Computer Architecture

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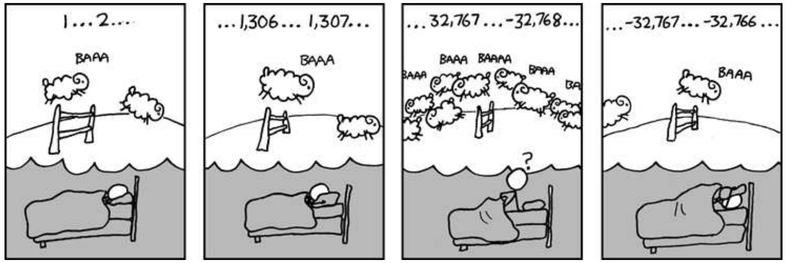
Keep in mind there are *two* PDFs available (of which this is the latter):

1. a PDF of examinable material used as lecture slides, and
2. a PDF of non-examinable, extra material:
   - the associated notes page may be pre-populated with extra, written explaination of material covered in lecture(s), plus
   - anything with a "grey'ed out" header/footer represents extra material which is useful and/or interesting but out of scope (and hence not covered).

Notes:

Notes:



https://xkcd.com/571

Notes:

- Claim: at least conceptually we could say that

$$123 \equiv \langle 3, 2, 1 \rangle,$$

  i.e., the decimal literal 123 is basically just a sequence of digits.

Notes:

- **Question**: given
  - a **bit** is a single binary digit, i.e., 0 or 1,
  - a **byte** is an 8-element sequence of bits, and
  - a **word** is a w-element sequence of bits

  and so, e.g.,

  $$01111011 \equiv \langle 1, 1, 0, 1, 1, 1, 1, 0 \rangle,$$

  what do these things *mean* ... what do they *represent*?
- **Answer**: anything *we* decide they do!

Notes:

- **Concept**:

$$\underbrace{\hat{X}}_{\text{the representation of } X} \;\; \underbrace{\mapsto}_{\text{maps to}} \;\; \underbrace{X}_{\text{the value of } X}$$

  i.e., we need

  1. a concrete representation that we can write down, plus
  2. a mapping that yields the correct value *and* is consistent (in both directions).

Notes:

- Agenda:
  1. useful properties of bit-sequences,
  2. positional number systems $\rightsquigarrow$ standard integer representations.

Notes:

Part 1: useful properties of bit-sequences

Definition

A given literal, say

$$X = 1111011,$$

can be interpreted in *two* ways:

1. A **little-endian** ordering is where we read bits in a literal from right-to-left, i.e.,

   $$X_{LE} = \langle X_0, X_1, X_2, X_3, X_4, X_5, X_6 \rangle = \langle 1, 1, 0, 1, 1, 1, 1 \rangle,$$

   where

   - the Least-Significant Bit (LSB) is the right-most in the literal (i.e., X_0), and
   - the Most-Significant Bit (MSB) is the left-most in the literal (i.e., $X_{n-1} = X_6$).

2. A **big-endian** ordering is where we read bits in a literal from left-to-right, i.e.,

   $$X_{BE} = \langle X_6, X_5, X_4, X_3, X_2, X_1, X_0 \rangle = \langle 1, 1, 1, 1, 0, 1, 1 \rangle,$$

   where

   - the Least-Significant Bit (LSB) is the left-most in the literal (i.e., $X_{n-1} = X_6$), and
   - the Most-Significant Bit (MSB) is the right-most in the literal (i.e., X_0).

Notes:

Part 1: useful properties of bit-sequences

Definition

Following the idea of vectorial Boolean function, given an n-element bit-sequence X, and an m-element bit-sequence Y we can clearly

1. overload $\oslash \in \{\neg\}$, i.e., write
$$R = \oslash X,$$
to mean
$$R_i = \oslash X_i$$
for $0 \leq i < n$,
2. overload $\ominus \in \{\wedge, \vee, \oplus\}$, i.e., write
$$R = X \ominus Y,$$
to mean
$$R_i = X_i \ominus Y_i$$
for $0 \leq i < n = m$, where if $n \neq m$, we pad either X or Y with 0 until the $n = m$.

Notes:

- Although they *look* similar, take care not to confuse the bit-wise operators `!`, `&&` and `||`. It's reasonable to think of the former as being used for *computation* and the latter for *conditions* (i.e., when a decision is needed).

Part 1: useful properties of bit-sequences

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for $0 \leq i < n = m$, where if $n \neq m$, we pad either X or Y with 0 until the $n = m$.

