香港科技大學
THE HONG KONG
UNIVERSITY OF SCIENCE
AND TECHNOLOGY

數學系
DEPARTMENT OF MATHEMATICS

FINAL EXAMINATION

Course Code:	MATH 1023
Course Title:	Honors Calculus I
Semester:	Fall 2020-21
Date and Time:	8:30AM - 11:30AM, 17 December 2020

Instructions

- Absolutely **NO DISCUSSION**, online or offline, with any person other than the instructor.
- Posting anything on social media or any public internet website is **NOT** allowed.
- It is an **OPEN-NOTES** and **OPEN-INTERNET** exam. You can do searching on the internet (including using WolframAlpha), but no posting and discussion.
- Only results discussed in lectures and tutorials, and results proved in homework can be directly quoted.
- You must **SHOW YOUR WORK** and **JUSTIFY YOUR STEPS** to receive credits in every problem in Part B.
- Some problems in Part B are structured into several parts. You can quote the results stated in the preceding parts to do the next part, even if you cannot do the preceding parts.
- One page 1 of your work, write the following statement and sign:

  > "I confirm that I have answered the questions using only materials specified approved for use in this examination, that all the answers are my own work, and that I have not received any assistance during the examination."
  >
  > Your signature

- You need to **SIGN** on the top right corner of **EVERY PAGE** of your work.

HKUST Academic Honor Code

Honesty and integrity are central to the academic work of HKUST. Students of the University must observe and uphold the highest standards of academic integrity and honesty in all the work they do throughout their program of study. As members of the University community, students have the responsibility to help maintain the academic reputation of HKUST in its academic endeavors. Sanctions will be imposed on students, if they are found to have violated the regulations governing academic integrity and honesty.

Part A - Short Questions (25 points)

Recommended timing: the whole Part A $<$ 30min.
Instruction: Write down your answers or solutions in your own answer sheets. **State the question and part numbers clearly**.

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1. Which mathematician below was never mentioned in lectures of this course? **[2]**
   A. L'Hospital
   B. Levi-Civita
   C. Lagrange
   D. Legendre
   E. Laplace

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2. Prove that **[6]**
$$\lim_{x\to 2} |x^2 - 9| = 5$$
from the ε, δ-definition of limits

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3. Suppose f is three times differentiable at 0, and as $x \to 0$, we have **[3]**
$$f(x) = \frac{\pi}{2} - \frac{1}{6}x^2 - \frac{3}{40}x^3 + o(x^3).$$
Choose the right choice in each bracket. No need to explain.
   (a) $(0, \frac{\pi}{2})$ is a local (maximum / minimum) point of f.
   (b) $f'''(0)$ is (positive / zero / negative).
   (c) $\left.\frac{d}{dx}\right|_{x=0} xf(x)$ is (positive / zero / negative).

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4. According to our lectures, which of the following was/were proved using Extreme Value Theorem (EVT)? List ALL correct answers - write the letters only. [Note: if EVT is used to prove (A), and (A) is used to prove (B), then (B) is also counted.] **[6]**
   (a) Intermediate Value Theorem
   (b) Mean Value Theorem
   (c) L'Hospital's Rule
   (d) Lagrange's Remainder Theorem
   (e) $f'(x) \geq 0 \ \ \forall x \in \mathbb{R} \implies f$ is increasing on $\mathbb{R}$.
   (f) f is increasing on $\mathbb{R} \implies f'(x) \geq 0 \ \ \forall x \in \mathbb{R}$.

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5. Suppose f is C^∞ on $\mathbb{R}$. Given that there exists a sequence $\{a_n\}_{n=1}^\infty$ of positive real numbers such that **[8]**
$$\lim_{n\to\infty} a_n = 0, \text{ and } f(a_n) = 0 \; \forall n \in \mathbb{N}.$$
Prove by induction that for any $k \in \mathbb{N} \cup \{0\}$, there exists a sequence $\left\{a_n^{(k)}\right\}_{n=1}^\infty$ such that $f^{(k)}\left(a_n^{(k)}\right) = 0$ for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n^{(k)} = 0$.

Part B - Long Questions (75 points): Answer ALL THREE problems

Recommended timing: Q1 $<$ 45min, Q2 $<$ 45min, Q3 $<$ 60min
Instructions: Write your solutions on your own answer sheets. **Clearly indicate the question and part numbers.**

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that

$$f'(x) = \cos^2\big(f(x)\big)\ \forall x \in \mathbb{R}.$$

(a) Prove by induction that for any $k \in \mathbb{N}$: **[12]**

$$\begin{cases} f^{(2k-1)}(x) = (-1)^{k-1}(2k-2)!\cos^{2k-1} f(x) \cdot \cos\big((2k-1)f(x)\big), \text{and} \\ f^{(2k)}(x) = (-1)^{k}(2k-1)!\cos^{2k} f(x) \cdot \sin\big(2kf(x)\big) \end{cases}$$

(b) Using (a), show that f is real analytic at every $a \in \mathbb{R}$. **[8]**

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2. Suppose $f : (0,\infty) \to \mathbb{R}$ is differentiable on $(0,\infty)$ and f' is strictly decreasing on $(0,\infty)$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Denote

$$S_n := a_1 + a_2 + \cdots + a_n \text{ and } H_n := \frac{1}{S_1} + \frac{1}{S_2} + \cdots + \frac{1}{S_n}.$$

(a) Prove that for each integer $N \geq 2$, we have: **[6]**

$$\sum_{n=2}^{N} f'(S_n) < \sum_{n=2}^{N} \frac{f(S_n) - f(S_{n-1})}{a_n} < \sum_{n=1}^{N-1} f'(S_n).$$

(b) Show that for each integer $N \geq 2$, we have **[6]**

$$H_N - \frac{1}{S_1} < \log\left(\left(\frac{S_2}{S_1}\right)^{\frac{1}{a_2}}\left(\frac{S_3}{S_2}\right)^{\frac{1}{a_3}}\cdots\left(\frac{S_N}{S_{N-1}}\right)^{\frac{1}{a_N}}\right) < H_N - \frac{1}{S_N}$$

(c) Show that the following limit exists: **[12]**

$$\lim_{N\to\infty} \frac{\log\left(\left(\frac{S_2}{S_1}\right)^{\frac{1}{a_2}}\left(\frac{S_3}{S_2}\right)^{\frac{1}{a_3}}\cdots\left(\frac{S_N}{S_{N-1}}\right)^{\frac{1}{a_N}}\right)}{H_N}.$$

[Hint: Consider two cases: (1) $\{H_n\}$ is bounded; (2) $\{H_n\}$ is unbounded]

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3. Consider a function $f : \mathbb{R} \to \mathbb{R}$ which is C^2 on $\mathbb{R}$ (i.e. $f''(x)$ exists for any $x \in \mathbb{R}$ and f'' is continuous on $\mathbb{R}$). Given that f satisfies the all of the following conditions:

   (i) $f(x) > 0$ for any $x \in \mathbb{R}$

   (ii) $f'(x) > 0$ for any $x \in \mathbb{R}$

   (iii) $f(-1) > 1$

   (iv) $f(-1)f'(-1) < 1$

   (v)
$$1 - \frac{f(x)f''(x)}{1 + f'(x)^2} = \frac{f(x)\big(1 + f'(x)^2\big)}{f(x) - xf'(x)} \text{ and } f(x) - xf'(x) \neq 0 \ \ \forall x \in \mathbb{R}.$$

   Denote $Q(x) := -\dfrac{f(x)f'(x) + x}{f(x) - xf'(x)}$.

   (a) Let $x_0 \in \mathbb{R}$, show that $Q(x_0) > 0$ if and only if $f''(x_0) > 0$. In the remaining parts (b)-(d), you may use the following **without** proofs: **[4]**

   - $Q(x_0) < 0$ if and only if $f''(x_0) < 0$; and
   - $Q(x_0) = 0$ if and only if $f''(x_0) = 0$.

   (b) Show that if $f''(x_0) = 0$, then $Q'(x_0) < 0$. **[8]**

   (c) Show that:

   i. $f''(x) < 0$ for any $x \geq 0$, and **[3]**

   ii. there **exists** a **unique** $x_* \in (-1, 0)$ such that $f''(x_*) = 0$. [Hint: for the uniqueness part it is helpful to use (a), (b) and consider the graph of Q.] **[12]**

   (d) Sketch the graph $y = f(x)$ on the interval $-1 \leq x \leq 1$. Your graph should illustrate conditions (i)-(iii) and results proved in (c). **[4]**

- End of Paper -