# Computer Architecture

Daniel Page

Department of Computer Science,
University Of Bristol,
Merchant Venturers Building,
Woodland Road,
Bristol, BS8 1UB. UK.
⟨csdsp@bristol.ac.uk⟩

October 14, 2024

Keep in mind there are *two* PDFs available (of which this is the latter):

1. a PDF of examinable material used as lecture slides, and
2. a PDF of non-examinable, extra material:
   - the associated notes page may be pre-populated with extra, written explaination of material covered in lecture(s), plus
   - anything with a "grey'ed out" header/footer represents extra material which is useful and/or interesting but out of scope (and hence not covered).

Notes:

Notes:

- Concept: consider

$$
\begin{array}{ccc}
\hat{x} & \mapsto & x \\
\hat{y} & \mapsto & y
\end{array}
$$

Notes:



COMS10015 lecture: week #7

- Concept: consider

$$
\begin{array}{ccccc}
\hat{x} & \mapsto & x & & \\
\hat{y} & \mapsto & y & & \\
 & & r & = & x \times y
\end{array}
$$

Notes:

© Daniel Page ⟨csdsp@bristol.ac.uk⟩
Computer Architecture
University of BRISTOL
git # c144985a @ 2024-10-14

- Concept: consider

$$\begin{array}{rcccl} & & \hat{x} & \mapsto & x \\ & & \hat{y} & \mapsto & y \\ f(\hat{x},\hat{y}) & = & \hat{r} & \mapsto & r = x \times y \end{array}$$

where $f$

1. has an action on $\hat{x}$ and $\hat{y}$ compatible with that of $\times$ on $x$ and $y$:
   - accepts $n$-bit
     - **multiplier** $\hat{y}$ (that "does the multiplying"), and
     - **multiplicand** $\hat{x}$ (that "is multiplied")

     as input, and
   - produces an $(2 \cdot n)$-bit **product** $\hat{r}$ as output,
2. is a Boolean function:

$$f : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}^{2 \cdot n}$$



Notes:

COMS10015 lecture: week #7

- Agenda: produce a design(s) for $f$, which
  1. functions correctly, and
  2. satisfies pertinent quality metrics (e.g., is efficient in time and/or space).

© Daniel Page ⟨csdsp@bristol.ac.uk⟩
Computer Architecture
University of BRISTOL
git # c144985a @ 2024-10-14

Notes:

## Quote

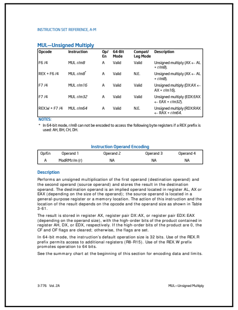
*I do not like × as a symbol for multiplication, as it is easily confounded with x; often I simply relate two quantities by an interposed dot and indicate multiplication by ZC · LM.*

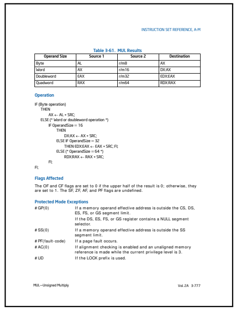
*– Leibniz (https://en.wikiquote.org/wiki/Gottfried_Leibniz)*

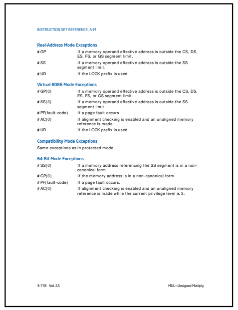


Notes:











Notes:

