- Concept: consider

$$\begin{array}{ccc} \hat{x} & \mapsto & x \\ \hat{y} & \mapsto & y \end{array}$$

- **Concept**: consider

$$\begin{array}{rcl} \hat{x} & \mapsto & x \\ \hat{y} & \mapsto & y \\ & & r = x \times y \end{array}$$

- **Concept**: consider

$$\begin{array}{rcrcrcl} & & \hat{x} & \mapsto & x & & \\ & & \hat{y} & \mapsto & y & & \\ f(\hat{x},\hat{y}) & = & \hat{r} & \mapsto & r & = & x \times y \end{array}$$

  where f

  1. has an action on $\hat{x}$ and $\hat{y}$ compatible with that of $\times$ on x and y:
     - accepts n-bit
       - **multiplier** $\hat{y}$ (that "does the multiplying"), and
       - **multiplicand** $\hat{x}$ (that "is multiplied")

       as input, and
     - produces an $(2 \cdot n)$-bit **product** $\hat{r}$ as output,
  2. is a Boolean function:

$$f : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}^{2 \cdot n}$$

- Agenda: produce a design(s) for f, which
  1. functions correctly, and
  2. satisfies pertinent quality metrics (e.g., is efficient in time and/or space).

Quote

I do not like $\times$ as a symbol for multiplication, as it is easily confounded with x; often I simply relate two quantities by an interposed dot and indicate multiplication by $ZC \cdot LM$.

– *Leibniz (https://en.wikiquote.org/wiki/Gottfried_Leibniz)*

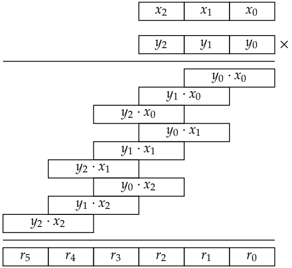
Example

$$
\begin{array}{lclcl|cccccc|l}
x & = & & 623_{(10)} & \mapsto & & & & 6 & 2 & 3 & \\
y & = & & 567_{(10)} & \mapsto & & & & 5 & 6 & 7 & \times \\
\hline
p_0 & = & 7 \cdot 3 \cdot 10^0 = & 21_{(10)} & \mapsto & & & & & 2 & 1 & \\
p_1 & = & 7 \cdot 2 \cdot 10^1 = & 140_{(10)} & \mapsto & & & & 1 & 4 & & \\
p_2 & = & 7 \cdot 6 \cdot 10^2 = & 4200_{(10)} & \mapsto & & & 4 & 2 & & & \\
p_3 & = & 6 \cdot 3 \cdot 10^1 = & 180_{(10)} & \mapsto & & & & 1 & 8 & & \\
p_4 & = & 6 \cdot 2 \cdot 10^2 = & 1200_{(10)} & \mapsto & & & 1 & 2 & & & \\
p_5 & = & 6 \cdot 6 \cdot 10^3 = & 36000_{(10)} & \mapsto & & 3 & 6 & & & & \\
p_6 & = & 5 \cdot 3 \cdot 10^2 = & 1500_{(10)} & \mapsto & & & 1 & 5 & & & \\
p_7 & = & 5 \cdot 2 \cdot 10^3 = & 10000_{(10)} & \mapsto & & 1 & 0 & & & & \\
p_8 & = & 5 \cdot 6 \cdot 10^4 = & 300000_{(10)} & \mapsto & 3 & 0 & & & & & \\
\hline
r & = & & 353241_{(10)} & \mapsto & 3 & 5 & 3 & 2 & 4 & 1 & \\
\hline
\end{array}
$$

Part 1: multiplication in theory (2)

Example (operand scanning)

Consider an example where where $|x| = |y| = 3$:

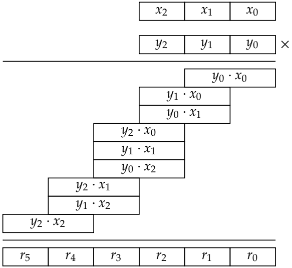


Notice that

1. an outer-loop steps through digits of y, say y_i,
2. an inner-loop steps through digits of x, say x_j.

Example (product scanning)

Consider an example where where $|x| = |y| = 3$:



Notice that

1. an outer-loop steps through digits of r, say r_i,
2. two inner-loops step through matching digits of x and y, say x_j and x_i.

Part 2: multiplication in practice: an algorithm (1)

Operand scanning

Algorithm (operand scanning)

Input: Two unsigned, base-b integers x and y
Output: An unsigned, base-b integer $r = x \cdot y$

$l_x \leftarrow |x|,\ l_y \leftarrow |y|,\ l_r \leftarrow l_x + l_y$
$r \leftarrow 0$
for $j = 0$ **upto** $l_y - 1$ **step** $+1$ **do**
  $c \leftarrow 0$
  for $i = 0$ **upto** $l_x - 1$ **step** $+1$ **do**
    $u \cdot b + v = t \leftarrow y_j \cdot x_i + r_{j+i} + c$
    $r_{j+i} \leftarrow v$
    $c \leftarrow u$
  end
  $r_{j+l_x} \leftarrow c$
end
return r

Example (operand scanning)

Consider a case where $b = 10$, $x = 623_{(10)}$ and $y = 567_{(10)}$:

j	i	r	c	y_i	x_j	$t = y_i \cdot x_i + r_{i+j} + c$	r'	c'
		$\langle 0,0,0,0,0,0 \rangle$						
0	0	$\langle 0,0,0,0,0,0 \rangle$	0	7	3	21	$\langle 1,0,0,0,0,0 \rangle$	2
0	1	$\langle 1,0,0,0,0,0 \rangle$	2	7	2	16	$\langle 1,6,0,0,0,0 \rangle$	1
0	2	$\langle 1,6,0,0,0,0 \rangle$	1	7	6	43	$\langle 1,6,3,0,0,0 \rangle$	4
0		$\langle 1,6,3,0,0,0 \rangle$	4				$\langle 1,6,3,4,0,0 \rangle$	
1	0	$\langle 1,6,3,4,0,0 \rangle$	0	6	3	24	$\langle 1,4,3,4,0,0 \rangle$	2
1	1	$\langle 1,4,3,4,0,0 \rangle$	2	6	2	17	$\langle 1,4,7,4,0,0 \rangle$	1
1	2	$\langle 1,4,7,4,0,0 \rangle$	1	6	6	41	$\langle 1,4,7,1,0,0 \rangle$	4
1		$\langle 1,4,7,1,0,0 \rangle$	4				$\langle 1,4,7,1,4,0 \rangle$	
2	0	$\langle 1,4,7,1,4,0 \rangle$	0	5	3	22	$\langle 1,4,2,1,4,0 \rangle$	2
2	1	$\langle 1,4,2,1,4,0 \rangle$	2	5	2	13	$\langle 1,4,2,3,4,0 \rangle$	1
2	2	$\langle 1,4,2,3,5,0 \rangle$	1	5	6	35	$\langle 1,4,2,3,5,0 \rangle$	3
2		$\langle 1,4,2,3,5,0 \rangle$	3				$\langle 1,4,2,3,5,3 \rangle$	3
		$\langle 1,4,2,3,5,3 \rangle$						

Part 2: multiplication in practice: an algorithm (3)
Product scanning

Algorithm (product scanning)

Input: Two unsigned, base-b integers x and y
Output: An unsigned, base-b integer $r = x \cdot y$

$l_x \leftarrow |x|, l_y \leftarrow |y|, l_r \leftarrow l_x + l_y$
$r \leftarrow 0, c_0 \leftarrow 0, c_1 \leftarrow 0, c_2 \leftarrow 0$
for $k = 0$ **upto** $l_x + l_y - 1$ **step** $+1$ **do**
  for $j = 0$ **upto** $l_y - 1$ **step** $+1$ **do**
    for $i = 0$ **upto** $l_x - 1$ **step** $+1$ **do**
      if $(j + i) = k$ **then**
        $u \cdot b + v = t \leftarrow y_j \cdot x_i$
        $c \cdot b + c_0 = t \leftarrow c_0 + v$
        $c \cdot b + c_1 = t \leftarrow c_1 + u + c$
        $c_2 \leftarrow c_2 + c$
      end
    end
  end
  $r_k \leftarrow c_0, c_0 \leftarrow c_1, c_1 \leftarrow c_2, c_2 \leftarrow 0$
end
$r_{l_x + l_y - 1} \leftarrow c_0$

Example (product scanning)

Consider a case where $b = 10$, $x = 623_{(10)}$ and $y = 567_{(10)}$:

k	j	i	r	c_2	c_1	c_0	y_i	x_j	$t = y_i \cdot x_i$	r'	c'_2	c'_1	c'_0
			$\langle 0,0,0,0,0,0 \rangle$	0	0	0							
0	0	0	$\langle 0,0,0,0,0,0 \rangle$	0	0	0	7	3	21	$\langle 0,0,0,0,0,0 \rangle$	0	2	1
0			$\langle 0,0,0,0,0,0 \rangle$	0	2	1				$\langle 1,0,0,0,0,0 \rangle$	0	0	2
1	0	1	$\langle 1,0,0,0,0,0 \rangle$	0	0	2	7	2	14	$\langle 1,0,0,0,0,0 \rangle$	0	1	6
1	1	0	$\langle 1,0,0,0,0,0 \rangle$	0	1	6	6	3	18	$\langle 1,0,0,0,0,0 \rangle$	0	3	4
1			$\langle 1,0,0,0,0,0 \rangle$	0	3	4				$\langle 1,4,0,0,0,0 \rangle$	0	0	3
2	0	2	$\langle 1,4,0,0,0,0 \rangle$	0	0	3	7	6	42	$\langle 1,4,0,0,0,0 \rangle$	0	4	5
2	1	1	$\langle 1,4,0,0,0,0 \rangle$	0	4	5	6	2	12	$\langle 1,4,0,0,0,0 \rangle$	0	5	7
2	2	0	$\langle 1,4,0,0,0,0 \rangle$	0	4	7	5	3	15	$\langle 1,4,0,0,0,0 \rangle$	0	7	2
2			$\langle 1,4,0,0,0,0 \rangle$	0	7	2				$\langle 1,4,2,0,0,0 \rangle$	0	0	7
3	1	2	$\langle 1,4,2,0,0,0 \rangle$	0	0	7	6	6	36	$\langle 1,4,2,0,0,0 \rangle$	0	4	3
3	2	1	$\langle 1,4,2,0,0,0 \rangle$	0	4	3	5	2	10	$\langle 1,4,2,0,0,0 \rangle$	0	5	3
3			$\langle 1,4,2,0,0,0 \rangle$	0	5	3				$\langle 1,4,2,3,0,0 \rangle$	0	0	5
4	2	2	$\langle 1,4,2,3,0,0 \rangle$	0	0	5	5	6	30	$\langle 1,4,2,3,0,0 \rangle$	0	3	5
4			$\langle 1,4,2,3,0,0 \rangle$	0	3	5				$\langle 1,4,2,3,5,0 \rangle$	0	0	3
			$\langle 1,4,2,3,5,0 \rangle$	0	0	3				$\langle 1,4,2,3,5,3 \rangle$	0	0	3
			$\langle 1,4,2,3,5,3 \rangle$										

- Idea:
  - multiplication *means* repeated addition, i.e.,

$$y \times x = \underbrace{x + x + \cdots + x}_{y \text{ terms}},$$

    so if $y = 14_{(10)}$ we have

$$y \times x = x + x + x + x + x + x + x + x + x + x + x + x + x + x.$$

  - expressing y in base-2, we can rewrite this as

$$\begin{array}{lcl} y \times x & = & (\sum_{i=0}^{n-1} y_i \cdot 2^i) \times x \\ & = & (y_{n-1} \cdot 2^{n-1} + \cdots + y_1 \cdot 2^1 + y_0 \cdot 2^0) \times x \\ & = & (y_{n-1} \cdot 2^{n-1} \cdot x) + \cdots + (y_1 \cdot 2^1 \cdot x) + (y_0 \cdot 2^0 \cdot x) \end{array}$$

