COMS10015 quick(ish) reference hand-out: Karnaugh map

An algorithm

Given the truth table for some n-input, 1-output Boolean function as input, application of a Karnaugh map proceeds as follows

1. Draw a rectangular $(p \times q)$-element grid, st.

   (a) $p \equiv q \equiv 0 \pmod 2$, and

   (b) $p \cdot q = 2^n$

   and each row and column represents one input combination; order rows and columns according to a **Gray code**.

2. Fill the grid elements with the output corresponding to inputs for that row and column.

3. Cover rectangular groups of adjacent 1 elements which are of total size 2^m for some m; groups can "wrap around" edges of the grid and overlap.

4. Translate each group into one term of an SoP form Boolean expression e where

   (a) *bigger* groups, and

   (b) *less* groups

   mean a simpler expression.

noting that

- We *only* ever have to consider an n-input, 1-output Boolean function, because we can decompose more complicated examples (i.e., n-input, m-output).

- For $n \geq 5$, we must alter this basic algorithm so it will produce correct output. Partly as a result, and partly since those functions a) get more complex but b) we see few (if any) cases where they are required, in COMS10015 you can assume $1 < n < 5$.

- Use of a Gray code simply means we reorder the intuitive labelling of rows and columns, i.e., instead of the left-hand side below we use the right-hand side:

integer sequence	Gray code sequence
$0_{(10)} \mapsto 000_{(2)}$	$0_{(10)} \mapsto 000_{(2)}$
$1_{(10)} \mapsto 001_{(2)}$	$1_{(10)} \mapsto 001_{(2)}$
$2_{(10)} \mapsto 010_{(2)}$	$3_{(10)} \mapsto 011_{(2)}$
$3_{(10)} \mapsto 011_{(2)}$	$2_{(10)} \mapsto 010_{(2)}$
$4_{(10)} \mapsto 100_{(2)}$	$6_{(10)} \mapsto 110_{(2)}$
$5_{(10)} \mapsto 101_{(2)}$	$7_{(10)} \mapsto 111_{(2)}$
$6_{(10)} \mapsto 110_{(2)}$	$5_{(10)} \mapsto 101_{(2)}$
$7_{(10)} \mapsto 111_{(2)}$	$4_{(10)} \mapsto 100_{(2)}$
$\vdots$	$\vdots$

  The reason for this is to ensure the label for adjacent rows (resp. columns) only differs in *one* bit; this allows the Karnaugh map to deliver an optimised expression, by allowing certain forms of group to be formed which would otherwise be disallowed.

- Rather than write the binary values themselves, we used a short-hand to avoid the font becoming too small and/or the diagram becoming too cluttered: a bar above (resp. to the left of) a column and (resp. a row) indicates where the associated variable is 1, with the absence of a bar indicating it is 0.

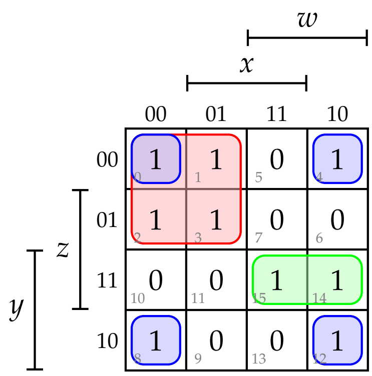
An example

Consider the 4-input, 1-output Boolean function

$$r = f(w, x, y, z)$$

described by the truth table below:

w	x	y	z	r
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	0
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
0	1	1	1	0
1	0	0	0	1
1	0	0	1	0
1	0	1	0	1
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	1



By applying the algorithm, the corresponding Karnaugh map results in an expression

$$\begin{array}{rl} r = & (\neg w \wedge \neg y) \vee \\ & (w \wedge y \wedge z) \vee \\ & (\neg x \wedge \neg z) \end{array}$$

because:

- The red group spans columns 0 and 1 and rows 0 and 1; provided $w = 0$ and $y = 0$ we specify *just* those cells, so the expression is $\neg w \wedge \neg y$. That is, $w = 0$ restricts us to columns 0 and 1 (columns 2 and 3 have $w = 1$) and $y = 0$ restricts us to rows 0 and 1 (rows 2 and 3 have $y = 1$). Note that the values of x and z don't matter: cells in the group hold the value 1 regardless of x and y.
- The green group spans columns 2 and 3 in row 2; provided $w = 1$, $y = 1$ and $z = 1$ we specify *just* those cells, so the expression is $w \wedge y \wedge z$. That is, $w = 1$ restricts us to columns 2 and 3 (columns 0 and 1 have $w = 0$) and $y = 1$ and $z = 1$ restricts us to row 2 (rows 0, 1 and 3 have at least one of $y = 0$ or $z = 0$). Note that the value of x doesn't matter: cells in the group hold the value 1 regardless of x.
- The blue group spans columns 0 and 3 and rows 0 and 3; provided $x = 0$ and $z = 0$ we specify *just* those cells, so the expression is $\neg x \wedge \neg z$. That is, $x = 0$ restricts us to columns 0 and 3 (columns 1 and 2 have $x = 1$) and $z = 0$ restricts us to rows 0 and 3 (rows 1 and 2 have $z = 1$). Note that the values of w and y don't matter: cells in the group hold the value 1 regardless of w and y.