Spontaneous synchronization for the Kuramoto model

Leonardo Hügens

(l.lobatodiasleitehugens@students.uu.nl)

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1 Introduction

The general concept of spontaneous synchronization appears all over nature, such as the group pulsating glow of fireflies or the simple fact that our hear beat consists of the uni-sonic contraction of our hear cells. The Kuramoto model is a mathematical framework which turns out to be ideal for the study of spontaneous synchronization, and consists in a set of phase oscillators which are coupled to each other with a certain strength. In this project I perform a theoretical analysis of the *mean-field Kuramoto model*, and some numerical simulations with varying frequency distributions and coupling strength values.

2 Theoretical Studies

The mean-field Kuramoto model is given by the following set of coupled differential equations:

$$\frac{d\theta_i}{dt}(t) = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin\left(\theta_j(t) - \theta_i(t)\right) \tag{1}$$

where $i = 1, \dots, N$, $\{\theta_i\}$ and $\{\omega_i\}$ are respectively the natural frequencies and the phases of the oscillators, and K is a coupling constant.

If we define the mean natural frequency as follows:

$$\bar{\omega} := \frac{1}{N}\sum_{i=1}^{N} \omega_i \tag{2}$$

it behaves as the derivative of the total mean phase, as intuitively expected, which we can see by explicitly differentiating it with respect to time:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{1}{N}\sum_{i=1}^{N}\theta_i(t)\right) &= \frac{1}{N}\sum_{i=1}^{N}\frac{d\theta_i}{dt}(t) \\
&= \frac{1}{N}\sum_{i=1}^{N}\left[\omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin\left(\theta_j(t)-\theta_i(t)\right)\right] \\
&= \frac{1}{N}\sum_{i=1}^{N}\omega_i + \frac{K}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sin\left(\theta_j(t)-\theta_i(t)\right) \\
&= \frac{1}{N}\sum_{i=1}^{N}\omega_i + \frac{K}{N^2}\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\sin\left(\theta_j(t)\right)\cos\left(\theta_i(t)\right) - \sum_{i=1}^{N}\sum_{j=1}^{N}\sin\left(\theta_i(t)\right)\cos\left(\theta_j(t)\right)\right] \\
&= \frac{1}{N}\sum_{i=1}^{N}\omega_i \\
&= \bar{\omega}
\end{aligned}
$$

This means that if at some point all the oscillators become synchronized, meaning that their phases are all the same (i.e. $\theta_i\left(t\right) = \bar{\theta}(t)\,\forall i$, they are *phase locked*) then the frequency of that bulk oscillation is $\bar{\omega}$, which is a constant by definition and can be determined just by calculating the average of all the natural frequencies of oscillators.

It can be insightful to perform a coordinate transformation to a reference frame which rotates with this mean natural frequency:

$$\phi_i(t) := \theta_i(t) - \bar{\omega}t \tag{3}$$

In this frame, the model becomes:

$$
\begin{aligned}
\frac{d\theta_i}{dt}(t) &= \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin\left(\theta_j(t)-\theta_i(t)\right) \\
\frac{d}{dt}\left(\phi_i(t)+\bar{\omega}t\right) &= \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin\left(\phi_j(t)+\bar{\omega}t-\phi_i(t)-\bar{\omega}t\right) \\
\frac{d\phi_i}{dt}(t) &= \left(\omega_i-\bar{\omega}\right) + \frac{K}{N}\sum_{j=1}^{N}\sin\left(\phi_j(t)-\phi_i(t)\right)
\end{aligned} \tag{4}
$$

2.1 Lyapunov function for studying the stability of the *equilibria*

The evolution of this system to a synchronized stable state can be studied in the formalism of Lyapunov stability. The Lyapunov stability theorem, as presented by [3] is:

Theorem 1. ***Lyapunov's stability theorem**: Considering the autonomous system*

$$\dot{\boldsymbol{x}} = f\left(\boldsymbol{x}\right) \tag{5}$$

where $f : D \to R^n$ is a locally Lipschitz map from a domain $D \subset R^n$ into R^n. Let $\boldsymbol{x} = 0$ be an equilibrium point, i.e. $f(\mathbf{0}) = 0$. Let $V : D \to R$ be a continuously differentiable function (called a Lyapunov function) such that

$$V(\mathbf{0}) = 0 \text{ and } V(\boldsymbol{x}) > 0 \text{ in } D - \{\mathbf{0}\} \tag{6}$$

$$\dot{V}(\boldsymbol{x}) \leq 0 \text{ in } D \tag{7}$$

Then $\boldsymbol{x} = \mathbf{0}$ is stable. Moreover, if

$$\dot{V}(\boldsymbol{x}) < 0 \text{ in } D - \{\mathbf{0}\} \tag{8}$$

then $\boldsymbol{x} = \mathbf{0}$ is asymptotically stable.

A Lyapunov function for the Kuramoto model, whose stability properties are studied in [4, 1] is:

$$\mathcal{H} := -\frac{K}{2N}\sum_{i,j}\cos(\phi_i - \phi_j) - \sum_{i=1}^{N}(\omega_i - \bar{\omega})\phi_i \tag{9}$$

It is not immediate to verify the conditions (6), but they are easily verifiable by adding a constant to the function, which is irrelevant for it to satisfy equation (7):

$$\begin{aligned}
\dot{\mathcal{H}} &= \sum_{i=1}^{N}\frac{\partial \mathcal{H}}{\partial \phi_i}\frac{d\phi_i}{dt} \\
&= \sum_{i=1}^{N}\frac{d\phi_i}{dt}\frac{\partial}{\partial \phi_i}\left[-\frac{K}{2N}\sum_{j,k}\cos(\phi_j - \phi_k) - \sum_{l=1}^{N}(\omega_l - \bar{\omega})\phi_l\right] \\
&= \sum_{i=1}^{N}\frac{d\phi_i}{dt}\left[\frac{K}{2N}\sum_{j,k}\sin(\phi_j - \phi_k)(\delta_{ij} - \delta_{ik}) - \sum_{l=1}^{N}(\omega_l - \bar{\omega})\delta_{il}\right] \\
&= \sum_{i=1}^{N}\frac{d\phi_i}{dt}\left[\frac{K}{2N}\left\{\sum_{j,k}\sin(\phi_j - \phi_k)\delta_{ij} - \sum_{j,k}\sin(\phi_j - \phi_k)\delta_{ik}\right\} - (\omega_i - \bar{\omega})\right] \\
&= \sum_{i=1}^{N}\frac{d\phi_i}{dt}\left[\frac{K}{2N}\left\{\sum_{k}\sin(\phi_i - \phi_k) - \sum_{j}\sin(\phi_j - \phi_i)\right\} - (\omega_i - \bar{\omega})\right] \\
&= -\sum_{i=1}^{N}\frac{d\phi_i}{dt}\left[\frac{K}{N}\sum_{j}\sin(\phi_j - \phi_i) + (\omega_i - \bar{\omega})\right] \\
&= -\sum_{i=1}^{N}\left[\frac{K}{N}\sum_{j}\sin(\phi_j - \phi_i) + (\omega_i - \bar{\omega})\right]\left[\frac{K}{N}\sum_{j}\sin(\phi_j - \phi_i) + (\omega_i - \bar{\omega})\right] \\
&= -\sum_{i=1}^{N}\left[\frac{K}{N}\sum_{j}\sin(\phi_j - \phi_i) + (\omega_i - \bar{\omega})\right]^2 \leq 0
\end{aligned}$$

Thus, the evolution of the system described converges to the minimum of $\mathcal{H}$. In order to determine the *extreme points* of $\mathcal{H}$, we must determine the conditions for which $\nabla \mathcal{H} = 0$:

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial \phi_i} &= 0 \\
\frac{K}{N} \sum_j \sin\left(\phi_j - \phi_i\right) + \left(\omega_i - \bar{\omega}\right) &= 0 \\
\sum_j \sin\left(\phi_j - \phi_i\right) &= -\frac{N}{K}\left(\omega_i - \bar{\omega}\right) \qquad (10)
\end{aligned}$$

This condition is better understood in terms of order parameters, which are typically scalar functions determined by the system's degrees of freedom that experience a clear change in behavior, also known as *phase transition.* In our case, such order parameter is the norm r of the *phasor* that results from adding all the individual phasors $e^{i\phi_i}$ of the system:

$$re^{i\Psi} := \frac{1}{N} \sum_{j=1}^{N} e^{i\phi_j} \qquad (11)$$

The vectorial addition present in this definition can be visually understood as gluing the tips of each phasor in the complex plane to each other, r being the norm of the phasor that goes from the origin to the tip of the last phasor in the ordered sum. Considering the situation in which all the oscillators are synchronized, i.e. $\theta_i\left(t\right) = \bar{\theta}(t) \, \forall i$, the $re^{i\Psi}$ phasor will be exactly superimposed with all the glued oscillators, but with norm 1. Because this is the only situation in which we can have $r = 1$, it's very easy to identify the phase transition corresponding the the synchronization in a r vs t plot for example, even if we have a very large number N of oscillators.

Because the real and imaginary parts of both sides of the definition need to be equal, this implies that

$$r\sin(\Psi - \phi_i) = \frac{1}{N} \sum_{j=1}^{N} \sin\left(\phi_j - \phi_i\right) \qquad (12)$$

$$r\cos(\Psi - \phi_i) = \frac{1}{N} \sum_{j=1}^{N} \cos\left(\phi_j - \phi_i\right) \qquad (13)$$

We can thus rewrite the stability conditions (10) is terms of this order parameter:

$$\begin{aligned}
\sum_j \sin\left(\phi_j - \phi_i\right) &= -\frac{N}{K}\left(\omega_i - \bar{\omega}\right) \\
Nr\sin(\Psi - \phi_i) &= -\frac{N}{K}\left(\omega_i - \bar{\omega}\right) \\
\left(\omega_i - \bar{\omega}\right) &= Kr\sin(\phi_i - \Psi) \\
\Delta\left(\omega_i\right) &= Kr\sin(\phi_i - \Psi) \qquad (14)
\end{aligned}$$

2.2 Model with N oscillators

Let's now consider for simplicity the case in which all the natural frequencies of the oscillators ω_i are the same - which implies that the mean natural frequency is the same as each individual frequency, $\bar{\omega} = \omega_i \, \forall i$ - and that $K = 1$. In this case, in the rotating frame previously discussed, the model (4) becomes:

$$\frac{d\phi_i}{dt}(t) = \frac{1}{N}\sum_{j=1}^{N} \sin\left(\phi_j(t) - \phi_i(t)\right) \tag{15}$$

$$\frac{d\phi_i}{dt}(t) = r\sin(\Psi\,(t) - \phi_i\,(t)) \tag{16}$$

where I used (13) to write it in terms of the order parameter. In the previous section, where I showed that $\nabla\mathcal{H} = 0$ for the Hamiltonian (9), I determined that in general

$$\frac{\partial\mathcal{H}}{\partial\phi_i} = -\frac{K}{N}\sum_{j} \sin\left(\phi_j - \phi_i\right) + \left(\omega_i - \bar{\omega}\right) \tag{17}$$

Thus, it turns out that $-\frac{\partial\mathcal{H}}{\partial\phi_i} = \frac{d\phi_i}{dt}$, i.e. our model is a gradient system [2]. Using our present case $\bar{\omega} = \omega_i \, \forall i$ for simplicity, this Hamiltonian can be rewritten as

$$\begin{aligned}
\mathcal{H} &= -\frac{1}{2N}\sum_{i,j} \cos\left(\phi_i - \phi_j\right) \\
&= -\frac{1}{2} r \sum_{i} \cos(\Psi - \phi_i) \\
&= -\frac{1}{2} r^2 \sum_{i} \cos(\Psi - \Psi) \\
&= -\frac{N}{2} r^2
\end{aligned}$$

where I used the identity (13) twice. We thus conclude that the Hamiltonian, which can be thought of the energy of a system, has it's minimum at the maximum r value, which we've argued before it's unique and it corresponds to the synchronized situation $\theta_i\,(t) = \bar{\theta}(t)\, \forall i$. This shows the usefulness of finding and order parameter and expressing our systems dynamics in terms of it.

In our case we can also show that the means phase $\frac{1}{N}\sum_{i=1}^{N}\theta_i(t)$ is conserved in time, as follows:

$$
\begin{aligned}
\frac{d\phi_i}{dt}(t) &= \frac{1}{N}\sum_{j=1}^{N}\sin\left(\phi_j-\phi_i\right)\\
\sum_{i=1}^{N}\frac{d\phi_i}{dt}(t) &= \frac{1}{N}\sum_{i,j=1}^{N}\sin\left(\phi_j-\phi_i\right)\\
\sum_{i=1}^{N}\frac{d\phi_i}{dt}(t) &= \sum_{i=1}^{N}\sum_{j=1}^{N}\sin\left(\theta_j(t)\right)\cos\left(\theta_i(t)\right)-\sum_{i=1}^{N}\sum_{j=1}^{N}\sin\left(\theta_i(t)\right)\cos\left(\theta_j(t)\right)\\
\implies \frac{d}{dt}\left[\frac{1}{N}\sum_{i=1}^{N}\phi_i(t)\right] &= 0\\
\implies \frac{1}{N}\sum_{i=1}^{N}\phi_i(t) &= \frac{1}{N}\sum_{i=1}^{N}\phi_i(t=0)
\end{aligned}
$$

3 Numerical studies of the Kuramoto model

At this stage we proceed to evolving (1) in time, by choosing the natural frequencies ω_i with some distribution, imposing initial conditions $\{\theta_i\}$ and using the Euler method. This means that we pick a time-step dt which is sufficiently small compared to the other relevant quantities in the system, and perform an iterative use of the following equation:

$$\theta_i\left(t+dt\right)=\theta_i\left(t\right)+\frac{d\phi_i}{dt}dt \tag{18}$$

It is very insightful to note that this expression can also be written as

$$\theta_i\left(t+dt\right)=\theta_i\left(t\right)-\frac{\partial\mathcal{H}}{\partial\phi_i}dt \tag{19}$$

$$\boldsymbol{\theta}\left(t+dt\right)=\boldsymbol{\theta}\left(t\right)-\nabla\mathcal{H}\,dt \tag{20}$$

which means that if we have an N dimensional hyper plane with all the θ_i in orthogonal axes, and plot the correspondent $\mathcal{H}$ values in an orthogonal axis to that plane, the system will generally take infinitesimal steps in the direction of steepest descent, converging to the minimum of $\mathcal{H}$. This statement is intuitive but not absolute, as there can sometimes be local minima that 'trap' the system, stopping it from converging to the global minimum. In our case there is a global minimum as I've stated before, which corresponds to the *phase locking* of all the oscillators.

3.1 Normal distribution of the natural frequencies

Let's firstly consider the case in which we sample our frequencies ω_i from a normal distribution $N\left(0,1\right)$, and our initial phases from a uniform distribution. At first it is insightful to plot $\sin\left(\theta_i\right)$ vs t, to check if the result corresponds to our intuitive expectations, and make sure the code implementation of Euler's method is properly done. These plots are represented in Figure

1. Each 'rectangle' of the Figure corresponds to a K value, and has $N = 10$ curves which correspond to the $\sin(\theta_i)$ evolution of the oscillators, with $t \in [0, 100]$, $dt = 0.01$. The K valued rectangles range from top to bottom as $K \in [0, 5]$ with $dK = 0.5$. Initial ω_i and θ_i conditions were kept the same for all K values.

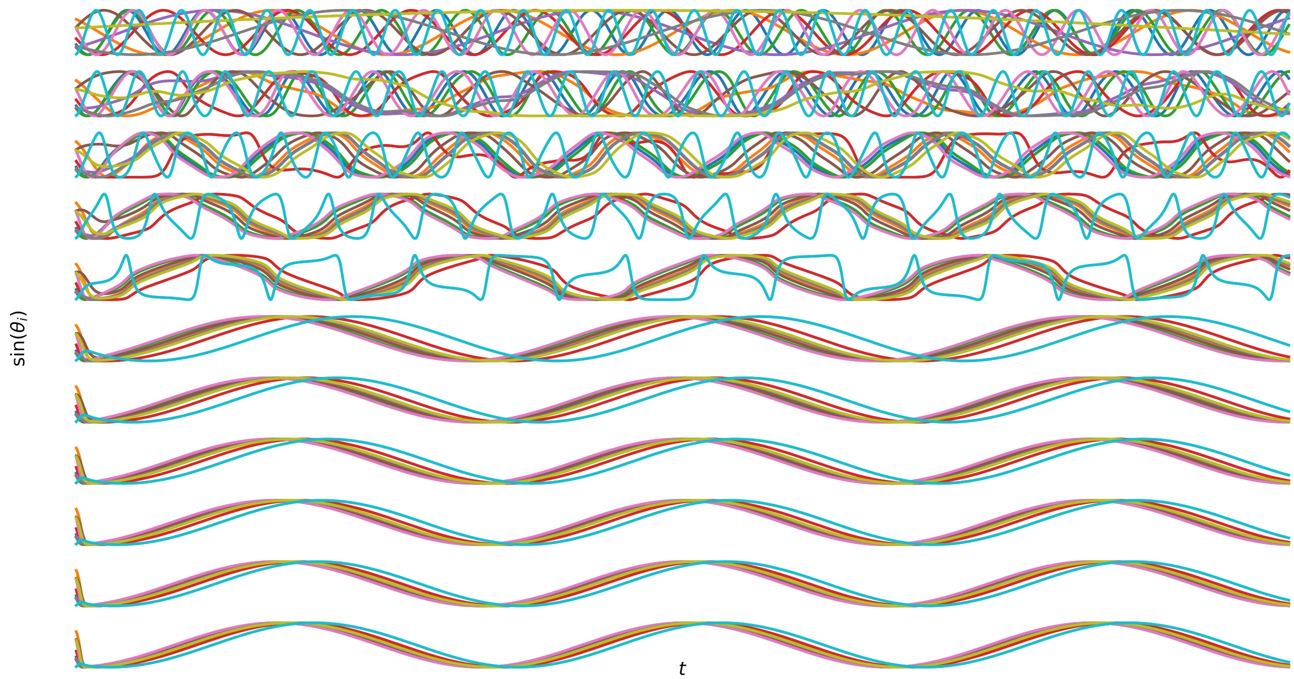

Figure 1: $\sin(\theta_i)$ vs t plots. $N = 10$. Normally distributed ω's.

For large K values, all the oscillators synchronize almost instantaneously, without even completing a full oscillation without it already becoming synchronized. It may be strange that in this plot we would expect the oscillations to become inexistent as K increases, since we're sampling ω_i from a normal distribution with average value 0, and yet we can clearly see that there are still visible oscillations for large K. This is simply because a variance of 1 is very significant for a $N = 10$ sample, and thus the average frequency is still large enough to be noticeable in the plot. It is also noticeable that there isn't much difference between the first K values in Figure 1, whereas the difference from the 4th to the 5th K values is clear, which hints us for the existence of a critical value K_c for which the system synchronizes.

A stationary synchronization theoretical analysis performed in detail in [1] tells us that if our natural frequencies ω_i follow a certain distribution $g(\omega)$ then our order parameter r obeys a consistency equation given by:

$$1 = K \int_{-\pi/2}^{\pi/2} \cos^2(\theta) g(Kr \sin\theta) d\theta \tag{21}$$

From this consistency equation it is possible to extract a critical value K_c for which the phase transition occurs, which is given by $K_c = 2\sqrt{2/\pi} \approx 1.6$ for a standard normal distribution $g(\omega) = e^{-\omega^2/2}/\sqrt{2\pi}$.

With this in mind, increasing the number of oscillators to $N = 1000$, *coarse-graining* the K values to $K \in [0, 5]$ with $dK = 0.2$ and only plotting the order parameter at the last time step $r_{t=100}$ vs K should clarify if there exists in fact a phase transition or not. In the following Figure 2 I plot several runs of this simulation, which differ by resampled ω_i and initial θ_i. I also plot a theoretical curve value pairs (r,K) correspond to solutions of the consistency equation (21).

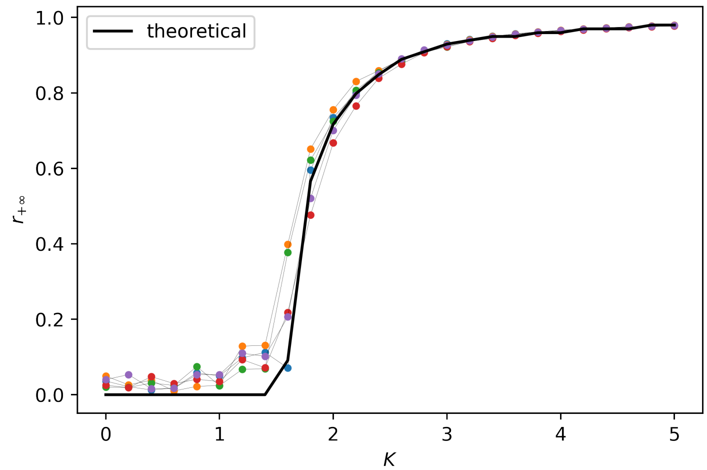

Figure 2: $r_{t=100}$ vs K plot for normal distributed ω_i.

We observe a high agreement between the simulations and what is predicted theoretically, a phase transition occurs for roughly $K \approx 1.6$. Because the derivation of the consistency equation (21) uses the $N \to +\infty$ limit, it is expected that we can improve the mentioned agreement even further by using arbitrarily larger number of oscillators in the simulation. Now, it is also important to remark that our 'theoretical' curve was also obtained by an algorithm that uses a finite set of r and K values, and it could be improved by decreasing the interval steps used in these sets. In the continuous limit, it is possible to identify a first order phase transition at K_c, which is unsafe to infer by analyzing our plot in Figure 2.

We now try to visualize further this phase transition, trying to catch how soon in the simulation it happens. Running the model for $K = 1$ and $K = 2$ and plotting $r\left(t\right)$, we obtain the plot in Figure 3.

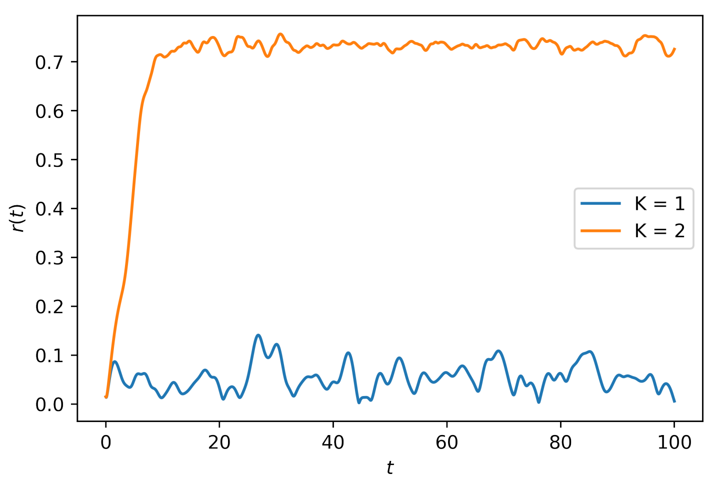

Figure 3: $r\left(t\right)$ for $K = 1, 2$.

These results are consistent with what is expected, as by analyzing 2 we can see that for $K = 1$ definitely shouldn't occur synchronization (identified by $r = 1$), and for $K = 2$ it should, and these stabilization r values seem to be roughly the same as those obtained in Figure 2.

3.2 Uniform distribution of the natural frequencies

We now turn to the case where we have a uniform distribution of the natural frequencies between $-\frac{1}{2}$ and $\frac{1}{2}$:

$$g(\omega) = \begin{cases} 1, & |\omega| \leq \frac{1}{2} \\ 0, & |\omega| > \frac{1}{2} \end{cases} \tag{22}$$

In Figure 4 I present the visually intuitive plot corresponding to this case.

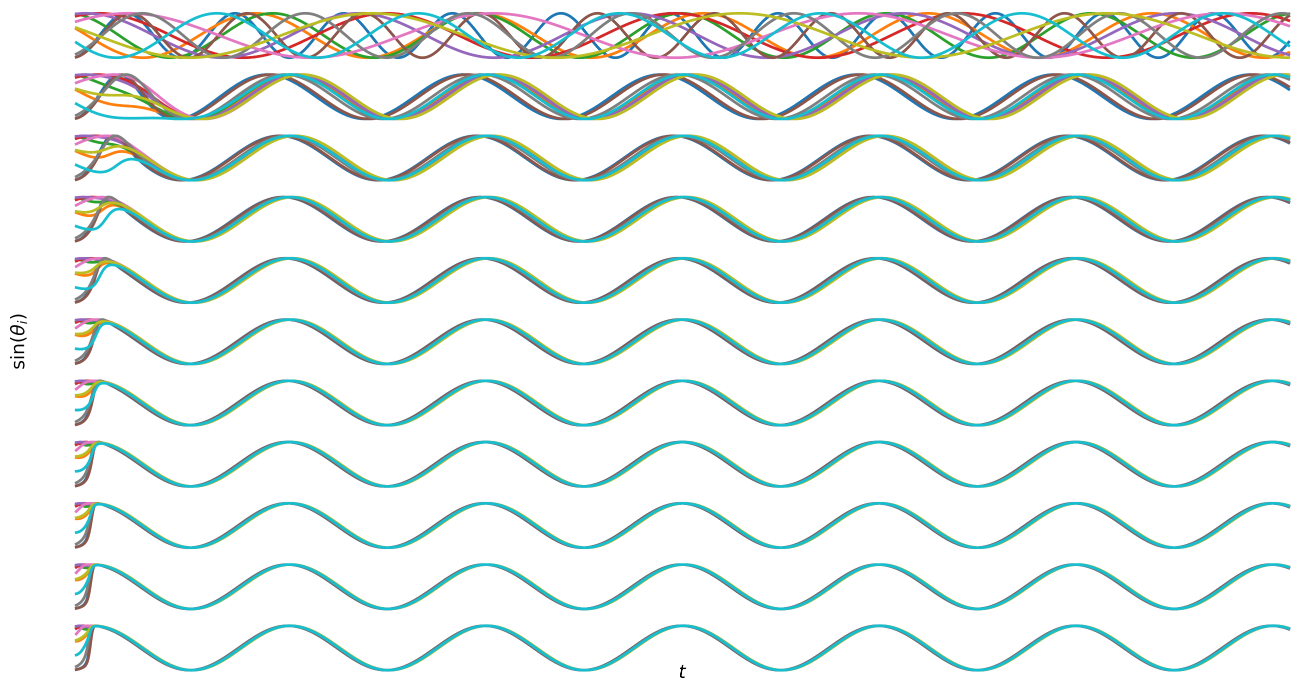

Figure 4: $\sin(\theta_i)$ vs t plots. $N = 10$. Uniformly distributed ω's.

Just from this plot it is possible to infer that the phase transition where all the oscillators synchronize will occur sooner (i.e. for a smaller K value).

Performing the much more rigorous procedure just as in the normally distributed ω's case, we obtain the following curves:

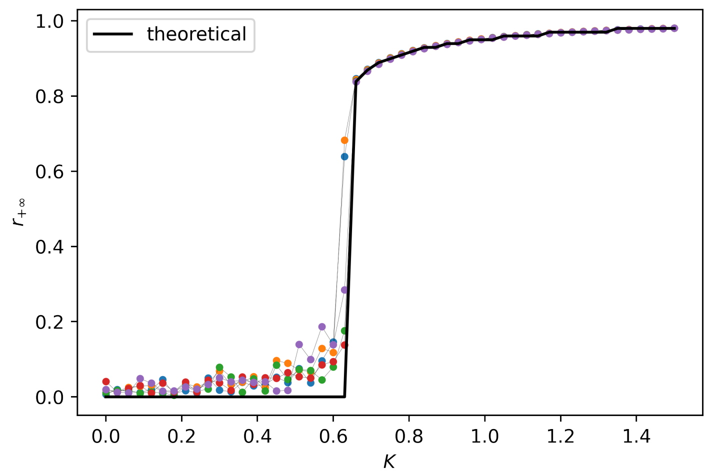

Figure 5: $r_{t=100}$ vs K plot for uniformly distributed ω_i.

Once more, we obtain a high agreement with the theoretical and empirical approach. We thus confirm that in this case the phase transition occurs at roughly $K_c \approx 0.65$, which confirms our suspicion from Figure 4.

Instead of resampling ω_i and the initial θ_i for each simulation as we did in the simulations represented in Figures 2 and 5, it would also be interesting to see if the t instant where the oscillators synchronize varies significantly with the initial conditions. With this in mind, I perform two simulations where I fix the initial θ_i's and the ω_i's respectively, with 10 runs for each. The results are plotted in Figure 6.

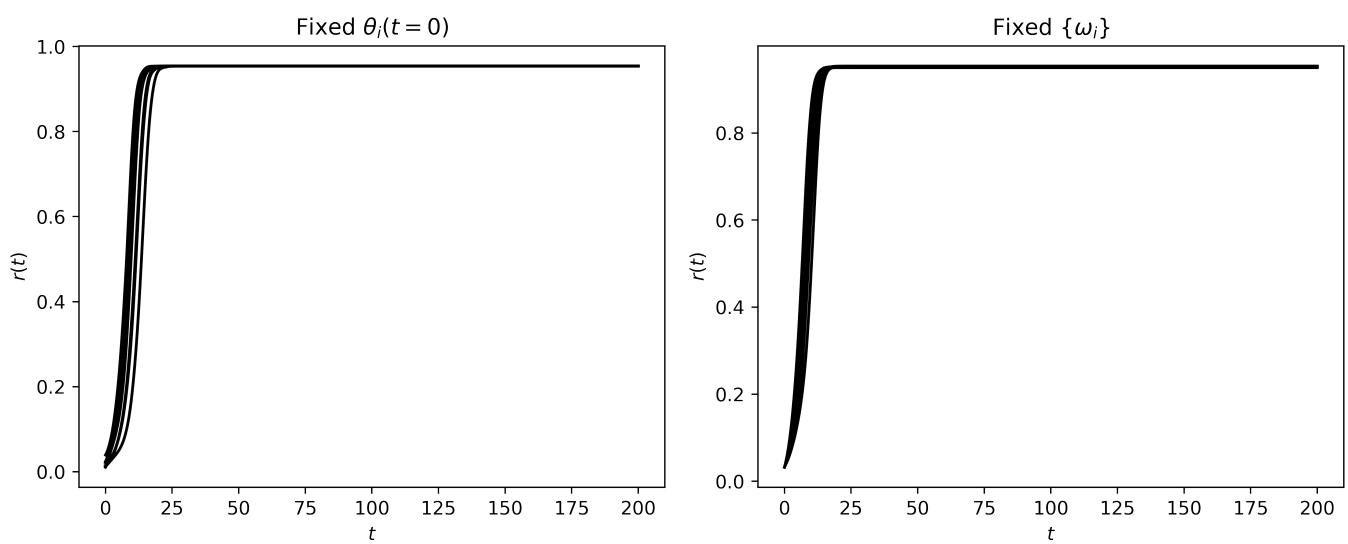

Figure 6: $r\,(t)$ for fixed $\theta_i\,(t=0)$ (left) and fixed $\{\omega_i\}$ (right).

We thus observe that the synchronization time is very stable with respect to the contribution from the variance in the $\{\omega_i\}$ and $\theta_i\,(t=0)$ samples in the case where both are sampled uniformly.

4 Conclusion

In this project we studied the mean-field Kuramoto model, namely with respect to the existence of a asymptotic stable solution, which corresponds to the synchronized state in which all the oscillators have the same phase trough time, identified by the order parameter $r = 1$. We also studied the K values for which the system evolves to this state, and discovered a phase transition, it being the existence of a critical K_c such that the synchronization only happens for $K > K_c$.

One of the main advantages of this model in the understanding of the synchronization of phase oscillators is the possibility of coming up with exact theoretical predictions like the consistency equation (21), which provide reassurance beyond what simulations procedures can bring.

Code used for the simulations

The python code used to perform the simulations is available at github.com/lhugens/kuramoto.

References

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