- Idea:
  - given $y = 14_{(10)} = 1110_{(2)}$ we can see that

$$\begin{array}{lclclclcl}
y \cdot x & = & y_0 \cdot x \cdot 2^0 & + & y_1 \cdot x \cdot 2^1 & + & y_2 \cdot x \cdot 2^2 & + & y_3 \cdot x \cdot 2^3 \\
 & = & 0 \cdot x \cdot 2^0 & + & 1 \cdot x \cdot 2^1 & + & 1 \cdot x \cdot 2^2 & + & 1 \cdot x \cdot 2^3 \\
 & = & 0 \cdot x & + & 2 \cdot x & + & 4 \cdot x & + & 8 \cdot x \\
 & = & 14 \cdot x & & & & & &
\end{array}$$

  - given $y = 14_{(10)} = 1110_{(2)}$ we can see that

$$\begin{array}{lcl}
y \cdot x & = & y_0 \cdot x + 2 \cdot (y_1 \cdot x + 2 \cdot (y_2 \cdot x + 2 \cdot (y_3 \cdot x + 2 \cdot (0)))) \\
 & = & 0 \cdot x + 2 \cdot (1 \cdot x + 2 \cdot (1 \cdot x + 2 \cdot (1 \cdot x + 2 \cdot (0)))) \\
 & = & 0 \cdot x + 2 \cdot (1 \cdot x + 2 \cdot (1 \cdot x + 2 \cdot (1 \cdot x + 0))) \\
 & = & 0 \cdot x + 2 \cdot (1 \cdot x + 2 \cdot (1 \cdot x + 2 \cdot (1 \cdot x))) \\
 & = & 0 \cdot x + 2 \cdot (1 \cdot x + 2 \cdot (1 \cdot x + 2 \cdot x)) \\
 & = & 0 \cdot x + 2 \cdot (1 \cdot x + 2 \cdot (3 \cdot x)) \\
 & = & 0 \cdot x + 2 \cdot (1 \cdot x + 6 \cdot x) \\
 & = & 0 \cdot x + 2 \cdot (7 \cdot x) \\
 & = & 0 \cdot x + 14 \cdot x \\
 & = & 14 \cdot x
\end{array}$$

  via application of **Horner's rule**.

- Idea: for $b = 2$ we now know

$$r \;=\; y \times x \;=\; \left(\sum_{i=0}^{n-1} y_i \cdot 2^i \right) \times x \;=\; \sum_{i=0}^{n-1} y_i \cdot x \cdot 2^i,$$

  plus
  - for any t,

$$y_i \cdot t \;=\; \begin{cases} 0 & \text{if } y_i = 0 \\ t & \text{if } y_i = 1 \end{cases}$$

  - for any t,

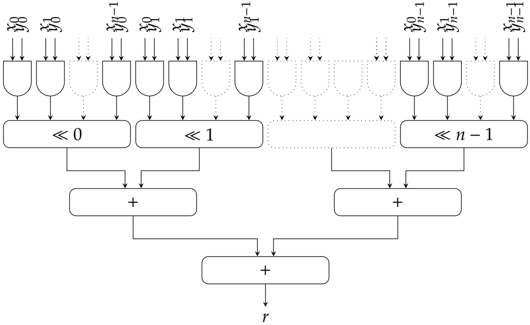
$$t \cdot 2^i \;\equiv\; t \ll i,$$

  so we can compute r via
  1. some AND gates to generate partial products (i.e., $y_i \cdot x$),
  2. some left-shift components to scale the partial products correctly (i.e., $y_i \cdot x \cdot 2^i$), and
  3. some adder components to sum the scaled partial products.

Part 3: multiplication in practice: a circuit (2)
A combinatorial, bit-parallel design

- Design:

Circuit



- Evaluation:

  −ve: requires a larger data-path

  +ve: requires a smaller control-path (i.e., none at all),

  +ve: requires less steps (i.e., 1),

  −ve: has a longer critical path (meaning each step is longer).

