Computer Architecture

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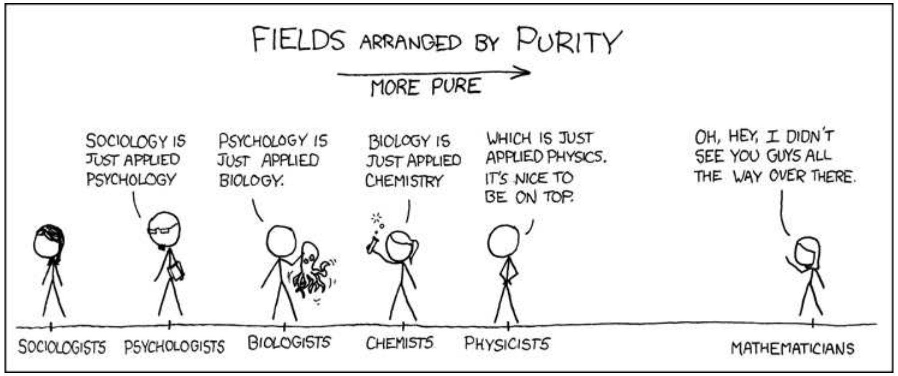
Keep in mind there are *two* PDFs available (of which this is the latter):

1. a PDF of examinable material used as lecture slides, and
2. a PDF of non-examinable, extra material:
   - the associated notes page may be pre-populated with extra, written explaination of material covered in lecture(s), plus
   - anything with a "grey'ed out" header/footer represents extra material which is useful and/or interesting but out of scope (and hence not covered).

Notes:

Notes:



https://xkcd.com/435

Notes:

- Agenda: an introduction to
  1. **propositional logic**,
  2. **Boolean algebra**, and
  3. application of, i.e., use-cases and rationale for the above within the context of COMS10015.

Notes:

Part 1: propositional logic (1)

- A **proposition** is basically a statement

  the temperature is 20°C

  this statement is false
  the temperature is too hot

Notes:

Part 1: propositional logic (1)

- A **proposition** is basically a statement

  the temperature is 20°C

  ~~this statement is false~~
  the temperature is too hot

  whose meaning
  1. can be **evaluated** to yield a **truth value**, i.e., **false** or **true**.

Notes:

Part 1: propositional logic (1)

- A **proposition** is basically a statement

  the temperature is $20°C$

  ~~this statement is false~~
  ~~the temperature is too hot~~

1. can be **evaluated** to yield a **truth value**, i.e., **false** or **true**,
2. must be unambiguous.

Notes:

Part 1: propositional logic (1)

- A **proposition** is basically a statement

  the temperature is $20°C$
  the temperature is $x°C$
  ~~this statement is false~~
  ~~the temperature is too hot~~

1. can be **evaluated** to yield a **truth value**, i.e., **false** or **true**,
2. must be unambiguous,
3. can include free **variables**.

Notes:

Part 1: propositional logic (1)

- A **proposition** is basically a statement

$$\begin{array}{lcl} f & = & \text{the temperature is } 20^{\circ}C \\ g(x) & = & \text{the temperature is } x^{\circ}C \\ & & \sout{\text{this statement is false}} \\ & & \sout{\text{the temperature is too hot}} \end{array}$$

1. can be **evaluated** to yield a **truth value**, i.e., **false** or **true**,
2. must be unambiguous,
3. can include free **variables**, and
4. can be represented using a short-hand variable or function, whereby free variables must be bound to concrete arguments before evaluation.

Notes:

Part 1: propositional logic (2)

- Single statements can be combined using various **connectives**, e.g.,

$$\text{the temperature is not } 20^{\circ}C$$

adding parentheses where needed to add clarity, so that

1. "not x" is denoted $\neg x$,

Notes:

Part 1: propositional logic (2)

- Single statements can be combined using various **connectives**, e.g.,

$$\neg(\text{the temperature is } 20^\circ C)$$

adding parentheses where needed to add clarity, so that

1. "not x" is denoted $\neg x$,

Notes:

Part 1: propositional logic (2)

- Single statements can be combined using various **connectives**, e.g.,

$$\text{the temperature is } 20^\circ C \text{ and it is sunny}$$

adding parentheses where needed to add clarity, so that

1. "not x" is denoted $\neg x$,
2. "x and y" is denoted $x \wedge y$,

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Notes:

- Single statements can be combined using various **connectives**, e.g.,

$$(\text{the temperature is } 20°C) \wedge (\text{it is sunny})$$

adding parentheses where needed to add clarity, so that

1. "not x" is denoted $\neg x$,
2. "x and y" is denoted $x \wedge y$,

Notes:

Part 1: propositional logic (2)

- Single statements can be combined using various **connectives**, e.g.,

$$\text{the temperature is } 20°C \text{ or it is sunny}$$

adding parentheses where needed to add clarity, so that

1. "not x" is denoted $\neg x$,
2. "x and y" is denoted $x \wedge y$,
3. "x or y" is denoted $x \vee y$, and usually called inclusive-or,

Notes:

Part 1: propositional logic (2)

- Single statements can be combined using various **connectives**, e.g.,

$$\text{(the temperature is } 20^\circ C) \vee \text{(it is sunny)}$$

  adding parentheses where needed to add clarity, so that

  1. "not x" is denoted $\neg x$,
  2. "x and y" is denoted $x \wedge y$,
  3. "x or y" is denoted $x \vee y$, and usually called inclusive-or,

Notes:

Part 1: propositional logic (2)

- Single statements can be combined using various **connectives**, e.g.,

$$\text{either the temperature is } 20^\circ C \text{ or it is sunny, but not both}$$

  adding parentheses where needed to add clarity, so that

  1. "not x" is denoted $\neg x$,
  2. "x and y" is denoted $x \wedge y$,
  3. "x or y" is denoted $x \vee y$, and usually called inclusive-or,
  4. "x or y but not x and y" is denoted $x \oplus y$, and usually called exclusive-or,

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Notes:

Part 1: propositional logic (2)

- Single statements can be combined using various **connectives**, e.g.,

$$(\text{the temperature is } 20^\circ C) \oplus (\text{it is sunny})$$

adding parentheses where needed to add clarity, so that

1. "not x" is denoted $\neg x$,
2. "x and y" is denoted $x \wedge y$,
3. "x or y" is denoted $x \vee y$, and usually called inclusive-or,
4. "x or y but not x and y" is denoted $x \oplus y$, and usually called exclusive-or,

Notes:

Part 1: propositional logic (2)

- Single statements can be combined using various **connectives**, e.g.,

$$\text{the temperature being } 20^\circ C \text{ implies that it is sunny}$$

adding parentheses where needed to add clarity, so that

1. "not x" is denoted $\neg x$,
2. "x and y" is denoted $x \wedge y$,
3. "x or y" is denoted $x \vee y$, and usually called inclusive-or,
4. "x or y but not x and y" is denoted $x \oplus y$, and usually called exclusive-or,
5. "x implies y" is denoted $x \Rightarrow y$, and sometimes written "if x then y", and

Notes:

Part 1: propositional logic (2)

- Single statements can be combined using various **connectives**, e.g.,

$$\text{(the temperature is } 20^\circ C\text{)} \Rightarrow \text{(it is sunny)}$$

adding parentheses where needed to add clarity, so that

1. "not x" is denoted $\neg x$,
2. "x and y" is denoted $x \wedge y$,
3. "x or y" is denoted $x \vee y$, and usually called inclusive-or,
4. "x or y but not x and y" is denoted $x \oplus y$, and usually called exclusive-or,
5. "x implies y" is denoted $x \Rightarrow y$, and sometimes written "if x then y", and

Notes:

Part 1: propositional logic (2)

- Single statements can be combined using various **connectives**, e.g.,

$$\text{the temperature is } 20^\circ C \text{ is equivalent to it being sunny}$$

adding parentheses where needed to add clarity, so that

1. "not x" is denoted $\neg x$,
2. "x and y" is denoted $x \wedge y$,
3. "x or y" is denoted $x \vee y$, and usually called inclusive-or,
4. "x or y but not x and y" is denoted $x \oplus y$, and usually called exclusive-or,
5. "x implies y" is denoted $x \Rightarrow y$, and sometimes written "if x then y", and
6. "x is equivalent to y" is denoted $x \equiv y$, and sometimes written "x if and only if y" or "x iff. y".

