Final Examination (Take Home)

MATH 1023: Honors Calculus I
Fall 2019, HKUST
February 17th, 2020, 16:30 - 19:30

INSTRUCTIONS

- This is a **Take Home** Exam. You are only allowed to consult all the materials available on the Canvas course website. Only results from the Final Summary can be quoted.
- Discussion with any person (online or offline) is **strictly prohibited**, and is a serious violation of the honor code. Posting related questions in any online forum, whether it is answered or not, is also a serious violation of the honor code.
- Answer **ALL** 4 problems on blank white paper, single-lined paper, or export from a whiteboard app on a tablet. If you use a tablet, you should only use a **single color** for answering. **DO NOT** use "Soc Paper" or other paper with patterns. 10 points penalty will be imposed. **DO NOT** LaTeX type solution. They will not be graded.
- **Scan and upload 4 PDF files, one for each problem, in a SINGLE submission.** Clearly label your filename (e.g. `Q1.pdf`, `Q2.pdf`,... and so on). This will help us grade. (We've got only two Ivan's to grade 100+ Exams......)
- Submit your solution by **19:45, February 17th, 2020**. **Double check your files!** We will only grade the latest submission. Late submission results in a 2-point penalty per minute late, according to the last timestamp.
- Please show your work clearly and **justify your answers**. Write neatly – answers which are illegible for the graders cannot be given credit. Scores are indicated in the square brackets.

Finally, remember to **submit the weighting** W on Canvas by the end time of the Exam.

Time allowed: 3 Hours

Turn to the next page for the instruction to sign the HKUST Academic Honor Code.

The HKUST Academic Honor Code

Honesty and integrity are central to the academic work of HKUST. Students of the University must observe and uphold the highest standards of academic integrity and honesty in all the work they do throughout their program of study.

As members of the University community, students have the responsibility to help maintain the academic reputation of HKUST in its academic endeavours.

Sanctions will be imposed on students, if they are found to have violated the regulations governing academic integrity and honesty.

Copy and sign the following **Honor Code** on the **top of first page** (i.e. Q1) of your answer sheet.

"I confirm that I will answer the questions using only materials specifically approved for use in this examination, that all the answers are my own work, and that I am taking this examination in isolation and will not receive any other assistance during the examination."

Student's Signature: ______________________________

Student Name: ______________________________

Student ID: ______________________________

Take a photo (`.jpg`) of your HKUST Student ID Card together with your answer sheets or tablet, clearly showing the Honor Code on the first page.

Upload and submit it before you start your Exam. You can use this to test the upload speed of your internet as well.

Failure to do so will have your Exam voided.

In this Exam, n always refers to positive integers, and $\log$ is the natural logarithm.

Do NOT use integration in this Exam.

PROBLEM 1. **[30]** Answer the following questions. Justify your answers.

(a) Give an example of a function with the following properties **(no justification are needed)**, or prove that it does not exist:

(i) **[2]** A continuous function $f(x)$ defined on $[0,1]$ with range $(0,1)$.

(ii) **[2]** A continuous function $f(x)$ defined on $(0,1)$ with range $[0,1]$.

(iii) **[2]** A function $f(x)$ defined on $\mathbb{R}$ such that $f^{(n)}(1) = n^2$ for all $n \geq 0$.

(iv) **[2]** A function $f(x)$ defined on $\mathbb{R}$ such that $f^{(1023)}(x)$ exists for all $x \in \mathbb{R}$ but not continuous on $\mathbb{R}$.

(b) **[2]** Find $a, b \in \mathbb{R}$ such that the following function is second order differentiable at $x = 0$:

$$f(x) = \begin{cases} \sin(ax + N\pi) - a\cos(x - N\pi) & x < 0 \\ x^3 D(x^N) + \sqrt{bx + 1} & x \geq 0 \end{cases}$$

where N is your student ID, and $D(x)$ is the Dirichlet function.

(c) **[5]** Find the global maximum and minimum (and their values) on $[1,2]$ of the function

$$f(x) = x^{\log \frac{x}{2}}.$$

(d) **[5]** For $n \geq 1$, let $P_n(x)$ be a polynomial and $a_n, b_n \in \mathbb{R}$ such that

$$x^{2n} = P_n(x)(x^2 - x + \frac{n-1}{n^2}) + a_n x + b_n.$$

For example, $\{a_n\} = \{1, \frac{1}{2}, \frac{7}{27}, ...\}$ and $\{b_n\} = \{0, -\frac{3}{16}, -\frac{62}{729}, ...\}$.

Find $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$.

(e) **[5]** Prove that

$$\frac{a}{b} \leq \frac{a^{b^2}}{b^{a^2}}$$

for $0 < a \leq b < 1$.

(f) **[5]** Let $f(x)$ be differentiable on $(0,1)$. Prove that if $\lim\limits_{x\to 0^+} f(x) = +\infty$, then $\lim\limits_{x\to 0^+} f'(x)$ diverges.

PROBLEM 2. **[24]** Evaluate the following limits.
(No need to prove rigorously, but state clearly the limit rules or theorems used.)

(a) **[8]** $\displaystyle\lim_{x\to 0} \frac{\sin^N x - \tan^N x}{x^{N+2}}$, where N is an integer.

(b) **[8]** $\displaystyle\lim_{n\to\infty} \sqrt{n^3}(\sqrt{n+1} - \sqrt{n+0} + \sqrt{n+2} - \sqrt{n+3})$.

(c) **[8]** $\displaystyle\lim_{x\to 0} \frac{a^{x^2} - b^{x^2}}{(a^x - b^x)^2}$, $a > b > 0$.

PROBLEM 3. **[20]** Fix an interval (a, b) and let $f(x)$ be differentiable on $\mathbb{R}$.

(a) **[12]** Let $m = \frac{1}{2}(a+b)$. Define $\alpha(t)$ and $\beta(t)$ on $[a, b]$ by

$$\alpha(t) = \begin{cases} a & a \le t \le m \\ 2t - b & m < t \le b \end{cases}, \qquad \beta(t) = \begin{cases} 2t - a & a \le t \le m \\ b & m < t \le b \end{cases}.$$

Then define $g(t)$ on $[a, b]$ by

$$g(t) = \begin{cases} \dfrac{f(\beta(t)) - f(\alpha(t))}{\beta(t) - \alpha(t)} & a < t < b \\ f'(a) & t = a \\ f'(b) & t = b \end{cases}.$$

Show that $g(t)$ is continuous on $[a, b]$.

(b) **[8]** Hence show that if γ is strictly between $f'(a)$ and $f'(b)$, then $\exists c \in (a, b)$ such that $f'(c) = \gamma$.

PROBLEM 4. **[26]** Consider the cubic polynomial

$$g(x) = x^3 - 3x.$$

Define a function $f(p)$ on $\mathbb{R}$ as follows:

- If $g(x) = p$ has only one root α, define $f(p) = \alpha^2$.
- Otherwise define $f(p)$ to be the product of the biggest and smallest root.

("Roots" in this question refer to real roots of $g(x) = p$.)

(a) **[4]** Find the ranges of p such that $g(x) = p$ has $1, 2, 3$ roots respectively.

(b) **[6]** Express $f(p)$ in terms of only one root, and hence find the minimum of $f(p)$.

(c) **[4]** Show that $f(p)$ is an even function.

(d) **[12]** Sketch the graph of $f(p)$. Explain and indicate clearly (with coordinates, if any) the x, y-intercepts, disruptions, local max/min, asymptotes, inflection points and convexity.

–End of Paper–