- Example: in C, we use the computational (or **bit-wise**) operators `~`, `&`, `|`, and `^` this way: they apply NOT, AND, OR, and XOR to corresponding bits in the operands.

Notes:

- Although they *look* similar, take care not to confuse the bit-wise operators `!`, `&&` and `||`. It's reasonable to think of the former as being used for *computation* and the latter for *conditions* (i.e., when a decision is needed).

Definition

Given two n-bit sequences X and Y, we can define some important properties named after Richard Hamming, a researcher at Bell Labs:

- The **Hamming weight** of X is the number of bits in X that are equal to 1, i.e., the number of times $X_i = 1$. This can be expressed as

$$\mathrm{HW}(X) = \sum_{i=0}^{n-1} X_i.$$

- The **Hamming distance** between X and Y is the number of bits in X that differ from the corresponding bit in Y, i.e., the number of times $X_i \neq Y_i$. This can be expressed as

$$\mathrm{HD}(X, Y) = \sum_{i=0}^{n-1} X_i \oplus Y_i.$$

Note that both quantities naturally generalise to non-binary sequences.

Notes:

Part 1: useful properties of bit-sequences

Definition

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$$\mathrm{HD}(X, Y) = \sum_{i=0}^{n-1} X_i \oplus Y_i.$$

Note that both quantities naturally generalise to non-binary sequences.

- Example: given $X = \langle 1, 0, 0, 1 \rangle$ and $Y = \langle 0, 1, 1, 1 \rangle$ we find that

$$\begin{array}{lcl} \mathrm{HW}(X) & = \sum_{i=0}^{n-1} X_i & = 1 + 0 + 0 + 1 = 2 \\ \mathrm{HD}(X, Y) & = \sum_{i=0}^{n-1} X_i \oplus Y_i = (1 \oplus 0) + (0 \oplus 1) + (0 \oplus 1) + (1 \oplus 1) & = 1 + 1 + 1 + 0 = 3 \end{array}$$

Notes:

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- Concept: a **positional number system** expresses the value of a number x using a base-b (or radix-b) expansion, i.e.,

$$\begin{array}{rcl} \hat{x} & = & \langle \hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{n-1} \rangle \\ & \mapsto & x \\ & = & \pm \sum_{i=0}^{n-1} \hat{x}_i \cdot b^i \end{array}$$

  where each $\hat{x}_i$
  - is one of n digits taken from the digit set $X = \{0, 1, \ldots, b-1\}$,
  - is "weighted" by some power of of the base b.

Notes:

Part 2: positional number systems $\leadsto$ standard integer representations (1)

- Beware!
  - for $b > 10$ we can't express $\hat{x}_i$ using a single Arabic numeral,
  - for $b = 16$, for example, we use letters instead:

$$\begin{array}{lcl} A & \mapsto & 10 \\ B & \mapsto & 11 \\ C & \mapsto & 12 \\ D & \mapsto & 13 \\ E & \mapsto & 14 \\ F & \mapsto & 15 \end{array}$$

Notes:

Example

Consider an example where we

1. set $b = 10$, i.e., deal with **decimal** numbers, and
2. have $\hat{x}_i \in X = \{0, 1, \ldots, 10 - 1 = 9\}$.

This means we can write

$$
\begin{array}{lcl}
\hat{x} = 123 & = & \langle 3, 2, 1 \rangle_{(10)} \\
 & \mapsto & x \\
 & = & \sum_{i=0}^{n-1} \hat{x}_i \cdot 10^i \\
 & = & 3 \cdot 10^0 + 2 \cdot 10^1 + 1 \cdot 10^2 \\
 & = & 3 \cdot 1 \quad + 2 \cdot 10 \quad + 1 \cdot 100 \\
 & = & 123_{(10)}
\end{array}
$$

i.e., represent the value "one hundred and twenty three" in a variety of ways using different bases.

Notes:

Example

Consider an example where we

1. set $b = 2$, i.e., deal with **binary** numbers, and
2. have $\hat{x}_i \in X = \{0, 2 - 1 = 1\}$.

This means we can write

$$
\begin{array}{lcl}
\hat{x} = 1111011 & = & \langle 1, 1, 0, 1, 1, 1, 1 \rangle_{(2)} \\
 & \mapsto & x \\
 & = & \sum_{i=0}^{n-1} \hat{x}_i \cdot 2^i \\
 & = & 1 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3 + 1 \cdot 2^4 + 1 \cdot 2^5 + 1 \cdot 2^6 \\
 & = & 1 \cdot 1 \quad + 1 \cdot 2 \quad + 0 \cdot 4 \quad + 1 \cdot 8 \quad + 1 \cdot 16 + 1 \cdot 32 + 1 \cdot 64 \\
 & = & 123_{(10)}
\end{array}
$$

i.e., represent the value "one hundred and twenty three" in a variety of ways using different bases.