Notes:

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Part 1: propositional logic (2)

- Single statements can be combined using various **connectives**, e.g.,

$$\text{(the temperature is } 20^\circ C) \equiv \text{(it is sunny)}$$

  adding parentheses where needed to add clarity, so that

  1. "not x" is denoted $\neg x$,
  2. "x and y" is denoted $x \wedge y$,
  3. "x or y" is denoted $x \vee y$, and usually called inclusive-or,
  4. "x or y but not x and y" is denoted $x \oplus y$, and usually called exclusive-or,
  5. "x implies y" is denoted $x \Rightarrow y$, and sometimes written "if x then y", and
  6. "x is equivalent to y" is denoted $x \equiv y$, and sometimes written "x if and only if y" or "x iff. y".

Notes:

Part 1: propositional logic (2)

- You *might* see more formal terms or different notation for the *same* connectives:
  - $\neg$ is often termed logical **compliment** (or **negation**),
  - $\wedge$ is often termed logical **conjunction**,
  - $\vee$ is often termed logical (inclusive) **disjunction**,
  - $\oplus$ is often termed logical (exclusive) **disjunction**,
  - $\Rightarrow$ is often termed logical **implication**, and
  - $\equiv$ is often termed logical **equivalence**.

Notes:

Part 1: propositional logic (3)

- You can think of the same thing diagrammatically, i.e.,

$$r = (\text{the temperature is } 20^\circ C) \wedge (\text{it is sunny})$$

$$\equiv$$

but either way, the question is how do we **evaluate** the (compound) proposition (or **expression**) to produce a truth value?

Notes:

Part 1: propositional logic (4)

- Since each statement can only evaluate to **true** or **false**, we can enumerate all possible outcomes in a **truth table**, e.g., if

$$\begin{array}{rcl} x & = & \text{the temperature is } 20^\circ C \\ y & = & \text{it is sunny} \\ r & = & (\text{the temperature is } 20^\circ C) \wedge (\text{it is sunny}) \end{array}$$

then

inputs		output
x	y	r
false	**false**	**false**
false	**true**	**false**
true	**false**	**false**
true	**true**	**true**

- Note that
  1. each row details the output(s) associated with a given assignment to the inputs,
  2. if there are n inputs, the truth table will have 2^n rows.

Notes:

Part 1: propositional logic (5)

Definition

x	y	$\neg x$	$x \wedge y$	$x \vee y$	$x \oplus y$	$x \Rightarrow y$	$x \equiv y$
false	**false**	**true**	**false**	**false**	**false**	**true**	**true**
false	**true**	**true**	**false**	**true**	**true**	**true**	**false**
true	**false**	**false**	**false**	**true**	**true**	**false**	**false**
true	**true**	**false**	**true**	**true**	**false**	**true**	**true**

Notes:

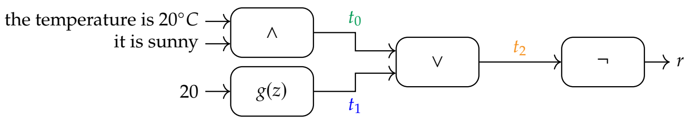
Part 1: propositional logic (6)

Example

Imagine that now

$$
\begin{array}{lcl}
x & = & \text{the temperature is } 20^\circ C \\
y & = & \text{it is sunny} \\
g(z) & = & \text{the temperature is } z^\circ C \\
r & = & \neg(((\text{the temperature is } 20^\circ C) \wedge (\text{it is sunny})) \vee (\text{the temperature is } z^\circ C))
\end{array}
$$

which we translate into the diagrammatic form



An example evaluation might be as follows:

inputs		intermediates			output
x	y	t_0	t_1	t_2	r
false	**false**	**false**	**false**	**false**	**true**
false	**true**	**false**	**false**	**false**	**true**
true	**false**	**false**	**true**	**true**	**false**
true	**true**	**true**	**true**	**true**	**false**

Notes:

Part 2: Boolean algebra (1)

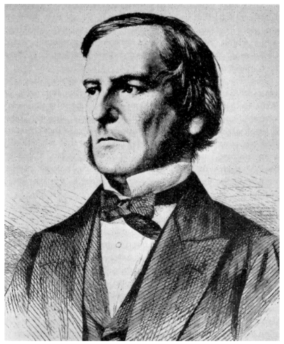
- Notice that
  1. in **elementary algebra**, for some number x we have that
     $$x + 0 = x$$
     and
     $$x \cdot 1 = x,$$
  2. in **set theory**, for some set x we have that
     $$x \cup \emptyset = x$$
     and
     $$x \cap \mathcal{U} = x,$$
     plus we've now demonstrated that
  3. in **propositional logic**, for some truth value x we have that
     $$x \vee \textbf{false} = x$$
     and
     $$x \wedge \textbf{true} = x.$$