## Part 1: multiplication in theory (1)

### Example

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $x\ =$ | $623_{(10)}$ | $\mapsto$ | | | | 6 | 2 | 3 | |
| $y\ =$ | $567_{(10)}$ | $\mapsto$ | | | | 5 | 6 | 7 | $\times$ |
| $p_0 = 7 \cdot 3 \cdot 10^0 =$ | $21_{(10)}$ | $\mapsto$ | | | | | 2 | 1 | |
| $p_1 = 7 \cdot 2 \cdot 10^1 =$ | $140_{(10)}$ | $\mapsto$ | | | | 1 | 4 | | |
| $p_2 = 7 \cdot 6 \cdot 10^2 =$ | $4200_{(10)}$ | $\mapsto$ | | | 4 | 2 | | | |
| $p_3 = 6 \cdot 3 \cdot 10^1 =$ | $180_{(10)}$ | $\mapsto$ | | | | 1 | 8 | | |
| $p_4 = 6 \cdot 2 \cdot 10^2 =$ | $1200_{(10)}$ | $\mapsto$ | | | 1 | 2 | | | |
| $p_5 = 6 \cdot 6 \cdot 10^3 =$ | $36000_{(10)}$ | $\mapsto$ | | 3 | 6 | | | | |
| $p_6 = 5 \cdot 3 \cdot 10^2 =$ | $1500_{(10)}$ | $\mapsto$ | | | 1 | 5 | | | |
| $p_7 = 5 \cdot 2 \cdot 10^3 =$ | $10000_{(10)}$ | $\mapsto$ | | 1 | 0 | | | | |
| $p_8 = 5 \cdot 6 \cdot 10^4 =$ | $300000_{(10)}$ | $\mapsto$ | 3 | 0 | | | | | |
| $r\ =$ | $353241_{(10)}$ | $\mapsto$ | 3 | 5 | 3 | 2 | 4 | 1 | |

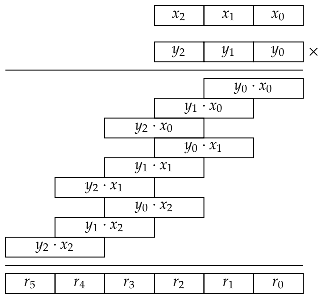


Notes:

## Part 1: multiplication in theory (2)

### Example (operand scanning)

Consider an example where where $|x| = |y| = 3$:

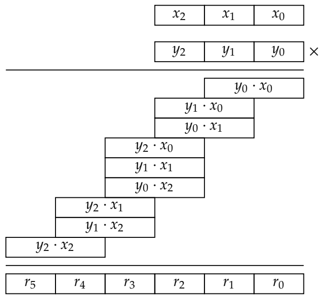



Notice that

1. an outer-loop steps through digits of $y$, say $y_i$,
2. an inner-loop steps through digits of $x$, say $x_j$.

### Example (product scanning)

Consider an example where where $|x| = |y| = 3$:




Notice that

1. an outer-loop steps through digits of $r$, say $r_i$,
2. two inner-loops step through matching digits of $x$ and $y$, say $x_j$ and $x_i$.



Notes:

# Part 2: multiplication in practice: an algorithm (1)

## Operand scanning

### Algorithm (operand scanning)

**Input:** Two unsigned, base-$b$ integers $x$ and $y$
**Output:** An unsigned, base-$b$ integer $r = x \cdot y$

$l_x \leftarrow |x|$, $l_y \leftarrow |y|$, $l_r \leftarrow l_x + l_y$
$r \leftarrow 0$
**for** $j = 0$ **upto** $l_y - 1$ **step** $+1$ **do**
    $c \leftarrow 0$
    **for** $i = 0$ **upto** $l_x - 1$ **step** $+1$ **do**
        $u \cdot b + v = t \leftarrow y_j \cdot x_i + r_{j+i} + c$
        $r_{j+i} \leftarrow v$
        $c \leftarrow u$
    **end**
    $r_{j+l_x} \leftarrow c$
**end**
**return** $r$

Notes:



# Part 2: multiplication in practice: an algorithm (2)

## Operand scanning

### Example (operand scanning)

Consider a case where $b = 10$, $x = 623_{(10)}$ and $y = 567_{(10)}$:

