Proof2：The Analysis of Jacobian matrix Astringency

From the analysis of Lyapunov Function Stability (stated in **proof 1**), the state equation:

$$\frac{d\boldsymbol{X}}{dt}=\begin{bmatrix}\frac{dx_j(t)}{dt}\\ \frac{dx_k(t)}{dt}\end{bmatrix}=\begin{bmatrix}-(\frac{1}{\tau}+\frac{W_{ij}}{C_m}\sigma(x_i(t)))x_j(t)+\frac{W_{ij}}{C_m}\sigma(x_i(t))E+\frac{x_{leak}}{\tau})\\ -(\frac{1}{\tau}+\frac{W_{ik}}{C_m}\sigma(x_i(t)))x_k(t)+\frac{W_{ik}}{C_m}\sigma(x_i(t))E+\frac{x_{leak}}{\tau})\end{bmatrix}+\begin{bmatrix}R_j\\ R_k\end{bmatrix}$$

We can define the Jacobian matrix of this state equation as follows:

$$\boldsymbol{J}=\frac{1}{\partial\boldsymbol{X}}(\frac{d\boldsymbol{X}}{dt})=\begin{bmatrix}\frac{1}{\partial x_j}(\frac{dx_j(t)}{dt}) & \frac{1}{\partial x_k}(\frac{dx_j(t)}{dt})\\ \frac{1}{\partial x_j}(\frac{dx_k(t)}{dt}) & \frac{1}{\partial x_k}(\frac{dx_k(t)}{dt})\end{bmatrix}$$

$$=\begin{bmatrix}-(\frac{1}{\tau}+\frac{W_{ij}}{C_m}\sigma(x_i(t))) & 0\\ 0 & -(\frac{1}{\tau}+\frac{W_{ik}}{C_m}\sigma(x_i(t)))\end{bmatrix}$$

Coincidentally, $\boldsymbol{J}$ is a diagonal matrix, which means that the eigenvalues of the Jacobian matrix are the diagonal elements:

$$\lambda_1=-(\frac{1}{\tau}+\frac{W_{ij}}{C_m}\sigma(x_i(t)))$$

$$\lambda_2=-(\frac{1}{\tau}+\frac{W_{ik}}{C_m}\sigma(x_i(t)))$$

The real part of the eigenvalues of the Jacobian matrix determines the convergence rate of the system. The larger the absolute value of the real part of the eigenvalues, the faster the system converges. Therefore, the convergence rate of the system can be judged by comparing the real parts of the eigenvalues. Specifically:

$$\lambda_{max}=max\{|\lambda_1|,\ |\lambda_2|\}=max\{\left|-(\frac{1}{\tau}+\frac{W_{ij}}{C_m}\sigma(x_i(t)))\right|,\ \left|-(\frac{1}{\tau}+\frac{W_{ik}}{C_m}\sigma(x_i(t)))\right|\}$$

Obviously, for the Pre-“pruning & rewiring” system:

$$\lambda_{pre-max}=\frac{1}{\tau}+\frac{W_{ij}}{C_m}\sigma(x_i(t))$$

And for the Post-“pruning & rewiring” system:

$$\lambda_{post-max}=\frac{1}{\tau}+\frac{W_{ik}}{C_m}\sigma(x_i(t))$$

Considering that $W_{ik}\gg W_{ij}$, we get:

$$\lambda_{post-max}>\lambda_{pre-max}$$

In other words, the system state of the Jacobian matrix with the pruning & rewiring algorithm has a larger maximum real part of the eigenvalues, indicating that the convergence rate of this new algorithm is superior to the original algorithm. This mathematically proves the effectiveness of this seemingly simple biological pruning algorithm.