Notes:

Part 2: Boolean algebra (2)

Thou must
1. work with the set $\mathbb{B} = \{0, 1\}$ of **binary** digits, using 0 and 1 instead of **false** and **true**,
2. shorten every statement into either a **variable** *or* **function**,
3. use **unary operators**, e.g., $\neg$ (or NOT), and **binary operators**, e.g., $\wedge$ and $\vee$ (or AND and OR), to form **expressions**,
4. manipulate said expressions according to some axioms (or rules),

then call the result **Boolean algebra**.



https://en.wikipedia.org/wiki/File:George_Boole.jpg

Notes:

- Put more concretely, we now have
  1. a set of operators specified by

Definition

x	y	$\neg x$	$x \wedge y$	$x \vee y$	$x \oplus y$	$x \Rightarrow y$	$x \equiv y$
0	0	1	0	0	0	1	1
0	1	1	0	1	1	1	0
1	0	0	0	1	1	0	0
1	1	0	1	1	0	1	1

Notes:

- The precedence levels for our suite of Boolean operators is
  1. $\neg$,
  2. $\wedge$,
  3. $\vee$

  meaning, for example, that we resolve an $\wedge$ before and $\vee$ (and sometimes say $\wedge$ "binds more tightly" to operands than $\vee$).

Part 2: Boolean algebra (3)

- Put more concretely, we now have

  2. a set of axioms that allow manipulation of expressions comprised of said operators, i.e.,

Definition

Name	Axiom(s)	Name	Axiom(s)
commutativity	$x \wedge y \equiv y \wedge x$	commutativity	$x \vee y \equiv y \vee x$
association	$(x \wedge y) \wedge z \equiv x \wedge (y \wedge z)$	association	$(x \vee y) \vee z \equiv x \vee (y \vee z)$
distribution	$x \wedge (y \vee z) \equiv (x \wedge y) \vee (x \wedge z)$	distribution	$x \vee (y \wedge z) \equiv (x \vee y) \wedge (x \vee z)$

plus other rules such as **precedence** (to deal with ambiguity in the absence of parentheses).

Notes:

- The precedence levels for our suite of Boolean operators is
  1. $\neg$,
  2. $\wedge$,
  3. $\vee$

  meaning, for example, that we resolve an $\wedge$ before and $\vee$ (and sometimes say $\wedge$ "binds more tightly" to operands than $\vee$).

- Put more concretely, we now have

2. a set of axioms that allow manipulation of expressions comprised of said operators, i.e.,

Definition

Name	Axiom(s)	Name	Axiom(s)
identity	$x \wedge 1 \equiv x$	identity	$x \vee 0 \equiv x$
null	$x \wedge 0 \equiv 0$	null	$x \vee 1 \equiv 1$
idempotency	$x \wedge x \equiv x$	idempotency	$x \vee x \equiv x$
inverse	$x \wedge \neg x \equiv 0$	inverse	$x \vee \neg x \equiv 1$

plus other rules such as **precedence** (to deal with ambiguity in the absence of parentheses).

Notes:

- The precedence levels for our suite of Boolean operators is
  1. $\neg$,
  2. $\wedge$,
  3. $\vee$

  meaning, for example, that we resolve an $\wedge$ before and $\vee$ (and sometimes say $\wedge$ "binds more tightly" to operands than $\vee$).

Part 2: Boolean algebra (3)

- Put more concretely, we now have

2. a set of axioms that allow manipulation of expressions comprised of said operators, i.e.,

Definition

Name	Axiom(s)	Name	Axiom(s)
absorption	$x \wedge (x \vee y) \equiv x$	absorption	$x \vee (x \wedge y) \equiv x$
de Morgan	$\neg(x \wedge y) \equiv \neg x \vee \neg y$	de Morgan	$\neg(x \vee y) \equiv \neg x \wedge \neg y$

plus other rules such as **precedence** (to deal with ambiguity in the absence of parentheses).

Notes:

- The precedence levels for our suite of Boolean operators is
  1. $\neg$,
  2. $\wedge$,
  3. $\vee$

  meaning, for example, that we resolve an $\wedge$ before and $\vee$ (and sometimes say $\wedge$ "binds more tightly" to operands than $\vee$).

Part 2: Boolean algebra (3)

- Put more concretely, we now have

2. a set of axioms that allow manipulation of expressions comprised of said operators, i.e.,

Definition

Name	Axiom(s)
equivalence	$x \equiv y \equiv (x \Rightarrow y) \wedge (y \Rightarrow x)$
implication	$x \Rightarrow y \equiv \neg x \vee y$
involution	$\neg\neg x \equiv x$

plus other rules such as **precedence** (to deal with ambiguity in the absence of parentheses).

Notes:

- The precedence levels for our suite of Boolean operators is
  1. $\neg$,
  2. $\wedge$,
  3. $\vee$

  meaning, for example, that we resolve an $\wedge$ before and $\vee$ (and sometimes say $\wedge$ "binds more tightly" to operands than $\vee$).

Part 2: Boolean algebra (4)

Standard forms

Definition

The fact there are AND and OR forms of most axioms hints at a more general underlying principle. Consider a Boolean expression e: the **principle of duality** states that the **dual expression** e^D is formed by

1. leaving each variable as is,
2. swapping each $\wedge$ with $\vee$ and vice versa, and
3. swapping each 0 with 1 and vice versa.

Of course e and e^D are different expressions, and clearly not equivalent; if we start with some $e \equiv f$ however, then we do still get $e^D \equiv f^D$.

Example

As an example, consider axioms for

1. distribution, e.g., if
$$e = x \wedge (y \vee z) \equiv (x \wedge y) \vee (x \wedge z)$$
then
$$e^D = x \vee (y \wedge z) \equiv (x \vee y) \wedge (x \vee z)$$
and
2. identity, e.g., if
$$e = x \wedge 1 \equiv x$$
then
$$e^D = x \vee 0 \equiv x.$$

Notes:

Definition

The de Morgan axiom can be turned into a more general principle. Consider a Boolean expression e: the **principle of complements** states that the **complement expression** $\neg e$ is formed by

1. swapping each variable x with the complement $\neg x$,
2. swapping each $\wedge$ with $\vee$ and vice versa, and
3. swapping each 0 with 1 and vice versa.

Example

As an example, consider that if

$$e = x \wedge y \wedge z,$$

then by the above we should find

$$f = \neg e = (\neg x) \vee (\neg y) \vee (\neg z).$$

Proof:

x	y	z	$\neg x$	$\neg y$	$\neg z$	e	f
0	0	0	1	1	1	0	1
0	0	1	1	1	0	0	1
0	1	0	1	0	1	0	1
0	1	1	1	0	0	0	1
1	0	0	0	1	1	0	1
1	0	1	0	1	0	0	1
1	1	0	0	0	1	0	1
1	1	1	0	0	0	1	0

Notes:

Definition

Consider a Boolean expression:

1. When the expression is written as a sum (i.e., OR) of terms which each comprise the product (i.e., AND) of variables, e.g.,

   $$\underbrace{(a \wedge b \wedge c)}_{\text{minterm}} \vee (d \wedge e \wedge f),$$

   it is said to be in **disjunctive normal form** or **Sum of Products (SoP)** form; the terms are called the **minterms**. Note that each variable can exist as-is *or* complemented using NOT, meaning

   $$\underbrace{(\neg a \wedge b \wedge c)}_{\text{minterm}} \vee (d \wedge \neg e \wedge f),$$

   is also a valid SoP expression.
2. When the expression is written as a product (i.e., AND) of terms which each comprise the sum (i.e., OR) of variables, e.g.,

   $$\underbrace{(a \vee b \vee c)}_{\text{maxterm}} \wedge (d \vee e \vee f),$$

   it is said to be in **conjunctive normal form** or **Product of Sums (PoS)** form; the terms are called the **maxterms**. As above each variable can exist as-is *or* complemented using NOT.

Notes:

Part 2: Boolean algebra (7)
Derived operators

- Concept: we can define various **derived operators** in terms of NOT, AND, and OR.
- Example:
  - "exclusive-OR" or **XOR**, such that

$$x \oplus y \;\; \equiv \;\; (\neg x \wedge y) \vee (x \wedge \neg y)$$

so

x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

Notes:

Part 2: Boolean algebra (7)
Derived operators

- Concept: we can define various **derived operators** in terms of NOT, AND, and OR.
- Example:
  - "NOT-AND" or **NAND**, such that

$$x \;\overline{\wedge}\; y \;\; \equiv \;\; \neg(x \wedge y)$$

so

x	y	$x \;\overline{\wedge}\; y$
0	0	1
0	1	1
1	0	1
1	1	0

  - "NOT-OR" or **NOR**, such that

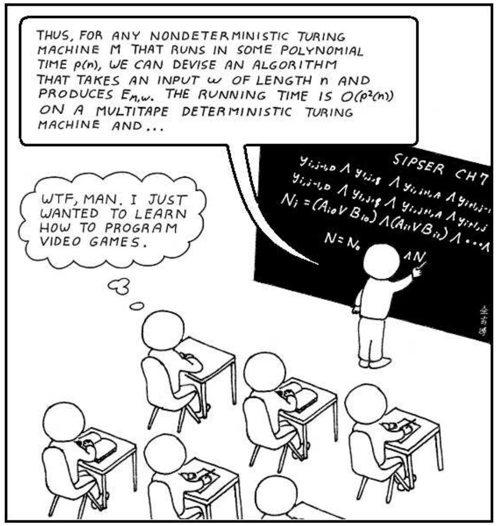
$$x \;\overline{\vee}\; y \;\; \equiv \;\; \neg(x \vee y)$$

so

x	y	$x \;\overline{\vee}\; y$
0	0	1
0	1	0
1	0	0
1	1	0

Notes:

Part 3: application (1)



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Notes:

Part 3: application (2)

- (Fairly) reasonable question(s):
  1. "I thought this was CS, not Maths!", and
  2. "why does *this* unit duplicate material in *other* units?".

Notes:

Part 3: application (2)

- (Fairly) reasonable question(s):
  1. "I thought this was CS, not Maths!", and
  2. "why does *this* unit duplicate material in *other* units?".
- Answer: it isn't, and it doesn't (well, not *too* much) ... note that
  - theoretical concepts, e.g., often have significant practical motivations or implications, and
  - it's perfectly reasonable to utilise **Electronic Design Automation (EDA)** [3] tools.

Notes:

Part 3: application (3)
Axiomatic manipulation $\rightsquigarrow$ optimisation

- Question: simplify the Boolean expression

$$(\neg(a \vee b) \wedge \neg(c \vee d \vee e)) \vee \neg(a \vee b)$$

into a form that contains the fewest operators possible.

Notes:

- Question: simplify the Boolean expression

$$(\neg(a \vee b) \wedge \neg(c \vee d \vee e)) \vee \neg(a \vee b)$$

into a form that contains the fewest operators possible.

- Solution #1: less steps.

$$\begin{array}{lllllll}
 & (\neg(a \vee b) & \wedge & \neg(c \vee d \vee e)) & \vee & \neg(a \vee b) & \\
= & \neg(a \vee b) & \vee & (\neg(a \vee b) & \wedge & \neg(c \vee d \vee e)) & \text{(commutativity)} \\
= & \neg(a \vee b) & & & & & \text{(absorption)}
\end{array}$$

Notes:

Part 3: application (3)
Axiomatic manipulation $\rightsquigarrow$ optimisation

- Question: simplify the Boolean expression

$$(\neg(a \vee b) \wedge \neg(c \vee d \vee e)) \vee \neg(a \vee b)$$

into a form that contains the fewest operators possible.

- Solution #2: more steps.

$$\begin{array}{lllllll}
 & (\neg(a \vee b) & \wedge & \neg(c \vee d \vee e)) & \vee & \neg(a \vee b) & \\
= & ((\neg a \wedge \neg b) & \wedge & \neg(c \vee d \vee e)) & \vee & \neg(a \vee b) & \text{(de Morgan)} \\
= & ((\neg a \wedge \neg b) & \wedge & (\neg c \wedge \neg d \wedge \neg e)) & \vee & \neg(a \vee b) & \text{(de Morgan)} \\
= & ((\neg a \wedge \neg b) & \wedge & (\neg c \wedge \neg d \wedge \neg e)) & \vee & (\neg a \wedge \neg b) & \text{(de Morgan)} \\
= & (\neg a \wedge \neg b) & \vee & ((\neg a \wedge \neg b) & \wedge & (\neg c \wedge \neg d \wedge \neg e)) & \text{(commutativity)} \\
= & (\neg a \wedge \neg b) & & & & & \text{(absorption)} \\
= & \neg(a \vee b) & & & & & \text{(de Morgan)}
\end{array}$$

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Notes:

Part 3: application (4)
Axiomatic manipulation $\rightsquigarrow$ optimisation

- Question: simplify the Boolean expression

$$(a \wedge b \wedge c) \vee (\neg a \wedge b) \vee (a \wedge b \wedge \neg c)$$

into a form that contains the fewest operators possible.

