Dirichlet function.** $D(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}. \end{cases}$
  - $f(x) = x^{n+1} D(x)$ is continuous, n-th order differentiable at $x = 0$ only. For $n \geq 2$, $f^{(n)}(x)$ does not exist $\forall x \in \mathbb{R}$.
  - $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous, differentiable at $x \in \mathbb{R}$, but $f'(x)$ not continuous at $x = 0$.
  - $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases} \in \mathcal{C}^\infty(\mathbb{R})$ with $f^{(n)}(0) = 0$ for all n.

2 Differentiation - Abstract

Definitions:

- **n-th order approximation at $x = a$.** A polynomial
$$P_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n$$
of degree n such that $\forall \epsilon > 0$, $\exists \delta > 0$ with
$$|x-a| < \delta \implies |f(x) - P_n(x)| \le \epsilon |x-a|^n.$$
(Equality holds when $x = a$.)
We write
$$f(x) = P_n(x) + o(x-a)^n$$
which means **remainder** $R_n(x) := f(x) - P_n(x)$ satisfies $R_n(a) = 0$ and
$$\lim_{x \to a} \frac{R_n(x)}{(x-a)^n} = 0.$$
- $\exists$ 0-th order approximation at $x = a \iff$ continuous at $x = a$.
- $\exists$ 1-st order (linear) approximation at $x = a \iff$ **differentiable at $x = a$**.
- $\exists$ 1-st order (linear) approximation at $x = \infty \iff$ **(horizontal/slant) asymptote**.
- $\exists$ n-th order approximation at $x = a \iff$ **n-th order differentiable at $x = a$**.
- n-th derivative: $f^{(n)}(x) := \dfrac{d}{dx} f^{(n-1)}(x), \qquad f^{(0)}(x) := f(x).$
- $f \in \mathcal{C}^n(\mathbb{R})$ if $f^{(n)}(x)$ exists AND continuous on $\mathbb{R}$.
- $f \in \mathcal{C}^\infty(\mathbb{R})$ (**smooth**) if $f^{(n)}(x)$ exists AND continuous on $\mathbb{R}$ for all n.

Properties:

- $f(x)$ differentiable at $x = a \iff$ **derivative (from First Principle)**
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
exists. (**one-sided derivative** $f'_\pm(a)$ defined for the limit $x \to a^\pm$ instead.)
- $f(x)$ differentiable at $x = a \implies f(x)$ continuous at $x = a$.
- $f(x)$ continuous at $x = a$ and $f'(x)$ exists near $x \approx a \implies f'(a) = \lim\limits_{x \to a} f'(x)$ if the limit exists.
(i.e. $f'(x)$ cannot have removable discontinuity at $x = a$.)
- $f(x)$ continuous at $x = a$ and $f'(x)$ exists near $x \approx a^\pm \implies f'_\pm(a) = \lim\limits_{x \to a^\pm} f'(x)$ if the limit exists.
(i.e. $f'(x)$ cannot have jump discontinuity at $x = a$.)
- n-th order approximation $P_n(x)$ is unique.
- $\exists$ n-th order approximation $\implies \exists$ m-th order approximation for $0 \le m \le n$ (by truncation):
$$P_n(x) = \sum_{k=0}^{n} c_k (x-a)^k \implies P_m(x) = \sum_{i=0}^{m} c_k (x-a)^k, \qquad 0 \le m \le n.$$
- $f^{(n)}(a)$ exists $\implies f^{(n-1)}(x)$ exists on $(a-\delta, a+\delta)$ for some $\delta > 0$
$\implies f^{(n-2)}(x)$ exists and differentiable on $(a-\delta, a+\delta)$ for some $\delta > 0$
$\implies f \in \mathcal{C}^{n-2}(a-\delta, a+\delta)$ for some $\delta > 0$.

3 Differentiation - Calculations

- **Differentiation Rules:**

Linearity	$(f \pm g)'(x) = f'(x) \pm g'(x)$	
	$(cf)'(x) = cf'(x),$	$c \in \mathbb{R}$
Leibniz Rule	$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$	
Quotient Rule	$(\frac{f}{g})'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$	
Chain Rule	$(g \circ f)'(x) = g'(f(x))f'(x)$	
Inverse	$(f^{-1})'(y) = \frac{1}{f'(x)}\Big\vert_{f(x)=y}$	

- **Implicit Differentiation.** Given a relationship $F(x, y) = 0$, treat $y = y(x)$ as a function of x and differentiate with chain rule.

- **Parametric Differentiation.** Given $y(t)$ and $x(t)$ so that $y = y(x)$ depends implicitly on x, then

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

- **L'Hôpital's Rule.** $f(x), g(x)$ differentiable on (a, b)
  - $g'(x) \neq 0$ near $x = a^+$,
  - $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$ or $\lim_{x \to a^+} |g(x)| = +\infty$,
  - $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$ converges.

  Then $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$.

  Variations:
  - Limits can be one-sided $\lim_{x \to a^\pm}$ or two-sided $\lim_{x \to a}$
  - a, L can be $\pm\infty$.

- **Taylor's Theorem (Taylor Series Expansion).**
  If $f^{(n)}(a)$ exists, then

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

  is an n-th order approximation for $f(x)$.

- **Lagrange Remainder Formula (Taylor Series Remainder Form).**
  Fixed $a < x$. If $f^{(n)}(a)$ exists and $f^{(n+1)}$ exists in (a, x), then $\exists \xi \in (a, x)$ s.t. the **remainder**

$$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(\xi)}{(n + 1)!}(x - a)^{n+1}.$$

  (Same formula for the case $x < a$ with $\xi \in (x, a)$.)

4 Derivative Tests

Definitions:

- $f : D \to \mathbb{R}$ has **global maximum** at $x = a$ if $f(x) \le f(a)$ for all $x \in D$.
- $f : D \to \mathbb{R}$ has **local maximum** at $x = a$ if $\exists \delta > 0$ such that $f(x) \le f(a)$ for all $x \in (a - \delta, a + \delta) \cap D$. (Similarly for **minimum**.)
- Extremum = maximum or minimum.
- Global extremum is local extremum.
- $f(x)$ is **increasing** on $[a, b]$ if $x < y \implies f(x) \le f(y)$.
- $f(x)$ is **strictly increasing** on $[a, b]$ if $x < y \implies f(x) < f(y)$. (Similarly for **decreasing**.)

Properties and Theorems:

- Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b).
  - $f'(x) \ge 0$ on $(a, b) \iff f(x)$ is increasing on $[a, b]$.
  - $f'(x) > 0$ on $(a, b) \iff f(x)$ is strictly increasing on $[a, b]$.
  - $f'(x) \le 0$ on $(a, b) \iff f(x)$ is decreasing on $[a, b]$.
  - $f'(x) < 0$ on $(a, b) \iff f(x)$ is strictly decreasing on $[a, b]$.
- **First derivative test.** If $f'(a)$ exists and $x = a$ is a local extremum, then $f'(a) = 0$.
- Local extremum of $f(x)$ defined on an interval I can occur at:
  - $x = a$ with $f'(a) = 0$,
  - endpoints of closed interval,
  - points where $f(x)$ is not differentiable.
- **Comparing functions ($x > a$).** Let $f(x), g(x)$ be continuous for $x \ge a$ and differentiable for $x > a$.
  - If $f(a) \ge g(a)$ and $f'(x) \ge g'(x)$ for $x > a$, then $f(x) \ge g(x)$ for $x > a$.
- **Comparing functions ($x < a$).** Let $f(x), g(x)$ be continuous for $x \le a$ and differentiable for $x < a$.
  - If $f(a) \ge g(a)$ and $f'(x) \le g'(x)$ for $x < a$, then $f(x) \ge g(x)$ for $x < a$.
- **Second derivative test.** If $f'(a) = 0$ and $f''(a)$ exists, then
  - $f''(a) > 0$: $x = a$ is local min,
  - $f''(a) < 0$: $x = a$ is local max.
- **Higher derivative test.** If $f(x) = f(a) + c(x - a)^n + o(x - a)^n$, then
  - n is odd: $x = a$ is not a local max/min,
  - n is even, $c > 0$: $x = a$ is local min,
  - n is even, $c < 0$: $x = a$ is local max.

5 Mean Value Theorems

- **Rolle's Theorem.** If $f(x)$ is continuous on $[a,b]$, differentiable on (a,b), and $f(a)=f(b)$, then $\exists c\in(a,b)$ such that $f'(c)=0$.
- **Mean Value Theorem (MVT).** If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then $\exists c\in(a,b)$ such that
$$\frac{f(b)-f(a)}{b-a}=f'(c).$$
- **Cauchy's MVT.** If $f(x),g(x)$ are continuous on $[a,b]$, differentiable on (a,b), $g(a)\neq g(b)$ and $g'(x)\neq 0$ on (a,b), then $\exists c\in(a,b)$ such that
$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}.$$
- **Darboux's Theorem.** If $f(x)$ is differentiable on $[a,b]$, then $f'(x)$ satisfies the Intermediate Value Property: if γ is a value strictly between $f'(a)$ and $f'(b)$, then there exists $c\in(a,b)$ such that $f'(c)=\gamma$.

6 Convex Functions

- $f(x)$ is **convex** on interval I if and only if for any $x,y,z\in I$ with $x\le z\le y$, any of the following holds
  - $f(z)\le L_{x,y}(z)$
  - slope $L_{x,z}\le$ slope $L_{x,y}$
  - slope $L_{x,y}\le$ slope $L_{z,y}$
  - slope $L_{x,z}\le$ slope $L_{z,y}$
  - $f(\lambda x+(1-\lambda)y)\le\lambda f(x)+(1-\lambda)f(y),\qquad 0\le\lambda\le 1$

  where
$$L_{x,y}(z)=f(x)+\frac{f(y)-f(x)}{y-x}(z-x)$$
  is the line joining $(x,f(x))$ with $(y,f(y))$.
- $f(x)$ is **concave** $\iff -f(x)$ is convex $\iff$ all inequalities above are reversed.
- If f' exists, then $f(x)$ is convex on $I\iff f'$ is increasing on I.
- If f'' exists, then $f(x)$ is convex on $I\iff f''\ge 0$ on I.
- $x=a$ is **inflection point** if $f(x)$ changes from convex to concave (or vice versa) near $x=a$.
- If $f(x)$ is convex (or concave) on an interval I, then it is continuous on I.
- $f(x)$ convex, $g(x)$ convex and increasing $\Longrightarrow g(f(x))$ convex.
- **Jensen's Inequality.** If $f(x)$ is convex, and $\lambda_1+\cdots+\lambda_n=1$, $0\le\lambda_i\le 1$, then
$$f(\lambda_1x_1+\cdots+\lambda_nx_n)\le\lambda_1f(x_1)+\cdots+\lambda_nf(x_n).$$
  (Inequality reverses if $f(x)$ is concave.)
- **HM-GM-AM-QM Inequality.**
$$\frac{n}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}}\le\sqrt[n]{a_1\cdots a_n}\le\frac{a_1+\cdots+a_n}{n}\le\sqrt{\frac{a_1^2+\cdots+a_n^2}{n}}.$$
- **Generalized Mean Inequality.** If $p>q>0$, then
$$\left(\frac{a_1^p+\cdots+a_n^p}{n}\right)^{\frac{1}{p}}\le\left(\frac{a_1^q+\cdots+a_n^q}{n}\right)^{\frac{1}{q}}$$