- **Idea**: for $b = 2$ we now know

$$r \;=\; y \times x \;=\; \left(\sum_{i=0}^{n-1} y_i \cdot 2^i \right) \times x \;=\; \sum_{i=0}^{n-1} y_i \cdot x \cdot 2^i,$$

  so we can compute r by evaluating the Horner expansion step-by-step in an "inside-out" order, applying

$$r \leftarrow \begin{cases} 2 \cdot r & \text{if } y_i = 0 \\ 2 \cdot r + x & \text{if } y_i = 1 \end{cases}$$

  so as to *accumulate* the result.

- **Idea**:

Algorithm

Input: Two unsigned, n-bit, base-2 integers x and y
Output: An unsigned, $2n$-bit, base-2 integer $r = y \cdot x$

```
r ← 0
for i = n − 1 downto 0 step −1 do
    r ← 2 · r
    if y_i = 1 then
        r ← r + x
    end
end
return r
```

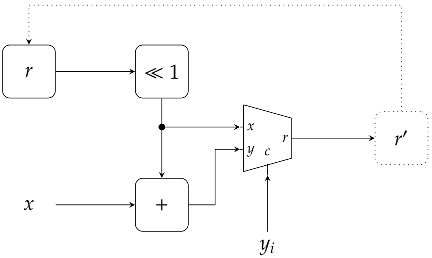
Example

Consider a case where $y = 14_{(10)} \mapsto 1110_{(2)}$:

i	r	y_i	r'	
	0			
3	0	1	x	$r' \leftarrow 2 \cdot r + x$
2	x	1	$3 \cdot x$	$r' \leftarrow 2 \cdot r + x$
1	$3 \cdot x$	1	$7 \cdot x$	$r' \leftarrow 2 \cdot r + x$
0	$7 \cdot x$	0	$14 \cdot x$	$r' \leftarrow 2 \cdot r$
	$14 \cdot x$			

- **Design**:

Circuit



- **Evaluation**:

  +ve: requires a smaller data-path
  −ve: requires a larger control-path (i.e., an entire FSM),
  −ve: requires more steps (i.e., n),
  +ve: has a shorter critical path (meaning each step is shorter).

Conclusions

- Take away points:
  1. Computer arithmetic is a broad, interesting (sub-)field:
     - it's a broad topic with a rich history,
     - there's usually a large design space of potential approaches,
     - they're often easy to understand at an intuitive, high level,
     - correctness and efficiency of resulting low-level solutions is vital and challenging.
  2. The strategy we've employed is important and (fairly) general-purpose:
     - explore and understand an approach in theory,
     - translate, formalise, and generalise the approach into an algorithm,
     - translate the algorithm, e.g., into circuit,
     - refine (or select) the circuit to satisfy any design constraints.

Additional Reading

- *Wikipedia: Computer Arithmetic*. URL: https://en.wikipedia.org/wiki/Category:Computer_arithmetic.
- D. Page. "Chapter 7: Arithmetic and logic". In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009.
- B. Parhami. "Part 3: Multiplication". In: *Computer Arithmetic: Algorithms and Hardware Designs*. 1st ed. Oxford University Press, 2000.
- W. Stallings. "Chapter 10: Computer arithmetic". In: *Computer Organisation and Architecture*. 9th ed. Prentice Hall, 2013.
- A.S. Tanenbaum and T. Austin. "Section 3.2.2: Arithmetic circuits". In: *Structured Computer Organisation*. 6th ed. Prentice Hall, 2012.

References

[1] *Wikipedia: Computer Arithmetic*. URL: https://en.wikipedia.org/wiki/Category:Computer_arithmetic (see p. 20).

[2] D. Page. "Chapter 7: Arithmetic and logic". In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009 (see p. 20).

[3] B. Parhami. "Part 3: Multiplication". In: *Computer Arithmetic: Algorithms and Hardware Designs*. 1st ed. Oxford University Press, 2000 (see p. 20).

[4] W. Stallings. "Chapter 10: Computer arithmetic". In: *Computer Organisation and Architecture*. 9th ed. Prentice Hall, 2013 (see p. 20).

[5] A.S. Tanenbaum and T. Austin. "Section 3.2.2: Arithmetic circuits". In: *Structured Computer Organisation*. 6th ed. Prentice Hall, 2012 (see p. 20).