THE ASTRONOMICAL JOURNAL VOLUME 72, NUMBER 2 MARCH 1967

Astronomical Refraction in a Polytropic Atmosphere

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(Received 29 November 1966)

The author's theory of astronomical refraction, based on a polytropic atmospheric model, is adapted here to automatic computers. An improved algorithm, involving a double power series in a small parameter, provides greater accuracy and faster convergence. The formulas are valid for all values of the zenith distance and the observer's elevation, extending to the most extreme cases of refraction below the horizon.

I. INTRODUCTION

A. The Scope of the Paper

THE author's previously published paper on astronomical refraction (1944) is based on a polytropic model of the atmosphere. After a lapse of 20 yr, this choice of a physical model still seems to be the "best compromise between accuracy and simplicity." The inherent simplicity of the temperature profile that is piecewise linear in the geopotential permits a mathematical representation of the refraction in terms of the incomplete beta and gamma functions. That such a model can be accurately fitted to meteorological data has been recognized by the National Committee for Aeronautics (NACA), and by its successor, the International Civil Aviation Organization (ICAO). The *U. S. Standard Atmosphere* (1962) formally adopts a piecewise polytropic distribution, composed of several spherical shells. For the calculation of refraction, two such shells suffice, with the second shell extended to infinity, as illustrated in Table I. The unit of length here is the *geopotential* kilometer; the value $h_1=11$ km corresponds to the so-called *tropopause.*

At the time the author's paper was published, electronic computers were not available, and there was little need for values of refraction at *zenith distances* exceeding 93°. With the advent of electronic machines and artificial satellites, the situation has drastically changed. To meet the changing needs, the following modifications of the 1944 theory are made in this paper:

(1) greater accuracy is achieved by the retention of the previously rejected second-order terms;

(2) the algorithm is based on a double power series, which converges rapidly for all values of the polytropic index $n\geq 1$;

(3) the refraction tables are replaced by a Fortran routine.

The routine is extremely flexible. The input consists of zenith distance z, pressure p, temperature T, altitude h above sea level, the temperature gradient T', and five geophysical constants, including the refractive index μ. Thus, the user of the routine is not restricted to the *U. S. Standard Atmosphere*, but may adjust the value of T' for seasonal and latitude variations (see *Supplements to the U. S. Standard*). The humidity correction of Sec. VE is optional; it can be easily incorporated into the routine if desired.

The new theory is valid in the entire domain $0\leq z\leq 180°$, $0\leq h<\infty$ of the arguments z and h, and satisfies the most crucial internal checks with an accuracy of better than 1″. The paper is almost entirely self-contained. Unnecessary duplication is avoided by references to the original 1944 paper, marked [1].

B. Physical Assumptions

Our atmospheric model is defined by the following physical assumptions:

(1) The Fermat principle of geometrical optics.

(2) A spherically symmetric distribution of density $\rho(r)$, where r is the distance from the center of the earth.

(3) The Gladstone–Dale relation

$$\mu-1=c\rho, \tag{1}$$

where c is an atomic constant depending only on the substance and the wavelength of light. This approximate form of the Clausius–Mossotti equation, $(\mu^2-1)/(\mu^2+2)=\frac{2}{3}c\rho$, is sufficiently accurate for our purpose, as noted in Sec. IIG.

(4) A perfect gas in a state of hydrostatic equilibrium in a Newtonian gravitational field.

(5) A polytropic distribution with index $n(r)$ a step-function, characterized as follows:

$$n=n_1<\infty,\quad a\leq r\leq r_1,$$
$$n=\infty,\qquad r_1\leq r<\infty.$$

Here a is the radius of the earth. In the *U. S. Standard Atmosphere*, $n_1=4.256$ and $r_1-a\equiv h_1=11$ km. The *simple polytrope*, with $n=$const, is treated in Secs. II and III of this paper in the cases $n<\infty$ and $n=\infty$, respectively. Section IV treats the *compound polytrope* of assumption (5), and Sec. V applies the result to the earth.

TABLE I. U. S. standard atmosphere (1962).

Height (km)	Temperature gradient T' (°C/km)	Polytropic index n	Remarks
$0\leq h\leq 11$	-6.5	4.256	troposphere
$11<h<\infty$	0	∞	stratosphere

C. General Theory

Under assumptions (1) and (2), the Euler differential equation of the variational calculus for a light ray has for its first integral

$$\mu r \sin\psi = \text{const}, \tag{2}$$

where ψ is the angle between the ray and the radius vector. The second integral is the well-known expression for the *astronomical refraction* R,

$$R=\int_1^{\mu_0}\left[\left(\frac{\mu r \csc z}{\mu_0 r_0}\right)^2-1\right]^{-\frac{1}{2}} d\log\mu. \tag{3}$$

Here z is the *apparent zenith distance;* the subscript 0, here and elsewhere, denotes the values at the observer's station.

Let the quantities α, u, s, and ω be defined by

$$\begin{aligned} \alpha &\equiv \mu_0-1,\\ 1-u &\equiv \mu_0 r_0/\mu r,\\ s &\equiv 1-r_0/r,\\ \omega &\equiv 1-\rho/\rho_0. \end{aligned} \tag{4}$$

Then assumption (3) leads to

$$\begin{aligned} \mu &= 1+\alpha(1-\omega),\\ d\log\mu &= -\alpha[1-\alpha(1-\omega)]d\omega+O(\alpha^3),\\ \mu_0/\mu &= 1+\alpha\omega+O(\alpha^2), \end{aligned} \tag{5}$$

where the small quantity α is of the order 3×10^{-4} under standard conditions. With the aid of (5.3), the expression for $1-u$ becomes

$$1-u=(1-s)(1+\alpha\omega)+O(\alpha^2), \tag{6}$$

and the substitution into (3) yields for the refraction integral

$$R=\alpha\int_0^1(1-u)(1-\alpha+\alpha\omega)\Delta^{-1}d\omega+O(\alpha^3), \tag{7}$$

$$\Delta=\cot^2 z+2u-u^2.$$

For a polytropic distribution this integral assumes special forms, exhibited in Secs. II and III.

II. SIMPLE POLYTROPE, $n<\infty$

A. Refraction Integral for $n<\infty$

Consider a polytrope for which $1\leq n<\infty$ and which extends to its natural boundary, where the absolute temperature decreases to zero. As shown in [1], assumptions (4) and (5) for such a polytrope lead to

$$\rho/\rho_0=(T/T_0)^n=[1-g_0h'/(n+1)\mathcal{R}T_0]^n. \tag{8}$$

Here T is the absolute temperature, g the acceleration of gravity, $\mathcal{R}$ the gas constant for air, and h' the *geopotential height* measured from the observer's station. The latter is related to the actual height h by means of

$$(1+h/r_0)(1-h'/r_0)=1, \tag{9}$$

which leads to

$$s=1-r_0/r=h'/r_0. \tag{10}$$

Let x and γ be defined by

$$\begin{aligned} x &\equiv T/T_0,\\ \gamma &\equiv [r_0g_0/2(n+1)\mathcal{R}T_0]^{\frac{1}{2}}. \end{aligned} \tag{11}$$

Then (4.4), (8), and (10) lead to

$$\begin{aligned} \omega &= 1-x^n,\\ s &= \tfrac{1}{2}\gamma^{-2}(1-x). \end{aligned} \tag{12}$$

Let us further define quantities β, β_1, and β_2 by

$$\begin{aligned} \beta &\equiv 2\alpha\gamma^2,\\ \beta_1 &\equiv \beta-\alpha,\\ \beta_2 &\equiv \tfrac{1}{4}\gamma^{-2}(1-\beta_1). \end{aligned} \tag{13}$$

Since $\alpha\sim3\times10^{-4}$ and $\gamma\sim8$ for the earth, it follows that γ^{-2} is a small quantity of $O(\alpha^{\frac{1}{2}})$, and that the β's are of the same order. With $\alpha^2\sim10^{-7}\sim0''\!.02$, we would be justified in rejecting terms of $O(\alpha^2)$ in our developments. However, because of the singularity in the integrand of (7) at $\Delta=0$ and the occurrence of large exponents in various binomial expansions, such terms may acquire large coefficients. In order to achieve an accuracy of 1 sec of arc, we observe the precaution of retaining terms of $O(\alpha)$ in the integrand, which is already accompanied by a factor α outside the integral sign. In some of our developments we exceed this requirement by retaining terms of a higher order. This is done for the purpose of reducing the size of the corrections that compensate for the various approximations adopted in the course of this work.

Accordingly, in view of (6) and (12), we write u in the form

$$\begin{aligned} u &= \tfrac{1}{2}\gamma^{-2}(1-x)-\alpha(1-x^n)[1-\tfrac{1}{2}\gamma^{-2}(1-x)]\\ &= 2\beta_2[1-x-\beta_1(1-\beta_1)^{-1}(x-x^n)-\alpha(x-x^{n+1})]. \end{aligned} \tag{14}$$

With x as the independent variable, the substitution from (12.1) converts (7) into

$$R=\alpha n\int_0^1(1-u)(1-\alpha+\alpha\omega)\Delta^{-1}x^{n-1}dx, \tag{15}$$

where the functions $u(x)$, $\omega(x)$, and $\Delta(x)$ are defined by (14), (12.1), and (7.2). Since the polytrope extends to a finite height, corresponding to $x=0$, $u=\frac{1}{2}\gamma^{-2}=O(\alpha^{\frac{1}{2}})$, we approximate (15) by

$$R=\alpha n\int_0^1\Delta^{-1}x^{n-1}dx. \tag{16}$$

The resulting error will be removed by the corrections δ_1 and δ_2 of Sec. IIG.

B. Change of Variable

Consider three successive transformations $x\to\xi_1$, $\xi_1\to\xi_2$, and $\xi_2\to\xi$ defined by

$$\begin{aligned}\xi_1&=x+\beta_1(1-\beta_1)^{-1}(x-x^n),\\ \xi_2&=\xi_1+\alpha(\xi_1-\xi_1{}^{n+1}),\\ \xi&=\xi_2-\beta_2(1-\beta_2)^{-1}(\xi_2-\xi_2{}^2).\end{aligned}\tag{17}$$

Then the expressions u and $2u-u^2$, appearing in (14) and (9), can be written to $O(\alpha)$ as

$$\begin{aligned}u&=2\beta_2(1-\xi_2),\\ 2u-u^2&=4\beta_2(1-\beta_2)(1-\xi).\end{aligned}\tag{18}$$

Let us next define Γ and a transform θ of z by

$$\begin{aligned}\Gamma&\equiv[4\beta_2(1-\beta_2)]^{-\frac{1}{2}}\\ &=\gamma[(1-\beta_1)(1-\beta_2)]^{-\frac{1}{2}},\\ \cot\theta&\equiv\Gamma\cot z.\end{aligned}\tag{19}$$

Then Δ^2 in (7) becomes

$$\Delta^2=\Gamma^{-2}(\csc^2\theta-\xi),\tag{20}$$

and the integral (16) takes the form

$$R=\alpha\Gamma n\int_0^1(\csc^2\theta-\xi)^{-\frac{1}{2}}x^{n-1}dx,\tag{21}$$

where x is an implicit function of ξ defined by (17).

C. Expansion into Series

If x is an implicit function of y defined by

$$x=y+\epsilon\varphi(x),\tag{22}$$

where ϵ is a constant, and $\varphi(x)$ and $f(x)$ are differentiable, then by a theorem of Lagrange

$$f(x)=\sum_0^\infty\frac{\epsilon^i}{i!}D^{i-1}[\varphi^i(y)f'(y)],\tag{23}$$

where $D\equiv d/dy$ (Whittaker and Watson 1952). Therefore,

$$f'(x)dx=\sum_0^\infty\frac{\epsilon^i}{i!}D^i[\varphi^i(y)f'(y)]dy.\tag{24}$$

In particular, if f and φ are of the form

$$\begin{aligned}f(x)&=x^p,\\ \varphi(x)&=x^q-x,\end{aligned}\tag{25}$$

we obtain with the aid of the binomial theorem,

$$x^{p-1}dx=\sum_{i=0}^\infty\sum_{j=0}^i\epsilon^i(-1)^k\frac{(m-1+i)!}{j!k!(m-1)!}y^{m-1}dy,\tag{26}$$

where m and k are defined by (cf. [1]),

$$\begin{aligned}m&\equiv p+j(q-1),\\ k&\equiv i-j.\end{aligned}\tag{27}$$

If we introduce the parameter κ defined by

$$\epsilon=\kappa(1-\kappa)^{-1},\tag{28}$$

and change the order of summation in (26), there follows

$$\begin{aligned}x^{p-1}dx&=\sum_{j=0}^\infty\epsilon^j\frac{(m-1+j)!}{j!(m-1)!}y^{m-1}\\ &\qquad\times\sum_{k=0}^\infty(-\epsilon)^k\frac{(m-1+j+k)!}{k!(m-1+j)!}dy\\ &=\sum_{j=0}^\infty\epsilon^j(1+\epsilon)^{-m-j}\binom{m-1+j}{j}y^{m-1}dy\\ &=\sum_{i=0}^\infty\kappa^i(1-\kappa)^m\binom{m-1+i}{i}y^{m-1}dy,\end{aligned}\tag{29}$$

with

$$m=p+i(q-1).\tag{30}$$

The result (29) is now applied to the three transformations of (17). In the first transformation,

$$x=x,\quad y=\xi_1,\quad p=n,\quad q=n,\quad \epsilon=\beta_1(1-\beta_1)^{-1},\tag{31}$$

so that

$$\begin{aligned}\kappa&=\beta_1,\\ m&=n+i(n-1)\equiv m_1.\end{aligned}\tag{32}$$

In the second transformation,

$$x=\xi_1,\quad y=\xi_2,\quad p=m_1,\quad q=n+1,\quad \epsilon=\alpha,\tag{33}$$

so that

$$\begin{aligned}\kappa&\sim\alpha,\\ m&=m_1+jn\equiv m_2.\end{aligned}\tag{34}$$

In the third transformation,

$$x=\xi_2,\quad y=\xi,\quad p=m_2,\quad q=2,\quad \epsilon=-\beta_2(1-\beta_2)^{-1},\tag{35}$$

and

$$\begin{aligned}\kappa&=-\beta_2(1-2\beta_2)^{-1},\\ 1-\kappa&=(1-\beta_2)(1-2\beta_2)^{-1},\\ m&=m_2+k.\end{aligned}\tag{36}$$

For the product $x\rightarrow\xi$ of the three transformations,

$$x^{n-1}dx=\sum_i\sum_j\sum_k \beta_1{}^i(1-\beta_1)^{m_1}\alpha^j(1-\alpha)^{m_2}(-\beta_2)^k(1-\beta_2)^m(1-2\beta_2)^{-k-m}M_{ijk}\xi^{m-1}d\xi,$$

$$m_1=n+i(n-1),\quad m_2=m_1+jn,\quad m=m_2+k, \tag{37}$$

$$M_{ijk}\equiv\binom{m_1-1+i}{i}\binom{m_2-1+j}{j}\binom{m-1+k}{k}.$$

Before we substitute this result into (20), we reduce it to a double power series, making use of the smallness of α.

D. Reduction to a Double Power Series

Since the coefficient of the integral in (21) is about 2500″, and since $\alpha\sim 3\times10^{-4}$, we retain in (37) only terms with $j=0$. The resulting error is removed by means of the correction δ_3 in Sec. IIG. We note that for $j=0$,

$$j=0;\quad m_2=m_1,\quad m=m_1+k. \tag{38}$$

In view of (13), the product $(1-\beta_1)(1-\alpha)$, which then occurs in (37) as a natural combination, can be approximated by

$$(1-\beta_1)(1-\alpha)=1-\beta+O(\alpha^{\frac{3}{2}}). \tag{39}$$

Hence (37) takes the form

$$x^{n-1}dx=\sum_i\sum_j \beta_1{}^i(1-\beta)^{m_1}(-\beta_2)^j(1-\beta_2)^m(1-2\beta_2)^{-j-m}M_{ij}\xi^{m-1}d\xi, \tag{40}$$

with

$$m_1=n+i(n-1),\quad m=m_1+j,$$
$$M_{ij}=\binom{m_1-1+i}{i}\binom{m-1+j}{j}. \tag{41}$$

In order to simplify the final result we introduce the following parameters:

$$C\equiv(1-\beta)(1-\beta_2)(1-2\beta_2)^{-1},$$
$$K\equiv 2\alpha\Gamma nC^n, \tag{42}$$
$$b_1\equiv\beta_1C^{n-1},$$
$$b_2\equiv-\beta_2(1-\beta_2)(1-2\beta_2)^{-2}.$$

Then (21) takes the form

$$R=K\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}b_1{}^ib_2{}^jM_{ij}\psi_m(\theta), \tag{43}$$

where $\psi_m(\theta)$ is defined by

$$\psi_m(\theta)=\frac{1}{2}\int_0^1(\csc^2\theta-\xi)^{-\frac{1}{2}}\xi^{m-1}|d\xi|. \tag{44}$$

E. Incomplete Beta Function

If $z>\frac{1}{2}\pi$, the variable ξ is not a monotonic function of the arc length. Indeed,

$$\begin{aligned}\max\xi&=1 &&\text{if}\quad z\leq\tfrac{1}{2}\pi\\ &=\csc^2\theta &&\text{if}\quad z>\tfrac{1}{2}\pi.\end{aligned} \tag{45}$$

It is therefore convenient to introduce a uniformizing variable λ defined by means of the substitution $\xi=\csc^2\theta\,\sin^2\lambda$. Then the integral (44) is transformed into

$$\psi_m(\theta)=\csc^{2m-1}\theta\int_0^\theta\sin^{2m-1}\lambda d\lambda. \tag{46}$$

For refraction below the horizon, where $\theta>\frac{1}{2}\pi$, and for an observer within the troposphere it may be convenient to use the *associate functions* $\bar\psi$ and ψ^*, defined by

$$\bar\psi_m(\theta)=\int_0^\theta\sin^{2m-1}\lambda d\lambda=\tfrac{1}{2}B(m,\tfrac{1}{2};\sin^2\theta),$$
$$\psi_m{}^*(\theta)=2\int_{\frac{1}{2}\pi}^\theta\sin^{2m-1}\lambda d\lambda=2\bar\psi_m(\theta)-2\bar\psi_m(\tfrac{1}{2}\pi). \tag{47}$$

Here B denotes the incomplete beta function. The following properties of the functions ψ, $\bar\psi$, and ψ^* are noted below:

$$\begin{aligned}
&(1)\quad \bar\psi_m(\theta)=\sin^{2m-1}\theta\,\psi_m(\theta),\\
&(2)\quad \bar\psi_m(\pi-\theta)=2\psi_m(\tfrac{1}{2}\pi)-\bar\psi_m(\theta),\\
&(3)\quad \psi_m{}^*(\pi-\theta)=2\psi_m(\tfrac{1}{2}\pi)-2\bar\psi_m(\theta),\\
&(4)\quad \psi_m(0)=0,\quad \psi_m(\pi)=\infty,\\
&(5)\quad \bar\psi_m(0)=0,\quad \bar\psi_m(\pi)=2\psi_m(\tfrac{1}{2}\pi),\\
&(6)\quad \psi_m{}^*(\tfrac{1}{2}\pi)=0,\quad \psi_m{}^*(\pi)=2\psi_m(\tfrac{1}{2}\pi),\\
&(7)\quad \psi_m(\tfrac{1}{2}\pi)=\bar\psi_m(\tfrac{1}{2}\pi)=\tfrac{1}{2}B(m,\tfrac{1}{2}),\\
&(8)\quad \psi_{m+1}(\theta)=(2m+1)^{-1}[-\cot\theta+2m\csc^2\theta\,\psi_m(\theta)].
\end{aligned} \tag{48}$$

Incidentally, the recursion formula (8) corrects a misprint on p. 172 of [1]. A convenient series for the calculation of ψ_m, also derived in [1], is

$$\psi_m(\theta)=\frac{1}{m}\sum_{s=0}^{\infty}\prod_{r=0}^{s}\frac{m-r}{m+r}\tan^{2s+1}\frac{\theta}{2}, \tag{49}$$

which converges for $\theta\leq\frac{1}{2}\pi$. For noninteger m, the series alternates for $r>m$; faster convergence near $\theta=\frac{1}{2}\pi$ is obtained by summing pairs of consecutive terms.

F. Weather Factors

The parameters K, b_1, b_2, and Γ, constant along the ray, are functions of the two independent parameters α and γ. The latter depend on the absolute temperature T_0, the barometric pressure p_0, and the elevation h_0 of the observer's station above the sea level. Let us choose T_{00}, p_{00}, and $h_0=0$ as the set of *standard conditions*, and denote the corresponding values of the various parameters by the symbol 00. Furthermore, let us measure T and p in terms of T_{00} and p_{00} as units.

Analogously to (4.3) we define S and S_0 by

$$S=1-a/r,\quad S_0=1-a/r_0=h_0/r_0=h_0'/a,\quad r_0=a+h_0. \tag{50}$$

Clearly, $S\geq s$, with

$$\begin{aligned} s&<S \quad\text{if}\quad h_0>0,\\ s&=S \quad\text{if}\quad h_0=0,\\ 0&\leq S\leq\tfrac{1}{2}\gamma^{-2}. \end{aligned} \tag{51}$$

The last line is a consequence of (12). We are therefore justified in treating S as a small quantity of $O(\alpha^{\frac{1}{2}})$.

In (43), R has been expressed in terms of the parameters K, b_1, and b_2. We now trace the effect on these parameters that arises from a departure of T_0, p_0, and S_0 from the standard conditions,

$$T_0=1,\quad p_0=1,\quad S_0=0. \tag{52}$$

For this purpose it is convenient to introduce the expressions V_1, V_2, W_1, and W_2 defined as follows:

$$\begin{aligned} V_1&=p_0T_0^{-n-1}=v_1,\\ V_2&=[(1-S_0)(1-\beta)(1-2\beta_2)(1-\beta_1)^{-1}]^{\frac{1}{2}},\\ W_1&=[CT_0]^{\frac{1}{2}}=[T_0(1-\beta)(1-\beta_2)(1-2\beta_2)^{-1}]^{\frac{1}{2}},\\ W_2&=(1-2\beta_2)^{-1}. \end{aligned} \tag{53}$$

We define the *weather factors* v_1, v_2, w_1, and w_2 by

$$v_i\equiv V_i/V_i^{00},\quad w_i\equiv W_i/W_i^{00},\quad i=1,2. \tag{54}$$

Our notation indicates that V_i^{00} and W_i^{00} are to be obtained from (53) by the replacement of the β's by their standard values β^{00}, and the substitution (52). Clearly, when (52) holds all the weather factors reduce to unity.

From (1), (4.1), and the perfect gas law we deduce

$$\alpha=\alpha_{00}\rho_0=\alpha_{00}p_0T_0^{-1}=\alpha_{00}v_1T_0^{n}. \tag{55}$$

With the aid of (51) and (11), it then follows that

$$\begin{aligned} r_0&=a(1-S_0)^{-1},\\ g_0&=g_{00}(1-S_0)^2,\\ \gamma&=\gamma_{00}(1-S_0)^{\frac{1}{2}}T_0^{-\frac{1}{2}}. \end{aligned} \tag{56}$$

The definitions (19), (42), (53), and (54) now yield

$$\begin{aligned} \Gamma&=\gamma_{00}V_2W_2/W_1=\Gamma_{00}v_2w_2/w_1,\\ C&=W_1^2T_0^{-1}=C_{00}w_1^2T_0^{-1},\\ K&=2\alpha\Gamma nC^n=K_{00}v_1v_2w_1^{2n-1}w_2. \end{aligned} \tag{57}$$

A convenient expression for v_2 follows from (56.3). Noting from (13) that

$$\begin{aligned} 1-\beta_1&=(1-\beta)(1-\alpha)^{-1}+O(\alpha^2),\\ 2\beta_2&=\tfrac{1}{2}\gamma^{-2}-\alpha+\tfrac{1}{2}\alpha\gamma^{-2}, \end{aligned} \tag{58}$$

we deduce with the aid of (53),

$$\begin{aligned} V_2^2&=(1-S_0)(1-\alpha)(1-2\beta_2)+O(\alpha^2)\\ &\sim(1-S_0)(1-\tfrac{1}{2}\gamma^{-2}),\\ v_2&=\{[2\gamma_{00}^2(1-S_0)-T_0](2\gamma_{00}^2-1)^{-1}\}^{\frac{1}{2}}+O(\alpha^2). \end{aligned} \tag{59}$$

For the calculation of the w's and the b's we need the following relations:

$$\begin{aligned} \beta&=2\alpha\gamma^2=\beta_{00}T_0^{n-1}v_1(1-S_0),\\ \beta_1&=(2\gamma^2-1)\alpha=\beta_1^{00}T_0^{n-1}v_1v_2^2,\\ \beta_2&\sim\tfrac{1}{4}\Gamma^{-2}=\beta_2^{00}(w_1/w_2v_2)^2, \end{aligned} \tag{60}$$

where the last line follows from (19.1).

Since $\beta_2\sim3\times10^{-3}$, an approximate value of w_1^2 and accurate values of $1-\beta_2$ and $1-2\beta_2$ are

$$\begin{aligned} w_1^2&\sim F\equiv[T_0-\beta_{00}T_0^{n}v_1(1-S_0)](1-\beta_{00})^{-1},\\ 1-\beta_2&=1-\beta_2^{00}F,\\ 1-2\beta_2&=1-2\beta_2^{00}F, \end{aligned} \tag{61}$$

where F is defined by (61.1). Hence the w's of (54) take the form

$$\begin{aligned} w_2&=(1-2\beta_2^{00})(1-2\beta_2^{00}F)^{-1},\\ w_1&=w_2^{\frac{1}{2}}F^{\frac{1}{2}}, \end{aligned} \tag{62}$$

if we reject terms of order $\beta_2^2\sim10^{-5}$. The b's, in view of (42), (19), (53.4), and (60.2), can be written as

$$\begin{aligned} b_1&=\beta_1C^{n-1}=b_1^{00}v_1v_2^2w_1^{2n-2},\\ b_2&=-(2\Gamma/W_2)^{-2}=b_2^{00}v_2^{-2}w_1^2. \end{aligned} \tag{63}$$

All the parameters appearing in (43) have now been expressed in terms of absolute constants and the initial conditions (T_0,p_0,S_0). The weather factors can be calculated by (53.1), (59.2), and (62). The substitution from (63) and (57) into (43) and (19) now yields the basic refraction formula

$$R=K_{00}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_1^i b_2^j M_{ij}\psi_m(\theta)w_1^{2m-1}v_1^{i+1}v_2^{2i-2j+1}, \tag{64}$$

$$\cot\theta=\Gamma_{00}w_1^{-1}w_2v_2\cot z,$$

where we have dropped the superscript 00 belonging to the b's in order to save writing. A factor w_2 has been omitted in (64.1) for the reason that is explained in Sec. IIG, where the corresponding correction δ_4 is evaluated.

G. Corrections

Three approximations have been introduced in the derivation of the basic refraction formula. The compensating corrections δ_i for $i=1$, 2, 3 are evaluated below:

(1) *The omission of the factor* $1-u$ *in* (16). The correction δ_1 is obtained by the multiplication of (40) by $-u$, followed by the substitution of the result into (21), and setting $i=j=0$. In view of (18),

$$u=2\beta_2(1-\xi_2)\sim 2\beta_2(1-\xi); \tag{65}$$

hence

$$\delta_1=-\beta_2K\int_0^1(\csc^2\theta-\xi)^{\frac{1}{2}}(\xi^{n-1}-\xi^n)d\xi. \tag{66}$$

With the aid of (44), (43), and the weather factors of Sec. IIF, we find that

$$\delta_1=-2\beta_2^{00}K_{00}Fv_1w_1^{2n-1}(\psi_n-\psi_{n+1}). \tag{67}$$

(2) *The omission of the factor* $1-\alpha+\alpha\omega$ *in* (16). Proceeding as in (1), and approximating ω by $\omega=1-x^n\sim 1-\xi^n$, we find that the contribution to (40) is $-\alpha\xi^{2n-1}d\xi$. In view of (55) and (64),

$$\delta_2=-\alpha_{00}K_{00}T_0^nv_1^2w_1^{2n-1}\psi_{2n}. \tag{68}$$

(3) *The omission of the terms of* (37) *with* $j\geq 1$. The dominant terms of δ_3 correspond to $_4=1$ and $i=k=0$. Then

$$m_1=n,\quad m_2=m=2n,\quad M=2n, \tag{69}$$

and the contribution to (40) is $2n\alpha\xi^{2n-1}d\xi$. By a comparison with (2) we find

$$\delta_3=-2n\delta_2. \tag{70}$$

Since δ_3 may amount to several seconds, we reduce the size of the total correction by the following device. First, we remove the factor w_2 in (57.3), reducing the latter equation to

$$K=K_{00}v_1v_2w_1^{2n-1}. \tag{71}$$

Second, we redefine C_{00} by means of

$$C_{00}=(1-\beta_1^{00})(1-\beta_2^{00})(1-2\beta_2^{00})^{-1}. \tag{72}$$

The latter expression is derived from the original C_{00} of (53.3) by the substitution $\beta^{00}\rightarrow\beta_1^{00}$. The compensating corrections δ_4 and δ_5 are now calculated.

(4) *The omission of the factor* w_2 *in* (71). From (62.1) we have the approximation

$$w_2=1+2\beta_2^{00}(F-1). \tag{73}$$

The multiplication of (43) by $2\beta_2^{00}(F-1)$ and the substitution $i=j=0$ leads to

$$\delta_4=2\beta_2^{00}K_{00}(F-1)v_1w_1^{2n-1}\psi_n. \tag{74}$$

(5) *The replacement of* β *by* β_1 *in* C_{00}. In view of (43), (42.2), and (39), the effect of this replacement is to multiply R by the expression

$$[(1-\beta_1^{00})/(1-\beta_{00})]^n\sim(1+\alpha_{00})^n=1+n\alpha_{00}+\cdots. \tag{75}$$

Therefore, δ_5 is obtained by the multiplication of (43) by $-n\alpha_{00}$, followed by the substitution $i=j=0$. The result is

$$\delta_5=-n\alpha_{00}K_{00}v_1w_1^{2n-1}\psi_n. \tag{76}$$

The total correction can now be written

$$\delta R=\delta_1+\delta_2+\delta_3+\delta_4+\delta_5=K_{00}v_1\{2\beta_2^{00}(w_1^{2n+1}\psi_{n+1}-w_1^{2n-1}\psi_n)+\alpha_{00}[(2n-1)v_1w_1^{4n-1}\psi_{2n}-nw_1^{2n-1}\psi_n]\}, \tag{77}$$

where we have used the approximations

$$F\sim T_0\sim w_1^2. \tag{78}$$

Since $\delta R=-0''.45$ for $z=90°$ under standard conditions, and is reduced to $-0''.32$ if the exact Clausius–Mossotti equation is used in place of (3), we do not include this correction in the algorithm of Sec. VB.

H. Refraction below the Horizon

As shown in [1], functions $\psi_m(\theta)$ are defined on the interval $0\leq\theta<\pi$, and can be used to calculate the refraction of celestial objects below the horizon, with $\frac{1}{2}\pi<z\leq\pi$. The associate function $\bar{\psi}$ of (47) is most suitable for this purpose, inasmuch as it is bounded at $\theta=\pi$. If we define $\bar{w}_1$ by

$$\bar{w}_1=w_1\csc\theta, \tag{79}$$

then (48.1) implies that

$$w_1^{2m-1}\psi_m=\bar{w}_1^{2m-1}\bar{\psi}_m. \tag{80}$$

The calculation of $\bar{\psi}_m$ can be based on (48.1), (48.2), and (49). To calculate $\bar{w}_1$ it is convenient to use the formula

$$\bar{w}_1=[w_1^2+(\Gamma_{00}v_2w_2\cot z)^2]^{\frac{1}{2}}, \tag{81}$$

derived from (79) and (64.2). By the substitution (80), R of (64) can be put in the form

$$R=K_{00}\sum_i\sum_j b_1{}^ib_2{}^jM_{ij}\bar{\psi}_m(\theta)\bar{w}_1{}^{2m-1}v_1{}^{i+1}v_2{}^{2i-2j+1}, \tag{82}$$

$$\sin\theta=w_1/\bar{w}_1,$$

most suitable for $\theta\geq\frac{1}{2}\pi$.

I. Internal Check

We define the *grazing ray* as one that is tangent to the circle $r=a$. Consider two observers, O at sea-level and O$'$ at altitude, both observers viewing the same grazing ray. Let z' be the zenith distance measured at O$'$, and let u_0 correspond to location O$'$ as *measured* from O. We assert that

$$\sin z'=1-u_0. \tag{83}$$

The proof is based on (2), which can be written as

$$\sin\psi=(1-u)\sin z, \tag{84}$$

if we make use of (4.2). We now apply this result to our situation by noting that $z=\frac{1}{2}\pi$ and that z' equals ψ or $\pi-\psi$ according as the observer O$'$ looks up or down. In either case (83) follows immediately. The value R_* that belongs to the grazing ray at a given point is called the *apparent horizontal refraction*. The two values R and $\bar{R}$ of refraction corresponding to a pair of values z and $\pi-z$ of the zenith distance for the same p_0, T_0, and S_0 are said to be *supplementary*. Because of the symmetry of the ray with respect to the *line of apsides*,

$$R_*'+\bar{R}_*'=2R_*, \tag{85}$$

where primed quantities refer to O$'$, and the unprimed to O. This relation provides a crucial test of any refraction theory; we proceed to apply this test to our basic formulas (64) and (82).

Generally, R is a function of z and the weather factors v_1, v_2, w_1, and w_2. We first show that v_1 and v_2 characterize the atmosphere as a whole, and are independent of the observer.

Indeed, for polytrope [1] we have the relation

$$p'/p=(T'/T)^{n+1}.$$

The conclusion

$$v_1'=v_1 \tag{86}$$

now follows from the definition (53.1). To show that $v_2'=v_2$, we put $r_0=a$, $s=S$, and $g_0=g_{00}$ into (10), (11.2), and (8), obtaining

$$\begin{aligned}S&=h'/a,\\ \gamma&=\gamma_{00}T_0{}^{-\frac{1}{2}},\\ T&=T_0-2\gamma_{00}{}^2S.\end{aligned} \tag{87}$$

For our two observers, this temperature profile yields

$$\begin{aligned}T&=T_0, &&\text{if}\quad S=S_0=0,\\ T&=T_0'=T_0-2\gamma_{00}{}^2S_0, &&\text{if}\quad S=S_0.\end{aligned} \tag{88}$$

In view of (59.2), the conclusion follows immediately, as

$$v_2=[(2\gamma_{00}{}^2-T_0)(2\gamma_{00}{}^2-1)^{-1}]^{\frac{1}{2}}=v_2'. \tag{89}$$

R can also be expressed as a function of the four independent arguments (z,p_0,T_0,S_0), which we refer to as the *initial conditions*. For observer O, let us assume the values

$$z=\tfrac{1}{2}\pi,\quad p_0=1,\quad T_0=1,\quad S_0=0. \tag{90}$$

Then (54) and (64) yield

$$v_1=v_2=w_1=w_2=1,\quad \theta=\tfrac{1}{2}\pi, \tag{91}$$

and

$$R_*=K_{00}\sum\sum b_1{}^ib_2{}^jM_{ij}\psi_m(\tfrac{1}{2}\pi). \tag{92}$$

In the test of (85) we also need the relations

$$\begin{aligned}F'&=\xi_2,\\ w_1'&=\xi^{\frac{1}{2}},\end{aligned} \tag{93}$$

where the symbols on the right refer to location O$'$ as measured from O. To prove (93.1), we make use of (17), (12.2), (13) and note that all the β's in (17) assume their standard values in consequence of (90), and that $x=T_0'$. From (61.1) now follows

$$\begin{aligned}F'&=\{x-\beta_{00}x^n[1-\tfrac{1}{2}\gamma_{00}{}^{-2}(1-x)]\}(1-\beta_{00})^{-1}\\ &=[x-\beta_{00}x^n+\alpha_{00}(x^n-x^{n+1})](1-\beta_{00})^{-1}.\end{aligned} \tag{94}$$

On the other hand, from (17)

$$\begin{aligned}\xi_2=x&+\beta_1{}^{00}(1-\beta_1{}^{00})^{-1}(x-x^n)\\ &+\alpha_{00}(x-x^{n+1})+O(\alpha\beta).\end{aligned} \tag{95}$$

Since $\beta_1=\beta-\alpha$, the conclusion (93.1) follows, if we reject quantities of order $\alpha\beta\sim10^{-5}$. To prove (93.2), we use (62) and (93.1), which lead to

$$\begin{aligned}(w_1')^2&\sim[1-2\beta_2{}^{00}(1-F')]^{\frac{1}{2}}F'\\ &\sim[1-\beta_2{}^{00}(1-\xi_2)]\xi_2\\ &=\xi+0(\beta_2{}^2).\end{aligned} \tag{96}$$

The conclusion (93.2) follows if we reject terms of order $\beta_2{}^2\sim10^{-5}$.

For observer O$'$, viewing the same grazing ray, the initial conditions are

$$\sin z'=1-u_0,\quad p_0'=x^{n+1},\quad T_0'=x,\quad S_0'=\tfrac{1}{2}\gamma_{00}{}^2(1-x), \tag{97}$$

where x is arbitrary, and $u_0(x)$ is defined by (14). With the aid of (86), (89), (91), (93), and (62), we find

$$
\begin{aligned}
v_1'&=v_2'=1,\\
w_1'&=\xi^{\frac{1}{2}}(x), \qquad (98)\\
w_2'&=1-2\beta_2^{00}[1-\xi_2(x)],
\end{aligned}
$$

to the order of accuracy adopted here. It is to be noted that the values of w_2' and z' are connected by the relations (98.3), (83), (18), and (19), leading to

$$
\begin{aligned}
\sin z'&=1-u=w_2', \qquad (99)\\
\cos^2 z'&=2u-u^2=\Gamma^{-2}(1-\xi).
\end{aligned}
$$

For both values of z' we use the formula (82). A simple calculation of $\bar{w}_1'$ and θ now yields

$$\bar{w}_1'=1, \quad \sin\theta=\xi^{\frac{1}{2}}. \qquad (100)$$

If we put

$$\theta_1=\sin^{-1}\xi^{\frac{1}{2}}, \qquad (101)$$

then

$$
\begin{aligned}
\theta&=\theta_1, &&\text{if}\quad z'\le\tfrac{1}{2}\pi \qquad (102)\\
\theta&=\pi-\theta_1, &&\text{if}\quad z'>\tfrac{1}{2}\pi.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
R_*'&=K_{00}\sum\sum b_1^i b_2^j M_{ij}\bar{\psi}_m(\theta_1), \qquad (103)\\
\bar{R}_*'&=K_{00}\sum\sum b_1^i b_2^j M_{ij}\bar{\psi}_m(\pi-\theta_1).
\end{aligned}
$$

With the aid of (48.2), the relation (85) is verified immediately.

As a curiosity, consider the extreme case where O′ is located on the boundary of the polytrope. There $x=0$, $T_0'=0$ so that (99.1), (98.3), and (17) lead to the initial conditions

$$\sin z'=1-2\beta_2^{00}, \quad p_0'=0, \quad T_0'=0, \quad S_0'=\tfrac{1}{2}\gamma_{00}^{-2}. \qquad (104)$$

It follows that

$$
\begin{aligned}
&w_1'=0, \quad \xi=0, \quad \theta_1=0, \qquad (105)\\
&R_*'=0, \quad \bar{R}_*'=2R_*,
\end{aligned}
$$

and the relation (85) is again satisfied.

Assumption (90) of standard pressure and temperature on the ground is not a restriction, inasmuch as the arbitrary p_{00} and T_{00} can always be redefined so that (90) holds. Thus the proof that our theory satisfies the fundamental relation (85) is perfectly general, as well as rigorous.

The proof depends on the *characteristic* of the refraction formulas in that ψ_m is always associated with the factor w_1^{2m-1} in the basic equations as well as in the corrections.

III. SIMPLE POLYTROPE, $n<\infty$

A. Refraction Integral for $n=\infty$

As $n\to\infty$, (8) reduces to

$$\rho/\rho_0=e^{-x}, \qquad (106)$$

where x has been redefined as

$$x=g_0h'/\mathcal{R}T_0. \qquad (107)$$

We also redefine γ and β by

$$
\begin{aligned}
\gamma&\equiv[r_0g_0/2\mathcal{R}T_0]^{\frac{1}{2}}, \qquad (108)\\
\beta&\equiv 2\alpha(1-\alpha)\gamma^2.
\end{aligned}
$$

Then (12) is replaced by

$$
\begin{aligned}
\omega&=1-e^{-x}, \qquad (109)\\
s&=h'/r_0=\tfrac{1}{2}\gamma^{-2}x,
\end{aligned}
$$

and the substitution into (6) yields

$$u=s-\alpha(1-s)(1-e^{-2\gamma^2 s}). \qquad (110)$$

Noting that $1-\alpha+\alpha\omega=(1-\alpha)(1+\alpha\omega)+O(\alpha^2)$, we exhibit the refraction integral (7) in the form

$$
\begin{aligned}
R&=\beta\int_0^1(1-u)e^{-2\gamma^2 s}\Delta^{-1}ds,\\
\Delta^2&=\cot^2 z+2u-u^2. \qquad (111)
\end{aligned}
$$

In the integrand we have suppressed the factor $1+\alpha\omega$. That the resulting error is negligible is shown in Sec. IIIG. The limits of integration have been obtained by the substitutions $h=0$ and $h=\infty$ into (9) and (10).

B. Expansion into Series

Since s is an implicit function of u defined by (110), Theorem (24) applies. If we substitute the following expressions,

$$
\begin{aligned}
&x=s, \qquad y=u,\\
&f=e^{-2\gamma^2 s}, \quad \phi=(1-s)(1-e^{-2\gamma^2 s}), \qquad (112)\\
&\epsilon=\alpha(1-\alpha),
\end{aligned}
$$

into (24) and make use of the binomial theorem, we obtain

$$e^{-2\gamma^2 s}ds=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{1}{i!}\epsilon^i(-1)^j\binom{i}{j}D^i[(1-u)^i e^{-2\gamma^2(j+1)u}]du. \qquad (113)$$

Since $2\gamma^2\sim 800$ for the earth, we can approximate (113) by applying the operator D to the exponential factor only. That the resulting error is, indeed, negligible is shown in Sec. IIIG. Accordingly, (113) is replaced by

$$e^{-2\gamma^2 s}ds=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{1}{i!}(-1)^{i+j}\binom{i}{j}[\beta(j+1)(1-u)]^i e^{-2\gamma^2(j+1)u}du, \qquad (114)$$

where we have made use of (108.2). If we now put $k=i-j$ and change the order of summation in (114), we obtain

$$e^{-2\gamma^2 s}ds=\sum_{j=0}^{\infty}\frac{1}{j!}[\beta(j+1)(1-u)]^j e^{-2\gamma^2(j+1)u}\sum_{k=0}^{\infty}\frac{1}{k!}[-\beta(j+1)(1-u)]^k du$$

$$=\sum_{j=0}^{\infty}\frac{1}{j!}[\beta(j+1)(1-u)]^j e^{-\beta(j+1)}e^{-2\gamma^2(j+1)(1-\alpha)u}du. \tag{115}$$

Upon the substitution into (111), R assumes the form

$$R=\Gamma^{-1}\sum_{0}^{\infty} b^{i+1}M_i I_i, \tag{116}$$

where we have defined

$$\Gamma\equiv\gamma(1-\alpha)^{\frac{1}{2}},$$
$$b\equiv\beta e^{-\beta},$$
$$M_i\equiv\frac{1}{i!}(i+1)^{i-\frac{1}{2}}, \tag{117}$$
$$I_i\equiv(i+1)^{\frac{1}{2}}\Gamma\int_0^1[(1-u)\exp(-2\Gamma^2 u)]^{i+1}\Delta^{-1}du.$$

Whether or not u is a monotonic function of the arc length can be ascertained from (85). Generally, $u\leq 1$ and ψ decreases monotonically from value z at the station to value zero at infinity. Two cases are distinguished: if $z\leq\frac{1}{2}\pi$, u increases monotonically from 0 to 1; if $z>\frac{1}{2}\pi$, u first decreases from 0 to

$$\min u=1-\csc z, \tag{118}$$

attaining this minimum at the perigee with $\psi=\frac{1}{2}\pi$, and then increases to 1. In summary,

$$\begin{aligned}\min u&=0 &&\text{if}\quad z\leq\tfrac{1}{2}\pi,\\ &=1-\csc z &&\text{if}\quad z>\tfrac{1}{2}\pi.\end{aligned} \tag{119}$$

The integral I_i of (117) is of the Laplace type, and can be evaluated by the expansion of u about the *critical point* (Erdelyi 1956), where u attains its minimum. Since $2\gamma^2$ is a very large number, rapid convergence is assured. This method is adopted in Secs. IIIC and F.

C. Case $z\leq\frac{1}{2}\pi$

If $z\leq\frac{1}{2}\pi$, the inequality $|u|\leq 1$ permits the transformation of the independent variable by means of

$$u_1=u-\tfrac{1}{2}u^2, \tag{120}$$

which reduces Δ^2 in (111) to

$$\Delta^2=\cot^2 z+2u_1. \tag{121}$$

The use of (24) with the substitutions

$$x=u,\quad y=u_1,\quad \epsilon=\tfrac{1}{2}, \tag{122}$$
$$f(x)=[(1-u)\exp(-2\Gamma^2 u)]^{i+1},\quad \phi(x)=u^2$$

leads to

$$f'(u)=-(i+1)e^{-2\Gamma^2 u(i+1)}[2\Gamma^2(1-u)^{i+1}+(1-u)^i], \tag{123}$$

and

$$(1-u)^{i+1}e^{-2\Gamma^2 u(i+1)}du=\sum_0^{\infty}\frac{1}{2^j j!}D^j[u_1^{2j}(1-u_1)^{i+1}e^{-2\Gamma^2 u_1(i+1)}]du_1, \tag{124}$$

if we drop the small terms

$$\tfrac{1}{2}\Gamma^{-2}[(1-u)^i e^{-2\Gamma^2 u(i+1)}du-(1-u_1)^i e^{-2\Gamma^2(i+1)u_1}du_1]. \tag{125}$$

We further approximate (124) by setting $j=0$ and replacing $1-u_1$ by its maximum value 1. Then the right-hand member of (124) reduces to

$$e^{-2\Gamma^2 u_1(i+1)}du_1, \tag{126}$$

and I_i in (117) takes the form

$$I_i=(i+1)^{\frac{1}{2}}\Gamma\int_0^{\frac{1}{2}}e^{-2\Gamma^2}u_1(i+1)\Delta^{-1}du_1 \tag{127}$$

with $\Delta(u_1)$ defined by (121).

We define the parameter Z and function Ψ by

$$Z\equiv\Gamma\cot z,$$

$$\Psi(x)\equiv e^{x^2}\int_x^{\infty}e^{-t^2}dt, \tag{128}$$

and introduce a new variable t of integration by

$$\Delta^2=\cot^2 z+2u_1=t^2/\Gamma^2(i+1). \tag{129}$$

Then

$$(i+1)^{\frac{1}{2}}\Gamma\Delta^{-1}du_1=dt, \tag{130}$$

and the range $t_1 \leq t \leq t_2$ of integration is given by

$$t_1 = Z(i+1)^{\frac{1}{2}}, \qquad t_2 = \Gamma \csc z (i+1)^{\frac{1}{2}}. \tag{131}$$

The substitution from (129) and (130) converts I_i into

$$I_i = \Psi(t_1) - e^{-\Gamma^2(i+1)}\Psi(t_2). \tag{132}$$

Since Γ^2 is a very large number, we drop the last term, and write

$$I_i = \Psi[Z(i+1)^{\frac{1}{2}}]. \tag{133}$$

Then R in (116) assumes the form

$$R = \Gamma^{-1} \sum_0^\infty b^{i+1} M_i \Psi[Z(i+1)^{\frac{1}{2}}], \qquad Z = \Gamma \cot z. \tag{134}$$

All the approximations of this section are justified in Sec. IIIG.

D. Incomplete Gamma Function

The function Ψ of (128) is related to the incomplete gamma function by

$$\Psi(x) = \tfrac{1}{2} e^{x^2}[\Gamma(\tfrac{1}{2}) - \Gamma(\tfrac{1}{2}; x^2)]. \tag{135}$$

For the calculation of Ψ, two series expansions are available,

$$\Psi(x) = e^{x^2}[\tfrac{1}{2}\pi^{\frac{1}{2}} - \sum_0^\infty (-1)^i x^{2i+1}/i!(2i+1)], \qquad \Psi(x) = \frac{1}{2x} \sum_0^N (-1)^i (2i-1)!!/(2x^2)^i, \tag{136}$$

where $(2i-1)!! = 1\cdot 3\cdot 5\cdots(2i-1)$ with $(-1)!! = 1$. The first series is useful for $x \leq 4$; the second series is a divergent asymptotic series, which for $x > 4$ will yield an accuracy of 10^{-6} if optimally truncated at $i = N$.

For refraction below the horizon, with $Z < 0$, it is convenient to use the *associate functions* $\bar{\Psi}$ and Ψ^*,

$$\bar{\Psi}(x) = \int_x^\infty e^{-t^2} dt, \qquad \Psi^*(x) = -2\int_0^x e^{-t^2} dt, \tag{137}$$

which are bounded for all x.

The following properties of the three functions are noted below:

$$\begin{aligned}
&(1) \quad \bar{\Psi}(x) = e^{-x^2}\Psi(x),\\
&(2) \quad \bar{\Psi}(-x) = 2\Psi(0) - \bar{\Psi}(x),\\
&(3) \quad \Psi^*(-x) = 2\Psi(0) - 2\bar{\Psi}(x),\\
&(4) \quad \Psi(\infty) = 0, \quad \Psi(-\infty) = \infty,\\
&(5) \quad \bar{\Psi}(\infty) = 0, \quad \bar{\Psi}(-\infty) = 2\Psi(0),\\
&(6) \quad \Psi^*(0) = 0, \quad \Psi^*(-\infty) = 2\Psi(0),\\
&(7) \quad \Psi(0) = \bar{\Psi}(0) = \tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \tfrac{1}{2}\pi^{\frac{1}{2}},\\
&(8) \quad \Psi'(x) = 1 - 2x\Psi(x) < 0.
\end{aligned} \tag{138}$$

E. Weather Factors

The departure of p_0, T_0, and S_0 from standard conditions is expressed in terms of the following weather factors, defined below:

$$\begin{aligned}
v_1 &\equiv p_0 T_0^{-1} \exp(2\gamma_{00}^2 S_0 T_0^{-1}),\\
v &\equiv [T_0^{-1}(1-S_0)(1-\alpha_{00}p_0T_0^{-1})(1-\alpha_{00})^{-1}]^{\frac{1}{2}},\\
v_2 &\equiv p_0 T_0^{-1} v^2,\\
w &\equiv v_2 \exp[\beta_{00}(1-v_2)].
\end{aligned} \tag{139}$$

Referring to (108), (117), and proceeding as in Sec. IIF, we obtain

$$\begin{aligned}
\alpha &= \alpha_{00} p_0 T_0^{-1},\\
\Gamma &= \Gamma_{00} v,\\
\beta &= \beta_{00} v_2,\\
b &= b_{00} w.
\end{aligned} \tag{140}$$

Following the convention of Sec. IIF, we drop the subscript 00 of b. The substitution from (140) into (134) now yields

$$R = (\Gamma_{00}v)^{-1} \sum_0^\infty b^{i+1} M_i w^{i+1} \Psi[Z(i+1)^{\frac{1}{2}}], \qquad Z = \Gamma_{00} v \cot z. \tag{141}$$

If we introduce the parameter K defined by

$$K = b\Gamma^{-1} = 2\alpha\gamma(1-\alpha)^{\frac{1}{2}} e^{-\beta}, \tag{142}$$

R can be exhibited in the form

$$R = K_{00} v^{-1} \sum_0^\infty b^i M_i w^{i+1} \Psi[Z(i+1)^{\frac{1}{2}}], \tag{143}$$

which is analogous to that of (64).

As in Sec. II, all the weather factors reduce to unity for $p_0 = 1$, $T_0 = 1$, $S_0 = 0$. Furthermore, it follows from (106), (109), and the perfect gas law that v_1 is independent of the observer. This factor is used only in Sec. IIIG, dealing with the corrections.

F. Case $z \geq \frac{1}{2}\pi$

If $z > \frac{1}{2}\pi$, we resort to the transformation

$$u = 1 - \csc z + t^2/2\Gamma^2(i+1). \tag{144}$$

Clearly, t is monotonic in the range $t_1 \leq t \leq t_2$ with

$$t_1 = -\Gamma[2(\csc z - 1)]^{\frac{1}{2}}(i+1)^{\frac{1}{2}}, \tag{145}$$
$$t_2 = \Gamma(2 \csc z)^{\frac{1}{2}}(i+1)^{\frac{1}{2}}.$$

Upon the substitution (144), Δ^2 of (111) becomes

$$\Delta^2 = [t^2 \csc z/\Gamma^2(i+1)][1 - t^2 \sin z/4\Gamma^2(i+1)], \tag{146}$$

leading to

$$(i+1)^{\frac{1}{2}}\Gamma\Delta^{-1}du = \sin^{\frac{1}{2}}z[1 + t^2 \sin z/8\Gamma^2(i+1) + \cdots]dt. \tag{147}$$

Let Z be now redefined by

$$Z \equiv \pm\Gamma[2(\csc z - 1)]^{\frac{1}{2}}, \tag{148}$$

with the plus sign corresponding to $z < \frac{1}{2}\pi$ and the minus sign to $z > \frac{1}{2}\pi$. Then the integral I_i of (117) becomes

$$I_i = \sin^{\frac{1}{2}}z(\csc z\, e^{Z^2})^{i+1}[\bar{\Psi}(t_1) - \bar{\Psi}(t_2) + \epsilon_i], \tag{149}$$

with $\bar{\Psi}$ defined in (137), and

$$\epsilon_i = [-(3+4i)\sin z/8(i+1)\Gamma^2 + \cdots]\int_{t_1}^{t_2} t^2 e^{-t^2}dt, \tag{150}$$

up to terms of order Γ^{-2}. Since Γ^2 is large, we drop the term ϵ_i and $\bar{\Psi}(t_2)$ in (149), and write

$$I_i = \sin^{\frac{1}{2}}z(\csc z\, e^{Z^2})^{i+1}\bar{\Psi}[Z(i+1)^{\frac{1}{2}}]. \tag{151}$$

Then R in (116) assumes the form

$$R = \Gamma^{-1}\sin^{\frac{1}{2}}z\sum_0^\infty (b \csc z\, e^{Z^2})^{i+1}M_i\bar{\Psi}[Z(i+1)^{\frac{1}{2}}]. \tag{152}$$

In the light of Sec. IIIE, the effect of the "weather" can be described in terms of the factors $\bar{v}$ and $\bar{w}$ defined by

$$\bar{v} = v \csc^{\frac{1}{2}}z, \tag{153}$$
$$\bar{w} = w \csc z\, e^{Z^2}.$$

The substitution from (153) and (140) into (152) and (148) yields

$$R = (\Gamma_{00}\bar{v})^{-1}\sum_0^\infty b^{i+1}M_i\bar{w}^{i+1}\bar{\Psi}[Z(i+1)^{\frac{1}{2}}],$$
$$Z = -\Gamma_{00}\bar{v}[2(1-\sin z)]^{\frac{1}{2}}, \tag{154}$$

where the minus sign of Z corresponds to $z \geq \frac{1}{2}\pi$. Clearly, the result agrees with (141) for $z = \frac{1}{2}\pi$. In terms of the parameter K of (142), we can exhibit R as

$$R = K_{00}\bar{v}^{-1}\sum_0^\infty b^i M_i \bar{w}^{i+1}\bar{\Psi}[Z(i+1)^{\frac{1}{2}}]. \tag{155}$$

The approximation introduced into (151) is justified in the next section.

G. Corrections

Five approximations have been introduced into the derivation of the basic refraction formulas. The corresponding corrections δ_i are evaluated below. The last two assume a distinct form when $z > \frac{1}{2}\pi$, and are marked δ_4' and δ_5'.

(1) *The omission of the factor* $1 + \alpha\omega$ *in* (111). The term $\alpha\omega$ contributes to (113)

$$\alpha(1 - e^{2\gamma^2 s})e^{-2\gamma^2 s}ds. \tag{156}$$

We show that this contribution is canceled by the corresponding contribution under (2), so that $\delta_1 + \delta_2 = 0$.

(2) *The omission of the terms resulting from the differentiation of the factor* $(1-u)^i$ *in* (113).

The dominant correction terms in (114) correspond to $i=1$ and $j=0$, 1, with D acting on $1-u$ only. The result of this operation is precisely the negative of (156).

(3) *The omission of the expression* (125) *in the summation* (124). In view of (120), Taylor series expansion of (125) with $i=0$ yields

$$-\tfrac{1}{2}u_1^2 e^{-2\Gamma^2 u_1}du_1. \tag{157}$$

The resulting correction in R is

$$\delta_3 = -\tfrac{1}{2}\beta\int_0^{\frac{1}{2}} u_1^2 e^{-2\Gamma^2 u_1}\Delta^{-1}du_1$$
$$= -\tfrac{1}{8}\Gamma_{00}^{-4}K_{00}(wv^{-5})e^{Z^2}\int_Z^\infty (t^2 - Z^2)^2 e^{-t^2}dt. \tag{158}$$

From (139),

$$wv^{-5} \sim v_1 v^{-3}\exp(-2\gamma_{00}^2 T_0^{-1}S_0) \tag{159}$$
$$\sim v_1 T_0^{\frac{3}{2}}(1-S_0)^{-\frac{3}{2}}\exp(-2\gamma_{00}^2 T_0^{-1}S_0).$$

On the interval [0,1], the function

$$f(S) = (1-S)^{-\frac{3}{2}}\exp(-2\gamma_{00}^2 T_0^{-1}S_0) \tag{160}$$

has a singularity at $S=1$, and a minimum at

$$S = 1 - \tfrac{3}{4}\gamma_{00}^{-2}. \tag{161}$$

To avoid the singularity we restrict the range of S_0 to

$$0 \leq S_0 \leq 1 - \tfrac{3}{4}\gamma_{00}^{-2}T_0 \sim 0.998, \tag{162}$$

which is equivalent to

$$0 \leq h \leq 500a. \tag{163}$$

Clearly, such a restriction is of no practical consequence. On the interval (162), f is maximized at $S_0 = 0$, assuming the value 1. The Z-dependent factor in (158), for $Z \geq 0$, is maximized at $Z=0$, assuming the value $\frac{3}{8}\pi^{\frac{1}{2}}$. Hence

$$\delta_3 \leq \tfrac{3}{64}\Gamma_{00}^{-4}K_{00}\pi^{\frac{1}{2}}v_1 T_0^{\frac{3}{2}} \sim 0''.001, \tag{164}$$

where we have inserted the numerical values of Sec. VA and put $v_1 = 1$, $T_0 = 1$, which correspond to standard conditions on the ground. We may set $\delta_3 = 0$.

(4) *The omission of the terms of* (124) *with* $j \geq 1$ *and the replacement of* $(1-u)^i$ by 1.

The leading correction term of (124), corresponding to $j=1$, $i=0$, is

$$\tfrac{1}{2}D[u_1^2(1-u_1)e^{-2\Gamma^2 u_1}]du_1 \sim -\Gamma^2 u_1^2 e^{-2\Gamma^2 u_1}du_1. \quad (165)$$

The latter expression is equal to that in (157) multiplied by $2\Gamma^2$. In view of (158),

$$\delta_4 = -\tfrac{1}{4}\gamma_{00}^{-2}K_{00}(wv^{-3})e^{Z^2}\int_Z^\infty (t^2-Z^2)^2 e^{-t^2}dt, \quad (166)$$

where we have used the approximation $\gamma\sim\Gamma$ and inserted the appropriate weather factors. Proceeding as in (3), we can show that wv^{-3} is nonsingular on the interval (162). Integration by parts converts the Z-dependent factor into

$$I_1 = \tfrac{1}{4}Z(3-2Z^2)+\tfrac{1}{4}(3-4Z^2+4Z^4)\Psi(Z). \quad (167)$$

In view of (136), this expression is bounded for all $Z\geq 0$. Hence

$$\delta_4 = -\tfrac{1}{4}\Gamma_{00}^{-2}K_{00}(wv^{-3})I_1(Z). \quad (168)$$

A bound on δ_4 given by

$$|\delta_4| \leq \tfrac{3}{32}\Gamma_{00}^{-2}K_{00}\pi^{\frac{1}{2}}v_1 T_0^{\frac{1}{2}} \sim 0\overset{''}{.}9 \quad (169)$$

corresponds to the standard horizontal refraction, $Z=0$ and $S_0=0$.

(5) *The omission of the last term of* (132).

The term in question is maximized at $z=\frac{1}{2}\pi$ and $i=0$, which corresponds to

$$\delta I_0 \sim -e^{-\Gamma^2}\Psi(\Gamma\ \mathrm{csc}z) \sim -e^{-\Gamma^2}/2\Gamma\ \mathrm{csc}z, \quad (170)$$

where we have made use of the asymptotic series (136.2). The resulting correction is

$$\delta_5 = -K_{00}(wv^{-2})e^{-\Gamma_{00}^2 v^2}/2\Gamma_{00}v\ \mathrm{csc}z. \quad (171)$$

From (139) it follows that

$$\begin{aligned} wv^{-2} &\sim v_1 \exp(-2\gamma_{00}^2 T_0^{-1}S_0) \\ &\sim v_1 \exp[2\gamma_{00}^2(v^2-T_0^{-1})]. \end{aligned} \quad (172)$$

Hence,

$$\delta_5 = -\tfrac{1}{2}K_{00}\sin z\ v_1 \exp(-2\gamma_{00}^2 T_0^{-1})(e^{\gamma_{00}^2 v^2}/\gamma_{00}v). \quad (173)$$

It follows from (162) that

$$\tfrac{1}{2}\sqrt{3}\gamma_{00}^{-1} \leq v \leq T_0^{-\frac{1}{2}}. \quad (174)$$

On this interval of v, the last factor of (173) enclosed in parentheses is maximized at $v=T_0^{-\frac{1}{2}}$. Thus a bound on δ_5 is given by

$$|\delta_5| \leq \tfrac{1}{2}K_{00}v_1(e^{-\gamma_{00}^2 T_0^{-1}}/\gamma_{00}T_0^{-\frac{1}{2}}). \quad (175)$$

Since $\gamma_{00}\sim 20$ and $T_0\sim 1$, we may set $\delta_5=0$.

(4′) *The omission of the term* ϵ_i *in* (149).

The dominant term of the correction corresponds to $i=0$, with

$$\epsilon_0 = -\tfrac{3}{8}\Gamma^{-2}\sin z\int_Z^\infty t^2 e^{-t^2}dt, \quad (176)$$

leading to

$$\delta_4' = -\tfrac{3}{8}\Gamma_{00}^{-2}K_{00}(\bar{w}\bar{v}^{-3})\int_Z^\infty t^2 e^{-t^2}dt. \quad (177)$$

Integration by parts converts the integral into

$$I(Z) = \tfrac{1}{2}[Ze^{-Z^2}+\Psi(Z)]. \quad (178)$$

For $Z\leq 0$, the integral is maximized at $Z=-\infty$, assuming the value $\frac{1}{2}\pi^{\frac{1}{2}}$. The bound given by

$$|\delta_4'| \leq \tfrac{3}{16}\Gamma_{00}^{-2}K_{00}\pi^{\frac{1}{2}} \sim 1\overset{''}{.}8 \quad (179)$$

is exactly twice that in (169). It corresponds to a grazing ray with the observer O′ at infinity, and is in accord with the fundamental relation (85).

(5′) *The omission of the term* $\Psi(t_2)$ *in* (149). For $i=0$ the correction is

$$\begin{aligned} \delta_5' &= -K_{00}\bar{v}^{-1}\bar{w}\Psi(t_2), \\ t_2 &= \Gamma_{00}v(2\ \mathrm{csc}z)^{\frac{1}{2}} \sim \gamma_{00}\bar{v}\sqrt{2}. \end{aligned} \quad (180)$$

From (136.2) and (138.1) it follows that

$$\Psi(t_2) \sim e^{-t_2^2}/2t_2 = \exp(-2\gamma_{00}^2\bar{v}^2)/2\gamma_{00}\bar{v}\sqrt{2}. \quad (181)$$

From (153), (154), and (139),

$$\begin{aligned} \bar{w}/\bar{v}^2 &= v_1 \exp(-2\gamma_{00}^2 T_0^{-1}S_0+Z^2), \\ Z^2 &= 2\gamma_{00}^2\bar{v}^2(1-\sin z), \\ \bar{v}^2 &\sim T_0^{-1}(1-S_0)\ \mathrm{csc}z. \end{aligned} \quad (182)$$

On collecting results we find

$$\delta_5' = -K_{00}v_1 \exp(-2\gamma_{00}^2 T_0^{-1})/2\gamma_{00}T_0^{-\frac{1}{2}}\sqrt{2}. \quad (183)$$

As in (5), we may set $\delta_5'=0$.

The total correction can therefore be written as

$$\begin{aligned} \delta_1 R &= -\tfrac{1}{4}\gamma_{00}^{-2}K_{00}(wv^{-3})I_1(Z) \quad \text{if} \quad z\leq\tfrac{1}{2}\pi, \\ \delta_2 R &= -\tfrac{3}{8}\gamma_{00}^{-2}K_{00}(\bar{w}\bar{v}^{-3})I_2(Z) \quad \text{if} \quad z>\tfrac{1}{2}\pi, \end{aligned} \quad (184)$$

where I_1 and I_2 are defined by (167) and (179), respectively. Clearly, the two corrections are equal at $z=\frac{1}{2}\pi$.

In application to the earth, the density in the stratosphere is at most $\frac{1}{4}$ of its standard value on the ground. In these circumstances, the largest correction will not exceed $0\overset{''}{.}5$. We do not include these corrections in the algorithm of Sec. VB.

H. Internal Check

Proceeding from the fundamental relation (85), we extend the range of the validity of formula (155). For observer O, we again assume

$$z=\tfrac{1}{2}\pi, \quad p_0=1, \quad T_0=1, \quad S_0=0. \quad (185)$$

Then (139) and (141) yield

$$v_1=v_2=v=w=1, \quad Z=0, \quad (186)$$

and

$$R_* = K_{00}\Psi(0)\sum_0^\infty b^i M_i. \quad (187)$$

For observer O′, viewing the same grazing ray in the reverse direction, (83) and (8) lead to

$$\sin z'=1-u_0=(1-S_0')(1+\alpha_{00}\omega_0'), \tag{188}$$

where we have put $s=S_0'$ and $\alpha=\alpha_{00}$ in consequence of (51.2) and assumption (185). Under the same assumption, (4.4) and the perfect gas law $p=\rho T$ lead to

$$\omega_0'=1-\rho_0'. \tag{189}$$

The density and the pressure profiles are then obtained with the aid of (106) and (109) as

$$\rho=p=\exp(-2\gamma_{00}^2 S). \tag{190}$$

Thus we are led to the following set of initial conditions at O′:

$$\begin{aligned} z'&=\pi-\sin^{-1}[(1-S_0)(1+\alpha_{00}\omega_0')],\\ p_0'&=\exp(-2\gamma_{00}^2 S_0')=\rho_0',\\ T_0'&=1, \end{aligned} \tag{191}$$

with $\omega_0'=1-\rho_0'$ and S_0' arbitrary. Now (139), (153), (154), and (188) yield

$$\begin{aligned} v'&=(1-u_0)^{\frac{1}{2}},\\ v_2'&=\rho_0'(1-u_0)=(1-u_0)\exp(-2\gamma_{00}^2 S_0'),\\ w'&=(1-u_0)\exp[-2\gamma_{00}^2 S_0'+\beta_{00}(1-v_2')],\\ \bar{v}'&=1,\\ \bar{w}'&=\exp[Z^2-2\gamma_{00}^2 S_0'+\beta_{00}(1-v_2')]. \end{aligned} \tag{192}$$

We now observe that

$$\begin{aligned} Z^2&=2\Gamma_{00}^2(1-\sin z')=2\gamma_{00}^2(1-\alpha_{00})u_0,\\ 1-v_2'&=\omega_0'+\rho_0' S_0'-\alpha_{00}(1-S_0')\rho_0'\omega_0',\\ \beta_{00}&=2\alpha_{00}(1-\alpha_{00})\gamma_{00}^2. \end{aligned} \tag{193}$$

On collecting these results we calculate the argument of the exponential function in (192.5) as

$$-2\gamma_{00}^2\alpha_{00}^2[S_0'+(1-S_0')\rho_0'\omega_0']+\cdots. \tag{194}$$

Its absolute value is maximized when $S_0'=1$, assuming the value $2\gamma_{00}^2\alpha_{00}^2\sim 10^{-4}$. If we neglect this term, the error in R will be at most 0″.4, for an observer at infinity. Thus, we may put

$$\bar{v}'=\bar{w}'=1; \tag{195}$$

then (155) yields

$$R_*'=K_{00}\sum_0^\infty b^i M_i\Psi(Z), \tag{196}$$

where Z is given by the *negative* square root of the expression (148).

From (187), (85), and (138.2) we now calculate

$$\bar{R}_*'=K_{00}\sum_0^\infty b^i M_i\Psi(-Z). \tag{197}$$

In view of the definition (148), a change in the sign of Z is equivalent to the replacement of z by $\pi-z$. We now conclude that (155) is valid for zenith distance $\pi-z$ if it is valid for $z\geq\frac{1}{2}\pi$, and if the backward continuation of the ray is not obstructed by the earth. Accordingly, the range of the validity of (155) can be extended to

$$\begin{aligned} \chi&\leq z\leq\pi-\chi,\\ \chi&=\sin^{-1}(1-u_0). \end{aligned} \tag{198}$$

We make use of this result in (215.2) of Sec. IVC.

IV. COMPOUND POLYTROPE

A. Transition Point

The Euler differential equation with the initial conditions (z,p_0,T_0,S_0) uniquely determines the ray trajectory and its intersection with the tropopause $r=a+h_1$. At this point of discontinuity of the polytropic index $n(r)$, the variables (ψ,p,T,S) are of class C^0. We proceed to calculate the values $\psi_1=z_1$, p_1, T_1, S_1 that the variables assume at that point.

The height of the tropopause above the station is given by

$$h=h_1-h_0. \tag{199}$$

Making use of (51), (7), (8), (55), (84), and noting that $\psi_0=z$ and $\psi=z_1$, we calculate

$$\begin{aligned} S_1&=h_1/(a+h_1),\\ s_1&=(h_1-h_0)/(a+h_1),\\ u_1&=s_1-\alpha_{00}p_0T_0^{-1}(1-s_1)\omega(s_1)+O(\alpha^2),\\ \sin z_1&=(1-u_1)\sin z, \end{aligned} \tag{200}$$

where subscript 1 refers to quantities associated with the tropopause. The form of the function $\omega(s)$ depends on whether $h_0<h_1$ or $h_0>h_1$. The two cases are treated in Secs. IVB and C.

B. Observer within the Troposphere

If $h_0<h_1$, the results of Sec. II apply. From (12), (56.3), (11), and the perfect gas law we find

$$\begin{aligned} x_1&=1-2\gamma^2 s_1=1-2\gamma_{00}^2(1-S_0)T_0^{-1}s_1,\\ \omega(s_1)&=1-x_1^n,\\ p_1&=p_0x_1^{n+1},\\ T_1&=T_0x_1. \end{aligned} \tag{201}$$

From (199), (200.2), and (51) it follows that

$$0\leq s_1\leq\tfrac{1}{2}\gamma^{-2}<1. \tag{202}$$

Therefore, (200.3) implies that

$$0\leq u_1<1. \tag{203}$$

We conclude from (200.4) that $\sin z_1\leq 1$ and z_1 is real. Since z_1 lies in the first quadrant, the solution is completed by adjoining to (201) the equation

$$z_1=\sin^{-1}[(1-u_1)\sin z]. \tag{204}$$

The branch of the ray in the troposphere contributes to astronomical refraction the difference

$$R_1(z,p_0,T_0,S_0)-R_1(z_1,p_1,T_1,S_1), \tag{205}$$

where the function R_1 is calculated by the basic formula (78) or (82). At the transition point, the ray enters the stratosphere and continues there to infinity. This branch of the ray contributes to refraction

$$R_2(z_1,p_1,T_1,S_1), \tag{206}$$

where the function R_2 is calculated by the basic formula (143). The total refraction is the sum of the expressions (205) and (206).

C. Observer within the Stratosphere

If $h_0>h_1$, the results of Sec. III apply. From (109), (56.3), and the perfect gas law for an isothermal atmosphere, we deduce

$$\begin{aligned} \omega(s_1)&=1-\exp(-2\gamma^2 s_1)\\ &=1-\exp[-2\gamma_{00}{}^2(1-S_0)T_0{}^{-1}s_1], \qquad (207)\\ T_1&=T_0,\\ p_1&=p_0[1-\omega(s_1)]. \end{aligned}$$

Since the quantities s_1 and $\omega(s_1)$ are negative, so is u_1 in (200). Consequently, two cases must be distinguished, according as $\sin z_1>1$ or $\sin z_1\leq 1$.

If $\sin z_1>1$, then z_1 is imaginary. The ray lies entirely in the stratosphere, and the refraction is given by

$$R_2(z,p_0,T_0,S_0). \tag{208}$$

If $\sin z_1\leq 1$, then z_1 is real and satisfies the inequality $\frac{1}{2}\pi\leq z_1\leq\pi$. The ray enters the troposphere, re-enters the stratosphere at the *second* transition point (z_2,p_2,T_2,S_2), and continues in the stratosphere to infinity. Because of the symmetry of the ray,

$$z_2=\pi-z_1\leq\tfrac{1}{2}\pi, \quad p_2=p_1, \quad T_2=T_1, \quad S_2=S_1. \tag{209}$$

The two branches in the stratosphere contribute to R the difference

$$R_2(z,p_0,T_0,S_0)-R_2(z_1,p_1,T_1,S_1) \tag{210}$$

and

$$R_2(\pi-z_1,\ p_1,\ T_1,\ S_1), \tag{211}$$

respectively. The branch in the troposphere contributes the difference

$$R_1(z_1,p_1,T_1,S_1)-R_1(\pi-z_1,\ p_1,\ T_1,\ S_1). \tag{212}$$

Let us define

$$\begin{aligned} \delta_1(z)&=R_1(z)-R_1(\pi-z), \qquad (213)\\ \delta_2(z)&=R_2(z)-R_2(\pi-z), \end{aligned}$$

with the deleted arguments p, T, S understood to be the same in both terms. Then the total refraction, which is the sum of the expressions (210), (211), and (212), can be written

$$R=R_2(z)+\delta_1(z_1)-\delta_2(z_1). \tag{214}$$

The two δ's can be conveniently expressed in terms of the associate functions ψ^* and Ψ^*. From the definitions (19.3) and (148) of the functions $\theta(z)$ and $Z(z)$, it follows that

$$\begin{aligned} \theta(\pi-z)&=\pi-\theta(z), \qquad (215)\\ Z(\pi-z)&=-Z(z). \end{aligned}$$

In view of the first three equations of (48) and the first three equations of (138), we note that

$$\begin{aligned} \bar{\psi}_m(\theta)-\bar{\psi}_m(\pi-\theta)&=\psi_m{}^*(\theta), \qquad (216)\\ \bar{\Psi}(Z)-\bar{\Psi}(-Z)&=\Psi_*(Z). \end{aligned}$$

We conclude that δ_1 and δ_2 can be calculated directly from (82) and (155) with the replacement of $\bar{\psi}$ and $\bar{\Psi}$ by ψ^* and Ψ_*, respectively. This procedure is used in the algorithm of Sec. VB.

The ambiguous case $h_0=h_1$ is resolved as follows. If $z>\frac{1}{2}\pi$, proceed as in Sec. IVB; if $z\leq\frac{1}{2}\pi$, use (208) of Sec. IVC. This rule is a consequence of the fact that $\beta<1$ for the earth, which implies that the curvature of the ray is less than $1/r$.

V. APPLICATION TO THE EARTH

A. Geophysical Constants

Refraction depends on five geophysical constants and five parameters. The former comprise the refractive index μ_{00} for a selected wavelength at standard temperature and pressure, the radius a of the earth, the acceleration g_{00} of gravity at sea level, the gas constant $\mathfrak{R}$ for air, and the height h_1 of the tropopause. The latter comprise the zenith distance z, the temperature T_0, the pressure p_0, the altitude h_0 at the observer's station, and the assumed temperature gradient T' in the troposphere. The standard temperature and pressure are chosen as

$$\begin{aligned} T_{00}&=273.15°\mathrm{K}, \qquad (217)\\ p_{00}&=760\ \mathrm{mm\ Hg}. \end{aligned}$$

Hereafter, to save writing, we drop the subscripts 00 referring to standard conditions, and the subscript 0 referring to the observer's station.

The five constants and a standard value of T', expressed in mks system of units, appear in Table II.

TABLE II. Geophysical constants.

	Revised values	1944 values	units
μ	1.00029241	1.00029429	...
a	6.37839×10^6	6.37840×10^6	m
g	9.80655	9.81	m/sec^2
$\mathfrak{R}$	2.87053×10^2	2.87×10^2	m^2/sec^2/°C
h_1	11.019×10^3	variable	m
T'	-0.006500	-0.005694	°C/m'

The symbol m' in the last column represents the geopotential meter. Of course, the user of the Fortran routine may choose his own values of μ and T'. Since T' is strongly affected by seasonal and latitude variations, one cannot expect accurate results for $z \geq 87°$ unless T' is known with sufficient accuracy. In the absence of such information, he may use the standard value, $T' = -0.0065$, when $z < 87°$.

The polytropic index n can be calculated by the formula

$$n+1 = -g/\mathcal{R}T' = -0.034163/T', \tag{218}$$

which follows from (8).

In the first column of Table II, the value of μ is for the wavelength of 5780 Å; the remaining values belong to the *U. S. Standard Atmosphere* (1962). The next column gives the constants used in the author's 1944 refraction tables, with μ adjusted so as to compensate for the inclusion of the factor $1+\alpha_0$ in Eq. (13) of [1]. The value of T' corresponds to $n=5$. The latter choice was partly governed by the convenience of an integral value of n for a desk machine calculation. This column is used here only for the purposes of comparison and check.

There are only three independent constants, inasmuch as a, g, and $\mathcal{R}$ enter the theory only in the combination γ.

B. The Algorithm

The calculation is divided into blocks labeled in Latin and Greek characters, and joined in a schematic flow chart in Fig. 1. The various blocks are described below.

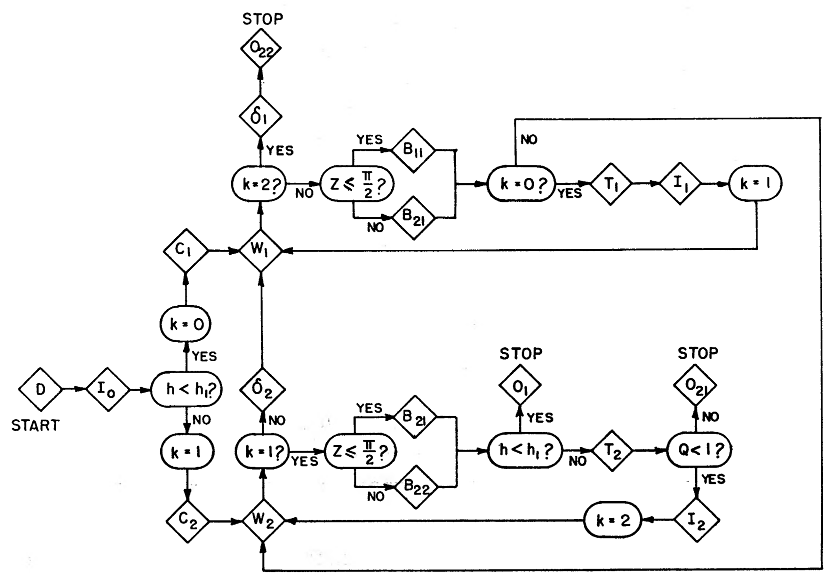

FIG. 1. Flow chart.

D Data

z_* degrees of angle
p_* mm Hg
T_* degrees Centigrade
h meters
T' degrees Centigrade per geopotential meter

I_0 Initial conditions

$z = \pi z_*/180$
$p = p_*/p_{00}$
$T = 1 + T_*/T_{00}$
$S = h/(a+h)$

C_1 Derived constants for the troposphere

$$n=-g/\mathcal{R}T'-1$$
$$\alpha=\mu-1$$
$$\gamma=[ag/2(n+1)\mathcal{R}T_{00}]^{\frac{1}{2}}$$
$$\beta=2\alpha\gamma^2$$
$$\beta_1=\beta-\alpha$$
$$\beta_2=(1-\beta_1)/4\gamma^2$$
$$\Gamma=[4\beta_2(1-\beta_2)]^{-\frac{1}{2}}$$
$$C=(1-\beta_1)(1-\beta_2)(1-2\beta_2)^{-1}$$
$$K=2\alpha\Gamma nC^n\times 206,265''$$
$$b_1=\beta_1C^{n-1}$$
$$b_2=-\beta_2(1-\beta_2)(1-2\beta_2)^{-2}$$

W_1 Weather factors and coefficients for the troposphere

$$v_1=pT^{-n-1}$$
$$v_2=\{[2\gamma^2(1-S)-T](2\gamma^2-1)^{-1}\}^{\frac{1}{2}}$$
$$F=[T-\beta T^n v_1(1-S)](1-\beta)^{-1}$$
$$w_2=(1-2\beta_2)(1-2\beta_2F)^{-1}$$
$$w_1=w_2^{\frac{1}{2}}F^{\frac{1}{2}}$$
$$m=n+i(n-1)+j$$
$$M_{ij}=\binom{n(i+1)-1}{i}\binom{n(i+1)+2j-1}{j}$$

Basic refraction for the troposphere

B_{11} if $z\leq\frac{1}{2}\pi$:

$$\theta_1=\cot^{-1}(\Gamma w_1^{-1}w_2v_2|\cot z|)$$
$$R_1=K\sum_i\sum_j b_1^ib_2^jM_{ij}\psi_m(\theta_1)w_1^{2m-1}v_1^{i+1}v_2^{2i-2j+1}$$
with ψ_m calculated from (49)

- -

B_{12} if $z\geq\frac{1}{2}\pi$

$$\bar{w}_1=[w_1^2+(\Gamma w_2v_2\cot z)^2]^{\frac{1}{2}}$$
$$\theta_1=\sin^{-1}(w_1/\bar{w}_1)$$
$$\theta=\pi-\theta_1$$
$$R_1=K\sum\sum b_1^ib_2^jM_{ij}\bar{\psi}_m(\theta)\bar{w}_1^{2m-1}v_1^{i+1}v_2^{2i-2j+1},$$
with $\bar{\psi}$ calculated from (48.3) and (49)

δ_1 Contribution δ_1

Write the preceding equation with R_1 and $\bar{\psi}$ replaced by δ_1 and ψ^*, respectively. Calculate ψ^* from (48.2), (48.1), and (49).

T_1 Transition point for $h<h_1$

$$s_1=(h_1-h)/(a+h_1)$$
$$x_1=1-2\gamma^2(1-S)T^{-1}s_1$$
$$\omega_1=1-x_1^n$$
$$u_1=s_1-\alpha pT^{-1}(1-s_1)\omega_1$$
$$Q=(1-u_1)\sin z$$
$$\chi=\sin^{-1}Q$$

I_1 New initial conditions

$$z\leftarrow\chi$$
$$T\leftarrow Tx_1$$
$$p\leftarrow px_1^{n+1}$$
$$S\leftarrow h_1/(a+h_1)$$

C_2 Derived constants for the stratosphere

$\alpha=\mu-1$
$\gamma=[ag/2\mathcal{R}T_{00}]^{\frac{1}{2}}$
$\Gamma=\gamma(1-\alpha)^{\frac{1}{2}}$
$\beta=2\alpha\Gamma^2$
$b=\beta e^{-\beta}$
$K=b\Gamma^{-1}\times 206\,265''$

W_2 Weather factors and coefficients for the stratosphere

$\rho=pT^{-1}$
$v_2=pT^{-2}(1-S)(1-\alpha\rho)(1-\alpha)^{-1}$
$v=(v_2/\rho)^{\frac{1}{2}}$
$w=v_2\exp[\beta(1-v_2)]$
$M_i=(1/i!)(i+1)^{i-\frac{1}{2}}$

Basic refraction for the stratosphere

B_{21} if $z\leq\frac{1}{2}\pi$
$Z=\Gamma v\cot z$
$R_2=Kv^{-1}\sum b^iM_iw^{i+1}\Psi[Z(i+1)^{\frac{1}{2}}]$,
with Ψ calculated from (136);

- - - - -

B_{22} if $z>\frac{1}{2}\pi$

$\bar{v}=v\csc z$
$Z=-\Gamma\bar{v}[2(1-\sin z)]^{\frac{1}{2}}$
$\bar{w}=w\csc z\exp Z^2$
$R_2=K\bar{v}^{-1}\sum_i b^iM_i\bar{w}^{i+1}\bar{\Psi}[Z(i+1)^{\frac{1}{2}}]$,
with $\bar{\Psi}$ calculated from (138.1), (138.2), and (136).

δ_2 Contribution δ_2

Write preceding equation with R_2 and $\bar{\Psi}$ replaced by δ_2 and Ψ_*, respectively. Calculate Ψ_* from (138.1), (138.2), and (136) for $Z<0$.

T_2 Transition point for $h>h_1$

$s_1=(h_1-h)/(a+h_1)$
$\omega_1=1-\exp[-2\gamma^2(1-S)T^{-1}s_1]$
$u_1=s_1-\alpha pT^{-1}(1-s_1)\omega_1$
$Q=(1-u_1)\sin z$
$\chi=\sin^{-1}Q$
Here χ is needed only if $Q<1$

I_2 New initial conditions if $\sin z_1<1$

$z\leftarrow\pi-\chi$
$p\leftarrow p(1-\omega_1)$
$T\leftarrow T$
$S\leftarrow h_1/(a+h_1)$

Output O

O_1 if $h<h_1$
$R=R_1(z)-R_1(z_1)+R_2(z_1)$

- - - - -

O_{21} if $h>h_1$ and $Q\geq 1$
$R=R_2(z)$

- - - - -

O_{22} if $h>h_1$ and $Q<1$
$R=R_2(z)+\delta_1-\delta_2$

In the flow chart of Fig. 1, the input is the data of Box D and the geophysical constants of Table II. The output is box O.

A Fortran routine, prepared by B. Rodin of the Computing Laboratory, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland, is available on request.

C. Examples

To illustrate the use of the new algorithm we shall calculate the standard horizontal refraction with the old constants of Table II.

$$D \qquad z_*=90°, \quad p_*=760, \quad T_*=0, \quad h=0, \quad T'=-0.005694$$

$$I_0 \qquad z=\tfrac{1}{2}\pi, \quad p=1, \quad T=1, \quad S=0,$$

$$C_1 \qquad \begin{array}{ll} n=5 & \alpha=0.00029429 \\ \gamma=8.15788 & \beta=0.039171 \\ \beta_1=0.038877 & \beta_2=0.003650 \\ \Gamma=8.33627 & C=0.964619 \\ b_1=0.033660 & b_2=-0.00365 \\ K=4226''.2 & \end{array}$$

$$W_1 \qquad v_1=v_2=w_1=w_2=1$$

$$m=\begin{bmatrix} 5 & 9 & 13 & 17 & 21 & 25 & 29 \\ 6 & 10 & 14 & 18 & 22 & 26 & 30 \\ 7 & 11 & 15 & 19 & 23 & 27 & 31 \end{bmatrix}$$

$$M_{ij}=\begin{bmatrix} 1 & 9 & 91 & 969 & 10626 & 11876\times10 & 13449\times10^2 \\ 6 & 90 & 1274 & 17422 & 23377\times10 & 30876\times10^2 & 40347\times10^3 \\ 28 & 594 & 10920 & 184110 & 29328\times10^2 & 44891\times10^3 & \end{bmatrix}$$

$$B_{11} \qquad \theta_1=\tfrac{1}{2}\pi$$

$$\psi_m=\begin{bmatrix} .406345 & .29954 & .24817 & .21653 & .19455 & .17763 & .16433 \\ .36941 & .28377 & .23898 & .21034 & .19003 & .17400 & .16145 \\ .34099 & .27026 & .23074 & .20466 & .18561 & .17059 & .15871 \end{bmatrix}$$

$$M_{ij}b_1{}^ib_2{}^j=\begin{bmatrix} 1.00000 & .30294 & .10310 & 3695 & 1364 & 513 & 196 \\ -02190 & -1105 & -527 & -243 & -110 & -49 & -21 \\ 37 & 27 & 16 & 10 & 5 & 3 & 1 \end{bmatrix}.$$

On a desk calculator, we took $i=0, 1, \cdots, 6$; $j=0, 1, 2$. The summation gave $R_{11}=2204''.0$, with an extrapolated remainder $\Delta R=0''.5$. Therefore,

$$R_1=2204''.5,$$

in agreement with the output of an electronic machine. We terminate the illustration at this point; the stratosphere contributes

$$R_2(z_1)-R_1(z_1)=2'',$$

so that the total refraction is

$$R_*=2206''.5.$$

On the other hand, the 1944 paper gave

$$\begin{aligned} R_1&=2205''.9 \\ \epsilon_1&=\ -1''.0 \\ \epsilon_2&=\ +2''.0 \\ \hline R_*&=2206''.9, \end{aligned}$$

where ϵ_1 and ϵ_2 are corrections. It thus appears that the improvements in the algorithm introduced in this paper do not alter the value of refraction by more than $0''.5$ for $z=90°$. It is only for refraction below the horizon that the improvement becomes significant.

An internal check of the theory, based on Eq. (85), is illustrated in Table III.

The effect of the temperature gradient and the rapid growth of refraction with z are seen in Tables IV and V, respectively.

D. Numerical Integration

A comparison of the theory with the results of numerical integration appears in Table VI. The initial conditions belong to Greenbaum (1954), who used the author's theory with the revised constants of Table II. Unfortunately, Greenbaum's published values of theoretical refraction are erroneous, as pointed out by this author in private communications (1964, 1966). Column 1 of Table VI gives the correct values, furnished by the Ballistic Research Laboratories Electronic and Scientific Computer following the Fortran routine of Sec. VB. The results agree precisely with column 2, based on numerical quadrature of the refraction integral (3). The last column is due to Simon (1966), who applied quadrature to a nonlinear fit of the Rocket Panel Data (1952), exhibited in Table VII. A linear fit of this data yields a temperature gradient of -6.56 °C/km, which is essentially the same as that of the U. S. Standard. The discrepancy of $2''$ at $z=89°$ must therefore be ascribed to a strong nonlinearity of the temperature profile data near the ground. If any statistical significance should be attached to such a departure from the assumed polytropic model, its effect on refraction could be treated by the method of perturbations. This, however, lies beyond the scope of this paper.

Table III. Refraction for a grazing ray (old constants).

z (deg)	p (mm)	T (°C)	h (m)	Refraction (sec of arc)
87.9202	392.570	-28.462	5000	661.0
92.0798	392.570	-28.462	5000	3751.8
90.0000	760	0	0	2206.4

Table IV. Refraction at sea level. $p=760$ mm, $T=0$°C, $h=0$.

$-T'\times 10^3 \rightarrow$ z \ $n \rightarrow$	6.5 4.256	6.4 4.338	6.3 4.423	6.2 4.510	6.1 4.600	6.0 4.694
85°	614″	614″	614″	614″	614″	615″
86	732	732	733	733	733	733
87	898	898	898	899	899	899
88	1142	1143	1143	1144	1144	1144
89	1524	1525	1526	1527	1528	1529
90	2163	2166	2169	2173	2176	2179

E. Humidity Correction

The humidity correction has not been included in the algorithm of Sec. VB. It affects the refractivity α and the gas constant $\mathcal{R}$ of air. For a mixture of air and water vapor, these two constants are to be replaced by $\bar{\alpha}$ and $\bar{\mathcal{R}}$. A simple theory of the dielectric (e.g., Joos 1932) leads to

$$\bar{\alpha}=\alpha[1-(1-\alpha'/\alpha)p'/p], \qquad (219)$$
$$\bar{\mathcal{R}}=\mathcal{R}[1-(1-M'/M)p'/p],$$

Table V. Grazing ray refraction (old constants).

h (m)	z (deg)	p (mm)	T (°C)	R (sec of arc)
0	90.0000	760.000	0	2206″
1000	90.9142	669.687	-5.696	2980
2000	91.2990	588.404	-11.390	3269
3000	91.5977	515.746	-17.083	3470
4000	91.8530	450.653	-22.773	3627
5000	92.0798	392.570	-28.462	3751
6000	92.2869	340.879	-34.149	3853
7000	92.4790	295.003	-39.834	3938
8000	92.6592	254.443	-45.517	4011
9000	92.8295	218.583	-51.199	4070
10000	92.9916	187.073	-56.880	4122

where the unprimed quantities refer to dry air, and the primed quantities to water vapor, with p' being the water vapor pressure at the observer's station, and M the mean molecular weight. For our choice of wavelength

$$\alpha'=2.52\times 10^{-4} \qquad \alpha=2.92\times 10^{-4} \qquad (220)$$
$$M'=18.016, \qquad M=28.964$$

(*International Critical Tables* 1930). Then (219) becomes

$$\bar{\alpha}=\alpha(1-0.138p'/p), \qquad (221)$$
$$\bar{\mathcal{R}}=\mathcal{R}(1-0.378p'/p),$$

in agreement with Lorentz (1880).

Since relative humidity is known to decrease rapidly with altitude, the correction is more meaningful if in case $z>90°$ the ratio p'/p refers not to the station but to the ray perigee. In the absence of reliable information regarding the value of this ratio when the station is high above the ground and $z>90°$, it may be preferable to dispense with this correction entirely.

Table VI. Refraction at sea level $p=762$ mm, $T=26.67$°C, $h=0$.

z	Theory $n=4.256$ R	Integration *U. S. Standard* R	Integration Rocket Panel data R
85°	555″	555″	554″
86	659	659	659
87	805	805	804
88	1015	1015	1015
89	1337	1337	1339
90	1859	1859	

Table VII. Rocket Panel data (1952).

h (km)	T (°C)
1.216	291.0
2	282.0
4	272.0
6	260.0
8	245.0
10	230.8
12	219.5

VI. DISCUSSION OF THE RESULTS

In the derivation of the basic refraction formulas, it has been assumed that the polytropic index n is a constant along the branch of the ray within the troposphere. For the purposes of calculation, it has been furthermore assumed that n can be determined from the local temperature gradient T' at the observer's station. Although neither of these assumptions is true, it is a fact that for $z \leq 80°$ refraction is largely a function of initial conditions at the observer's station, and is insensitive to atmospheric distribution. It is only for $z > 80°$ that departures from a polytropic structure along the path of the ray may have an appreciable effect. To paraphrase the original paper, "a crucial observational check of the theory must await the time when the refraction data for $z > 80°$ become more abundant and more reliable." Such a check must be statistical in nature. Two questions can be asked: (1) How accurately does the theory predict the observed mean refraction for given z, p, T, and h? (2) How large is the observed scattering about the mean?

ACKNOWLEDGMENT

The research reported in this article was partially supported by the Air Force Office of Scientific Research Grant AFOSR 397–67 and by the Office of Naval Research Contract ONR 609(50).

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