Notes:

Part 3: application (4)
Axiomatic manipulation $\rightsquigarrow$ optimisation

- Question: simplify the Boolean expression

$$(a \wedge b \wedge c) \vee (\neg a \wedge b) \vee (a \wedge b \wedge \neg c)$$

into a form that contains the fewest operators possible.
- Solution:

$$\begin{array}{llllll}
 & (a \wedge b \wedge c) & \vee & (\neg a \wedge b) & \vee & (a \wedge b \wedge \neg c) & \\
= & (a \wedge b \wedge c) & \vee & (a \wedge b \wedge \neg c) & \vee & (\neg a \wedge b) & \text{(commutativity)} \\
= & (a \wedge b) & \wedge & (c \vee \neg c) & \vee & (\neg a \wedge b) & \text{(distribution)} \\
= & (a \wedge b) & \wedge & 1 & \vee & (\neg a \wedge b) & \text{(inverse)} \\
= & (a \wedge b) & & & \vee & (\neg a \wedge b) & \text{(identity)} \\
= & b & \wedge & (a \vee \neg a) & & & \text{(distribution)} \\
= & b & \wedge & 1 & & & \text{(inverse)} \\
= & b & & & & & \text{(identity)}
\end{array}$$

Notes:

Part 3: application (5)

Axiomatic manipulation $\rightsquigarrow$ optimisation

Quote

If I designed a computer with 200 chips, I tried to design it with 150. And then I would try to design it with 100. I just tried to find every trick I could in life to design things real tiny.

– *Wozniak*

Quote

So I took 20 chips off their board; I bypassed 20 of their chips.

– *Wozniak*

Notes:

- The quotes relate to design and implementation of a (floppy) disk controller for the Apple II computer (circa 1977); there is an obvious focus on efficiency, which is credited as allowing the controller to be commercially viable. A detailed overview of the overarching anecdote is available via

  https://en.wikipedia.org/wiki/Disk_II

  or

  https://apple2history.org/history/ah05/

  The moral is that, in reality, "it works", while important, may not be good enough: meeting various other (market-driven) quality metrics (e.g., efficiency, physical size, power consumption, etc.) is often vital rather than simply attractive.

Part 3: application (6)

Axiomatic manipulation $\rightsquigarrow$ optimisation

https://en.wikipedia.org/wiki/File:Shugart_SA400.jpg

https://en.wikipedia.org/wiki/File:Interface_Card_-_Disk_II_Interface_Apple2.jpg

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Notes:

Part 3: application (7)

Practical use-cases $\rightsquigarrow$ richer specification

- Concept: truth tables can accommodate **don't care** entries, e.g.,

x	y	r
?	0	1
0	1	**?**
1	1	0

  such that
  - a **?** (rather than 0 or 1) means we "don't care" ($\neq$ "don't know"),
  - on the LHS, for an *in*put,
    - **?** is a wildcard (or short-hand),
    - it means 0 *and* 1,
    - we've compressed two truth table rows into one.
  - on the RHS, for an *out*put,
    - **?** is a choice,
    - it means 0 *or* 1,
    - we can select which one to, e.g., optimise the associated expression.

Notes:

Part 3: application (8)

Universality $\rightsquigarrow$ manufacturability

- Fact: NAND and NOR are **functionally complete** (or **universal**), e.g.,

$$
\begin{array}{lclcl}
\neg x & & & \equiv & x \overline{\wedge} x \\
x \wedge y & & & \equiv & (x \overline{\wedge} y) \overline{\wedge} (x \overline{\wedge} y) \\
x \vee y & \equiv & \neg x \overline{\wedge} \neg y & \equiv & (x \overline{\wedge} x) \overline{\wedge} (y \overline{\wedge} y)
\end{array}
$$

  which we can prove via

x	y	$x \overline{\wedge} y$	$x \overline{\wedge} x$	$y \overline{\wedge} y$	$(x \overline{\wedge} y) \overline{\wedge} (x \overline{\wedge} y)$	$(x \overline{\wedge} x) \overline{\wedge} (y \overline{\wedge} y)$
0	0	1	1	1	0	0
0	1	1	1	0	0	1
1	0	1	0	1	0	1
1	1	0	0	0	1	1

  $\therefore$ *any* Boolean function can be expressed using a *single* operator.

Notes:

Part 3: application (9)
Universality $\rightsquigarrow$ manufacturability

- Question: translate

$$x \wedge (y \vee z)$$

into a version using NAND only.

Notes:

Part 3: application (9)
Universality $\rightsquigarrow$ manufacturability

- Question: translate

$$x \wedge (y \vee z)$$

into a version using NAND only.
- Solution #1: apply the identities *naively* to get

$$\begin{array}{cl} & x \wedge (y \vee z) \\ = & x \wedge ((y \,\overline{\wedge}\, y) \,\overline{\wedge}\, (z \,\overline{\wedge}\, z)) \\ = & (x \,\overline{\wedge}\, ((y \,\overline{\wedge}\, y) \,\overline{\wedge}\, (z \,\overline{\wedge}\, z))) \,\overline{\wedge}\, (x \,\overline{\wedge}\, ((y \,\overline{\wedge}\, y) \,\overline{\wedge}\, (z \,\overline{\wedge}\, z))) \end{array}$$

Notes:

Part 3: application (9)

Universality $\rightsquigarrow$ manufacturability

- Question: translate

$$x \wedge (y \vee z)$$

  into a version using NAND only.
- Solution #2: apply the identities *intelligently* to get

$$\begin{array}{rl} & x \wedge (y \vee z) \\ = & x \wedge ((y \overline{\wedge} y) \overline{\wedge} (z \overline{\wedge} z)) \\ = & t \overline{\wedge} t \end{array}$$

  where $t = x \overline{\wedge} ((y \overline{\wedge} y) \overline{\wedge} (z \overline{\wedge} z))$ is a common sub-expression [2].

Notes:

Conclusions

- Take away points:
  1. The design of computational devices, e.g., micro-processors, *isn't* ad hoc: Boolean algebra offers a theoretical basis for reasoning about computational devices (and computation) in practice.

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Notes:

Conclusions

- Take away points:

2. Boolean algebra is a (somewhat) cosmetic extension of what you already know.
3. Keep in mind that
   - *any* Boolean function f which can be expressed by a truth table can be computed using an associated Boolean expression,
   - a Boolean expression is composed of Boolean operators,
   - *if* we (physically) implement the Boolean operators, we can implement the Boolean expression and hence compute f.

Notes:

Conclusions

- Take away points:

4. We'll focus on *application* (i.e., *use*) vs. *theory* (e.g., *study*) of Boolean algebra from here on.
5. Keep in mind that
   - "it works" $\neq$ "it works *well*",
   - using automation is fine *iff.* you know the underlying theory,
   - using brute-force is fine *iff.* you know the underlying theory,
   - Boolean algebra > Boolean axioms: concepts that *seem* of interest in theory alone, *can* be important if/when applied in practice.

Notes:

Additional Reading

- *Wikipedia: Boolean algebra*. URL: https://en.wikipedia.org/wiki/Boolean_algebra.
- D. Page. "Chapter 1: Mathematical preliminaries". In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009.
- W. Stallings. "Chapter 11: Digital logic". In: *Computer Organisation and Architecture*. 9th ed. Prentice Hall, 2013.
- A.S. Tanenbaum and T. Austin. "Section 3.1: Gates and Boolean algebra". In: *Structured Computer Organisation*. 6th ed. Prentice Hall, 2012.

Notes:

References

[1] *Wikipedia: Boolean algebra*. URL: https://en.wikipedia.org/wiki/Boolean_algebra (see p. 113).

[2] *Wikipedia: Common sub-expression elimination*. URL: https://en.wikipedia.org/wiki/Common_subexpression_elimination (see pp. 103, 105).

[3] *Wikipedia: Electronic Design Automation (EDA)*. URL: https://en.wikipedia.org/wiki/Electronic_design_automation (see pp. 79, 81).

[4] D. Page. "Chapter 1: Mathematical preliminaries". In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009 (see p. 113).

[5] W. Stallings. "Chapter 11: Digital logic". In: *Computer Organisation and Architecture*. 9th ed. Prentice Hall, 2013 (see p. 113).

[6] A.S. Tanenbaum and T. Austin. "Section 3.1: Gates and Boolean algebra". In: *Structured Computer Organisation*. 6th ed. Prentice Hall, 2012 (see p. 113).

Notes: