Problem 1.

(1) First, we have: $W(t)$ is a standard Brownian motion

$\forall s, t \geq 0$, we have $W(s+t) - W(s) \sim N(0, t)$

Therefore, consider cs and ct, by definition,

we have $W(cs+ct) - W(cs) \sim N(0, ct)$

By the property of Gaussian's variance, we have:

$$c^{-\frac{1}{2}}[W(cs+ct) - W(cs)] \sim N(0, t)$$

$$\Rightarrow c^{-\frac{1}{2}}W(cs+ct) - c^{-\frac{1}{2}}W(cs) \sim N(0, t)$$

Then, $c^{-\frac{1}{2}}W(0) = 0$ since $W(0)=0$ by definition.

Since $W(t_1)-W(t_0)$, $W(t_2)-W(t_1)$ ... are mutually independent. By viewing ct_1, ct_0, ct_2 ... as new t_1', t_0', t_2'.

According to $W(t)$'s property, $W(ct_1)-W(ct_0)$, $W(ct_2)-W(ct_1)$ ... are mutualy independent. Since the same $c^{-\frac{1}{2}}$ is a coefficient which won't influence its independence. So we have:

$c^{-\frac{1}{2}}W(ct_1) - c^{-\frac{1}{2}}W(ct_0),\ c^{-\frac{1}{2}}W(ct_2) - c^{-\frac{1}{2}}W(ct_1)$ are mutually independent.

$c^{-\frac{1}{2}}W(ct)$ is also almost surely as $W(t)$ does since $c>0$ doesn't influence

Therefore, $c^{-\frac{1}{2}}W(ct)$ is also a standard Brownian motion.

(2) $X(s+t) - X(s) = W(c+s+t) - W(c) - W(c+s) + W(c)$

$= W(c+s+t) - W(c+s) \sim N(0,t)$

$X(t_2) - X(t_1) = W(t_2+t) - W(t_1+t)$

since $W(t_2) - W(t_1),\ W(t_1) - W(t_0)$ are mutually independent.

By viewing $t_2+t,\ t_1+t$ as new t_2', t_1',

$X(t_2) - X(t_1),\ X(t_1) - X(t_0) \ldots$ are also mutually independent.

$X(0) = W(c) - W(c) = 0$.

Therefore, $\{X(t) : t \geq 0\}$ is a standard Brownian motion.

for $\forall\, 0 \leq t \leq c \leq c+t$, we have: $W(t) - W(0)$ and

$W(c+t) - W(c)$ are independent by the property of $W(t)$

Therefore, since $W(t) - W(0) = W(t)$, $W(c+t) - W(c) = X(t)$,

we have: $\{X(t) : t \geq 0\}$ is independent of $\{W(t) : 0 \leq t \leq c\}$

(3)

$$\Pr[W(1)>0 \mid W(\tfrac{1}{2})>0] = \frac{\Pr[W(1)>0 \wedge W(\frac{1}{2})>0]}{\Pr[W(\frac{1}{2})>0]}$$

$W(\frac{1}{2}) = W(\frac{1}{2}) - W(0) \sim N(0, \frac{1}{2}) = \frac{1}{\sqrt{\pi}} e^{-x^2}$

So $\Pr[W(\frac{1}{2})>0] = \frac{1}{2}$

$$\Pr[W(1)>0 \wedge W(\tfrac{1}{2})>0] = \int_0^\infty \Pr[W(1)>0 \mid W(\tfrac{1}{2})=x] \cdot \frac{1}{\sqrt{\pi}} e^{-x^2} dx$$

while $\Pr[W(1)>0 \mid W(\frac{1}{2})=x] = \Pr[W(1) - W(\frac{1}{2}) > -x]$

Since $W(1) - W(\frac{1}{2}) \sim N(0, \frac{1}{2})$, $\Pr[W(1) - W(\frac{1}{2}) > -x] = \int_{-x}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} dy$

So $\Pr[W(1)>0 \wedge W(\frac{1}{2})>0] = \int_0^\infty \int_{-x}^\infty \frac{1}{\sqrt{\pi}} e^{-y^2} dy \cdot \frac{1}{\sqrt{\pi}} e^{-x^2} dx$

$$= \frac{1}{\pi} \cdot \int_0^\infty \left(\int_{-x}^\infty e^{-y^2} dy \right) e^{-x^2} dx = \frac{3}{8}\pi \cdot \frac{1}{\pi} = \frac{3}{8}$$

Therefore, $\Pr[W(1)>0 \mid W(\frac{1}{2})>0] = \frac{3}{4}$

Problem 2.

(1) $\Pr[X(t) \le \delta] = \Pr[\mu t + \sigma W(t) \le \delta] = \Pr[W(t) \le \frac{\delta - \mu t}{\sigma}]$

$$= \Pr\left[\frac{W(t)}{\sqrt{t}} \le \frac{\delta - \mu t}{\sigma\sqrt{t}}\right]$$

Since $W(t)$ is a standard Brownian motion with $W(t) \sim N(0,t)$, we have $\frac{W(t)}{\sqrt{t}} \sim N(0,1) \sim \xi$

Therefore, we have $\Pr[X(t) \le \delta] = \Pr[\xi \le \frac{\delta - \mu t}{\sigma\sqrt{t}}]$

(2) $E[T] = E[\int_0^\infty \mathbf{1}[X(t) \in [0,\delta]] dt] = \int_0^\infty \Pr[0 \le X(t) \le \delta] dt$

$$= \int_0^\infty \Pr[0 \le \mu t + \sigma W(t) \le \delta] dt = \int_0^\infty \Pr\left[\frac{-\mu t}{\sigma} \le W(t) \le \frac{\delta - \mu t}{\sigma}\right] dt$$

$$= \int_0^\infty \Pr\left[\frac{\mu t - \delta}{\sigma\sqrt{t}} \le -\frac{W(t)}{\sqrt{t}} \le \frac{\mu\sqrt{t}}{\sigma}\right] dt$$

since $-\frac{W(t)}{\sqrt{t}}$ also $\sim N(0,1) \sim \xi$

we have: $E[T] = \int_0^\infty \Pr\left[\frac{\mu t - \delta}{\sigma\sqrt{t}} \le \xi \le \frac{\mu\sqrt{t}}{\sigma}\right] dt$

(3) $\frac{\mu t - \delta}{\sigma\sqrt{t}} \le \xi \le \frac{\mu\sqrt{t}}{\sigma} \Rightarrow \sqrt{t} \ge \frac{\sigma\xi}{\mu}$, $\mu t - \sigma\xi\sqrt{t} - \delta \le 0$

from $\mu t - \sigma\xi\sqrt{t} - \delta \le 0$, we have: $0 \le \sqrt{t} \le \frac{\sigma\xi + \sqrt{\sigma^2\xi^2 + 4\mu\delta}}{2\mu}$

therefore, we have: $\left(\frac{\sigma\xi}{\mu}\right)^2 \le t \le \left(\frac{\sigma\xi + \sqrt{\sigma^2\xi^2 + 4\mu\delta}}{2\mu}\right)^2$

$$\text{So } f(\delta, x) = \left(\frac{\sigma x}{2\mu} + \frac{\sqrt{\sigma^2 x^2 + 4\mu\delta}}{2\mu}\right)^2$$

where $f(0,\xi) = \left(\frac{\sigma\xi}{\mu}\right)^2$, $f(\delta,\xi) = \left(\frac{\sigma\xi + \sqrt{\sigma^2\xi^2 + 4\mu\delta}}{2\mu}\right)^2$

Proved.

$$(4)\ E[f(\delta,\xi)] = E\left[\left(\frac{\sigma\xi+\sqrt{\sigma^2\xi^2+4\mu\delta}}{2\mu}\right)^2\right] = \frac{\sigma^2}{4\mu^2}E[\xi^2] + \frac{\sigma^2}{4\mu^2}E[\xi^2]$$

$$+\frac{\delta}{\mu} + \frac{\sigma}{2\mu^2}E[\xi\sqrt{\sigma^2\xi^2+4\mu\delta}]$$

Since $\xi \sim N(0,1)$, $E[\xi]^2 - E[\xi]^2 = Var[\xi]$

$E[\xi]^2 = 1$

Since $\xi \in [0,1]$ is symmetry, $\xi\sqrt{\sigma^2\xi^2+4\mu\delta}$ is odd,

$E[\xi\sqrt{\sigma^2\xi^2+4\mu\delta}] = 0$

Therefore, $E[f(\delta,\xi)] = \frac{\sigma^2}{2\mu^2} + \frac{\delta}{\mu}$

$$(5)\ E[T] = \int_0^\infty Pr[f(0,\xi) \le t \le f(\delta,\xi)]\,dt$$

$$= \int_0^\infty Pr[t \le f(\delta,\xi)]\,dt - \int_0^\infty Pr[t \le f(0,\xi)]\,dt$$

$$= E[f(\delta,\xi)] - E[f(0,\xi)]$$

$$= \frac{\sigma^2}{2\mu^2} + \frac{\delta}{\mu} - \frac{\sigma^2}{2\mu^2} = \frac{\delta}{\mu}$$

NO Refference