1. Random choose T samples, Let X_i denote the result of i-th sample, namely

$X_i = 1[\text{the i-th sample is } 1,\ \text{o.w. } 0]$

$X_i \sim Ber(p(1))$, use $X = \sum_{i=1}^{T} X_i$ to denote the total number of 1,

$\hat{p}(1) = \frac{X}{T}$

$Pr[\hat{p}(1) > p(1) + \varepsilon] = pr[X > (p(1)+\varepsilon)\cdot T] = Pr[X > (1+\frac{\varepsilon}{p(1)})\cdot EX]$

By Chenoff Bound: $Pr[X > (1+\frac{\varepsilon}{p(1)})EX] \le \exp(-\frac{(\frac{\varepsilon}{p(1)})^2}{3}\cdot EX) \le \delta$

$\Rightarrow \exp(-\frac{(\frac{\varepsilon}{p(1)})^2}{3}\cdot T\cdot p(1)) \le \delta$. sufficient to pick $T \ge \frac{3p(1)}{\varepsilon^2}\log\frac{1}{\delta}$

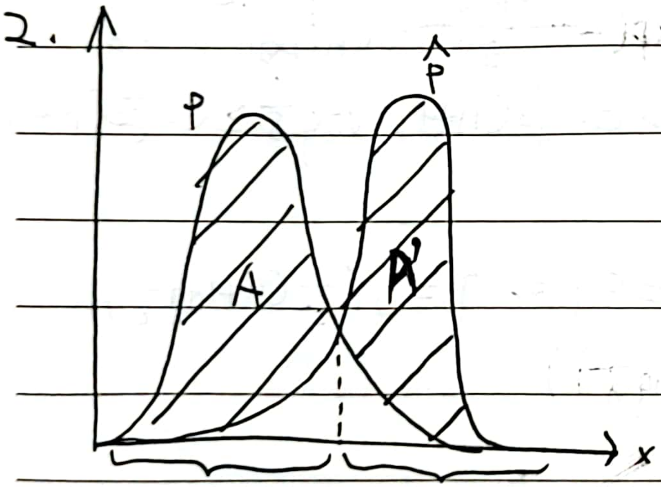
Similarly, $Pr[\hat{p}(1) < p(1) - \varepsilon] = pr[X < (p(1)-\varepsilon)\cdot T] = Pr[X < (1-\frac{\varepsilon}{p(1)})\cdot EX]$

$\le \exp(-\frac{(\frac{\varepsilon}{p(1)})^2}{2}\cdot EX) \le \delta \Rightarrow \exp(-\frac{(\frac{\varepsilon}{p(1)})^2}{2}\cdot T\cdot p(1)) \le \delta$. pick $T \ge \frac{2p(1)}{\varepsilon^2}\log\frac{1}{\delta}$

Combine the two results, $Pr[dist(p,\hat{p}) \ge \varepsilon] = pr[(\hat{p}(1)-p(1) > \varepsilon) \cup (\hat{p}(1)-p(1) < -\varepsilon)]$

where $T \ge \frac{5p(1)}{\varepsilon^2}\log\frac{1}{\delta}$. Since $p(1) \in [0,1]$, $T = O(\frac{1}{\varepsilon^2}\log\frac{1}{\delta})$

2.



$B := \{x : p(x) \ge \hat{p}(x)\}$ $B^c := \{x \mid \hat{p}(x) \ge p(x)\}$

令$p(x) \ge \hat{p}(x)$的域为B，其余为B^c

令$p(x) \ge \hat{p}(x)$包围的面积为A，反之为A'

则$dist(p,\hat{p}) = \frac{1}{2}(A+A')$

而$\max_{S\subseteq[n]} \sum_{j\in S}(p(j)-\hat{p}(j)) = \max\{A, A'\}$

此时，只需证A=A'即可证$dist(p,\hat{p}) = \max_{S\subseteq[n]} \sum_{j\in S}(p(j)-\hat{p}(j))$

Proof: $p(A) + p(A') = \hat{p}(A) + \hat{p}(A') = 1$

$\Rightarrow p(A) - \hat{p}(A) = \hat{p}(A') - p(A')$

而A的面积$=|p(A)-\hat{p}(A)|$, $A' = |\hat{p}(A')-p(A')|$

$A = A'$，得证

$n+1$

3. Similar to 1. $X_i \sim Ber(p(S))$ (在S为1, o.w.为0), $X = \sum_{i=1}^{T} X_i$

$\hat{p}(S) = \frac{X}{T}$

$Pr[p(S) - \hat{p}(S) \geq \varepsilon] = Pr[X < (1 - \frac{\varepsilon}{p(S)}) EX] \leq exp(-\frac{(\frac{\varepsilon}{p(S)})^2}{2} \cdot EX)$

$= exp(-\frac{(\frac{\varepsilon}{p(S)})^2}{2} \cdot T \cdot p(S)) = exp(-\frac{1}{2p(S)} \cdot \varepsilon^2 \cdot T)$

since $p(S)$ is a constant, we can pick $c = \frac{1}{2p(S)}$

~~4. $dist(p, \hat{p}) = \max_{S \subseteq [n]} \sum_{j \in S} (p(j) - \hat{p}(j)) = \max [p(S) - \hat{p}(S)]$~~

~~$Pr[dist(p, \hat{p}) \geq \varepsilon] = Pr[\max(p(S) - \hat{p}(S)) \geq \varepsilon] \geq Pr[p(S) - \hat{p}(S) \geq \varepsilon]$~~

4. $dist(p, \hat{p}) = \frac{1}{2} \sum_{i \in [n]} |p(i) - \hat{p}(i)|$, $Pr[dist(p, \hat{p}) \geq \varepsilon] =$

$Pr[\frac{1}{2} \sum_{i \in [n]} |p(i) - \hat{p}(i)| \geq \varepsilon] \leq Pr[\frac{1}{2} \cdot n \cdot \max_{i \in [n]} |p(i) - \hat{p}(i)| \geq \varepsilon]$

let $i = j$ when $|p(j) - \hat{p}(j)|$ is maximal

$\therefore Pr[dist(p, \hat{p}) \geq \varepsilon] \leq Pr[\frac{1}{2} \cdot n \cdot |p(j) - \hat{p}(j)| \geq \varepsilon]$

① $Pr[\frac{1}{2} \cdot n \cdot (\hat{p}(j) - p(j)) \geq \varepsilon] = Pr[X > (p(j) + \frac{2\varepsilon}{n}) \cdot T]$

$= Pr[X > (1 + \frac{2\varepsilon}{n \cdot p(j)}) \cdot T \cdot p(j)] \leq exp(-\frac{T \cdot p(j)}{3} \cdot (\frac{\varepsilon}{n \cdot p(j)})^2) < \delta$

$\Rightarrow T > c \cdot \frac{1}{\varepsilon^2} \cdot n^2 \cdot \log \frac{1}{\delta}$, $\xRightarrow{n \geq 2} T > c \cdot \frac{1}{\varepsilon^2} (n+1) \cdot \log \frac{1}{\delta} \geq c \cdot \frac{1}{\varepsilon^2} (n + \log \frac{1}{\delta})$

$= O(\frac{1}{\varepsilon^2} (n + \log \frac{1}{\delta}))$

② Similarly, $Pr[\frac{1}{2} n \cdot (p(j) - \hat{p}(j)) \geq \varepsilon] \Rightarrow T = O(\frac{1}{\varepsilon^2} (n + \log \frac{1}{\delta}))$

So the overall $T = O(\frac{1}{\varepsilon^2} (n + \log \frac{1}{\delta}))$

5. $P_0(\varepsilon_2) = Pr[\max_{1 \leq t \leq T^*} |\sum_{i=1}^{t} \omega_i - \frac{t}{2}| \leq \sqrt{2T^*}] = Pr[\max_{1 \leq t \leq T^*} |\sum_{i=1}^{t} (\omega_i - \frac{1}{2})| \leq \sqrt{2T^*}]$

Because it's a fair coin, $E(\omega_i - \frac{1}{2}) = 0$ for every $i \in [T^*]$

By the theorem, $Pr[\max_{1 \leq t \leq T^*} |\sum_{i=1}^{t} (\omega_i - \frac{1}{2})| \geq \sqrt{2T^*}] \leq 2 exp(-\frac{2 \cdot 2T^*}{T^* \cdot 1})$

while $2 \cdot e^{-4} \leq \frac{1}{4}$, $Pr[\max_{1 \leq t \leq T^*} |\sum_{i=1}^{t} (\omega_i - \frac{1}{2})| \leq \sqrt{2T^*}] \geq \frac{3}{4}$, $P_0(\varepsilon_2) \geq \frac{3}{4}$

7. $P_1(\mathcal{E}_1) \geq P_1(\mathcal{E}_1 \cap \mathcal{E}_2)$. clearly holds since $\mathcal{E}_2$ is an extra condition

to prove $P_1(\mathcal{E}_1 \cap \mathcal{E}_2) \geq P_0(\mathcal{E}_1 \cap \mathcal{E}_2) \cdot \min\limits_{\omega \in \mathcal{E}_1 \cap \mathcal{E}_2} \frac{P_1(\omega)}{P_0(\omega)}$ is to prove:

$$\frac{P_1(\mathcal{E}_1 \cap \mathcal{E}_2)}{P_0(\mathcal{E}_1 \cap \mathcal{E}_2)} \geq \min_{\omega \in \mathcal{E}_1 \cap \mathcal{E}_2} \frac{P_1(\omega)}{P_0(\omega)}$$

We can rewrite $\frac{P_1(\mathcal{E}_1 \cap \mathcal{E}_2)}{P_0(\mathcal{E}_1 \cap \mathcal{E}_2)}$ as $\frac{P_1(\omega_1)+P_1(\omega_2)+\ldots+P_1(\omega_n)}{P_0(\omega_1)+P_0(\omega_2)+\ldots+P_0(\omega_n)}$ where $\omega_1, \omega_2, \ldots, \omega_n \in \mathcal{E}_1 \cap \mathcal{E}_2$, let ω_t to minimize $\frac{P_1(\omega)}{P_0(\omega)}$ so that $\frac{P_1(\omega_t)}{P_0(\omega_t)}$ is minimum

$$\Rightarrow \min_{\omega \in \mathcal{E}_1 \cap \mathcal{E}_2} \frac{P_1(\omega)}{P_0(\omega)} = \frac{P_1(\omega_t)}{P_0(\omega_t)} = \frac{P_1(\omega_t)+P_1(\omega_t)+\ldots+P_1(\omega_t)}{\underbrace{P_0(\omega_t)+P_0(\omega_t)+\ldots+P_0(\omega_t)}_{n \text{个} P}}$$

lemma: set a constant $\frac{b}{a}$, two varibles $\frac{d}{c}$ and $\frac{d'}{c'}$, where $\frac{d}{c} > \frac{d'}{c'}$

we have $\frac{b+d}{a+c} > \frac{b+d'}{a+c'}$ (clearly)

we can apply the lemma to above inequility: $\frac{P_1(\omega_t)+P_1(\omega_1)}{P_0(\omega_t)+P_0(\omega_1)} \geq \frac{P_1(\omega_t)+P_1(\omega_t)}{P_0(\omega_t)+P_0(\omega_t)}$

repeatedly add $\frac{P_1(\omega)}{P_0(\omega)}$, we have $\frac{P_1(\omega_1)+P_1(\omega_2)+\ldots+P_1(\omega_t)+\ldots+P_1(\omega_n)}{P_0(\omega_1)+P_0(\omega_2)+\ldots+P_0(\omega_t)+\ldots+P_0(\omega_n)} \geq \frac{P_1(\omega_t)+\ldots+P_1(\omega_t)}{P_0(\omega_t)+\ldots+P_0(\omega_t)} = \frac{P_1(\omega_t)}{P_0(\omega_t)}$

$$\Rightarrow \frac{P_1(\mathcal{E}_1 \cap \mathcal{E}_2)}{P_0(\mathcal{E}_1 \cap \mathcal{E}_2)} \geq \min_{\omega \in \mathcal{E}_1 \cap \mathcal{E}_2} \frac{P_1(\omega)}{P_0(\omega)} \Rightarrow P_1(\mathcal{E}_1 \cap \mathcal{E}_2) \geq P_0(\mathcal{E}_1 \cap \mathcal{E}_2) \cdot \min_{\omega \in \mathcal{E}_1 \cap \mathcal{E}_2} \frac{P_1(\omega)}{P_0(\omega)}$$

So, $P_1(\mathcal{E}_1) \geq P_1(\mathcal{E}_1 \cap \mathcal{E}_2) \geq P_0(\mathcal{E}_1 \cap \mathcal{E}_2) \cdot \min\limits_{\omega \in \mathcal{E}_1 \cap \mathcal{E}_2} \frac{P_1(\omega)}{P_0(\omega)}$

since $P_0(\mathcal{E}_1) \geq 1-\delta$, $P_0(\mathcal{E}_2) \geq \frac{3}{4}$ and $\mathcal{E}_1, \mathcal{E}_2$ independent

$P_0(\mathcal{E}_1 \cap \mathcal{E}_2) \geq \frac{3}{4} - \frac{3}{4}\delta \geq \frac{3}{4} - \delta \geq \frac{1}{2}$, $\min\limits_{\omega \in \mathcal{E}_1 \cap \mathcal{E}_2} \frac{P_1(\omega)}{P_0(\omega)} > 2\delta$

So $P_1(\mathcal{E}_1) \geq P_0(\mathcal{E}_1 \cap \mathcal{E}_2) \cdot \min\limits_{\omega \in \mathcal{E}_1 \cap \mathcal{E}_2} \frac{P_1(\omega)}{P_0(\omega)} > \frac{1}{2} \cdot 2\delta = \delta$

So we can conclude that no $(\varepsilon, \delta, T^*)$ algorithm exists

无讨论/参考