Notes:

Part 2: positional number systems ⇝ standard integer representations (2)

Example

Consider an example where we

1. set $b = 8$, i.e., deal with **octal** numbers, and
2. have $\hat{x}_i \in X = \{0, 1, \ldots, 8 - 1 = 7\}$.

This means we can write

$$
\begin{array}{lcl}
\hat{x} = 173 & = & \langle 3, 7, 1 \rangle_{(8)} \\
 & \mapsto & x \\
 & = & \sum_{i=0}^{n-1} \hat{x}_i \cdot 8^i \\
 & = & 3 \cdot 8^0 + 7 \cdot 8^1 + 1 \cdot 8^2 \\
 & = & 3 \cdot 1 \;\; + 7 \cdot 8 \;\; + 1 \cdot 64 \\
 & = & 123_{(10)}
\end{array}
$$

i.e., represent the value "one hundred and twenty three" in a variety of ways using different bases.

Notes:

Part 2: positional number systems ⇝ standard integer representations (2)

Example

Consider an example where we

1. set $b = 16$, i.e., deal with **hexadecimal** numbers, and
2. have $\hat{x}_i \in X = \{0, 1, \ldots, 16 - 1 = 15\}$.

This means we can write

$$
\begin{array}{lcl}
\hat{x} = 7B & = & \langle B, 7 \rangle_{(16)} \\
 & \mapsto & x \\
 & = & \sum_{i=0}^{n-1} \hat{x}_i \cdot 16^i \\
 & = & 11 \cdot 16^0 + 7 \cdot 16^1 \\
 & = & 11 \cdot 1 \;\;\; + 7 \cdot 16 \\
 & = & 123_{(10)}
\end{array}
$$

i.e., represent the value "one hundred and twenty three" in a variety of ways using different bases.

Notes:

- Problem: we want to represent and perform various operations on elements of $\mathbb{Z}$, *but*
  1. it's an an infinite set, and
  2. so far we've ignored the issue of sign.
- Solution: in C, for example, we get

$$
\begin{array}{rclcl}
\texttt{unsigned char} & \simeq & \texttt{uint8\_t} & \mapsto & \{ 0, \ldots, +2^8 - 1 \} \\
\texttt{char} & \simeq & \texttt{int8\_t} & \mapsto & \{ -2^7, \ldots, 0, \ldots, +2^7 - 1 \}
\end{array}
$$

but why *these*, and how do they work?

Notes:

Definition

An unsigned integer can be represented in n bits by using the natural binary expansion. That is, we have

$$
\begin{array}{rcl}
\hat{x} & = & \langle \hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{n-1} \rangle \\
 & \mapsto & x \\
 & = & \sum_{i=0}^{n-1} \hat{x}_i \cdot 2^i
\end{array}
$$

for $\hat{x}_i \in \{0, 1\}$, which yields

$$0 \le x \le 2^n - 1.$$

Notes:

Example ($n = 8$)

11111111	$\mapsto$	$1 \cdot 2^7 + 1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$	$=$	$+255_{(10)}$
	$\vdots$		$\vdots$	
10000101	$\mapsto$	$1 \cdot 2^7 + 0 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$	$=$	$+133_{(10)}$
	$\vdots$		$\vdots$	
10000000	$\mapsto$	$1 \cdot 2^7 + 0 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$	$=$	$+128_{(10)}$
01111111	$\mapsto$	$0 \cdot 2^7 + 1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$	$=$	$+127_{(10)}$
	$\vdots$		$\vdots$	
01111011	$\mapsto$	$0 \cdot 2^7 + 1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$	$=$	$+123_{(10)}$
	$\vdots$		$\vdots$	
00000001	$\mapsto$	$0 \cdot 2^7 + 0 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$	$=$	$+1_{(10)}$
00000000	$\mapsto$	$0 \cdot 2^7 + 0 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$	$=$	$+0_{(10)}$

Notes:

- **Fact**:
  - each hexadecimal digit $x_i \in \{0, 1, \ldots, 15\}$,
  - four bits gives $2^4 = 16$ possible combinations, so
  - each hexadecimal digit can be thought of as a short-hand for four binary digits.
- **Example**: we can perform the following translation steps

$$
\begin{array}{lclllll}
8AC & = & \langle & C, & A, & 8, & \rangle_{(16)} \\
 & = & \langle & \langle 0, 0, 1, 1 \rangle_{(2)}, & \langle 0, 1, 0, 1 \rangle_{(2)}, & \langle 0, 0, 0, 1 \rangle_{(2)} & \rangle_{(16)} \\
 & = & \langle & 0, 0, 1, 1, & 0, 1, 0, 1, & 0, 0, 0, 1 & \rangle_{(16)} \\
 & \mapsto & 2220_{(10)} & & & &
\end{array}
$$

  such that in C, for example,

$$\texttt{0x8AC} = 2220_{(10)}.$$

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Notes:

- Fact: left-shift (resp. right-shift) of some x by y digits is equivalent to multiplication (resp. division) by b^y.
- Example: taking $b = 2$ we find that

$$\begin{array}{rcl} x \times 2^y & = & (\sum_{i=0}^{n-1} x_i \cdot 2^i) \times 2^y \\ & = & \sum_{i=0}^{n-1} x_i \cdot 2^i \times 2^y \\ & = & \sum_{i=0}^{n-1} x_i \cdot 2^{i+y} \\ & = & x \ll y \end{array}$$

and

$$\begin{array}{rcl} x / 2^y & = & (\sum_{i=0}^{n-1} x_i \cdot 2^i) / 2^y \\ & = & \sum_{i=0}^{n-1} x_i \cdot 2^i / 2^y \\ & = & \sum_{i=0}^{n-1} x_i \cdot 2^{i-y} \\ & = & x \gg y \end{array}$$

such that in C, for example,

$$\begin{array}{lclcrcl} \texttt{0x8AC << 2} & \mapsto & 2220_{(10)} \times 2^2 & = & 8880_{(10)} & \mapsto & \texttt{0x22B0} \\ \texttt{0x8AC >> 2} & \mapsto & 2220_{(10)} \; / \; 2^2 & = & 555_{(10)} & \mapsto & \texttt{0x22B} \end{array}$$

Notes:

- Problem: set the i-th bit of some x, i.e., x_i, to 1.
- Solution: compute

$$x \vee (1 \ll i).$$

Example

If $x = 0011_{(2)}$ and $i = 2$ then we compute

$$\begin{array}{lclclcl} x & \vee & (& 1 & \ll & i &) \\ 0011_{(2)} & \vee & (& 1 & \ll & 2 &) \\ 0011_{(2)} & \vee & & 0100_{(2)} & & & \\ 0111_{(2)} & & & & & & \end{array}$$

meaning initially $x_2 = 0$, then we changed it so $x_2 = 1$.

Notes:

- Problem: set the i-th bit of some x, i.e., x_i, to 0.
- Solution: compute

$$x \wedge \neg(1 \ll i).$$

Example

If $x = 0111_{(2)}$ and $m = 2$ then we compute

$$\begin{array}{lccclcc}
x & \wedge & \neg & (& 1 & \ll & i &) \\
0111_{(2)} & \wedge & \neg & (& 1 & \ll & 2 &) \\
0111_{(2)} & \wedge & \neg & (& 0100_{(2)} & & &) \\
0111_{(2)} & \wedge & & & 1011_{(2)} & & & \\
0011_{(2)} & & & & & & &
\end{array}$$

meaning initially $x_2 = 1$, then we changed it so $x_2 = 0$.

Notes:

- Problem: extract the i-th bit of some x, i.e., x_i.
- Solution: compute

$$(x \gg i) \wedge 1.$$

Example

If $x = 0011_{(2)}$, then

1. if $i = 2$ we compute

$$\begin{array}{clcccccc}
(& x & \gg & i &) & \wedge & 1 \\
(& 0011_{(2)} & \gg & 2 &) & \wedge & 1 \\
(& 0000_{(2)} & & &) & \wedge & 1 \\
 & 0000_{(2)} & & & & &
\end{array}$$

   meaning $x_2 = 0$, or

2. if $i = 0$ we compute

$$\begin{array}{clcccccc}
(& x & \gg & i &) & \wedge & 1 \\
(& 0011_{(2)} & \gg & 0 &) & \wedge & 1 \\
(& 0011_{(2)} & & &) & \wedge & 1 \\
 & 0001_{(2)} & & & & &
\end{array}$$

   meaning $x_0 = 1$.

Notes:

- Problem: extract an m-bit sub-word (i.e., m contiguous bits) starting at the i-th bit of some x.
- Solution: compute

$$(x \gg i) \wedge ((1 \ll m) - 1).$$

Example

If $x = 1011_{(2)}$, $m = 2$ and $i = 1$ then we want to extract the sub-word $\langle x_1, x_2 \rangle$

$$\begin{array}{cccccccccccccc}
(& x & \gg & i &) & \wedge & (& (& 1 & \ll & m &) & - & 1 &) \\
(& 1011_{(2)} & \gg & 1 &) & \wedge & (& (& 1 & \ll & 2 &) & - & 1 &) \\
(& 0101_{(2)} & & &) & \wedge & (& (& 0100_{(2)} & & &) & - & 1 &) \\
(& 0101_{(2)} & & &) & \wedge & (& & 0011_{(2)} & & & & & &) \\
 & 0001_{(2)} & & & & & & & & & & & & &
\end{array}$$

meaning $\langle x_1, x_2 \rangle = \langle 1, 0 \rangle$ as expected.

Notes:

Definition

A signed integer can be represented in n bits by using the **sign-magnitude** approach; 1 bit is reserved for the sign (0 means positive, 1 means negative) and $n-1$ for the magnitude. That is, we have

$$\begin{array}{rcl}
\hat{x} & = & \langle \hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{n-1} \rangle \\
 & \mapsto & x \\
 & = & (-1)^{\hat{x}_{n-1}} \cdot \sum_{i=0}^{n-2} \hat{x}_i \cdot 2^i
\end{array}$$

for $\hat{x}_i \in \{0, 1\}$, which yields

$$-2^{n-1} + 1 \leq x \leq +2^{n-1} - 1.$$

Note that there are two representations of zero (i.e., $+0$ and -0).

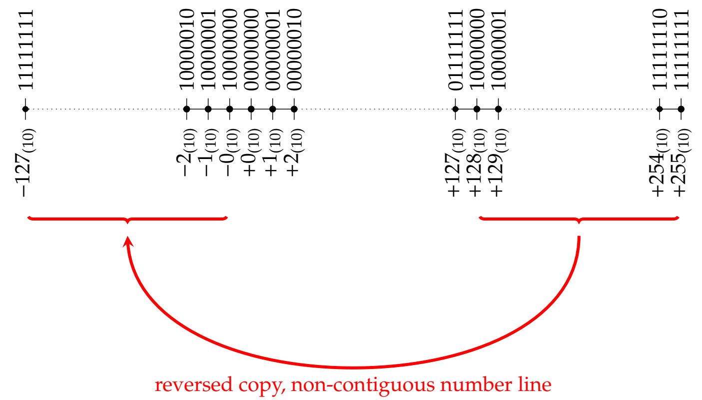
Notes:

Example ($n = 8$)

01111111	$\mapsto$	$(-1)^0$	$\cdot$	(	$1 \cdot 2^6$	$+ 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$	)	=	$+127_{(10)}$
$\vdots$								$\vdots$	
01111011	$\mapsto$	$(-1)^0$	$\cdot$	(	$1 \cdot 2^6$	$+ 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$	)	=	$+123_{(10)}$
$\vdots$								$\vdots$	
00000001	$\mapsto$	$(-1)^0$	$\cdot$	(	$0 \cdot 2^6$	$+ 0 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$	)	=	$+1_{(10)}$
00000000	$\mapsto$	$(-1)^0$	$\cdot$	(	$0 \cdot 2^6$	$+ 0 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$	)	=	$+0_{(10)}$
10000000	$\mapsto$	$(-1)^1$	$\cdot$	(	$0 \cdot 2^6$	$+ 0 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$	)	=	$-0_{(10)}$
10000001	$\mapsto$	$(-1)^1$	$\cdot$	(	$0 \cdot 2^6$	$+ 0 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$	)	=	$-1_{(10)}$
$\vdots$								$\vdots$	
11111011	$\mapsto$	$(-1)^1$	$\cdot$	(	$1 \cdot 2^6$	$+ 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$	)	=	$-123_{(10)}$
$\vdots$								$\vdots$	
11111111	$\mapsto$	$(-1)^1$	$\cdot$	(	$1 \cdot 2^6$	$+ 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$	)	=	$-127_{(10)}$

Notes:

Example ($n = 8$)



Notes:

Definition

A signed integer can be represented in n bits by using the **two's-complement** approach; the basic idea is to weight the $(n-1)$-th bit using -2^{n-1} rather than $+2^{n-1}$, and all other bits as normal. That is, we have

$$\begin{array}{rcl} \hat{x} &=& \langle \hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{n-1} \rangle \\ &\mapsto& x \\ &=& \hat{x}_{n-1} \cdot -2^{n-1} + \sum_{i=0}^{n-2} \hat{x}_i \cdot 2^i \end{array}$$

for $\hat{x}_i \in \{0,1\}$, which yields

$$-2^{n-1} \leq x \leq +2^{n-1} - 1.$$

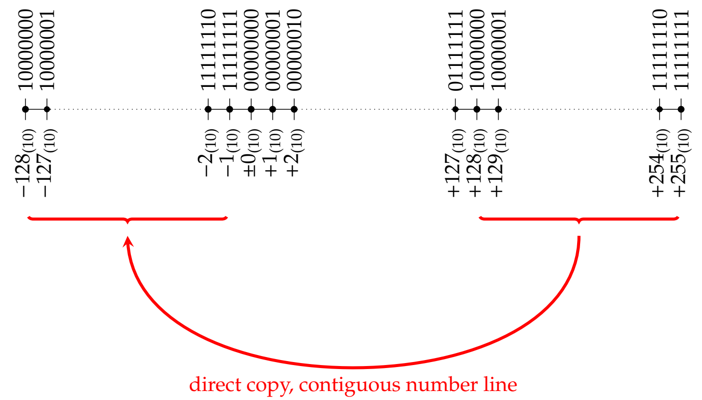
Notes:

Example ($n = 8$)

$$\begin{array}{lclcr} 01111111 & \mapsto & 0 \cdot -2^7 + 1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 & = & +127_{(10)} \\ & \vdots & & \vdots & \\ 01111011 & \mapsto & 0 \cdot -2^7 + 1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 & = & +123_{(10)} \\ & \vdots & & \vdots & \\ 00000001 & \mapsto & 0 \cdot -2^7 + 0 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 & = & +1_{(10)} \\ 00000000 & \mapsto & 0 \cdot -2^7 + 0 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 & = & +0_{(10)} \\ 11111111 & \mapsto & 1 \cdot -2^7 + 1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 & = & -1_{(10)} \\ & \vdots & & \vdots & \\ 10000101 & \mapsto & 1 \cdot -2^7 + 0 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 & = & -123_{(10)} \\ & \vdots & & \vdots & \\ 10000000 & \mapsto & 1 \cdot -2^7 + 0 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 & = & -128_{(10)} \end{array}$$

Notes:

Example ($n = 8$)



Notes:

Conclusions

- Take away points:
  1. *We* control what bit-sequences mean: we can interpret an instance of the C `char` data-type as
     - a signed 8-bit integer, *or*
     - a generic object which can take one of 2^8 states,

     and, as a result, can represent *anything*, e.g.,
     - a pixel within an image,
     - a character within a document,
     - a number within a matrix,
     - ...
  2. Beyond this, knowing about various standard representations is important and useful in a general sense.

Notes:

Additional Reading

- *Wikipedia: Numeral system*. URL: `https://en.wikipedia.org/wiki/Numeral_system`.
- D. Page. "Chapter 1: Mathematical preliminaries". In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009.
- B. Parhami. "Part 1: Number representation". In: *Computer Arithmetic: Algorithms and Hardware Designs*. 1st ed. Oxford University Press, 2000.
- W. Stallings. "Chapter 9: Number systems". In: *Computer Organisation and Architecture*. 9th ed. Prentice Hall, 2013.
- A.S. Tanenbaum and T. Austin. "Appendix A: Binary numbers". In: *Structured Computer Organisation*. 6th ed. Prentice Hall, 2012.

Notes:

References

[1] *Wikipedia: Numeral system*. URL: `https://en.wikipedia.org/wiki/Numeral_system` (see p. 69).

[2] D. Page. "Chapter 1: Mathematical preliminaries". In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009 (see p. 69).

[3] B. Parhami. "Part 1: Number representation". In: *Computer Arithmetic: Algorithms and Hardware Designs*. 1st ed. Oxford University Press, 2000 (see p. 69).

[4] W. Stallings. "Chapter 9: Number systems". In: *Computer Organisation and Architecture*. 9th ed. Prentice Hall, 2013 (see p. 69).

[5] A.S. Tanenbaum and T. Austin. "Appendix A: Binary numbers". In: *Structured Computer Organisation*. 6th ed. Prentice Hall, 2012 (see p. 69).

Notes: