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ASTRONOMY AND ASTROPHYSICS

Expressions for the Precession Quantities Based upon the IAU (1976) System of Astronomical Constants

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Summary. Expressions for the precession quantities enabling one to precess to and from an arbitrary epoch are developed as a function of the fundamental astronomical constants. The expressions with numerical values of the coefficients are given relative to epoch J2000.0 in Table 5 for the IAU (1976) System of Astronomical Constants adopted at the XVI General Assembly of the IAU in Grenoble. They must be used with the introduction of the new constants into the ephemerides and in constructing the new fundamental reference system, the FK5. Finally, the developments presented here are applicable for revising revelant precession quantities whenever the system of astronomical constants is changed.

Key words: precession expressions — reference system — astronomical constants

Introduction

In comparing astronomical observations with calculated places of celestial objects, reductions have to be made to refer either the observed or the calculated positions to the same reference coordinate system. Such reductions include such well-known effects as aberration, parallax, precession and nutation. Although rather lengthy numerical expressions are usually given for the parameters describing precession, the expressions, in fact, only depend upon a rather limited set of basic parameters or fundamental constants and are thus amenable to revision.

In this paper we will examine the structure of the expressions usually employed in calculating the effects of precession and we will outline the method by which the expressions are revised to account for changes in the fundamental astronomical constants. It will be shown that the basic set of parameters, upon which the lengthy polynomials for calculating the mean obliquity of date and the elements of the precession matrix depend, consists of the mean obliquity and the speed of general precession in longitude at a fixed epoch $\mathscr{E}_0$ together with the system of planetary masses. We will also present the new precession quantities at epoch J2000.0 (JED 2451545.0) which result from the revision of astronomical constants adopted by the International Astronomical Union (IAU) at the XVI General Assembly in Grenoble (Transactions IAU 16B, 1977).

Attention should be drawn to the fact that, as it was expressed by Newcomb (1906, p. 226), "There is no formula by which the actual positions of the two poles [those of the ecliptic and the equator] can be expressed rigorously for any time. But their instantaneous motions, which appear as derivatives of the elements of position relative to the time, may be expressed numerically through a period of several centuries before and after any epoch."

The development of the usual precession quantities depends upon the dynamical motion of the ecliptic pole relative to a fixed ecliptic, due to planetary perturbations, and it depends upon the dynamical motion of the celestial pole due to luni-solar torques on the oblate earth. The former effect is treated by Newcomb (1894) who calculates the components of the earth's angular momentum upon a fixed ecliptic of 1850.0, while the latter effect (Newcomb, 1906; Andoyer, 1911; deSitter and Brouwer, 1938) is treated by *inferring* Newcomb's "Precessional Constant" P_0 (which depends upon the internal structure of the earth) from an observationally-determined value of the "general precession" (more rigorously called the "speed of general precession in longitude"). Based on Newcomb's values, Andoyer (1911) has derived the expressions for the precession quantities referred to the basic epoch of 1850.0; in the annual volumes of the *Connaissance des Temps* the same expressions reduced to 1900.0 are given.

In this paper we have divided the development into sections on Definitions, Motion of the Ecliptic Pole, Formulation for a Basic Epoch, Formulation for an Arbitrary Epoch, concluding with sections on Ecliptic Motion Relative to Basic Epoch, Numerical Results and Summary.

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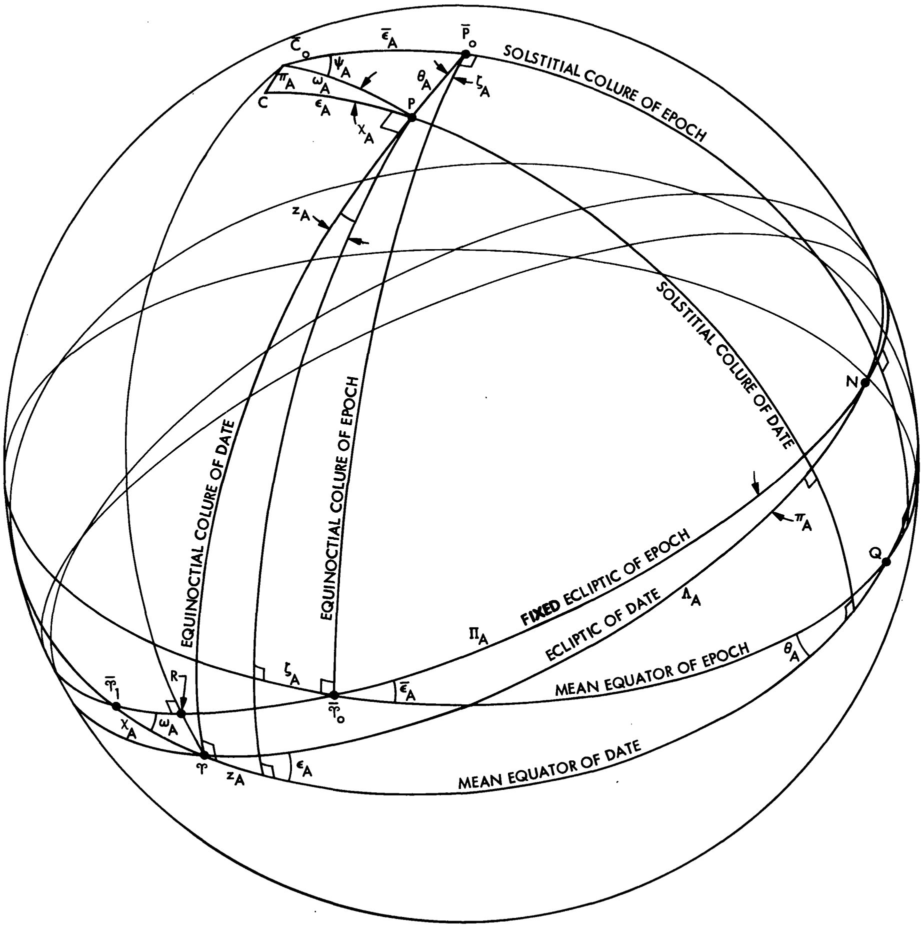

Fig. 1. Celestial sphere depicting mean ecliptics and equators at epochs $\mathscr{E}_F$ and $\mathscr{E}_D$. Ecliptic poles are represented by $\bar{C}_0$ and C with $\bar{P}_0$ and P representing the pole of the equator at the two epochs. Light lines represent curves on the back side of the celestial sphere. The point R is employed in Newcomb's definition of general precession (see text)

Definitions

In developing the precession quantities, three epochs often occur: a basic epoch ($\mathscr{E}_0$), an arbitrary fixed epoch ($\mathscr{E}_F$) and the mean epoch of date ($\mathscr{E}_D$). It is apparent that at least $\mathscr{E}_0$ and $\mathscr{E}_D$ must be present, but it is somewhat less obvious why one also requires a second fixed epoch $\mathscr{E}_F$. It often happens that although one might have expressions for $\mathscr{E}_D$ relative to $\mathscr{E}_0$, he requires the expressions for $\mathscr{E}_D$ relative to $\mathscr{E}_F$. For example, in the currently employed precession quantities, $\mathscr{E}_0$ is 1900.0 (the basic epoch for which the fundamental constants are defined), yet most astronomers employ another epoch $\mathscr{E}_F$ (viz. 1950.0) in analyzing and discussing their data. Similarly, Newcomb (1894) develops expressions for the motion of the ecliptic pole relative to the ecliptic of 1850.0 ($\mathscr{E}_0$), yet we might require the comparable values relative to 1900.0 ($\mathscr{E}_F$) where the fundamental constants are defined. Or, as in the current situation, our basic epoch $\mathscr{E}_0$ will be J2000.0, and we must find a means of relating to Newcomb's results for the motion of the ecliptic pole relative to 1850.0. It is for the purpose of transferring from one fixed epoch ($\mathscr{E}_0$) to another ($\mathscr{E}_F$) that it is convenient to develop the expression quantities relative to two fixed epochs. One first solves the easier problem of expressing the quantities for $\mathscr{E}_D$ relative to $\mathscr{E}_0$ and one then develops the expressions for $\mathscr{E}_D$ relative to $\mathscr{E}_F$.

In Figure 1 we depict the celestial sphere with ecliptics and equators shown for two epochs $\mathscr{E}_F$ and $\mathscr{E}_D$. $\bar{P}_0$ and P represent the mean pole of earth's equator at the fixed epoch $\mathscr{E}_F$ and the epoch of date $\mathscr{E}_D$ while $\bar{C}_0$ and C represent the ecliptic pole at those two epochs. Light lines represent portions of great circles on the back side of the sphere. The vernal equinox at $\mathscr{E}_F$ is denoted by $\bar{\Upsilon}_0$ while the mean equinox of date is denoted by Υ. The relevant precession quantities are also depicted in the figure and the symbols are further defined in Table 1. In Figure 2 we present a polar diagram of the precession quantities. As given in Column 2 of Table 1, we have attempted to develop a consistent method for describing the accumulated precessional angles and their rates. We have attempted to develop a system of notation which does not contain any ambiguity, which has a logical structure, and which retains as many elements of the conventional notation as possible. In the past there have been numerous conflicting sets of symbols employed and much confusion exists in the notation for precessional quantities.

Since it often is important to carefully distinguish between the angle due to the accumulated precessional displacement (α_A) from a fixed epoch, and the instantaneous rate of change (α) of a precession quantity, we have developed the following method of notation. Unless otherwise noted, the time is regarded as dynamical time and is measured in Julian centuries of length 36525 days.

Let T represent the time from $\mathscr{E}_0$ to $\mathscr{E}_F$ (e.g. $\mathscr{E}_0$ being 1900.0, and $\mathscr{E}_F$ being 1950.0). Let t represent the time from $\mathscr{E}_F$ to the mean epoch of date $\mathscr{E}_D$, and let τ represent time from $\mathscr{E}_0$ to $\mathscr{E}_D$, so that $\tau = T + t$. Let the symbol α_A represent the accumulated precessional displacement in an angle from epoch $\mathscr{E}_F$ to the mean epoch of date $\mathscr{E}_D$. Let $\bar{\alpha}_A$ represent the analogous displacement from $\mathscr{E}_0$ to $\mathscr{E}_F$, and let $\tilde{\alpha}_A$ represent the displacement from $\mathscr{E}_0$ to $\mathscr{E}_D$, each displacement being measured in its relevant system. Then in general we seek to obtain expressions for the quantities

$$\begin{aligned}
\alpha_A &= (\alpha_1 + \alpha_2 T + \alpha_3 T^2)t + (\alpha_1' + \alpha_2' T)t^2 + \alpha_1'' t^3 \\
\bar{\alpha}_A &= \alpha_A(T=0, t=T) = \alpha_1 T + \alpha_1' T^2 + \alpha_1'' T^3 \\
\tilde{\alpha}_A &= \alpha_A(T=0, t=\tau) = \alpha_1 \tau + \alpha_1' \tau^2 + \alpha_1'' \tau^3
\end{aligned} \tag{1}$$

which describe the accumulated precessional displacements. Although $\bar{\alpha}_A$ and $\tilde{\alpha}_A$ are of the same class (both being measured from $T=0$, i.e. $\mathscr{E}_0$), and hence only one usually would be sufficient, we explicitly distinguish between the accumulated angles from $\mathscr{E}_0$ to $\mathscr{E}_F$ ($\bar{\alpha}_A$) and from $\mathscr{E}_0$ to $\mathscr{E}_D(\tilde{\alpha}_A)$ in the subsequent developments. It may be noted that our formulation is also applicable in precessing from one fixed equinox at $\mathscr{E}_1$ to another at $\mathscr{E}_2$ by identifying $\mathscr{E}_1$ with $\mathscr{E}_F$ and $\mathscr{E}_2$ with $\mathscr{E}_D$.

For quantities which cannot be regarded solely as displacements (i.e. $\Pi_A, \varepsilon_A, \omega_A$), we include additional terms, as given in Table 1, which are independent of t.

By the symbol α (no subscript) we will denote the instantaneous rate of change of the corresponding angle,

$$\alpha = \left(\frac{d\alpha_A}{dt}\right)_{t=0} = \alpha_1 + \alpha_2 T + \alpha_3 T^2 . \tag{2}$$

It is seen that in our notation, the power of time (T or t) present in the accumulated precessional displacement is represented by the sum of subscripts and superscripts (e.g. α_3, α_2' and α_1'' contain time to the third power in the accumulated precessional displacements, namely $T^2 t$, Tt^2 and t^3). The convention adopted for α is the usual one for precessional speeds, but it avoids confusion by clearly showing at which point it is evaluated. Note that for angles measured relative to an equinox, α is the speed relative to the equinox of date (since $t=0$), and is fundamentally different from $(d\alpha_A/dt)_{T=0}$ (*vide* Clemence, 1948 and errata). Similarly, denoting by α' and α'' (no subscripts) the quantities

$$\alpha' = \frac{1}{2!}\left(\frac{d^2\alpha_A}{dt^2}\right)_{t=0}$$

$$\alpha'' = \frac{1}{3!}\left(\frac{d^3\alpha_A}{dt^3}\right)_{t=0},$$

we may also write Equation (1) in the compact form

$$\alpha_A = \alpha t + \alpha' t^2 + \alpha'' t^3 .$$

It should be noted that for Π, ε and ω which designate instantaneous values, Equation (2) has to be replaced by $\alpha = (\alpha_A)_{t=0}$.

Various notations which have been employed by other authors are listed in Table 2. From the table (entry p_A) it is seen that the "general precession" (strictly, the speed of general precession in longitude) is represented by the symbol p, where

$$p = p_1 + p_2 T + p_3 T^2 \tag{3}$$

in accordance with conventional usage. Note that here we employ the symbol p_1 to represent the general precession at epoch $\mathscr{E}_0$, and not for speed of the luni-solar precession as other authors do. The new symbol for the speed of luni-solar precession (entry ψ_A) is $\psi = \psi_1 + \psi_2 T + \psi_3 T^2$ where ψ_1 is the speed at $\mathscr{E}_0$.

The equatorial precession quantities ζ_A, z_A and θ_A (see Figs. 1 and 2) are the angles most appropriate to precess from a fixed equinox and equator at epoch $\mathscr{E}_F$ to the mean equinox and equator of date $\mathscr{E}_D$, the transformation in equatorial rectangular coordinates being

$$(x, y, z)_{\mathscr{E}_D} = (x, y, z)_{\mathscr{E}_F} R(-\zeta_A) Q(\theta_A) R(-z_A) = (x, y, z)_{\mathscr{E}_F} A \tag{4}$$

where

$$R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$Q(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

Performing the matrix multiplications yields the elements of the matrix $A = R(-\zeta_A) Q(\theta_A) R(-z_A)$

$$\begin{aligned}
a_{11} &= \cos\zeta_A \cos\theta_A \cos z_A - \sin\zeta_A \sin z_A \\
a_{12} &= \cos\zeta_A \cos\theta_A \sin z_A + \sin\zeta_A \cos z_A \\
a_{13} &= \cos\zeta_A \sin\theta_A \\
a_{21} &= -\sin\zeta_A \cos\theta_A \cos z_A - \cos\zeta_A \sin z_A \\
a_{22} &= -\sin\zeta_A \cos\theta_A \sin z_A + \cos\zeta_A \cos z_A \\
a_{23} &= -\sin\zeta_A \sin\theta_A \\
a_{31} &= -\sin\theta_A \cos z_A \\
a_{32} &= -\sin\theta_A \sin z_A \\
a_{33} &= \cos\theta_A
\end{aligned} \tag{5}$$

which are given in numerical form (for Newcomb's value of general precession) in the *Explanatory Supplement* (p. 34). The precessional parameters m and n, used in describing the speed of precession in right ascension and declination, are represented by

$$m = \frac{d}{dt}(\zeta_A + z_A)_{t=0} = (\zeta_1 + z_1) + (\zeta_2 + z_2)T + (\zeta_3 + z_3)T^2$$

TABLE 1. NOMENCLATURE

Symbol	General Form*	Name	Reference Plane	Initial Point	Final Point
π_A	$(q_1 + q_2T + q_3T^2)t + (q_1' + q_2'T)t^2 + q_1''t^3$	Angle between ecliptics	Inclination	$\mathcal{E}_F$ ecliptic	$\mathcal{E}_D$ ecliptic
$\bar{\pi}_A$	$q_1T + q_1'T^2 + q_1''T^3$		Inclination	$\mathcal{E}_0$ ecliptic	$\mathcal{E}_F$ ecliptic
$\tilde{\pi}_A$	$q_1\tau + q_1'\tau^2 + q_1''\tau^3$		Inclination	$\mathcal{E}_0$ ecliptic	$\mathcal{E}_D$ ecliptic
Π_A	$x_0 + x_1T + x_2T^2 + (y_1 + y_2T)t + y_1't^2$	Longitude of node of moving ecliptic upon fixed ecliptic	$\mathcal{E}_F$ ecliptic	$\mathcal{E}_F$ equinox	$\mathcal{E}_D$ ecliptic
$\bar{\Pi}_A$	$x_0 + y_1T + y_1'T^2$		$\mathcal{E}_0$ ecliptic	$\mathcal{E}_0$ equinox	$\mathcal{E}_F$ ecliptic
$\tilde{\Pi}_A$	$x_0 + y_1\tau + y_1'\tau^2$		$\mathcal{E}_0$ ecliptic	$\mathcal{E}_0$ equinox	$\mathcal{E}_D$ ecliptic
$\sin \pi_A \sin \Pi_A$	$(s_1 + s_2T + s_3T^2)t + (s_1' + s_2'T)t^2 + s_1''t^3$	Expressions (via Newcomb) for planetary perturbations of ecliptic plane	$\mathcal{E}_F$ ecliptic		
$\sin \bar{\pi}_A \sin \bar{\Pi}_A$	$s_1T + s_1'T^2 + s_1''T^3$		$\mathcal{E}_0$ ecliptic		
$\sin \tilde{\pi}_A \sin \tilde{\Pi}_A$	$s_1\tau + s_1'\tau^2 + s_1''\tau^3$		$\mathcal{E}_0$ ecliptic		
$\sin \pi_A \cos \Pi_A$	$(c_1 + c_2T + c_3T^2)t + (c_1' + c_2'T)t^2 + c_1''t^3$	Expressions (via Newcomb) for planetary perturbations of ecliptic plane	$\mathcal{E}_F$ ecliptic		
$\sin \bar{\pi}_A \cos \bar{\Pi}_A$	$c_1T + c_1'T^2 + c_1''T^3$		$\mathcal{E}_0$ ecliptic		
$\sin \tilde{\pi}_A \cos \tilde{\Pi}_A$	$c_1\tau + c_1'\tau^2 + c_1''\tau^3$		$\mathcal{E}_0$ ecliptic		
ε_0	ε_0	Obliquity of the ecliptic	Inclination	$\mathcal{E}_0$ ecliptic	$\mathcal{E}_0$ equator
ε_A	$\bar{\varepsilon}_A + (\varepsilon_1 + \varepsilon_2T + \varepsilon_3T^2)t + (\varepsilon_1' + \varepsilon_2'T)t^2 + \varepsilon_1''t^3$		Inclination	$\mathcal{E}_D$ ecliptic	$\mathcal{E}_D$ equator
$\bar{\varepsilon}_A$	$\varepsilon_0 + \varepsilon_1T + \varepsilon_1'T^2 + \varepsilon_1''T^3$		Inclination	$\mathcal{E}_F$ ecliptic	$\mathcal{E}_F$ equator
ω_A	$\bar{\varepsilon}_A + (\omega_1' + \omega_2'T)t^2 + \omega_1''t^3$	Inclination of moving equator on fixed ecliptic	Inclination	$\mathcal{E}_F$ ecliptic	$\mathcal{E}_D$ equator
$\bar{\omega}_A$	$\varepsilon_0 + \omega_1'T^2 + \omega_1''T^3$		Inclination	$\mathcal{E}_0$ ecliptic	$\mathcal{E}_F$ equator
$\tilde{\omega}_A$	$\varepsilon_0 + \omega_1'\tau^2 + \omega_1''\tau^3$		Inclination	$\mathcal{E}_0$ ecliptic	$\mathcal{E}_D$ equator
ψ_A	$(\psi_1 + \psi_2T + \psi_3T^2)t + (\psi_1' + \psi_2'T)t^2 + \psi_1''t^3$	Luni-solar precession	$\mathcal{E}_F$ ecliptic	$\mathcal{E}_F$ equinox	$\mathcal{E}_D$ equator
$\bar{\psi}_A$	$\psi_1T + \psi_1'T^2 + \psi_1''T^3$		$\mathcal{E}_0$ ecliptic	$\mathcal{E}_0$ equinox	$\mathcal{E}_F$ equator
$\tilde{\psi}_A$	$\psi_1\tau + \psi_1'\tau^2 + \psi_1''\tau^3$		$\mathcal{E}_0$ ecliptic	$\mathcal{E}_0$ equinox	$\mathcal{E}_D$ equator
χ_A	$(\chi_1 + \chi_2T + \chi_3T^2)t + (\chi_1' + \chi_2'T)t^2 + \chi_1''t^3$	Planetary precession	$\mathcal{E}_D$ equator	$\mathcal{E}_F$ ecliptic	$\mathcal{E}_D$ equinox
χ_A	$\chi_1T + \chi_1'T^2 + \chi_1''T^3$		$\mathcal{E}_F$ equator	$\mathcal{E}_0$ ecliptic	$\mathcal{E}_F$ equinox
$\tilde{\chi}_A$	$\chi_1\tau + \chi_1'\tau^2 + \chi_1''\tau^3$		$\mathcal{E}_D$ equator	$\mathcal{E}_0$ ecliptic	$\mathcal{E}_D$ equinox

Symbol	General Form	Name	Reference Plane	Initial Point	Final Point
Λ_A	-	Ecliptic longitude of axis of rotation	$\mathcal{E}_D$ ecliptic	$\mathcal{E}_D$ equinox	$\mathcal{E}_F$ ecliptic
$\bar{\Lambda}_A$	-		$\mathcal{E}_F$ ecliptic	$\mathcal{E}_F$ equinox	$\mathcal{E}_0$ ecliptic
$\tilde{\Lambda}_A$	-		$\mathcal{E}_D$ ecliptic	$\mathcal{E}_D$ equinox	$\mathcal{E}_0$ ecliptic
$p_A = \Lambda_A - \Pi_A$	$\left(p_1 + p_2T + p_3T^2\right)t + (p_1' + p_2'T)t^2 + p_1''t^3$	General precession (Andoyer)	Λ: $\mathcal{E}_D$ ecliptic Π: $\mathcal{E}_F$ ecliptic	Λ: $\mathcal{E}_D$ equinox Π: $\mathcal{E}_E$ equinox	$\mathcal{E}_F$ ecliptic $\mathcal{E}_D$ ecliptic
$\bar{p}_A = \bar{\Lambda}_A - \bar{\Pi}_A$	$p_1T + p_1'T^2 + p_1''T^3$		Λ: $\mathcal{E}_F$ ecliptic Π: $\mathcal{E}_0$ ecliptic	Λ: $\mathcal{E}_F$ equinox Π $\mathcal{E}_0$ equinox	$\mathcal{E}_0$ ecliptic $\mathcal{E}_F$ ecliptic
$\tilde{p}_A = \tilde{\Lambda}_A - \tilde{\Pi}_A$	$p_1\tau + p_1'\tau^2 + p_1''\tau^3$		Λ: $\mathcal{E}_D$ ecliptic Π: $\mathcal{E}_0$ ecliptic	Λ: $\mathcal{E}_D$ equinox Λ: $\mathcal{E}_0$ equinox	$\mathcal{E}_0$ ecliptic $\mathcal{E}_D$ ecliptic
p_N	$p_A + (\eta_1' + \eta_2'T)t^2 + \eta_1''t^3$	General precession (Newcomb)	$\mathcal{E}_F$ ecliptic	$\mathcal{E}_F$ equinox	$R_D(C_F)$
$\bar{p}_N$	$\bar{p}_A + \eta_1'T^2 + \eta_1''T^3$		$\mathcal{E}_0$ ecliptic	$\mathcal{E}_0$ equinox	$R_F(C_0)$ See discussion
$\tilde{p}_N$	$\tilde{p}_A + \eta_1'\tau^2 + \eta_1''\tau^3$		$\mathcal{E}_0$ ecliptic	$\mathcal{E}_0$ equinox	$R_D(C_0)$
ζ_A	$\left(\zeta_1 + \zeta_2T + \zeta_3T^2\right)t + (\zeta_1' + \zeta_2'T)t^2 + \zeta_1''t^3$	Equatorial precession parameter. Description is for 90° − ζ_A	$\mathcal{E}_F$ equator	$\mathcal{E}_F$ equinox	$\mathcal{E}_D$ equator
$\bar{\zeta}_A$	$\zeta_1T + \zeta_1'T^2 + \zeta_1''T^3$		$\mathcal{E}_0$ equator	$\mathcal{E}_0$ equinox	$\mathcal{E}_F$ equator
$\tilde{\zeta}_A$	$\zeta_1\tau + \zeta_1'\tau^2 + \zeta_1''\tau^3$		$\mathcal{E}_0$ equator	$\mathcal{E}_0$ equinox	$\mathcal{E}_D$ equator
z_A	$\left(z_1 + z_2T + z_3T^2\right)t + (z_1' + z_2'T)t^2 + z_1''t^3$	Equatorial precession parameter. Description is for 90° + z_A	$\mathcal{E}_D$ equator	$\mathcal{E}_D$ equinox	$\mathcal{E}_F$ equator
$\bar{z}_A$	$z_1T + z_1'T^2 + z_1''T^3$		$\mathcal{E}_F$ equator	$\mathcal{E}_F$ equinox	$\mathcal{E}_0$ equator
$\tilde{z}_A$	$z_1\tau + z_1'\tau^2 + z_1''\tau^3$		$\mathcal{E}_D$ equator	$\mathcal{E}_D$ equinox	$\mathcal{E}_0$ equator
θ_A	$\left(\theta_1 + \theta_2T + \theta_3T^2\right)t + (\theta_1' + \theta_2'T)t^2 + \theta_1''t^3$	Equatorial precession parameter. Angle between equators	Inclination	$\mathcal{E}_F$ equator	$\mathcal{E}_D$ equator
$\bar{\theta}_A$	$\theta_1T + \theta_1'T^2 + \theta_1''T^3$		Inclination	$\mathcal{E}_0$ equator	$\mathcal{E}_F$ equator
$\tilde{\theta}_A$	$\theta_1\tau + \theta_1'\tau^2 + \theta_1''\tau^3$		Inclination	$\mathcal{E}_0$ equator	$\mathcal{E}_D$ equator

*Notes: (1) $\mathcal{E}_0$ represents the basic epoch (e.g., J2000.0)

$\mathcal{E}_F$ represents an arbitrary fixed epoch (e.g., J2050.0)

$\mathcal{E}_D$ represents the mean epoch of date

(2) T represents time from $\mathcal{E}_0$ to $\mathcal{E}_F$

t represents time from $\mathcal{E}_F$ to $\mathcal{E}_D$

$\tau = T + t$ represents time from $\mathcal{E}_0$ to $\mathcal{E}_D$

(3) The symbol $\alpha_A = \left(\alpha_1 + \alpha_2T + \alpha_3T^2\right)t + (\alpha_1' + \alpha_2'T)t^2 + \alpha_1''t^3$ represents the accumulated angle from $\mathcal{E}_F$ to $\mathcal{E}_D$

The symbol $\bar{\alpha}_A = \alpha_A(T = 0,\ t = T) = \alpha_1T + \alpha_1'T^2 + \alpha_1''T^3$ represents the accumulated angle from $\mathcal{E}_0$ to $\mathcal{E}_F$

The symbol $\tilde{\alpha}_A = \alpha_A(T = 0,\ t = \tau) = \alpha_1\tau + \alpha_1'\tau^2 + \alpha_1''\tau^3$ represents the accumulated angle from $\mathcal{E}_0$ to $\mathcal{E}_D$

(4) The symbol $\alpha = (d\alpha_A/dt)_{t=0}$ represents the instantaneous rate of change of α_A at time T

(5) The suffix A denotes the accumulated value, without exception.

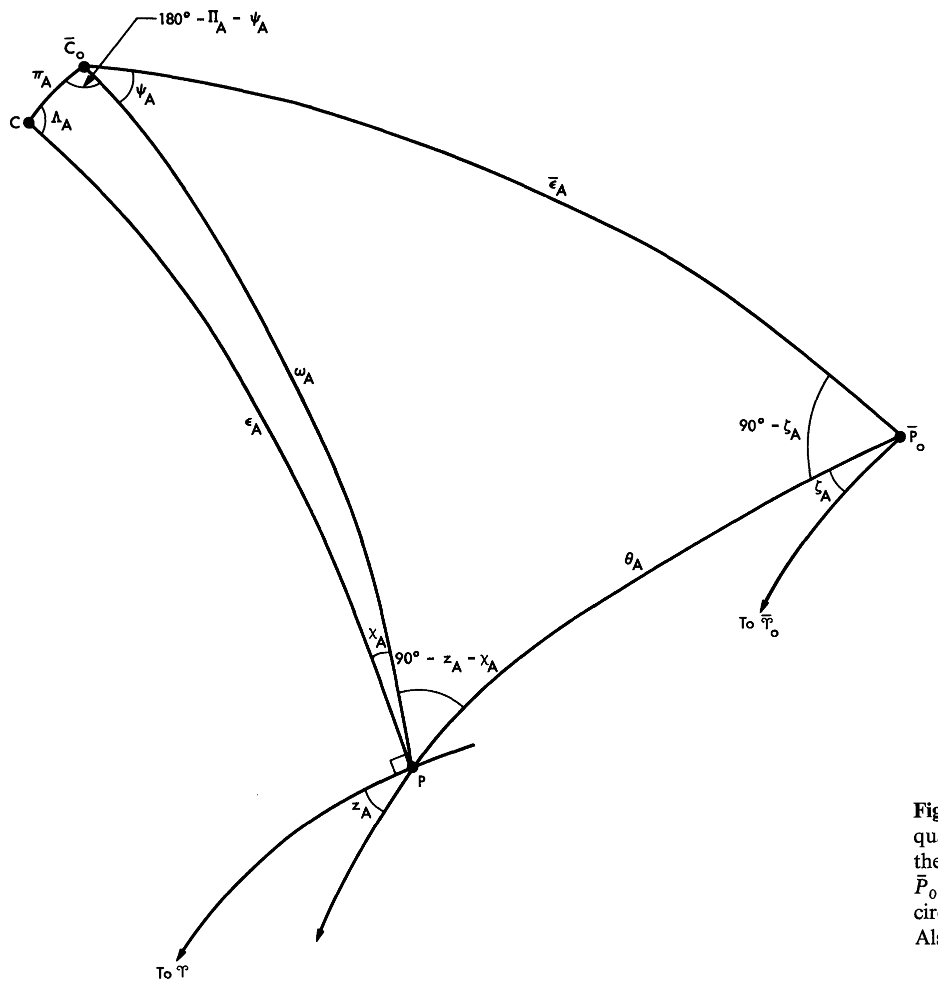

Fig. 2. Polar diagram of precession quantities depicted in Figure 1. Shown are the ecliptic poles $\bar{C}_0$, C and equatorial poles $\bar{P}_0$, P. The arc $\bar{P}_0P$ is a portion of a great circle passing through the celestial poles. Also shown are the equinoctial colures

and

$$n=\frac{d}{dt}(\theta_A)_{t=0}=\theta_1+\theta_2T+\theta_3T^2. \tag{6}$$

We will now proceed to develop the formulation which yields values for the various coefficients in the expressions for the precession quantities α_A.

Motion of Ecliptic Pole

The motion of the ecliptic pole at epoch $\mathscr{E}_D$ relative to the fixed ecliptic at the basic epoch $\mathscr{E}_0$ is generally described in terms of the angles $\tilde{\pi}_A$ and $\tilde{\Pi}_A$, in the form given in Table 1 which represents the angular momentum components,

$$\begin{aligned}\sin\tilde{\pi}_A\sin\tilde{\Pi}_A&=s_1\tau+s_1'\tau^2+s_1''\tau^3\\ \sin\tilde{\pi}_A\cos\tilde{\Pi}_A&=c_1\tau+c_1'\tau^3+c_1''\tau^3.\end{aligned} \tag{7}$$

These quantities s and c, coupled with ε_0 and p, enable one to calculate power series expressions for all the precession quantities. Although we will also give the individual components π_A and Π_A in our final expressions, it is to be understood that the more basic quantities are $\sin\pi_A\sin\Pi_A$ and $\sin\pi_A\cos\Pi_A$, the individual angles being determined by their trigonometric relations.

Newcomb (1894) derived values of $\kappa\sin L$ and $\kappa\cos L$, which are equivalent to $d(\sin\tilde{\pi}_A\sin\tilde{\Pi}_A)/d\tau$ and $d(\sin\tilde{\pi}_A\cos\tilde{\Pi}_A)/d\tau$, respectively. Newcomb's values, however, were expressed relative to the fixed 1850.0 ecliptic for epochs 1600, 1850 and 2100. Since 1850.0 will not, in general, be our basic epoch $\mathscr{E}_0$ we will employ $\sin\pi^*\sin\Pi^*$ and $\sin\pi^*\cos\Pi^*$ to represent Newcomb's data in the form

$$\begin{aligned}\sin\pi^*\sin\Pi^*&=s_1^*T_1+s_1^{*\prime}T_1^2+s_1^{*\prime\prime}T_1^3\\ \sin\pi^*\cos\Pi^*&=c_1^*T_1+c_1^{*\prime}T_1^2+c_1^{*\prime\prime}T_1^3\end{aligned} \tag{8}$$

where T_1 is measured in Julian centuries relative to 1850.0 (taken as JED 2396758.20358095). The epochs 1600 and 2100 are assumed to lie exactly 2.5 Julian centuries on either side of 1850.0. We subsequently will develop a means of obtaining $\sin\tilde{\pi}_A\sin\tilde{\Pi}_A$ and $\sin\tilde{\pi}_A\cos\tilde{\Pi}_A$ relative to the fixed basic epoch $\mathscr{E}_0$, given the values relative to 1850.0.

The planetary components $\kappa\sin L$ and $\kappa\cos L$ are presented in Table 3 for Newcomb's planetary masses at epochs 1600, 1850 and 2100. The values in parentheses

TABLE 2. EQUIVALENT NOTATIONS FOR DISPLACEMENTS AND RATES*

NEW	Bessel	Opp	Newcomb	Andoyer	deBall	IAU	W&C	Lieske	ARI	CdT	AE
Displacements											
π_A		π	k	(k)	π_τ		π_1	$\bar{\pi}_1$	(π)	k	
Π_A		Π	$180°-N$	(φ)	Π_τ		Π_1	$\bar{\Pi}_1$	(Π)	φ	
ϵ_o		ϵ_o	ϵ_o	ϵ_o	ϵ_o			ϵ_o	ϵ_o		
ϵ_A		ϵ		(ϵ)	$(\epsilon)_\tau$		ϵ	$\bar{\epsilon}$			
ω_A		ϵ'	ϵ_1	(ϵ_1)	$(\epsilon')_\tau$		ϵ_1	$\bar{\epsilon}_1$	(ϵ_1)	ω_1	ϵ_1
ψ_A		l'	ψ	(ψ)	$(-\psi)_\tau$		ψ_1	$\bar{\psi}$	(ψ_1)	ψ	
χ_A		a	λ	(χ)	$(a)_\tau$		a	$\bar{\lambda}$	(a)	χ	
Λ_A				(ω)				$\bar{\Lambda}$			
$p_A(=\Lambda_A-\Pi_A)$		l		$(\omega)-(\varphi)$	$(\Lambda)_\tau$		Ψ	$\bar{\Lambda}-\bar{\Pi}_1$	(ψ)	λ	
ζ_A		p	ζ_o	$90°-(\rho)$	p	ζ	ζ_o	$\bar{\zeta}_o$	$90°-(N)$	$90°-\rho$	ζ_o
z_A		$m-p$	z	$(\mu)-90°$	$\mathfrak{m}-p$		z	$\bar{z}$	$(N)+(m)-90°$	$\mu+\rho-90°$	z
θ_A		n	θ	(j)	$\mathfrak{n}$	ν	J	$\bar{\theta}$	(n)	j	θ
$\bar{\pi}_A$	π	(π)		k	π	π		π_1			π
$\bar{\Pi}_A$	Π	(Π)		φ	Π	Π		Π_1			Π
$\bar{\epsilon}_A$	ω	(ϵ)	ϵ	ϵ	(ϵ)	ϵ	ϵ^o	$\bar{\epsilon}_o$	ϵ	ω	ϵ
$\bar{\psi}_A$	ψ	(l')		ψ	$(-\psi)$			ψ			
$\bar{\chi}_A$	λ	(a)		χ	(a)			λ			
$\bar{\Lambda}_A$				ω				Λ			
$\bar{p}_A$	ψ_1	(l)		$\omega-\varphi$	(Λ)			$\Lambda-\Pi_1$			
$\bar{\zeta}_A$	z'	(p)		$90°-\rho$				ζ_o			
$\bar{z}_A$	z	$(m)-(p)$		$\mu-90°$				z			
$\bar{\theta}_A$	θ	(n)		j				θ			
Rates $\left[\left(\frac{d\alpha_A}{dt}\right)_{t=0}\right]$											
ψ		L_1'	p	f	$-\psi$	p_1	f_1	$\bar{f}$	ψ_1	p_l	ψ'
χ		A_1'	λ'	g	a	p_2	g_1	$\bar{g}$	a		λ'
p		L_1	l	h	Λ	p	h_1	$\bar{h}$	ψ	p_g	p
ζ		$\frac{1}{2}m_1$	$\frac{1}{2}m$	$\frac{1}{2}r$	$\frac{1}{2}m_1$		$\frac{1}{2}m$	$\frac{1}{2}\bar{r}$	$\frac{1}{2}m$	$\frac{1}{2}m$	$\frac{1}{2}m$
z		$\frac{1}{2}m_1$	$\frac{1}{2}m$	$\frac{1}{2}r$	$\frac{1}{2}m_1$		$\frac{1}{2}m$	$\frac{1}{2}\bar{r}$	$\frac{1}{2}m$	$\frac{1}{2}m$	$\frac{1}{2}m$
θ		n_1	n	s	n_1		n	$\bar{\omega}$	n	n	n

*Sources

NEW:	Notation used in this paper
Bessel:	Bessel, F. W. 1830, Tabulae Regiomontanae.
Opp:	Oppolzer, Th. R. v. 1882, Lehrbuch zur Bahnbestimmung der Kometen und Planeten, 1. Bd., 2. Aufl., Leipzig.
Newcomb:	Newcomb, S. 1906, A Compendium of Spherical Astronomy, New York.
Andoyer:	Andoyer, H. 1911, Bull. Astr. 28, 67.
deBall:	de Ball, L. 1912, Lehrbuch der Sphärischen Astronomie, Leipzig.
IAU:	International Astronomical Union, Transactions 6, 346 (1938).
W & C:	Woolard, E. W. and G. M. Clemence, 1966, Spherical Astronomy.
Lieske:	Lieske, J. H. 1967, NASA Technical Report 32–1044.
ARI:	Astronomisches Rechen-Institut Heidelberg; cf. Kopff, A. and F. Gondolatsch 1950, Grundbegriffe der Sphärischen Astronomie.
CdT:	Connaissance des Temps (Bureau des Longitudes, Paris), annual volumes from 1914 onwards.
AE:	Astronomical Ephemeris, in particular: Explanatory Supplement to the A.E., London, 1961.

are the original numbers of Newcomb (1894, p. 377), while the extended precision values have been derived by repeating Newcomb's calculations. The entries for Pluto are due to Clemence (1948). The extended precision values are needed for two reasons: (a) Newcomb's original values were only given to 0".001 which results in truncation difficulties with the higher-order terms upon integrating $\kappa \sin L$ and $\kappa \cos L$ to obtain $\sin \pi^* \sin \Pi^*$ and $\sin \pi^* \cos \Pi^*$, and (b) round-off errors are introduced in calculating the components of $\kappa \sin L$ and $\kappa \cos L$ for revised values of planetary masses. In addition, it is impossible to duplicate Andoyer's (1911) expressions for the precession quantities if one employs Newcomb's components listed in Table 3. The reason is that Andoyer employed the pre-rounded sum of the components, which is no longer available. (If one sums the components, excluding Pluto, for $\kappa \sin L$ at epoch 1600, for example, he finds +4".274, but Newcomb listed the sum

Table 3. Newcomb values of $\frac{d}{dt}\left(\sin\pi^* \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} \Pi^*\right)$ for 3 epochs[a]

Planet	Newcomb inverse Mass M_N^{-1}	$\kappa \sin L = \frac{d}{dt}(\sin\pi_A^* \sin\Pi_A^*)$			$\kappa \cos L = \frac{d}{dt}(\sin\pi_A^* \cos\Pi_A^*)$		
		1600	1850	2100	1600	1850	2100
Mercury	7500000	+0.247266	+0.250818	+0.254359	−0.211715	− 0.209848	− 0.207971
		(+0.247	+0.251	+0.254	− 0.212	− 0.210	− 0.208)
Venus	410000	+6.789930	+7.411897	+8.031934	−28.472881	−28.332065	−28.185002
		(+6.790	+7.412	+8.032	−28.473	−28.332	−28.185)
Mars	3093500	+0.617570	+0.634309	+0.650891	− 0.734839	− 0.719136	− 0.702991
		(+0.617	+0.634	+0.651	− 0.735	− 0.719	− 0.703)
Jupiter	1047.88	−2.804002	−2.511528	−2.224385	−16.169696	−16.046939	−15.919290
		(−2.804	−2.511	−2.224	−16.170	−16.047	−15.919)
Saturn	3501.6	−0.574201	−0.542325	−0.510169	− 1.311104	− 1.318862	− 1.325625
		(−0.574	−0.542	−0.510	− 1.310	− 1.318	− 1.325)
Uranus	22756	+0.002028	+0.002375	+0.002720	− 0.007909	− 0.007873	− 0.007831
		(+0.002	+0.002	+0.003	− 0.008	− 0.008	− 0.008)
Neptune	19540	−0.003798	−0.003694	−0.003589	− 0.004388	− 0.004377	− 0.004364
		(−0.004	−0.004	−0.004	− 0.004	− 0.004	− 0.004)
Pluto	360000	−0.0004	−0.0004	−0.0004	− 0.0012	− 0.0012	− 0.0012

[a] Units are arcseconds per Julian century and are referred to the fixed ecliptic of 1850.0. Values listed in parentheses are Newcomb's original numbers taken from *Astron. Pap.* **5**, Part 4, page 377 (1894)

as +4″275.) For these reasons it was found necessary to repeat Newcomb's calculations and to derive the extended precision results listed in Table 3. In redeveloping Newcomb's (1894) values, we employed his basic data regarding planetary orbits (pp. 336–338). Newcomb's values result from employing the disturbing function (p. 305)

$$F = \tfrac{1}{2}\{P_{00}^{00} + (e^2 + e'^2)P_{00}^{20} + e^4 P_{00}^{40} + \ldots\} + \{ee' P_{-11}^{11} + e^3 e' P_{11}^{31} + \ldots\}\cos(\omega - \omega') + \ldots \tag{9}$$

where $P_{jj'}^{nn'}$ are Newcomb operators and where e, e' and ω, ω' are the eccentricities and arguments of perihelion, respectively, for inner and outer planets. We employed Newcomb's general formulation for the operators $P_{jj'}^{nn'}$ in terms of orbital elements, rather than the low-precision numerical values ("Special Values") which he also listed[1].

In redeveloping the extended precision expressions listed in Table 3 it was also found that Newcomb's calculated results for Saturn employed the wrong mass. Apparently (see p. 336 entry $\log M = 3.58865$, implying $M = 3878.4$) Newcomb employed the wrong mass μ in calculating the mass factor for Saturn's effect on the earth,

$$M = n_\oplus \alpha m_{\text{♄}}/\mu \tag{10}$$

where $n_\oplus$ is the earth's sidereal mean motion, $\alpha = a_\oplus / a_{\text{♄}}$ and $\mu = 1 + m_{\text{♄}}$. The correct equation is $\mu = 1 + m_\oplus$, as used for other calculations. This error accounts for the Saturn disagreement in Table 3 between Newcomb's original value and our extended precision value. Newcomb's original result corresponds to a Saturn mass of $m_{\text{♄}}^{-1} = 3503$. With the preceding discrepancies taken into account, we are able to duplicate Andoyer's series if we employ the masses given in Table 3.

To derive the series for $\sin\pi^* \sin\Pi^*$ and $\sin\pi^* \cos\Pi^*$ from the data in Table 3, we proceed as follows. Denote by α_{-1}, α_0 and α_1 the sum of the planetary contributions to $\kappa \sin L$ for epochs 1600, 1850 and 2100, respectively. Let β_j denote similar results for $\kappa \cos L$. If the mass of a planet relative to the sun is denoted by M (the inverse masses M^{-1} are listed in Table 3 for Newcomb and in Table 4 for the recently adopted IAU values), then, for example, we will have for $\frac{d}{dT_1}(\sin\pi^* \sin\Pi^*)$ at epoch 1600 referred to the ecliptic of 1850.0,

$$\begin{aligned}\alpha_{-1} = {} & 0''247266 \cdot (7\,500\,000 M_{\text{☿}}) + 6''789930 \cdot (410\,000 M_{\text{♀}}) \\ & + 0''617570 \cdot (3\,093\,500 M_{\text{♂}}) \\ & - 2''804002 \cdot (1047.88 M_{\text{♃}}) \\ & - 0''574201 \cdot (3501.6 M_{\text{♄}}) + 0''002028 \cdot (22\,756 M_{\text{♂}}) \\ & - 0''003798 \cdot (19\,540 M_{\text{♆}}) - 0''0004 \cdot (360\,000 M_{\text{♇}}).\end{aligned} \tag{11}$$

We can then obtain, at time T_1 from 1850.0, the power series expansion, relative to the 1850.0 ecliptic, from Stirling's interpolation formula as

$$\begin{aligned}\sin\pi^* \sin\Pi^* &= \alpha_0 T_1 + \tfrac{1}{10}(\alpha_1 - \alpha_{-1})T_1^2 + \tfrac{2}{75}(\alpha_1 - 2\alpha_0 + \alpha_{-1})T_1^3 \\ \sin\pi^* \cos\Pi^* &= \beta_0 T_1 + \tfrac{1}{10}(\beta_1 - \beta_{-1})T_1^2 + \tfrac{2}{75}(\beta_1 - 2\beta_0 + \beta_{-1})T_1^3 .\end{aligned} \tag{12}$$

[1] The relevant operators are given by Newcomb on pages 350–351, 356–357, 363–367 for planetary perturbations on the earth. The following typographical errors were corrected:

Venus p. 357	$P'^{22}_{0-2}: 922\sigma$	should read $922\sigma^2$
Mars p. 363	$P^{20}_{00}: -3683\sigma^2$	should read $-36.83\sigma^2$
Mars p. 364	$P''^{14}_{-1-3}: -12M\sigma^4$	should read $-1.2M\sigma^4$
Jupiter p. 364	$P^{20}_{00}: -200.3m\sigma^2$	should read $-209.3m\sigma^2$
Uranus p. 367	$P'^{13}_{-1-1}: -2.1\sigma^2$	should read $-2.1m\sigma^2$
Neptune p. 367	$P^{22}_{00}:$	should read $1.255m - 7.6m\sigma^2$

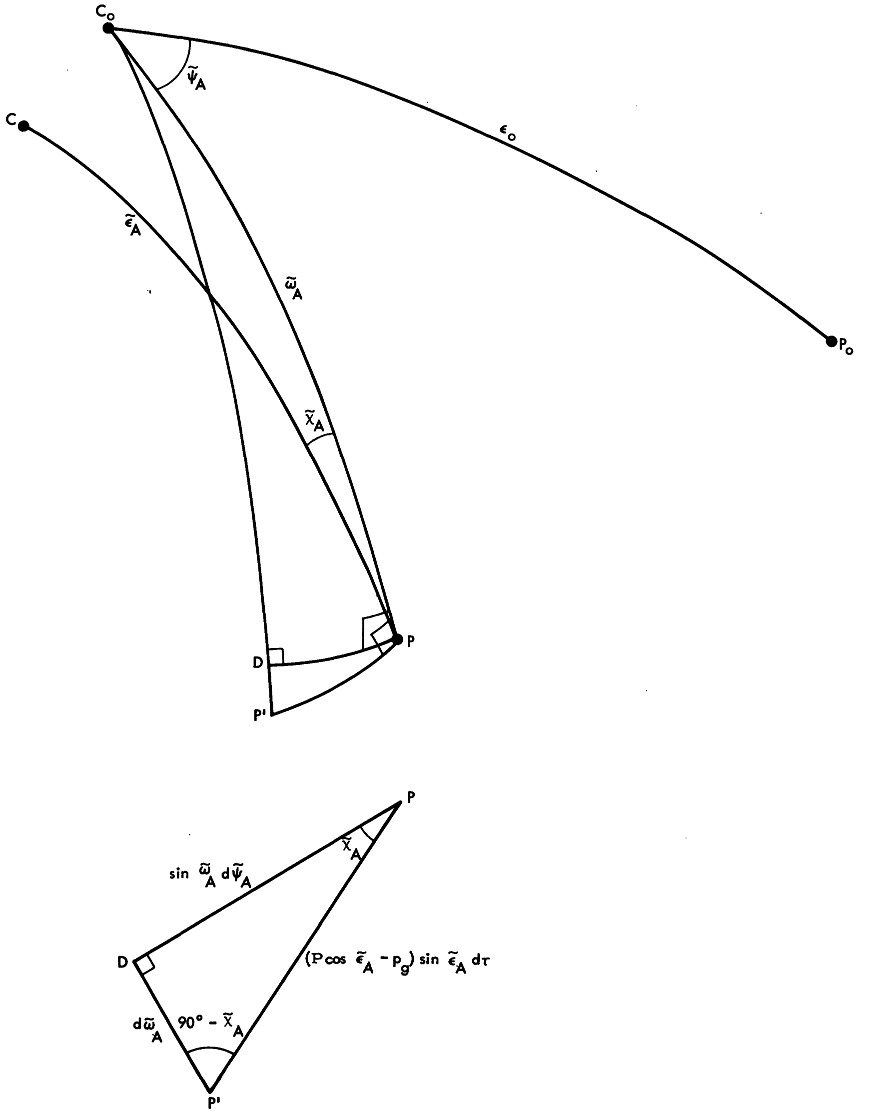

Fig. 3. Polar precessional diagram for epoch $\mathscr{E}_D$ relative to $\mathscr{E}_0$. The enlarged portion depicts the spherical triangle *PDP'*. The celestial pole *P* moves in a small circle *PP'* about the ecliptic pole of date *C*

These expressions define the quantities s_1^*, $s_1^{*\prime}$, $s_1^{*\prime\prime}$ and c_1^*, $c_1^{*\prime}$, $c_1^{*\prime\prime}$ of Equation (8). Previous questions as to the accuracy of Newcomb's expressions for the motion of the ecliptic pole have been largely answered by Lieske's (1970) theoretical comparison via a numerical integration, and on the observational side, by Duncombe and van Flandern (1976).

We subsequently will develop means to refer $\sin\pi^* \sin\Pi^*$ and $\sin\pi^* \cos\Pi^*$ to the basic epoch $\mathscr{E}_0$. At this stage, however, we may assume that the parameters s_1, s_1', s_1'', c_1, c_1', and c_1'' appearing in Equation (7) are known at epoch $\mathscr{E}_0$. We next need to develop the motion of the equatorial pole and to derive the precession quantities for epoch $\mathscr{E}_D$ relative to $\mathscr{E}_0$.

Formulation for Basic Epoch

In Figure 3 we depict the equatorial pole P_0 and ecliptic pole C_0 at epoch $\mathscr{E}_0$, along with the equatorial (*P*) and ecliptic (*C*) poles at epoch $\mathscr{E}_D$. The celestial pole of date *P* moves in a small circle about the ecliptic pole *C* with speed $(\mathrm{P}\cos\tilde{\varepsilon}_A - p_g)\sin\tilde{\varepsilon}_A$ where P is Newcomb's "Precessional Constant" (Newcomb, 1906, p. 228). Hence, in the interval $d\tau$, we have the geometry given in triangle *DPP'* of Figure 3. The small parameter p_g is the so-called geodesic precession (de Sitter and Brouwer, 1938), given by them as

$$p_g = \tfrac{3}{2} k^2 (1 - e_\oplus^2) n_\oplus \tag{13}$$

where k is the constant of aberration. The value of p_g, used here is

$$p_g = 1''\!.92 \quad \text{per century}.$$

As shown by Barker and O'Connell (1970), this small relativistic effect is one-half the earth's relativistic motion of its longitude of perihelion [See their Equations (14), (28) and (76a)].

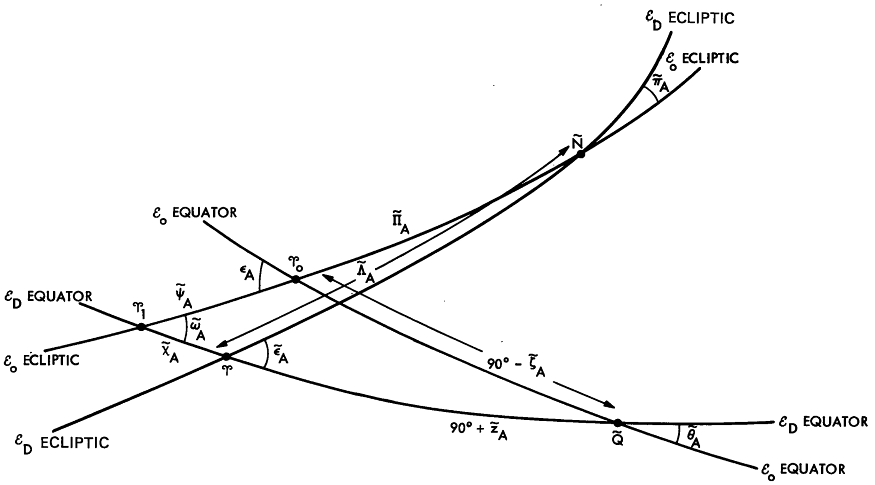

Fig. 4. Definitions of ecliptic-equator precession quantities for mean epoch of date $\mathcal{E}_D$ relative to fixed basic epoch $\mathcal{E}_0$. The vernal equinox at $\mathcal{E}_0$ is represented by Υ_0, while Υ represents the mean equinox at epoch $\mathcal{E}_D$

From the geometry depicted in Figure 3 one can write the dynamical equation (Lieske, 1967)

$$\left.\begin{aligned}
\sin\tilde{\omega}_A \frac{d\tilde{\psi}_A}{d\tau} &= (\mathrm{P}\cos\tilde{\varepsilon}_A - p_g)\sin\tilde{\varepsilon}_A \cos\tilde{\chi}_A \\
\frac{d\tilde{\omega}_A}{d\tau} &= (\mathrm{P}\cos\tilde{\varepsilon}_A - p_g)\sin\tilde{\varepsilon}_A \sin\tilde{\chi}_A \\
\frac{d\tilde{\varepsilon}_A}{d\tau} &= \cos(\tilde{\Lambda}_A - \tilde{\Pi}_A)\frac{d}{d\tau}(\sin\tilde{\pi}_A \cos\tilde{\Pi}_A) \\
&\quad - \sin(\tilde{\Lambda}_A - \tilde{\Pi}_A)\frac{d}{d\tau}(\sin\tilde{\pi}_A \sin\tilde{\Pi}_A) \\
&\quad + 2\sin^2\frac{\tilde{\pi}_A}{2}\cos\tilde{\Lambda}_A \frac{d\tilde{\pi}_A}{d\tau} \\
&\text{and, for the equatorial parameters,} \\
\frac{d\tilde{\theta}_A}{d\tau} &= (\mathrm{P}\cos\tilde{\varepsilon}_A - p_g)\sin\tilde{\varepsilon}_A \cos\tilde{z}_A \\
\sin\tilde{\theta}_A \frac{d\tilde{\zeta}_A}{d\tau} &= (\mathrm{P}\cos\tilde{\varepsilon}_A - p_g)\sin\tilde{\varepsilon}_A \sin\tilde{z}_A .
\end{aligned}\right\} \quad (14)$$

The various angles are depicted in Figure 4 for epoch $\mathcal{E}_D$ relative to epoch $\mathcal{E}_0$.

From Equations (7) and (11)—(14) it is seen that if one is given the set of planetary masses, together with the constants ε_0, P and p_g at epoch $\mathcal{E}_0$, then one can solve Equation (14) for the other precession parameters and thus obtain power series for the angles $\tilde{\alpha}_A$ as listed in Table 1. Newcomb's "Precessional Constant" P is a function of the moments of inertia of the earth and also of the elements of the earth's orbit (de Sitter and Brouwer, 1938). Because it has been impossible, so far, to calculate P from its theoretical dependence upon geodetic parameters, P has to be inferred from observationally determined values of the general precession p. Hence P is replaced by the general precession p in the list of astronomical constants. There is a slight dependence of P upon time ($\mathrm{P} = \mathrm{P}_0 + \mathrm{P}_1 t$), the variation being approximately $\mathrm{P}_1 = -0\overset{''}{.}00369$ per century, which is due mainly to changes in the eccentricity of the earth's orbit. From the work of de Sitter and Brouwer (1938) it can be shown that the centennial variation is

$$\mathrm{P}_1 = -0\overset{''}{.}00001 - 7.313 \times 10^{-7} p - 2\overset{''}{.}5 \times 10^{-3} v \qquad (15)$$

where p is the general precession (units: arc seconds per century) and where v is defined by the mass of the moon relative to the earth (μ),

$$\frac{M}{E} = \mu = 0.0123\,(1 + v). \qquad (16)$$

The resultant series one obtains for $\tilde{\alpha}_A$ will not be presented in this section, but are a part (viz. α_1, α_1', α_1'') of the terms given later relative to $\mathcal{E}_F$. The basic developments of de Sitter and Brouwer (1938), Woolard (1953), Woolard and Clemence (1966), Lieske (1967), and Hristov (1970), all depend upon the original work of Andoyer (1911).

Before proceeding to the development relative to an arbitrary fixed epoch $\mathcal{E}_F$, we will briefly discuss several definitions of the accumulated angle sometimes called the general precesion in longitude. As employed in this paper, and originally developed by Andoyer, we denote the *accumulated angle* p_A (which corresponds to the instantaneous rate p of general precession) of "general precession in longitude" by the symbol $p_A = \Lambda_A - \Pi_A$. Our measure is thus defined by the subtraction of two angles measured in different planes. As may be seen from Figure 1 and Table 1, Λ_A is measured along the mean ecliptic of date from the equinox of date to the point N, which is the intersection of the ecliptic of date with the fixed ecliptic, while the angle Π_A is measured along the fixed ecliptic from the fixed equinox to the point N. We prefer Andoyer's definition since in any rigorous reduction for precession one will generally employ rotation matrices, or their equivalent, and our angles Π_A and Λ_A implicitly are calculated. However, Newcomb (1906, p. 234; see also Woolard and Clemence., 1966, p. 237) employs the definition shown in Figure 5, where p_N is the arc $\Upsilon_0 R$. Newcomb thus defines the accumulated general

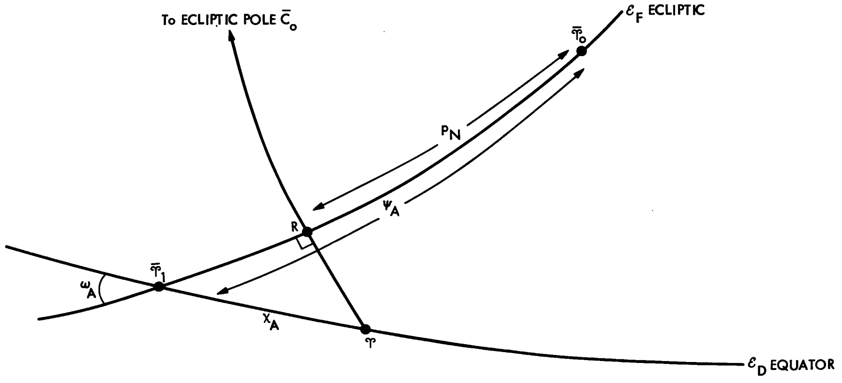

Fig. 5. Diagram showing Newcomb's measure of the accumulated general precession in longitude p_N: the arc, measured along the fixed ecliptic, from the fixed equinox $\bar{\Upsilon}_0$ to the point R, which is the intersection with the fixed ecliptic of the great circle from $\bar{C}_0$ to Υ

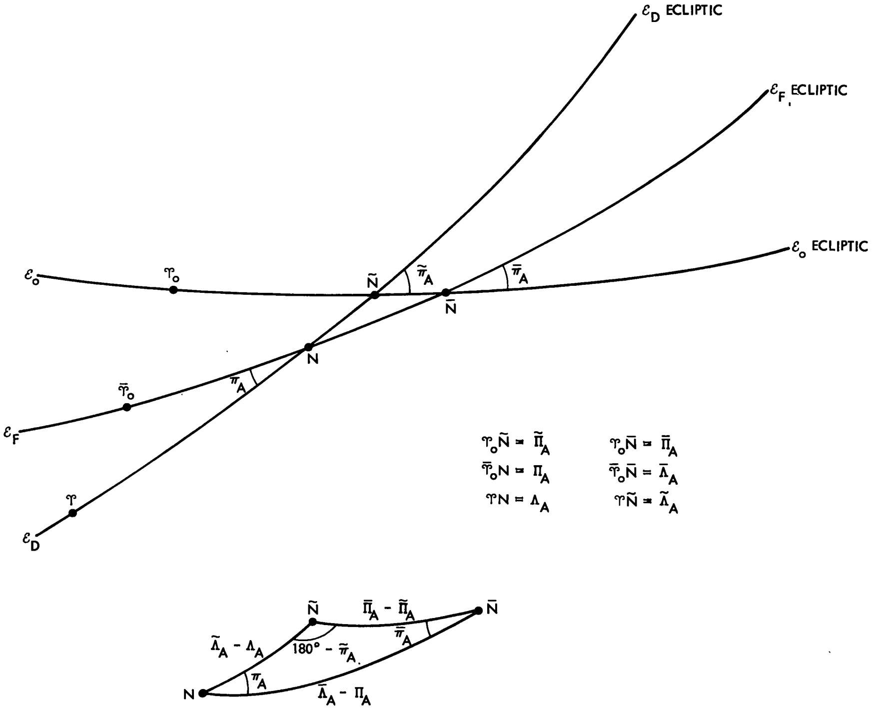

Fig. 6. Diagram showing the ecliptic precession quantities for three epochs $\mathscr{E}_0$, $\mathscr{E}_F$ and $\mathscr{E}_D$

precession as measured along the fixed ecliptic from the fixed equinox $\bar{\Upsilon}_0$ to the point R, which is the intersection with the fixed ecliptic, of the great circle from the equinox of date Υ to the fixed ecliptic pole $\bar{C}_0$ at epoch $\mathscr{E}_F$. The two definitions differ only by $0''.0005\,t^2$ but we prefer Andoyer's definition since it places more emphasis on the *accumulated* precession via Eulerian angles (and is more readily handled by rotation matrices), while Newcomb's expression is perhaps more readily understood in discussing instantaneous rates. With either definition, the instantaneous rates are identical, so that the fundamental constant p represents both interpretations. Aside from the accumulated precessional displacements p_A and p_N, no other quantities are affected by adopting Andoyer's definition of general precession.

From the geometry shown in Figure 5 it is seen that the two expressions for the accumulated angles p_A (our recommended definition, due to Andoyer) and p_N (Newcomb's definition) are related by

$$\tan(\psi_A - p_N) = \cos\omega_A \tan\chi_A \tag{17}$$

and, as given in Tables 1 and 5, by

$$p_N = p_A + (\eta'_1 + \eta'_2 T)t^2 + \eta''_1 t^3$$

where the η values [see Equation (22) and Table 5] are all less than $0''.0005$.

Formulation for an Arbitrary Epoch

Having, in principle, expressions for the precession quantities at epoch $\mathscr{E}_D$ relative to the basic epoch $\mathscr{E}_0$, we can proceed to develop them relative to an arbitrary fixed epoch $\mathscr{E}_F$. The ecliptics and relevant angles at epochs $\mathscr{E}_D$, $\mathscr{E}_F$ and $\mathscr{E}_0$ are depicted in Figure 6, while the equatorial

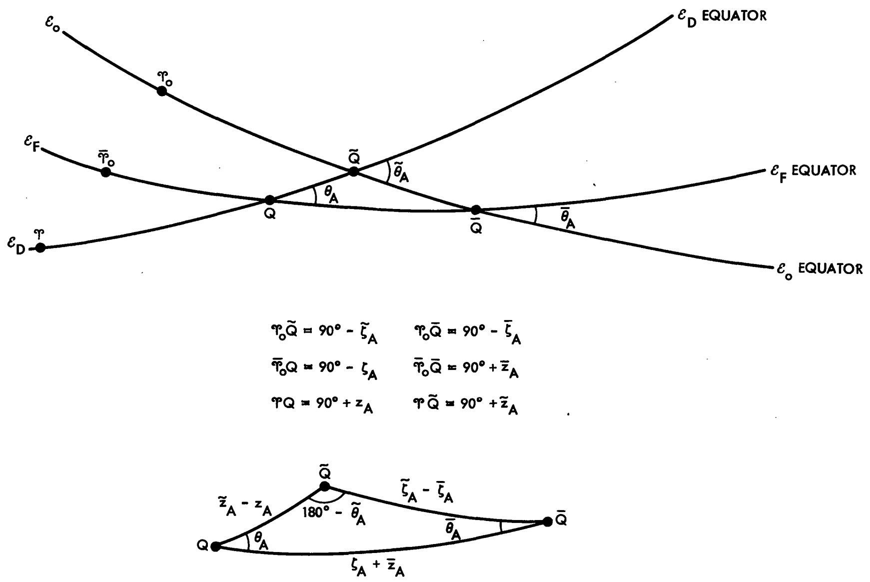

Fig. 7. Diagram showing the equatorial precession quantities for three epochs $\mathscr{E}_0$, $\mathscr{E}_F$ and $\mathscr{E}_D$

quantities for similar epochs are shown in Figure 7. The development proceeds as follows.

From the expressions developed in the preceding section, one in principle has the known quantities $\sin\tilde{\pi}_A \sin\tilde{\Pi}_A$, $\sin\tilde{\pi}_A \cos\tilde{\Pi}_A$, $\tilde{\varepsilon}_A$, p, P_1 and p_g. From the expressions in Equation (7) we can also write the known series for $\sin\bar{\pi}_A \sin\bar{\Pi}_A$ and $\sin\bar{\pi}_A \cos\bar{\Pi}_A$ at epoch $\mathscr{E}_F$ relative to epoch $\mathscr{E}_0$ as

$$\begin{aligned}\sin\bar{\pi}_A \sin\bar{\Pi}_A &= s_1 T + s_1' T^2 + s_1'' T^3\\ \sin\bar{\pi}_A \cos\bar{\Pi}_A &= c_1 T + c_1' T^2 + c_1'' T^3\,,\end{aligned} \tag{18}$$

and similarly, for the accumulated precession $\bar{p}_A$,

$$\bar{p}_A = \bar{\Lambda}_A - \bar{\Pi}_A = p_1 T + p_1' T^2 + p_1'' T^3\,. \tag{19}$$

From the triangle $N\tilde{N}\bar{N}$ in Figure 6 we can relate $\sin\pi_A \sin\Pi_A$ and $\sin\pi_A \cos\Pi_A$ at epoch $\mathscr{E}_D$ relative to $\mathscr{E}_F$ to the known series for the quantities given in Equations (18) and (19). From the well-known relations of spherical trigonometry we have

$$\begin{aligned}\sin\pi_A \sin(\bar{\Lambda}_A - \Pi_A) &= \sin\tilde{\pi}_A \sin(\bar{\Pi}_A - \tilde{\Pi}_A)\\ \sin\pi_A \cos(\bar{\Lambda}_A - \Pi_A) &= \sin\tilde{\pi}_A \cos\bar{\pi}_A \cos(\bar{\Pi}_A - \tilde{\Pi}_A)\\ &\quad - \sin\bar{\pi}_A \cos\tilde{\pi}_A\end{aligned} \tag{20}$$

which yields the desired expressions

$$\begin{aligned}\sin\pi_A \sin\Pi_A = {}&(\sin\tilde{\pi}_A \sin\tilde{\Pi}_A)\cos(\bar{\Lambda}_A - \bar{\Pi}_A)\\ &+ (\sin\tilde{\pi}_A \cos\tilde{\Pi}_A)\sin(\bar{\Lambda}_A - \bar{\Pi}_A)\\ &- 2\sin^2\frac{\bar{\pi}_A}{2}\sin\tilde{\pi}_A \sin\bar{\Lambda}_A \cos(\bar{\Pi}_A - \tilde{\Pi}_A)\\ &- \sin\bar{\pi}_A \sin\bar{\Lambda}_A\left(1 - 2\sin^2\frac{\tilde{\pi}_A}{2}\right)\end{aligned}$$

and

$$\begin{aligned}\sin\pi_A \cos\Pi_A = {}&(\sin\tilde{\pi}_A \cos\tilde{\Pi}_A)\cos(\bar{\Lambda}_A - \bar{\Pi}_A)\\ &- (\sin\tilde{\pi}_A \sin\tilde{\Pi}_A)\sin(\bar{\Lambda}_A - \bar{\Pi}_A)\\ &- 2\sin^2\frac{\bar{\pi}_A}{2}\sin\tilde{\pi}_A \cos\bar{\Lambda}_A \cos(\bar{\Pi}_A - \tilde{\Pi}_A)\\ &- \sin\bar{\pi}_A \cos\bar{\Lambda}_A\left(1 - 2\sin^2\frac{\tilde{\pi}_A}{2}\right).\end{aligned} \tag{21}$$

The coefficients s_1, s_2, s_3, etc. for $\sin\pi_A \sin\Pi_A$ and $\sin\pi_A \cos\Pi_A$ described in Table 1 may be derived from the known series. From the differential equations in Equation (14) and the geometry shown in Figures 6 and 7, one finds the following expressions for the coefficients of the precession quantities [Andoyer (1911); or Lieske (1967), who uses Andoyer's development but introduces p_g] using the notations of Table 1:

Equation 22

Constant Terms

$x_0 = \tan^{-1}(s_1/c_1)$

Coefficients of t

$\varepsilon_1 = c_1$

$\chi_1 = s_1 \csc\varepsilon_0$

$\psi_1 = p_1 + \chi_1 \cos\varepsilon_0$

$P_0 = (p_1 + p_g)\sec\varepsilon_0 + \chi_1$

$z_1 - \zeta_1 = 0$

$z_1 + \zeta_1 = \psi_1 \cos\varepsilon_0 - \chi_1$

$\theta_1 = \psi_1 \sin\varepsilon_0$

$q_1 = (s_1^2 + c_1^2)^{1/2}$

$y_1 = (c_1 s_1' - s_1 c_1')/q_1^2$

Coefficients of t^2

$\varepsilon'_1 = c'_1 - s_1 p_1/2$

$\omega'_1 = s_1\psi_1/2$

$\psi'_1 = c_1\psi_1 \cot 2\varepsilon_0 + (\mathrm{P}_1 \cos\varepsilon_0 - c_1 p_g \tan\varepsilon_0)/2$

$\chi'_1 = (s'_1 + c_1 p_1)\csc\varepsilon_0$

$p'_1 = \psi'_1 - \chi'_1 \cos\varepsilon_0 + s_1 c_1/2$

$\eta'_1 = -s_1 c_1/2$

$z'_1 - \zeta'_1 = (\psi'_1\chi_1 - \psi_1\chi'_1)/3\psi_1$

$z'_1 + \zeta'_1 = \psi'_1 \cos\varepsilon_0 - \chi'_1 = z_2 = \zeta_2$

$\theta'_1 = \psi'_1 \sin\varepsilon_0$

$q'_1 = (s_1 s'_1 + c_1 c'_1)/q_1$

$y'_1 = (c_1 s''_1 - s_1 c''_1)/q_1^2 - y_1(c'_1 + s_1 y_1)/c_1$

Coefficients of t^3

$\varepsilon''_1 = c''_1 - (2s'_1 p_1 + s_1 p'_1)/3 - c_1(p_1^2 - s_1^2 - c_1^2)/6$

$\omega''_1 = \sin\varepsilon_0(2\psi'_1\chi_1 + \psi_1\chi'_1)/3$

$3\psi''_1 = (c_1\mathrm{P}_1 + \varepsilon'_1\mathrm{P}_0)\cos 2\varepsilon_0 \csc\varepsilon_0 - \psi_1\omega'_1 \cot\varepsilon_0$
$\qquad - \psi_1(4c_1^2 + \chi_1^2)/2 - p_g(3c_1^2 + 2\varepsilon'_1 \cot\varepsilon_0)/2$

$\chi''_1 = (s''_1 + c'_1 p_1 + c_1 p'_1 - s_1\omega'_1 \cot\varepsilon_0 - s_1 p_1^2/2)\csc\varepsilon_0 + \chi_1^3/6$

$p''_1 = \psi''_1 - \chi''_1 \cos\varepsilon_0 + \sin\varepsilon_0[(\varepsilon'_1 + \omega'_1)\chi_1 + c_1\chi'_1]/2$

$2\eta''_1 = p_1(s_1^2 - c_1^2) - (s_1 c'_1 + c_1 s'_1)$

$z''_1 - \zeta''_1 =$ See entry after coefficients $z'_2 - \zeta'_2$ of Tt^2

$z''_1 + \zeta''_1 = \psi''_1 \cos\varepsilon_0 - \chi''_1 + \psi_1^2 \sin^2\varepsilon_0(\psi_1 \cos\varepsilon_0 - 3\chi_1)/12$

$\theta''_1 = \psi''_1 \sin\varepsilon_0 + \psi_1 \sin\varepsilon_0(3\chi_1^2 + 6\psi_1\chi_1 \cos\varepsilon_0$
$\qquad - \psi_1^2 \cos^2\varepsilon_0)/24$

$q''_1 = (2s_1 s''_1 + 2c_1 c''_1 + s_1'^2 + c_1'^2 - q_1'^2)/2q_1 + q_1^3/6$

Coefficients of T

$x_1 = 2y_1 + p_1$

Coefficients of Tt

$s_2 = 2s'_1 + c_1 p_1$

$c_2 = 2c'_1 - s_1 p_1$

$\varepsilon_2 = c_2 = 2\varepsilon'_1$

$\psi_2 = -c_1 \mathrm{P}_0 \sin\varepsilon_0 + \mathrm{P}_1 \cos\varepsilon_0$

$\chi_2 = s_2 \csc\varepsilon_0 - c_1\chi_1 \cot\varepsilon_0$

$p_2 = \psi_2 - \chi_2 \cos\varepsilon_0 + c_1\chi_1 \sin\varepsilon_0 = 2p'_1$

$z_2 - \zeta_2 = 0$

$z_2 + \zeta_2 = \psi_2 \cos\varepsilon_0 - c_1\psi_1 \sin\varepsilon_0 - \chi_2 = 2(z'_1 + \zeta'_1)$

$\theta_2 = \psi_2 \sin\varepsilon_0 + c_1\psi_1 \cos\varepsilon_0 = 2\theta'_1$

$q_2 = (s_1 s_2 + c_1 c_2)/q_1 = 2q'_1$

$y_2 = 3(c_1 s''_1 - s_1 c''_1)/q_1^2 - 4y_1(c'_1 + s_1 y_1)/c_1$

Coefficients of T^2t:

$s_3 = 3s''_1 + 2c'_1 p_1 + c_1 p'_1 - s_1(p_1^2 - s_1^2 - c_1^2)/2$

$c_3 = 3\varepsilon''_1$

$\varepsilon_3 = c_3 = 3\varepsilon''_1$

$\psi_3 = -\tfrac{1}{2}c_1^2\mathrm{P}_0 \cos\varepsilon_0 - \varepsilon'_1\mathrm{P}_0 \sin\varepsilon_0 - c_1\mathrm{P}_1 \sin\varepsilon_0$

$\chi_3 = s_3 \csc\varepsilon_0 - c_1\chi_2 \cot\varepsilon_0$
$\qquad + \chi_1(\tfrac{1}{2}c_1^2 - \varepsilon'_1 \cot\varepsilon_0)$

$p_3 = \psi_3 - \chi_3 \cos\varepsilon_0 + c_1\chi_2 \sin\varepsilon_0$
$\qquad + \varepsilon'_1\chi_1 \sin\varepsilon_0 + \tfrac{1}{2}c_1^2\chi_1 \cos\varepsilon_0$

$z_3 - \zeta_3 = 0$

$z_3 + \zeta_3 = \psi_3 \cos\varepsilon_0 - c_1\psi_2 \sin\varepsilon_0 - \chi_3$
$\qquad - \tfrac{1}{2}c_1^2\psi_1 \cos\varepsilon_0 - \varepsilon'_1\psi_1 \sin\varepsilon_0$

$\theta_3 = \psi_3 \sin\varepsilon_0 + c_1\psi_2 \cos\varepsilon_0 + \varepsilon'_1\psi_1 \cos\varepsilon_0$
$\qquad - \tfrac{1}{2}c_1^2\psi_1 \sin\varepsilon_0 = \theta'_2$

$q_3 = [s_2^2 + c_2^2 + 2(s_1 s_3 + c_1 c_3) - q_2^2]/2q_1$

Coefficients of T^2:

$x_2 = (c_1 s_3 - s_1 c_3)/q_1^2 - 2x_1(c'_1 + s_1 y_1)/c_1$

Coefficients of Tt^2:

$s'_2 = 3s''_1 + c'_1 p_1 + s_1(s_1^2 + c_1^2)/2$

$c'_2 = 3c''_1 - s'_1 p_1 + c_1(s_1^2 + c_1^2)/2$

$\varepsilon'_2 = c'_2 - (s_2 p_1 + s_1 p_2)/2 = 3\varepsilon''_1$

$\omega'_2 = (s_1\psi_2 - s_2\psi_1)/2$

$2\psi'_2 = \mathrm{P}_0 \csc\varepsilon_0[c_2 \cos 2\varepsilon_0 - c_1^2(\cot\varepsilon_0 + \sin 2\varepsilon_0)]$
$\qquad + c_1\mathrm{P}_1 \csc\varepsilon_0(\cos 2\varepsilon_0 - \sin^2\varepsilon_0)$
$\qquad + p_g \csc\varepsilon_0(c_1^2 \csc\varepsilon_0 - c_2 \cos\varepsilon_0)$

$\chi'_2 = (s'_2 + c_1 p_2 + c_2 p_1)\csc\varepsilon_0$
$\qquad - c_1 \cot\varepsilon_0 \csc\varepsilon_0(s'_1 + c_1 p_1)$

$p'_2 = \psi'_2 + c_1\chi'_1 \sin\varepsilon_0 - \chi'_2 \cos\varepsilon_0$
$\qquad + (s_2 c_1 + c_2 s_1)/2$

$\eta'_2 = -(s_2 c_1 + c_2 s_1)/2$

$z'_2 - \zeta'_2 = [\chi_1(\psi'_2 - \psi_2\psi'_1/\psi_1) + \psi'_1\chi_2 - \psi_1\chi'_2]/3\psi_1$

$z''_1 - \zeta''_1 = \tfrac{1}{2}(z'_2 - \zeta'_2)$

$z'_2 + \zeta'_2 = \psi'_1 \cos\varepsilon_0 - c_1\psi'_1 \sin\varepsilon_0 - \chi'_2 = z_3 + \zeta_3$

$q'_2 = (s_1 s'_2 + s'_1 s_2 + c_1 c'_2 + c'_1 c_2 - q'_1 q_2)/q_1 = q_3$

$\theta'_2 = \psi'_2 \sin\varepsilon_0 - c_1\psi'_1 \cos\varepsilon_0 = \theta_3$.

Employing the two argument form $\alpha_A(T, t)$, where the first argument represents $\mathscr{E}_F - \mathscr{E}_0$ and the second $\mathscr{E}_D - \mathscr{E}_F$, one can derive some of the identities given in Equation (22). Since $\tilde{\alpha}_A = \alpha_A(0, T+t)$, we can write, for example, $\tilde{\varepsilon}_A = \varepsilon_A(0, T+t)$ which from geometric considerations is also $\varepsilon_A(T, t)$. Thus, relations such as $\varepsilon_A(T, t) = \varepsilon_A(0, T+t) = \varepsilon_A(T+t, 0)$, $\pi_A(T, t) = -\pi_A(T+t, -t)$, $\theta_A(T, t) = -\theta_A(T+t, -t)$, $z_A(T, t) = -\zeta_A(T+t, -t)$ and $\zeta_A(T, t) = -z_A(T+t, -t)$ yield identities given in Equation (22).

Ecliptic Motion Relative to Basic Epoch

The expressions given above are the desired coefficients for all the precession quantities. From the development it is seen that they depend upon relatively few fundamental constants: the general precession p, the obliquity of the ecliptic ε_0 and the system of planetary masses. One slight problem, however, is still present. From Newcomb's development as given in Table 3 we have values for the coefficients describing the motion of the ecliptic pole relative to 1850.0. Usually, however, the other fundamental constants are referred to the basic epoch $\mathscr{E}_0$ which may be 1900.0 for the currently employed precession quantities or J2000.0 for our revised expressions. Hence, some method must be developed to update the values of $\sin\pi^* \sin\Pi^*$ and $\sin\pi^* \cos\Pi^*$ in Equations (8) and (12) to epoch $\mathscr{E}_0$, knowing only the astronomical constants ε_0 and p at epoch $\mathscr{E}_0$. One may iterate, using the results in Equation (22) to develop expressions for the quantities relative to 1850.0 and then update them to $\mathscr{E}_0$ which is T_1 Julian centuries from 1850.0 *with the proviso* that the relevant constants in Equation (22) refer to 1850.0 and T is replaced by T_1 in Table 1.

However, since the updating is really only sensitive to the value of general precession p, we can perform the iteration only once, employing some arbitrary value p^* and express the results for the updating of $\sin\pi^* \sin\Pi^*$ and $\sin\pi^* \cos\Pi^*$ to epoch $\mathscr{E}_0$ by employing $p=p^*+\Delta p^*$, the algebraic quantity Δp^* handling any difference from p^*. In performing the updating, it is necessary to introduce fourth powers of time to conserve precision. We will use

$$\sin\tilde{\pi}^* \sin\tilde{\Pi}^* = (s_1^* + s_2^* T_1 + s_3^* T_1^2 + s_4^* T_1^3)\tau + (s_1^{*\prime} + s_2^{*\prime} T_1 + s_3^{*\prime} T_1^2)\tau^2 + (s_1^{*\prime\prime} + s_2^{*\prime\prime} T_1)\tau^3 \tag{23}$$

$$\sin\tilde{\pi}^* \cos\tilde{\Pi}^* = (c_1^* + c_2^* T_1 + c_3^* T_1^2 + c_4^* T_1^3)\tau + (c_1^{*\prime} + c_2^{*\prime} T_1 + c_3^{*\prime} T_1^2)\tau^2 + (c_1^{*\prime\prime} + c_2^{*\prime\prime} T_1)\tau^3$$

where s_1^*, $s_1^{*\prime}$, $s_1^{*\prime\prime}$ and the corresponding c^*'s are defined in Equation (12). The quantities s_j^*, $s_j^{*\prime}$ and $s_j^{*\prime\prime}$ $(j>1)$ up to third order are given in Equation (22) (interpreted relative to 1850.0) and it is found that the fourth-order terms are

$$\begin{aligned}
s_4^* &= c_1 p_1'' + c_2 p_1' + c_2' p_1 - \tfrac{1}{6} c_1 p_1^3 + s_1'(2s_1^2 + \tfrac{1}{2} c_1^2) + \tfrac{3}{2} s_1 c_1 c_1' \\
s_3^{*\prime} &= c_2' p_1 + c_1' p_1' + s_1'(\tfrac{1}{2} p_1^2 + \tfrac{1}{2} c_1^2 + 3 s_1^2) + \tfrac{5}{2} s_1 c_1 c_1' \\
s_2^{*\prime\prime} &= c_1'' p_1 + s_1(s_1 s_1' + c_1 c_1')
\end{aligned}$$

and (24)

$$\begin{aligned}
c_4^* &= s_1 p_1'' - s_2 p_1' - s_2' p_1 + \tfrac{1}{6} s_1 p_1^3 + c_1'(\tfrac{1}{2} s_1^2 + 2c_1^2) + \tfrac{3}{2} s_1 c_1 s_1' \\
c_3^{*\prime} &= -s_2' p_1 - s_1' p_1' + c_1'(\tfrac{1}{2} p_1^2 + 3c_1^2 + \tfrac{1}{2} s_1^2) + \tfrac{5}{2} s_1 c_1 s_1' \\
c_2^{*\prime\prime} &= -s_1'' p_1 + c_1(s_1 s_1' + c_1 c_1')
\end{aligned}$$

with all terms being evaluated at 1850.0. The basic quantities required for our general expressions in Equation (22) relative to $\mathscr{E}_0$ and $\mathscr{E}_F$ are then, by inspection,

$$\begin{aligned}
s_1 &= s_1^* + s_2^* T_1 + s_3^* T_1^2 + s_4^* T_1^3 \\
s_1' &= s_1^{*\prime} + s_2^{*\prime} T_1 + s_3^{*\prime} T_1^2 \\
s_1'' &= s_1^{*\prime\prime} + s_2^{*\prime\prime} T_1
\end{aligned}$$

and (25)

$$\begin{aligned}
c_1 &= c_1^* + c_2^* T_1 + c_3^* T_1^2 + c_4^* T_1^3 \\
c_1' &= c_1^{*\prime} + c_2^{*\prime} T_1 + c_3^{*\prime} T_1^2 \\
c_1'' &= c_1^{*\prime\prime} + c_2^{*\prime\prime} T_1 .
\end{aligned}$$

We will evaluate the expressions in Equations (23) to (25) for the value $p^* = 5029''.0966$ per Julian century at J2000.0 using $p = p^* + \Delta p^*$ to handle any other values of general precession. The following equivalent values of p^* may be useful for various epochs in determining the appropriate Δp^*:

$$\begin{aligned}
p^* = {} & 5029''.0966 \text{ per Julian century at J2000.0} \\
& 5027''.878 \text{ per tropical century at 1950.0} \\
& 5026''.767 \text{ per tropical century at 1900.0} \\
& 5025''.656 \text{ per tropical century at 1850.0.}
\end{aligned} \tag{26}$$

The reference value p^* has been selected to coincide with the value recently adopted by the IAU. The value $\Delta p^* = -1''.127$ is appropriate for Newcomb's general precession of $p = 5025''.64$ per tropical century at 1900.0.

We find for the basic quantities in Equation (25) relative to $\mathscr{E}_0$ the following expressions:

$$\begin{aligned}
s_1 = {} & \alpha_0 + T_1[(\alpha_1 - \alpha_{-1})/5 + 0.02436\,5588\,\beta_0 - 0.000222\,690\,\Delta p^*] \\
& + T_1^2[-0.16029\,6815\alpha_0 + 2(\alpha_1 + \alpha_{-1})/25 + 0.00487\,312(\beta_1 - \beta_{-1}) - 8.49 \times 10^{-8} \Delta p^* - 0''.0002\,5207] \\
& + T_1^3[2''.2506 \times 10^{-5}] \\
s_1' = {} & (\alpha_1 - \alpha_{-1})/10 + T_1[2(\alpha_1 - 2\alpha_0 + \alpha_{-1})/25 + 0.00243\,656(\beta_1 - \beta_{-1}) + 2.72 \times 10^{-7} \Delta p^* \\
& + 1''.39 \times 10^{-7}] + T_1^2[-3''.2512 \times 10^{-5}] \\
s_1'' = {} & 2(\alpha_1 - 2\alpha_0 + \alpha_{-1})/75 + T_1[8''.201 \times 10^{-6}]
\end{aligned}$$

and (27)

$$\begin{aligned}
c_1 = {} & \beta_0 + T_1[(\beta_1 - \beta_{-1})/5 - 0.02436\,5588\alpha_0 - 0.00002\,583\,\Delta p^*] \\
& + T_1^2[-0.16029\,6815\beta_0 + 2(\beta_1 + \beta_{-1})/25 - 0.00487\,312(\alpha_1 - \alpha_{-1}) + 3.654 \times 10^{-6} \Delta p^* - 5''.386 \times 10^{-6}] \\
& + T_1^3[-2''.473 \times 10^{-6}] \\
c_1' = {} & (\beta_1 - \beta_{-1})/10 + T_1[2(\beta_1 - 2\beta_0 + \beta_{-1})/25 - 0.00243\,656(\alpha_1 - \alpha_{-1}) - 9.37 \times 10^{-7} \Delta p^* - 1''.220 \times 10^{-6}] \\
& + T_1^2[-3''.746 \times 10^{-6}] \\
c_1'' = {} & 2(\beta_1 - 2\beta_0 + \beta_{-1})/75 + T_1[4''.655 \times 10^{-6}]
\end{aligned}$$

where all quantities are expressed in arc seconds and where $T_1 = \mathscr{E}_0 - 1850.0$, expressed in Julian centuries.

The quantities α_j and β_j are dependent upon the system of planetary masses and are defined in Equations (11), (12) and Table 3. The value Δp^* is expressed in arc seconds per century.

Table 4. Astronomical constants employed in precession calculations

Obliquity at J2000.0 (JED 2451545.0)		$\varepsilon_0 = 23°26'21''.448$
Speed of general precession in longitude at J2000.0		$p = 5029''.0966$ per Julian century
Rate of change of Newcomb's Precessional Constant[a]		$P_1 = -0''.00369$ per Julian century
Geodesic precession[a]		$p_g = 1''.92$ per Julian century
Inverse Planetary Masses M^{-1}		
	Mercury	6023600
	Venus	408523.5
	Mars	3098710
	Jupiter	1047.355
	Saturn	3498.5
	Uranus	22869
	Neptune	19314
	Pluto	3000000

[a] Note that P_1 and p_g are not part of the IAU (1976) System of Astronomical Constants, but are derived from them via Equations (13) and (15)

Table 5. Numerical expressions for precession quantities[a]

$\sin\pi_A \sin\Pi_A$	$(4''.1976 - 0''.75250\,T + 0''.000431\,T^2)t + (0''.19447 + 0''.000697\,T)t^2 - 0.000179\,t^3$
$\sin\pi_A \cos\Pi_A$	$(-46''.8150 - 0''.00117\,T + 0''.005439\,T^2)t + (0''.05059 - 0''.003712\,T)t^2 + 0''.000344\,t^3$
π_A	$(47''.0029 - 0''.06603\,T + 0''.000598\,T^2)t + (-0''.03302 + 0''.000598\,T)t^2 + 0''.000060\,t^3$
Π_A	$174°52'34''.982 + 3289''.4789\,T + 0''.60622\,T^2 + (-869''.8089 - 0''.50491\,T)t + 0''.03536\,t^2$
ε_0	$23°26'21''.448$
$\bar{\varepsilon}_A$	$\varepsilon_0 - 46''.8150\,T - 0''.00059\,T^2 + 0''.001813\,T^3$
ε_A	$\bar{\varepsilon}_A + (-46''.8150 - 0''.00117\,T + 0''.005439\,T^2)t + (-0''.00059 + 0''.005439\,T)t^2 + 0''.001813\,t^3$
ω_A	$\bar{\varepsilon}_A + (0''.05127 - 0''.009186\,T)t^2 - 0''.007726\,t^3$
ψ_A	$(5038''.7784 + 0''.49263\,T - 0''.000124\,T^2)t + (-1''.07259 - 0''.001106\,T)t^2 - 0''.001147\,t^3$
χ_A	$(10''.5526 - 1''.88623\,T + 0''.000096\,T^2)t + (-2''.38064 - 0''.000833\,T)t^2 - 0''.001125\,t^3$
$\bar{p}_A$	$(5029''.0966 + 2''.22226\,T - 0''.000042\,T^2)t + (1''.11113 - 0''.000042\,T)t^2 - 0''.000006\,t^3$
p_N	$p_A + (0''.00048 - 0''.000085\,T)t^2 - 0''.000107\,t^3$
ζ_A	$(2306''.2181 + 1''.39656\,T - 0''.000139\,T^2)t + (0''.30188 - 0''.000345\,T)t^2 + 0''.017998\,t^3$
z_A	$(2306''.2181 + 1''.39656\,T - 0''.000139\,T^2)t + (1''.09468 + 0''.000066\,T)t^2 + 0''.018203\,t^3$
θ_A	$(2004''.3109 - 0''.85330\,T - 0''.000217\,T^2)t + (-0''.42665 - 0''.000217\,T)t^2 - 0''.041833\,t^3$

[a] Basic epoch $\mathscr{E}_0$ is J2000 (JED 2451545.0). The parameters T and t are $\mathscr{E}_F - \mathscr{E}_0$ and $\mathscr{E}_D - \mathscr{E}_F$, respectively, measured in Julian centuries of 36525 days, $T = [\mathrm{JED}(\mathscr{E}_F) - \mathrm{JED}(\mathscr{E}_0)]/36525$, $t = [\mathrm{JED}(\mathscr{E}_D) - \mathrm{JED}(\mathscr{E}_F)]/36525$

We now have the means available for obtaining numerical values for the precession quantities relative to any basic epoch $\mathscr{E}_0$. As noted earlier, the fundamental constants are the obliquity of the ecliptic ε_0 and the general precession p at epoch $\mathscr{E}_0$, coupled with the system of planetary masses which yield the basic quantities s and c of Equation (27). With these values one can evaluate all of the precession quantities of Table 1, as given by the formulas in Equation (22).

Numerical Results and Summary

Employing the fundamental astronomical constants listed in Table 4 we have evaluated the expressions of Equation (22) for the precession quantities and obtain the numerical results given in Table 5 for the basic epoch $\mathscr{E}_0$ of J2000.0. All the quantities are expressed in arc seconds and time in Julian centuries. The symbol T represents $\mathscr{E}_F - \mathrm{J2000.0}$ and t represents $\mathscr{E}_D - \mathscr{E}_F$. Normally the table will be used with $T = 0$ (viz. precession to and from J2000.0), but means are available via T to adopt another epoch $\mathscr{E}_F$. Often one is required to precess from one fixed equinox at epoch $\mathscr{E}_1$ to another fixed equinox at $\mathscr{E}_2$. It should be noted that our formulation is also applicable to this case if one identifies $\mathscr{E}_1$ with $\mathscr{E}_F$ and $\mathscr{E}_2$ with $\mathscr{E}_D$ in the preceding formulation. The values in Table 4 are those adopted by the XVI General Assembly of the IAU (*Transactions IAU*, 16B, 1977). The value of p results from Newcomb's value of general precession $p = 5025''.64$ per tropical century at 1900.0, coupled with Fricke's (1967, 1971) correction to luni-solar precession $\Delta\psi = +1''.10$ per century at 1900.0 and the correction to planetary precession at 1900.0, $\Delta\chi = -0''.029$, which is due to the revised system of masses. These corrections yield, for the general precession at 1900.0, $p = \psi - \chi\cos\varepsilon_0 = 5026''.7666$ per tropical century. This value corresponds to the newly adopted value $p = 5029''.0966$ per Julian century at $\mathscr{E}_0 = \mathrm{J2000.0}$, using the conventional values of 36524.2198781 days for the length of the tropical century and JED 2415020.31352 for the epoch 1900.0. The value newly adopted for the obliquity at J2000.0 ($\varepsilon_0 = 23°26'21''.448$) corresponds to the previously adopted value at 1900.0 ($\varepsilon_0 = 23°27'8''.26$) evaluated at J2000.0 using the revised rates of change.

The expressions at epoch $\mathscr{E}_F$ for the rates per Julian century of general precession in right ascension and declination are, from Equation (6),

$$m = \frac{d}{dt}(\zeta_A + z_A)|_{t=0} = 4612''.4362 + 2''.79312T - 0''.000278T^2$$

and

$$n = \frac{d}{dt}(\theta_A)|_{t=0} = 2004''.3109 - 0''.85330T - 0''.000217T^2. \qquad (28)$$

It is often convenient, especially in constructing star catalogues, to employ a Taylor series expansion for the precession reductions

$$\alpha_D = \alpha_F + t\left(\frac{d\alpha}{dt}\right)_{t=0} + \tfrac{1}{2}t^2\left(\frac{d^2\alpha}{dt^2}\right)_{t=0} + \ldots$$

$$\delta_D = \delta_F + t\left(\frac{d\delta}{dt}\right)_{t=0} + \tfrac{1}{2}t^2\left(\frac{d^2\delta}{dt^2}\right)_{t=0} + \ldots \qquad (29)$$

where (α_F, δ_F) represent the right ascension and declination at the catalogue equinox $(\mathscr{E}_F)$ and where (α_D, δ_D) represent similar quantities at an equinox $(\mathscr{E}_D)$ which is t centuries from the catalogue equinox. Woolard and Clemence (1966, pp. 278—279) list the components of the expansion as

$$\frac{d\alpha}{dt} = m + n\sin\alpha\tan\delta$$

$$\frac{d\delta}{dt} = n\cos\alpha$$

$$\frac{d^2\alpha}{dt^2} = n^2\sin 2\alpha[\tfrac{1}{2} + \tan^2\delta] + mn\tan\delta\cos\alpha + \frac{dm}{dt} + \frac{dn}{dt}\tan\delta\sin\alpha$$

$$\frac{d^2\delta}{dt^2} = -n^2\sin^2\alpha\tan\delta - mn\sin\alpha + \frac{dn}{dt}\cos\alpha. \qquad (30)$$

If we denote $\zeta_A + z_A$ by m_A and if we denote θ_A by n_A, then the series can be written in our notation of Equation (2) as

$$m_A = mt + \tfrac{1}{2}t^2\frac{dm}{dt} + \tfrac{1}{6}t^3\frac{d^2m}{dt^2} = mt + m't^2 + m''t^3$$

$$n_A = nt + \tfrac{1}{2}t^2\frac{dn}{dt} + \tfrac{1}{6}t^3\frac{d^2n}{dt^2} = nt + n't^2 + n''t^3 \qquad (31)$$

where

$$m = m_1 + m_2T + m_3T^2 = (\zeta_1 + z_1) + (\zeta_2 + z_2)T + (\zeta_3 + z_3)T^2$$

$$n = n_1 + n_2T + n_3T^2 = \theta_1 + \theta_2T + \theta_3T^2.$$

From Equation (31) one can readily deduce the values of dm/dt and dn/dt required in Equation (30) as twice the coefficient of t^2 in m_A and n_A. Because of the relationships among the coefficients given in Equation (22), it can readily be shown that the expressions for dm/dt and dn/dt can also be obtained from the expressions for m and n of Equation (28) as

$$\frac{dm}{dt} = \frac{dm}{dT}$$

$$\frac{dn}{dt} = \frac{dn}{dT} \qquad (32)$$

and it may be noted that the values given in the FK4 (Fricke and Kopff, 1963) were obtained in this manner.

In comparing our results with other expressions, the reader is cautioned to remember that our times are measured in Julian centuries and one must be careful to clearly distinguish the reference planes for the various angles. For example, the expression given in the *Explanatory Supplement* (p. 38) lists

$$\Pi = 173°57'.06 + 54'.77\,T_0$$

and it should be noted that the Π given there is measured relative to the *mean equinox of date*, with T_0 in tropical centuries from 1900.0. Our expression

$$\Pi_A = 174°52'34''.982 + 3289''.4789T + 0''.60622T^2 + (-869''.8089 - 0''.50491T)t + 0''.03536t^2$$

is measured relative to the *fixed equinox* $\mathscr{E}_F$.

The value of Π given there corresponds to our value $\Pi_A(T=\tau, t=0)$, since in general $\alpha_A(T=\tau, t=0)$ yields an angle measured relative to the fundamental plane of date, while $\alpha_A(T=0, t=\tau)$ or $\tilde{\alpha}_A$ yields the angle measured relative to the fundamental plane of epoch.

The expressions presented here are intended to be used with the introduction of the revised system of astronomical constants upon which they are based. The expressions and their derivation serve to thoroughly document the dependence of the precession quantities on the astronomical constants. It is hoped that the formulation will also provide suitable means for the revision of the precession quantities if in the future any changes in the system of constants will become necessary.

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