1. $f(x_1,x_2) = 10-2(x_1^4-2x_1^2x_2+x_2^2)$

$= -2x_1^4+4x_1^2x_2-2x_2^2+10$

$\frac{\partial f(x_1,x_2)}{\partial x_1} = -8x_1^3+8x_2x_1$, $\frac{\partial f(x_1,x_2)}{\partial x_2} = 4x_1^2-4x_2$

$\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} = -24x_1^2+8x_2$, $\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} = -4$

$\frac{\partial^2 f(x_1,x_2)}{\partial x_2 \partial x_1} = 8x_1 = \frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}$

Hessian matrix:

$$H = \begin{bmatrix} -24x_1^2+8x_2 & 8x_1 \\ 8x_1 & -4 \end{bmatrix}$$

$|H| = 96x_1^2-32x_2 -64x_1^2 = 32(x_1^2-x_2)$

let, $x_1=0$ and $x_2=1$, than $|H|<0$

so Hessian matrix is not positive semidefinite

Hence, $f(x_1,x_2)$ is not a convex function over the set S.

2.(a) By Jensen's inequality: $f(x)=f\left(\frac{b-x}{b-a}\cdot a+\frac{x-a}{b-a}\cdot b\right)\le \frac{b-x}{b-a}f(a)+\frac{x-a}{b-a}f(b)$. $\left(\frac{b-x}{b-a}\in[0,1],\ \frac{x-a}{b-a}\in[0,1].\ x\in[a,b]\right)$

(b) $f(x)\le\frac{(b-a)-(x-a)}{b-a}f(a)+\frac{x-a}{b-a}f(b)$

$\Rightarrow f(x)-f(a)\le\frac{x-a}{b-a}(f(b)-f(a))$

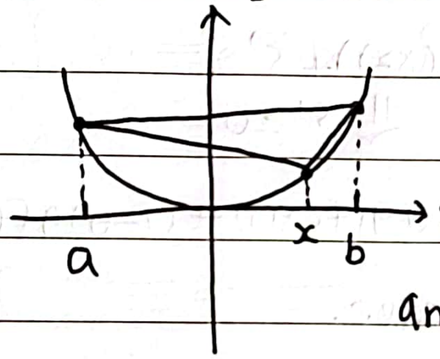
$\Rightarrow \frac{f(x)-f(a)}{x-a}\le\frac{f(b)-f(a)}{b-a}$

Similarly, $f(x)\le\frac{b-x}{b-a}f(a)+\frac{(b-a)-(b-x)}{b-a}f(b)$

$\Rightarrow f(b)-f(x)\ge\frac{b-x}{b-a}f(b)-\frac{b-x}{b-a}f(a)$

$\Rightarrow \frac{f(b)-f(x)}{b-x}\ge\frac{f(b)-f(a)}{b-a}$

Hence, $\frac{f(x)-f(a)}{x-a}\le\frac{f(b)-f(a)}{b-a}\le\frac{f(b)-f(x)}{b-x}$



the slope of the line ax is smaller than the line ab's and the slope of the line ab is smaller than the line xb's.

(c) let $x=a+\Delta x$, $\Delta x\to 0$

than we have $\frac{f(x)-f(a)}{x-a}=\frac{f(a+\Delta x)-f(a)}{a+\Delta x-a}=f'(a)$,

Similarly, we have $\frac{f(b-\Delta x)-f(b)}{b-\Delta x-b}=f'(b)$

Therefore, $f'(a)\le\frac{f(b)-f(a)}{b-a}\le f'(b)$

(d) let $b\to a+$, than we have: $f'(a)\le f'(a+)$

let $b=a+\Delta x$, $\Delta x\to 0+$, $\therefore f'(a)\le f'(a+\Delta x)$

$\Rightarrow \frac{f'(a+\Delta x)-f'(a)}{\Delta x}\ge\frac{0}{\Delta x}=0$, $\Rightarrow f''(a)\ge 0$

Similarly, we have $\frac{f'(b+\Delta x)-f'(b)}{\Delta x}\ \cancel{\le 0}\ \ge 0$, $f''(b)\ \cancel{\le 0}\ \ge 0$

Therefore, $f''(a)\ge 0$ and $f''(b)\ge 0$

$\log v_i + 1$

3. Concave.

we have $g(f(x)) = x$, with domain $(f(a), f(b))$ and $a<x<b$.

Consider $g(\theta f(x_1) + (1-\theta) f(x_2))$, $a<x_1<x_2<b$, $0 \le \theta \le 1$

Since $f(x)$ is convex and increasing, we have $f(\theta x_1 + (1-\theta)x_2) \le \theta f(x_1) + (1-\theta) f(x_2)$.

So $g(\theta f(x_1) + (1-\theta) f(x_2)) = g(f(\theta x_1 + (1-\theta)x_2) + s)$, $s \ge 0$.

$g(f(\theta x_1 + (1-\theta)x_2) + s) = \theta x_1 + (1-\theta)x_2 + s'$ because $f(x)$ in increasing, $f(\theta x_1 + (1-\theta)x_2) + s = f(\theta x_1 + (1-\theta)x_2 + s')$, $s' \ge 0$.

Finally we get $g(\theta f(x_1) + (1-\theta) f(x_2)) = \theta x_1 + (1-\theta)x_2 + s'$

$= \theta g(f(x_1)) + (1-\theta) g(f(x_2)) + s'$

$\Downarrow$ $s' \ge 0$

$g(\theta f(x_1) + (1-\theta) f(x_2)) \ge \theta g(f(x_1)) + (1-\theta) g(f(x_2))$, $0 \le \theta \le 1$

Therefore, g is concave.

4. $f(v) = \sum_{i=1}^{n} v_i \log v_i$, $f''(v) = \sum_{i=1}^{n} \left(\frac{1}{v_i}\right) > 0$ because $u, v \in R_{++}^{n}$

So $f(v)$ is a convex function and $f(u) \ge f(v) + \nabla f(v)^T (u-v)$ (Since it's differentiable).

Therefore, we have $f(u) - f(v) - \nabla f(v)^T (u-v) \ge 0$.

Because $f''(v) > 0$, $f(u) > f(v) + \nabla f(v)^T (u-v)$ when $u \ne v$.

$f(u) = f(v) + \nabla f(v)^T (u-v)$ only when $u = v$, which means $D_{KL}(u, v) = 0$

Therefore, $D_{KL} \ge 0$ and $D_{KL} = 0$ if and only if $u = v$.

$-\quad \frac{1}{x_2}\cdot\frac{1}{x_1} \quad -\frac{1}{x_1}\cdot\frac{1}{x_1^2} \quad \cancel{\frac{2}{x_1x}} \quad \frac{2}{x_2}\frac{1}{x_3}$

5. (a) $f''(x) = e^x > 0$, convex

(b) $f''_{x_1x_1} = f''_{x_2x_2} = 0$, $f''_{x_1x_2} = f''_{x_2x_1} = 1$

Hessian matrix $= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$|H| < 0$, $Tr(H) = 0$. So it's not positive semidefinite nor negative semidefinite

Therefore, f is neither convex nor concave

(c) $f''_{x_1x_1} = \frac{2}{x_1^3 x_2}$, $f''_{x_2x_2} = \frac{2}{x_2^3 x_1}$, $f''_{x_1x_2} = f''_{x_2x_1} = \frac{1}{x_1^2 x_2^2}$

$$H = \begin{bmatrix} \frac{2}{x_1^3 x_2} & \frac{1}{x_1^2 x_2^2} \\ \frac{1}{x_1^2 x_2^2} & \frac{2}{x_2^3 x_1} \end{bmatrix},$$

$|H| = \frac{3}{x_1^4 x_2^4} > 0$, $Tr(H) > 0$, positive semidefinite

Therefore, f is convex.

(d) $f''_{x_1x_1} = 0$, $f''_{x_2x_2} = \frac{2x_1}{x_2^3}$, $f''_{x_1x_2} = f''_{x_2x_1} = -\frac{1}{x_2^2}$

$$H = \begin{bmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} \end{bmatrix}, \quad |H| < 0. \ Tr(H) > 0$$

H is neither positive nor negative semidefinite

Therefore, f is neither convex nor concave

(e) $f''_{x_1x_1} = \alpha(\alpha-1)x_1^{\alpha-2}x_2^{1-\alpha}$, $f''_{x_2x_2} = (1-\alpha)(-\alpha)x_1^{\alpha}x_2^{-\alpha-1}$

$f''_{x_1x_2} = f''_{x_2x_1} = \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha}$

$|H| = 0$. $Tr(H) < 0$, H is negative semidefinite

Therefore, f is concave.