

# Computer Architecture

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Keep in mind there are *two* PDFs available (of which this is the latter):

1. a PDF of examinable material used as lecture slides, and
2. a PDF of non-examinable, extra material:
  - ▶ the associated notes page may be pre-populated with extra, written explanation of material covered in lecture(s), plus
  - ▶ anything with a “grey’ed out” header/footer represents extra material which is useful and/or interesting but out of scope (and hence not covered).

Notes:

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► **Concept:** consider

$$\begin{array}{lcl} \hat{x} & \mapsto & x \\ \hat{y} & \mapsto & y \end{array}$$

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$$\begin{array}{lcl} \hat{x} & \mapsto & x \\ \hat{y} & \mapsto & y \\ & & r = x + y \end{array}$$

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$$\begin{array}{rcl} \hat{x} & \mapsto & x \\ \hat{y} & \mapsto & y \\ f(\hat{x}, \hat{y}) = \hat{r} & \mapsto & r = x + y \end{array}$$

where  $f$

1. has an action on  $\hat{x}$  and  $\hat{y}$  compatible with that of  $+$  on  $x$  and  $y$ :

- accepts  $n$ -bit
  - **addend**  $\hat{x}$ , and
  - **addend**  $\hat{y}$

as input, and

- produces an  $(n + 1)$ -bit **sum**  $\hat{r}$  as output,

2. is a Boolean function:

$$f : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\}^{n+1}$$

Notes:

► **Agenda:** produce a design(s) for  $f$ , which

1. functions correctly, and
2. satisfies pertinent quality metrics (e.g., is efficient in time and/or space).

Notes:

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► Concept:

$$\begin{array}{rcl} x & = & 107_{(10)} \mapsto \begin{array}{ccc} 1 & 0 & 7 \\ 0 & 1 & 4 \end{array} + \\ y & = & 14_{(10)} \mapsto \hline c & = & \\ r & = & \hline \end{array}$$
$$\begin{array}{rcll} x & = & 107_{(10)} & \mapsto & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ y & = & 14_{(10)} & \mapsto & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & + \\ c & = & & & & & & & & & & & \\ r & = & & & & & & & & & & & \end{array}$$

1. this process matches our understanding of manual, “school-book” addition, *and*
2. the same process applies, irrespective of  $b$ .

## Part 1: addition in theory (1)

► Concept:

Example ( $b = 10$ )

$$\begin{array}{rcll} x & = & 107_{(10)} & \mapsto & 1 & 0 & 7 \\ y & = & 14_{(10)} & \mapsto & 0 & 1 & 4 + \\ c & = & & & \hline r & = & & & & 0 \end{array}$$

### Example ( $b = 2$ )

$$\begin{array}{rcll} x & = & 107_{(10)} & \mapsto \quad 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 1 \ 1 \\ y & = & 14_{(10)} & \mapsto \quad 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ + \\ c & = & & \\ r & = & & \end{array}$$

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4

+

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r

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0

1

0

2

1

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=

107<sub>(10)</sub>

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Part 2: addition in practice: an algorithm (1)

Algorithm

**Input:** Two unsigned,  $n$ -digit, base- $b$  integers  $x$  and  $y$ , and a 1-digit carry-in  $ci \in \{0, 1\}$   
**Output:** An unsigned,  $n$ -digit, base- $b$  integer  $r = x + y$ , and a 1-digit carry-out  $co \in \{0, 1\}$

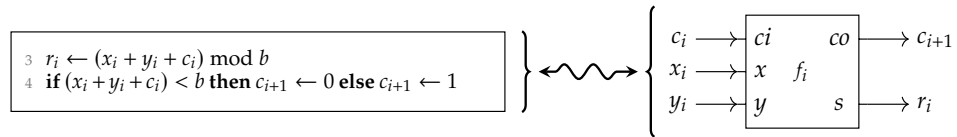
```
1  $r \leftarrow 0, c_0 \leftarrow ci$ 
2 for  $i = 0$  upto  $n - 1$  step  $+1$  do
3    $r_i \leftarrow (x_i + y_i + c_i) \bmod b$ 
4   if  $(x_i + y_i + c_i) < b$  then  $c_{i+1} \leftarrow 0$  else  $c_{i+1} \leftarrow 1$ 
5 end
6  $co \leftarrow c_n$ 
7 return  $r, co$ 
```

Notes:

### Part 3: addition in practice: a circuit (1)

#### ► Idea:

- for  $b = 2$ , it's clear from the algorithm that



- the loop body is therefore analogous to a Boolean function

$$f_i : \{0, 1\}^3 \rightarrow \{0, 1\}^2$$

specified by the following truth table

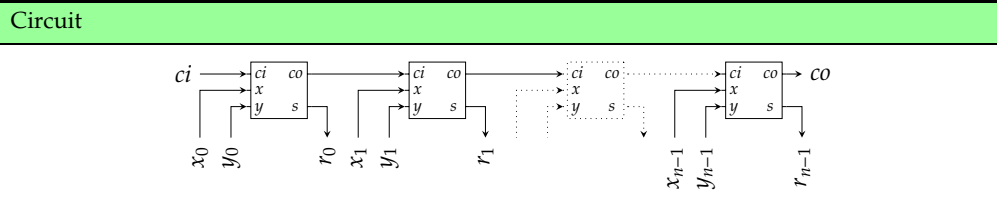
$ci$	$x$	$y$	$co$	$s$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Notes:

### Part 3: addition in practice: a circuit (1)

#### ► Idea:

- the loop bound is fixed, i.e.,  $n$  is some known constant, so we can unroll it to yield



which, now read left-to-right, mirrors the algorithm:

- the  $i$ -th instance  $f_i$  implements the  $i$ -th loop iteration,
- the connection, or **carry chain** between instances captures  $c$ ,
- those instances are termed **full adder** cells,
- this combination of them is termed a **ripple-carry adder**.

Notes:

## Part 3: addition in practice: a circuit (2)

### ► Beware:

- the magnitude of  $r = x + y$  can exceed what we can represent via  $\hat{r}$ :

$\hat{x}$  and  $\hat{y}$  are *unsigned*, and there is a carry-out  $\Rightarrow$  **carry** condition

$\hat{x}$  and  $\hat{y}$  are *signed*, and the sign of  $\hat{r}$  is incorrect  $\Rightarrow$  **overflow** condition

- to cope, we typically

1. detect the condition,
2. potentially take some action (e.g., try to “fix” the result somehow),
3. potentially signal the condition somehow (e.g., via a status register or some form of exception).

Notes:

## Part 3: addition in practice: a circuit (3)

### Example

Consider use of an *unsigned* representation:

$$\begin{array}{rclcl} x & = & 15_{(10)} & \mapsto & \begin{array}{cccc} 1 & 1 & 1 & 1 \end{array} \\ y & = & 1_{(10)} & \mapsto & \begin{array}{cccc} 0 & 0 & 0 & 1 \end{array} + \\ c & = & & & \begin{array}{cccc} 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array} \\ r & = & 0_{(10)} & \mapsto & \end{array}$$

Here, the carry-out indicates an error: the correct result  $r = 16$  is too large for  $n = 4$  bits.

### ► Note that

1. **detection**:

$$c_n = CO = 0 \Rightarrow \text{no carry}$$

$$c_n = CO = 1 \Rightarrow \text{carry}$$

2. **action**, e.g., **truncate** the result to  $n$  bits.

Notes:

Example	Example
Consider use of a <i>signed</i> representation: $\begin{array}{rcll} x & = & -1_{(10)} & \mapsto & \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 \end{array} \\ y & = & 1_{(10)} & \mapsto & \begin{array}{cccc} 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array} \\ c & = & & & \\ r & = & 0_{(10)} & \mapsto & \begin{array}{cccc} 0 & 0 & 0 & 0 \end{array} \end{array} +$ Irrespective of the carry-out, the signs of inputs and output make sense: there is no overflow, so $r = 0$ is correct.	Consider use of a <i>signed</i> representation: $\begin{array}{rcll} x & = & 7_{(10)} & \mapsto & \begin{array}{cccc} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 1 & 0 \end{array} \\ y & = & 1_{(10)} & \mapsto & \begin{array}{cccc} 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \end{array} \\ c & = & & & \\ r & = & -8_{(10)} & \mapsto & \begin{array}{cccc} 1 & 0 & 0 & 0 \end{array} \end{array} +$ Irrespective of the carry-out, the signs of inputs and output make no sense: there is an overflow, so $r = -8$ is incorrect.

- Note that
1. **detection:**

$x$ +ve	$y$ -ve	$\Rightarrow$	no overflow
$x$ -ve	$y$ +ve	$\Rightarrow$	no overflow
$x$ +ve	$y$ +ve	$r$ +ve	$\Rightarrow$ no overflow
$x$ +ve	$y$ +ve	$r$ -ve	$\Rightarrow$ overflow
$x$ -ve	$y$ -ve	$r$ +ve	$\Rightarrow$ overflow
$x$ -ve	$y$ -ve	$r$ -ve	$\Rightarrow$ no overflow

2. **action**, e.g., **clamp** (or **saturate**) the result to the largest magnitude representable in  $n$  bits.

Notes:

Conclusions

- Take away points:

1. Computer arithmetic is a broad, interesting (sub-)field:
  - it's a broad topic with a rich history,
  - there's usually a large design space of potential approaches,
  - they're often easy to understand at an intuitive, high level,
  - correctness and efficiency of resulting low-level solutions is vital and challenging.
2. The strategy we've employed is important and (fairly) general-purpose:
  - explore and understand an approach in theory,
  - translate, formalise, and generalise the approach into an algorithm,
  - translate the algorithm, e.g., into circuit,
  - refine (or select) the circuit to satisfy any design constraints.

Notes:

# Additional Reading

- ▶ [Wikipedia: Computer Arithmetic](https://en.wikipedia.org/wiki/Category:Computer_arithmetic). URL: [https://en.wikipedia.org/wiki/Category:Computer\\_arithmetic](https://en.wikipedia.org/wiki/Category:Computer_arithmetic).
- ▶ D. Page. “Chapter 7: Arithmetic and logic”. In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009.
- ▶ B. Parhami. “Part 2: Addition/subtraction”. In: *Computer Arithmetic: Algorithms and Hardware Designs*. 1st ed. Oxford University Press, 2000.
- ▶ W. Stallings. “Chapter 10: Computer arithmetic”. In: *Computer Organisation and Architecture*. 9th ed. Prentice Hall, 2013.
- ▶ A.S. Tanenbaum and T. Austin. “Section 3.2.2: Arithmetic circuits”. In: *Structured Computer Organisation*. 6th ed. Prentice Hall, 2012.

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# References

- [1] [Wikipedia: Computer Arithmetic](https://en.wikipedia.org/wiki/Category:Computer_arithmetic). URL: [https://en.wikipedia.org/wiki/Category:Computer\\_arithmetic](https://en.wikipedia.org/wiki/Category:Computer_arithmetic) (see p. 57).
- [2] D. Page. “Chapter 7: Arithmetic and logic”. In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009 (see p. 57).
- [3] B. Parhami. “Part 2: Addition/subtraction”. In: *Computer Arithmetic: Algorithms and Hardware Designs*. 1st ed. Oxford University Press, 2000 (see p. 57).
- [4] W. Stallings. “Chapter 10: Computer arithmetic”. In: *Computer Organisation and Architecture*. 9th ed. Prentice Hall, 2013 (see p. 57).
- [5] A.S. Tanenbaum and T. Austin. “Section 3.2.2: Arithmetic circuits”. In: *Structured Computer Organisation*. 6th ed. Prentice Hall, 2012 (see p. 57).

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