| $j$ | $i$ | $r$ | $c$ | $y_i$ | $x_j$ | $t = y_i \cdot x_i + r_{i+j} + c$ | $r'$ | $c'$ |
|---|---|---|---|---|---|---|---|---|
| | | $\langle 0,0,0,0,0,0 \rangle$ | | | | | | |
| 0 | 0 | $\langle 0,0,0,0,0,0 \rangle$ | 0 | 7 | 3 | 21 | $\langle 1,0,0,0,0,0 \rangle$ | 2 |
| 0 | 1 | $\langle 1,0,0,0,0,0 \rangle$ | 2 | 7 | 2 | 16 | $\langle 1,6,0,0,0,0 \rangle$ | 1 |
| 0 | 2 | $\langle 1,6,0,0,0,0 \rangle$ | 1 | 7 | 6 | 43 | $\langle 1,6,3,0,0,0 \rangle$ | 4 |
| 0 | | $\langle 1,6,3,0,0,0 \rangle$ | 4 | | | | $\langle 1,6,3,4,0,0 \rangle$ | |
| 1 | 0 | $\langle 1,6,3,4,0,0 \rangle$ | 0 | 6 | 3 | 24 | $\langle 1,4,3,4,0,0 \rangle$ | 2 |
| 1 | 1 | $\langle 1,4,3,4,0,0 \rangle$ | 2 | 6 | 2 | 17 | $\langle 1,4,7,4,0,0 \rangle$ | 1 |
| 1 | 2 | $\langle 1,4,7,4,0,0 \rangle$ | 1 | 6 | 6 | 41 | $\langle 1,4,7,1,0,0 \rangle$ | 4 |
| 1 | | $\langle 1,4,7,1,0,0 \rangle$ | 4 | | | | $\langle 1,4,7,1,4,0 \rangle$ | |
| 2 | 0 | $\langle 1,4,7,1,4,0 \rangle$ | 0 | 5 | 3 | 22 | $\langle 1,4,2,1,4,0 \rangle$ | 2 |
| 2 | 1 | $\langle 1,4,2,1,4,0 \rangle$ | 2 | 5 | 2 | 13 | $\langle 1,4,2,3,4,0 \rangle$ | 1 |
| 2 | 2 | $\langle 1,4,2,3,5,0 \rangle$ | 1 | 5 | 6 | 35 | $\langle 1,4,2,3,5,0 \rangle$ | 3 |
| 2 | | $\langle 1,4,2,3,5,0 \rangle$ | 3 | | | | $\langle 1,4,2,3,5,3 \rangle$ | 3 |
| | | $\langle 1,4,2,3,5,3 \rangle$ | | | | | | |

Notes:

## Algorithm (product scanning)

**Input:** Two unsigned, base-$b$ integers $x$ and $y$
**Output:** An unsigned, base-$b$ integer $r = x \cdot y$

$l_x \leftarrow |x|, l_y \leftarrow |y|, l_r \leftarrow l_x + l_y$
$r \leftarrow 0, c_0 \leftarrow 0, c_1 \leftarrow 0, c_2 \leftarrow 0$
**for** $k = 0$ **upto** $l_x + l_y - 1$ **step** $+1$ **do**
  **for** $j = 0$ **upto** $l_y - 1$ **step** $+1$ **do**
    **for** $i = 0$ **upto** $l_x - 1$ **step** $+1$ **do**
      **if** $(j + i) = k$ **then**
        $u \cdot b + v = t \leftarrow y_j \cdot x_i$
        $c \cdot b + c_0 = t \leftarrow c_0 + v$
        $c \cdot b + c_1 = t \leftarrow c_1 + u + c$
        $c_2 \leftarrow c_2 + c$
      **end**
    **end**
  **end**
  $r_k \leftarrow c_0, c_0 \leftarrow c_1, c_1 \leftarrow c_2, c_2 \leftarrow 0$
**end**
$r_{l_x + l_y - 1} \leftarrow c_0$



Notes:



## Example (product scanning)

Consider a case where $b = 10$, $x = 623_{(10)}$ and $y = 567_{(10)}$:

| $k$ | $j$ | $i$ | $r$ | $c_2$ | $c_1$ | $c_0$ | $y_i$ | $x_j$ | $t = y_i \cdot x_i$ | $r'$ | $c'_2$ | $c'_1$ | $c'_0$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $\langle 0,0,0,0,0,0 \rangle$ | 0 | 0 | 0 | | | | | | | |
| 0 | 0 | 0 | $\langle 0,0,0,0,0,0 \rangle$ | 0 | 0 | 0 | 7 | 3 | 21 | $\langle 0,0,0,0,0,0 \rangle$ | 0 | 2 | 1 |
| 0 | | | $\langle 0,0,0,0,0,0 \rangle$ | 0 | 2 | 1 | | | | $\langle 1,0,0,0,0,0 \rangle$ | 0 | 0 | 2 |
| 1 | 0 | 1 | $\langle 1,0,0,0,0,0 \rangle$ | 0 | 0 | 2 | 7 | 2 | 14 | $\langle 1,0,0,0,0,0 \rangle$ | 0 | 1 | 6 |
| 1 | 1 | 0 | $\langle 1,0,0,0,0,0 \rangle$ | 0 | 1 | 6 | 6 | 3 | 18 | $\langle 1,0,0,0,0,0 \rangle$ | 0 | 3 | 4 |
| 1 | | | $\langle 1,0,0,0,0,0 \rangle$ | 0 | 3 | 4 | | | | $\langle 1,4,0,0,0,0 \rangle$ | 0 | 0 | 3 |
| 2 | 0 | 2 | $\langle 1,4,0,0,0,0 \rangle$ | 0 | 0 | 3 | 7 | 6 | 42 | $\langle 1,4,0,0,0,0 \rangle$ | 0 | 4 | 5 |
| 2 | 1 | 1 | $\langle 1,4,0,0,0,0 \rangle$ | 0 | 4 | 5 | 6 | 2 | 12 | $\langle 1,4,0,0,0,0 \rangle$ | 0 | 5 | 7 |
| 2 | 2 | 0 | $\langle 1,4,0,0,0,0 \rangle$ | 0 | 4 | 7 | 5 | 3 | 15 | $\langle 1,4,0,0,0,0 \rangle$ | 0 | 7 | 2 |
| 2 | | | $\langle 1,4,0,0,0,0 \rangle$ | 0 | 7 | 2 | | | | $\langle 1,4,2,0,0,0 \rangle$ | 0 | 0 | 7 |
| 3 | 1 | 2 | $\langle 1,4,2,0,0,0 \rangle$ | 0 | 0 | 7 | 6 | 6 | 36 | $\langle 1,4,2,0,0,0 \rangle$ | 0 | 4 | 3 |
| 3 | 2 | 1 | $\langle 1,4,2,0,0,0 \rangle$ | 0 | 4 | 3 | 5 | 2 | 10 | $\langle 1,4,2,0,0,0 \rangle$ | 0 | 5 | 3 |
| 3 | | | $\langle 1,4,2,0,0,0 \rangle$ | 0 | 5 | 3 | | | | $\langle 1,4,2,3,0,0 \rangle$ | 0 | 0 | 5 |
| 4 | 2 | 2 | $\langle 1,4,2,3,0,0 \rangle$ | 0 | 0 | 5 | 5 | 6 | 30 | $\langle 1,4,2,3,0,0 \rangle$ | 0 | 3 | 5 |
| 4 | | | $\langle 1,4,2,3,0,0 \rangle$ | 0 | 3 | 5 | | | | $\langle 1,4,2,3,5,0 \rangle$ | 0 | 0 | 3 |
| | | | $\langle 1,4,2,3,5,0 \rangle$ | 0 | 0 | 3 | | | | $\langle 1,4,2,3,5,3 \rangle$ | 0 | 0 | 3 |
| | | | $\langle 1,4,2,3,5,3 \rangle$ | | | | | | | | | | |



Notes:

# Part 2: multiplication in practice: an algorithm (5)

## Repeated addition

- Idea:
  - multiplication *means* repeated addition, i.e.,

$$y \times x = \underbrace{x + x + \cdots + x}_{y \text{ terms}},$$

    so if $y = 14_{(10)}$ we have

$$y \times x = x + x + x + x + x + x + x + x + x + x + x + x + x + x.$$

  - expressing $y$ in base-2, we can rewrite this as

$$\begin{array}{rcl} y \times x & = & (\sum_{i=0}^{n-1} y_i \cdot 2^i) \times x \\ & = & (y_{n-1} \cdot 2^{n-1} + \cdots + y_1 \cdot 2^1 + y_0 \cdot 2^0) \times x \\ & = & (y_{n-1} \cdot 2^{n-1} \cdot x) + \cdots + (y_1 \cdot 2^1 \cdot x) + (y_0 \cdot 2^0 \cdot x) \end{array}$$



Notes:

# Part 2: multiplication in practice: an algorithm (5)

## Repeated addition

- Idea:
  - given $y = 14_{(10)} = 1110_{(2)}$ we can see that

$$\begin{array}{rcccccccc} y \cdot x & = & y_0 \cdot x \cdot 2^0 & + & y_1 \cdot x \cdot 2^1 & + & y_2 \cdot x \cdot 2^2 & + & y_3 \cdot x \cdot 2^3 \\ & = & 0 \cdot x \cdot 2^0 & + & 1 \cdot x \cdot 2^1 & + & 1 \cdot x \cdot 2^2 & + & 1 \cdot x \cdot 2^3 \\ & = & 0 \cdot x & + & 2 \cdot x & + & 4 \cdot x & + & 8 \cdot x \\ & = & 14 \cdot x & & & & & & \end{array}$$

