Adaptative Exponential Integrate-and-Fire Model

Leonardo Hügens

l.lobatodiasleitehugens@students.uu.nl

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system

$$
\begin{aligned}
C\dot{V} &= I - g_{\mathrm{L}}\left(V - E_{\mathrm{L}}\right) + g_{\mathrm{L}}\Delta_{\mathrm{T}}\mathrm{e}^{(V-V_{\mathrm{T}})/\Delta_{\mathrm{T}}} - w \\
\tau_w \dot{w} &= a\left(V - E_{\mathrm{L}}\right) - w
\end{aligned}
$$

consider the following transformation of coordinates:

$$
\begin{aligned}
\tau'_w &= \frac{\tau_w}{\tau_m} = \frac{g_L \tau_w}{C} \\
a' &= \frac{a}{g_L} \\
I' &= \frac{I}{g_L \Delta_T} + \left(1 + \frac{a}{g_L}\right)\frac{E_L - V_T}{\Delta_T} \\
t' &= \frac{t}{\tau_m} \\
b' &= \frac{b}{g_L \Delta_T} \\
V'_r &= \frac{V_r - V_T}{\Delta_T} \\
V'(t') &= \frac{V(t) - V_T}{\Delta_T} \\
w'(r') &= \frac{w(t) + a\left(E_L - V_T\right)}{g_L \Delta_T}
\end{aligned}
$$

the system becomes:

$$
\begin{cases}
\dot{V'} & = -V' + e^{V'} - w' + I' \\
\tau'_w \dot{w'} & = a'V' - w'
\end{cases}
$$

fixed points

$$
\begin{cases} 0 & = -V' + e^{V'} - w' + I' \\ 0 & = a'V' - w' \end{cases}
$$

$$
\Longrightarrow \begin{cases} -e^{V'} + (a'+1)\,V' & = I' \\ w' & = a'V' \end{cases}
$$

Considering the fixed point expression, let's define $F(V') = e^{V'} - (a'+1)\,V'$. Because $F''(V') = e^{V'} > 0\ \forall V'$, F is convex, which implies it has a minimum at a certain V'_{min}. Thus, we can split the existence of solution to the fixed point equation in 3 cases: if $F(V'_{\text{min}}) > -I'$, then there are no V' solutions, if $F(V'_{\text{min}}) = -I'$, there is only the solution $V' = V'_{\text{min}}$, and if $F(V'_{\text{min}}) < -I'$ there are two solutions, let's call them V_+ and V_-.

$$
e^{V'} - (a'+1)\,V' = -I'
$$

To write the first expression in the form $V_{\text{fixed}}(I)$, we can use Lambert's W function. The equation

$$
ye^{y} = x
$$

has solutions

$$
y = \begin{cases} W_0(x) & ,\text{if } x \geq 0 \\ W_0(x) \text{ and } W_{-1}(x) & ,\text{if } -\frac{1}{e} \leq x < 0 \end{cases}
$$

Thus, we have:

$$
\begin{aligned}
(a'+1)\,V' - I' &= e^{V'} \\
\left[(a'+1)\,V' - I'\right] e^{-V'} &= 1 \\
\left[-V' + \frac{I'}{(1+a')}\right] e^{-V'} e^{\frac{I}{(1+a')}} &= -\frac{1}{(1+a')} e^{\frac{I}{(1+a')}} \\
\left[-V' + \frac{I'}{(1+a')}\right] e^{\left[-V' + \frac{I'}{(1+a')}\right]} &= -\frac{1}{(1+a')} e^{\frac{I}{(1+a')}} \\
\Longrightarrow \left[-V' + \frac{I'}{(1+a')}\right] &= W_i\left[-\frac{1}{(1+a')} e^{\frac{I}{(1+a')}}\right] \\
V^{'} &= -\frac{I'}{(1+a')} - W_i\left[-\frac{1}{(1+a')} e^{\frac{I}{(1+a')}}\right]
\end{aligned}
$$

coming back to our original coordinates, we have

$$\begin{aligned}
V_{+} &= E_L + \frac{I}{g_L + a} - \Delta_T W_0 \left(-\frac{1}{1 + a/g_L} e^{\frac{I}{\Delta_T (g_L + a)}} + \frac{E_L - V_T}{\Delta_T} \right) \\
V_{-} &= E_L + \frac{I}{g_L + a} - \Delta_T W_{-1} \left(-\frac{1}{1 + a/g_L} e^{\frac{I}{\Delta_T (g_L + a)}} + \frac{E_L - V_T}{\Delta_T} \right)
\end{aligned}$$

Now, let's analyze the stability of the fixed point curves. In the simplified coordinates, the Jacobian matrix of this system is

$$\mathbf{J}'(V', w') = \begin{bmatrix} \frac{\partial \dot{V}'}{\partial V'} & \frac{\partial \dot{V}'}{\partial w'} \\ \frac{\partial \dot{w}'}{\partial V'} & \frac{\partial \dot{w}'}{\partial w'} \end{bmatrix} = \begin{bmatrix} e^{V'} - 1 & -1 \\ \frac{a'}{\tau'_w} & -\frac{1}{\tau'_w} \end{bmatrix}$$

Denoting $\Delta = \det\left(\mathbf{J}'(V', w')\right)$, we have:

$$\begin{aligned}
\Delta &= -\frac{e^{V'} - 1}{\tau'_w} + \frac{a'}{\tau'_w} \\
&= -\frac{e^{V'}}{\tau'_w} + \frac{1}{\tau'_w}\left(1 + a'\right)
\end{aligned}$$

It's derivative being $\frac{d\Delta}{dV'} = -\frac{e^{V'}}{\tau'_w} < 0 \; \forall V'$, since $\tau'_w > 0$, it's a monotonically decreasing function, and it's zero is $V^* = \log\left(1 + a'\right)$, which also corresponds to V_{min}, since $F'\left(V_{\text{min}}\right) = 0 \implies V_{\text{min}} = \log\left(1 + a'\right)$. Thus, in all the fixed points of the V_+ branch the determinant of the Jacobian matrix is negative, which means they are all saddle points, i.e. they are unstable fixed points. For the V_- branch, the determinant is positive, so to analyze the stability of the fixed points we need to analyze the trace of the Jacobian matrix $\tau = \mathrm{T}\left(\mathbf{J}'(V', w')\right)$, which is:

$$\tau = e^{V'} - 1 - \frac{1}{\tau'_w}$$

Thus, $\tau\left(V'_{\text{min}}\right) = \left(1 + a'\right) - \left(1 + \frac{1}{\tau'_w}\right) = a' - \frac{1}{\tau'_w}$, which means that when $a' < \frac{1}{\tau'_w} \iff a < \frac{C}{\tau_w}$ the fixed point at V_{min} is stable, and when $a < \frac{C}{\tau_w}$ it is stable.

Analyzing τ at the branch V'_-, we can use the fact that τ is an increasing function of V', and $V'_- < V'_{\text{min}} < V'_+$, which implies:

$$\tau\left(V'_-\right) \leq \tau\left(V'_{\text{min}}\right) = a' - \frac{1}{\tau'_w}$$

Thus, for case I, $a' < \frac{1}{\tau'_w} \iff a < \frac{C}{\tau_w}$, the entire V'_- branch is stable, which implies that two stable and unstable fixed points $\left(V'_+ \text{ and } V'_-\right)$ merge at V'_{min} and disappear, i.e. this is a saddle node bifurcation. The current at which this bifurcation happens is

$$
\begin{aligned}
I'_{rh} &= -e^{V'_{\min}} + (a'+1)\, V'_{\min} \\
&= -(1+a') + (1+a')\log(1+a') \\
\implies \frac{I^{I}_{rh}}{g_L \Delta_T} + \left(1+\frac{a}{g_L}\right)\frac{E_L - V_T}{\Delta_T} &= -\left(1+\frac{a}{g_L}\right) + \left(1+\frac{a}{g_L}\right)\log\left(1+\frac{a}{g_L}\right) \\
I^{I}_{\text{th}} &= (g_L + a)\left[V_T - E_L - \Delta_T + \Delta_T \log\left(1+\frac{a}{g_L}\right)\right]
\end{aligned}
$$

For case II, $a' > \frac{1}{\tau'_w}$, the trace τ may have positive values in the V_- brach generally, and the solutions to $\tau = 0$ are:

$$
V'_{rh} = \log\left(1+\frac{1}{\tau'_w}\right)
$$

Thus, because $V'_{rh} < V'_{\min}$, the initially stable point in the branch V'_- becomes unstable before the merging of the two branches, and so it corresponds to an Andronov-Hopf bifurcation. The current at which it happens is

$$
\begin{aligned}
I'_{rh} &= -e^{V'_{rh}} + (a'+1)\, V'_{rh} \\
&= -\left(1+\frac{1}{\tau'_w}\right) + \left(1+\frac{a}{g_L}\right)\log\left(1+\frac{1}{\tau'_w}\right) \\
\frac{I^{II}_{rh}}{g_L \Delta_T} + \left(1+\frac{a}{g_L}\right)\frac{E_L - V_T}{\Delta_T} &= -\left(1+\frac{C}{g_L \tau_w}\right) + \left(1+\frac{a}{g_L}\right)\log\left(1+\frac{C}{g_L \tau_w}\right) \\
I^{II}_{\text{rh}} &= (g_L + a)\left[V_T - E_L - \Delta_T + \Delta_T \log\left(1+\frac{\tau_m}{\tau_w}\right)\right] + \Delta_T\left(a - \frac{C}{\tau_w}\right)
\end{aligned}
$$