COMS10015 quick(ish) reference hand-out: Boolean algebra

Truth tables

Standard

x	y	$\neg x$	$x \wedge y$	$x \vee y$	$x \oplus y$	$x \Rightarrow y$	$x \equiv y$
0	0	1	0	0	0	1	1
0	1	1	0	1	1	1	0
1	0	0	0	1	1	0	0
1	1	0	1	1	0	1	1

Negative

x	y	$x \overline{\wedge} y$	$x \overline{\vee} y$	$x \overline{\oplus} y$
0	0	1	1	1
0	1	1	0	0
1	0	1	0	0
1	1	0	0	1

Axioms

Axioms in AND-form

commutativity	$x \wedge y$	$\equiv$	$y \wedge x$
association	$(x \wedge y) \wedge z$	$\equiv$	$x \wedge (y \wedge z)$
distribution	$x \wedge (y \vee z)$	$\equiv$	$(x \wedge y) \vee (x \wedge z)$
identity	$x \wedge 1$	$\equiv$	x
null	$x \wedge 0$	$\equiv$	0
idempotency	$x \wedge x$	$\equiv$	x
inverse	$x \wedge \neg x$	$\equiv$	0
absorption	$x \wedge (x \vee y)$	$\equiv$	x
de Morgan	$\neg(x \wedge y)$	$\equiv$	$\neg x \vee \neg y$

Axioms in OR-form

commutativity	$x \vee y$	$\equiv$	$y \vee x$
association	$(x \vee y) \vee z$	$\equiv$	$x \vee (y \vee z)$
distribution	$x \vee (y \wedge z)$	$\equiv$	$(x \vee y) \wedge (x \vee z)$
identity	$x \vee 0$	$\equiv$	x
null	$x \vee 1$	$\equiv$	1
idempotency	$x \vee x$	$\equiv$	x
inverse	$x \vee \neg x$	$\equiv$	1
absorption	$x \vee (x \wedge y)$	$\equiv$	x
de Morgan	$\neg(x \vee y)$	$\equiv$	$\neg x \wedge \neg y$

Misc

equivalence	$x \equiv y$	$\equiv$	$(x \Rightarrow y) \wedge (y \Rightarrow x)$
implication	$x \Rightarrow y$	$\equiv$	$\neg x \vee y$
involution	$\neg\neg x$	$\equiv$	x

Universality in NAND-form

$$
\begin{array}{lclcl}
\neg x & & & \equiv & x \overline{\wedge} x \\
x \wedge y & & & \equiv & (x \overline{\wedge} y) \overline{\wedge} (x \overline{\wedge} y) \\
x \vee y & \equiv & \neg x \overline{\wedge} \neg y & \equiv & (x \overline{\wedge} x) \overline{\wedge} (y \overline{\wedge} y)
\end{array}
$$

Universality in NOR-form

$$\begin{array}{lclcl}
\neg x & & & \equiv & x \overline{\vee} x \\
x \wedge y & \equiv & \neg x \overline{\vee} \neg y & \equiv & (x \overline{\vee} x) \overline{\vee} (y \overline{\vee} y) \\
x \vee y & & & \equiv & (x \overline{\vee} y) \overline{\vee} (x \overline{\vee} y)
\end{array}$$

Transformations and standard forms

Definition 1. *The fact there are AND and OR forms of most axioms hints at a more general underlying* **principle of duality***. Consider a Boolean expression e: the* **dual expression** e^D *is formed by*

1. *leaving each variable as is,*
2. *swapping each $\wedge$ with $\vee$ and vice versa, and*
3. *swapping each 0 with 1 and vice versa.*

Definition 2. *The de Morgan axiom can be generalised into a* **principle of complements***. Consider a Boolean expression e: the* **complement expression** $\neg e$ *is formed by*

1. *swapping each variable x with the complement $\neg x$,*
2. *swapping each $\wedge$ with $\vee$ and vice versa, and*
3. *swapping each 0 with 1 and vice versa.*

Definition 3. *Consider a Boolean function f with n inputs. When the expression for f is written as a sum (i.e., OR) of terms which each comprise the product (i.e., AND) of a number of inputs, it is said to be in* **disjunctive normal form** *or* **Sum of Products (SoP)** *form; the terms in this expression are called the* **minterms***. For example,*

$$\underbrace{(a \wedge b \wedge c)}_{minterm} \vee (d \wedge e \wedge f),$$

is in SoP form. Note that each variable can exist as-is or *complemented using NOT, meaning*

$$\underbrace{(\neg a \wedge b \wedge c)}_{minterm} \vee (d \wedge \neg e \wedge f),$$

is also a valid SoP expression.

Conversely, when the expression for f is written as a product (i.e., AND) of terms which each comprise the sum (i.e., OR) of a number of inputs, it is said to be in **conjunctive normal form** *or* **Product of Sums (PoS)** *form; the terms in this expression are called the* **maxterms***. For example,*

$$\underbrace{(a \vee b \vee c)}_{maxterm} \wedge (d \vee e \vee f),$$

is in PoS form. As above, each variable can exist as-is or *complemented using NOT.*