  - given $y = 14_{(10)} = 1110_{(2)}$ we can see that

$$\begin{array}{rcl} y \cdot x & = & y_0 \cdot x + 2 \cdot (\, y_1 \cdot x + 2 \cdot (\, y_2 \cdot x + 2 \cdot (\, y_3 \cdot x + 2 \cdot (\, 0 \,)\,)\,)\,) \\ & = & 0 \cdot x + 2 \cdot (\, 1 \cdot x + 2 \cdot (\, 1 \cdot x + 2 \cdot (\, 1 \cdot x + 2 \cdot (\, 0 \,)\,)\,)\,) \\ & = & 0 \cdot x + 2 \cdot (\, 1 \cdot x + 2 \cdot (\, 1 \cdot x + 2 \cdot (\, 1 \cdot x + 0 \,)\,)\,) \\ & = & 0 \cdot x + 2 \cdot (\, 1 \cdot x + 2 \cdot (\, 1 \cdot x + 2 \cdot (\, 1 \cdot x \,)\,)\,) \\ & = & 0 \cdot x + 2 \cdot (\, 1 \cdot x + 2 \cdot (\, 1 \cdot x + 2 \cdot x \,)\,) \\ & = & 0 \cdot x + 2 \cdot (\, 1 \cdot x + 2 \cdot (\, 3 \cdot x \,)\,) \\ & = & 0 \cdot x + 2 \cdot (\, 1 \cdot x + 6 \cdot x \,) \\ & = & 0 \cdot x + 2 \cdot (\, 7 \cdot x \,) \\ & = & 0 \cdot x + 14 \cdot x \\ & = & 14 \cdot x \end{array}$$

    via application of **Horner's rule**.



Notes:

- **Idea**: for $b = 2$ we now know

$$r \;=\; y \times x \;=\; \left( \sum_{i=0}^{n-1} y_i \cdot 2^i \right) \times x \;=\; \sum_{i=0}^{n-1} y_i \cdot x \cdot 2^i,$$

  plus
  - for any $t$,

$$y_i \cdot t \;=\; \begin{cases} 0 & \text{if } y_i = 0 \\ t & \text{if } y_i = 1 \end{cases}$$

  - for any $t$,

$$t \cdot 2^i \;\equiv\; t \ll i,$$

  so we can compute $r$ via
  1. some AND gates to generate partial products (i.e., $y_i \cdot x$),
  2. some left-shift components to scale the partial products correctly (i.e., $y_i \cdot x \cdot 2^i$), and
  3. some adder components to sum the scaled partial products.

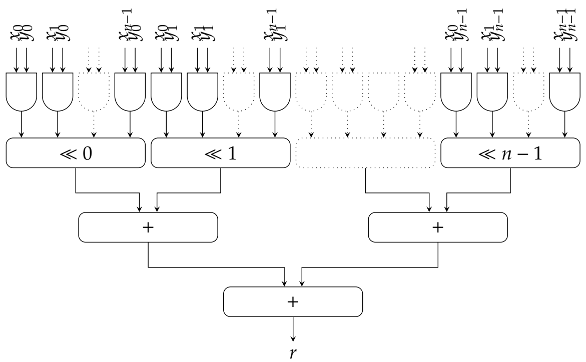


Notes:



- **Design**:

**Circuit**




- **Evaluation**:
  - −ve: requires a larger data-path
  - +ve: requires a smaller control-path (i.e., none at all),
  - +ve: requires less steps (i.e., 1),
  - −ve: has a longer critical path (meaning each step is longer).



Notes:

## Part 3: multiplication in practice: a circuit (3)

### An iterative, bit-serial design

- **Idea**: for $b = 2$ we now know

$$r \;=\; y \times x \;=\; \left( \sum_{i=0}^{n-1} y_i \cdot 2^i \right) \times x \;=\; \sum_{i=0}^{n-1} y_i \cdot x \cdot 2^i,$$

  so we can compute $r$ by evaluating the Horner expansion step-by-step in an "inside-out" order, applying

$$r \leftarrow \begin{cases} 2 \cdot r & \text{if } y_i = 0 \\ 2 \cdot r + x & \text{if } y_i = 1 \end{cases}$$

  so as to *accumulate* the result.



Notes:

## Part 3: multiplication in practice: a circuit (3)

### An iterative, bit-serial design

- **Idea**:

**Algorithm**

**Input:** Two unsigned, $n$-bit, base-2 integers $x$ and $y$
**Output:** An unsigned, $2n$-bit, base-2 integer $r = y \cdot x$

$r \leftarrow 0$
**for** $i = n - 1$ **downto** $0$ **step** $-1$ **do**
    $r \leftarrow 2 \cdot r$
    **if** $y_i = 1$ **then**
        $r \leftarrow r + x$
    **end**
**end**
**return** $r$

**Example**

Consider a case where $y = 14_{(10)} \mapsto 1110_{(2)}$:

| $i$ | $r$ | $y_i$ | $r'$ | |
|---|---|---|---|---|
| | $0$ | | | |
| 3 | $0$ | 1 | $x$ | $r' \leftarrow 2 \cdot r + x$ |
| 2 | $x$ | 1 | $3 \cdot x$ | $r' \leftarrow 2 \cdot r + x$ |
| 1 | $3 \cdot x$ | 1 | $7 \cdot x$ | $r' \leftarrow 2 \cdot r + x$ |
| 0 | $7 \cdot x$ | 0 | $14 \cdot x$ | $r' \leftarrow 2 \cdot r$ |
| | $14 \cdot x$ | | | |

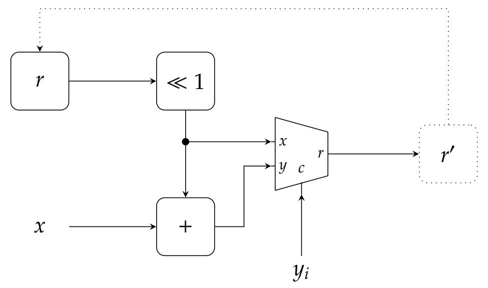


Notes:

## Part 3: multiplication in practice: a circuit (4)
An iterative, bit-serial design

- **Design**:

**Circuit**




- **Evaluation**:

  +ve: requires a smaller data-path
  −ve: requires a larger control-path (i.e., an entire FSM),
  −ve: requires more steps (i.e., $n$),
  +ve: has a shorter critical path (meaning each step is shorter).



Notes:

## Conclusions

- **Take away points**:
  1. Computer arithmetic is a broad, interesting (sub-)field:
     - it's a broad topic with a rich history,
     - there's usually a large design space of potential approaches,
     - they're often easy to understand at an intuitive, high level,
     - correctness and efficiency of resulting low-level solutions is vital and challenging.
  2. The strategy we've employed is important and (fairly) general-purpose:
     - explore and understand an approach in theory,
     - translate, formalise, and generalise the approach into an algorithm,
     - translate the algorithm, e.g., into circuit,
     - refine (or select) the circuit to satisfy any design constraints.



Notes:

# Additional Reading

- *Wikipedia: Computer Arithmetic*. URL: https://en.wikipedia.org/wiki/Category:Computer_arithmetic.
- D. Page. "Chapter 7: Arithmetic and logic". In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009.
- B. Parhami. "Part 3: Multiplication". In: *Computer Arithmetic: Algorithms and Hardware Designs*. 1st ed. Oxford University Press, 2000.
- W. Stallings. "Chapter 10: Computer arithmetic". In: *Computer Organisation and Architecture*. 9th ed. Prentice Hall, 2013.
- A.S. Tanenbaum and T. Austin. "Section 3.2.2: Arithmetic circuits". In: *Structured Computer Organisation*. 6th ed. Prentice Hall, 2012.

Notes:



# References

[1] *Wikipedia: Computer Arithmetic*. URL: https://en.wikipedia.org/wiki/Category:Computer_arithmetic (see p. 45).

[2] D. Page. "Chapter 7: Arithmetic and logic". In: *A Practical Introduction to Computer Architecture*. 1st ed. Springer, 2009 (see p. 45).

[3] B. Parhami. "Part 3: Multiplication". In: *Computer Arithmetic: Algorithms and Hardware Designs*. 1st ed. Oxford University Press, 2000 (see p. 45).

[4] W. Stallings. "Chapter 10: Computer arithmetic". In: *Computer Organisation and Architecture*. 9th ed. Prentice Hall, 2013 (see p. 45).

[5] A.S. Tanenbaum and T. Austin. "Section 3.2.2: Arithmetic circuits". In: *Structured Computer Organisation*. 6th ed. Prentice Hall, 2012 (see p. 45).

Notes: