Problem 1.

1. $E[X_{n+1}|X_1,\dots,X_n]=0$

$E[Z_{n+1}|X_1,\dots,X_n]=E[Z_n\cdot e^{\lambda X_{n+1}}|X_1,\dots,X_n]=Z_n\cdot E[e^{\lambda X_{n+1}}|X_1,\dots,X_n]$

$\because e^x\ge x+1$ and λ is a positive real number,

$\therefore Z_n\cdot E[e^{\lambda X_{n+1}}|X_1,\dots,X_n]\ge Z_n\cdot E[\lambda X_{n+1}+1|X_1,\dots,X_n]$

$=Z_n(1+\lambda E[X_{n+1}|X_1,\dots,X_n])=Z_n$

$\therefore E[Z_{n+1}|X_1,\dots,X_n]\ge Z_n$

Therefore, $\{Z_n\}_{n\in\{0\}\cup[N]}$ is a submartingale

2. for any $s>0$, $Pr[\max_{1\le n\le N} Z_n\ge e^{\lambda s}]\le \frac{E[Z_n]}{e^{\lambda s}}$

from 1. we have known that $\{Z_n\}$ is a submartingale.
Then we fix N and define $T\triangleq\min\{k\ge0\,|\,Z_k\ge e^{\lambda s}\}\wedge N$. T is a stopping time bounded above by N.

we have: $\{\max Z_n\ge e^{\lambda s}\}=\{Z_T\ge e^{\lambda s}\}$

Thus, $Pr[\max Z_n\ge e^{\lambda s}]=Pr[Z_T\ge e^{\lambda s}]\le 1\le E[\frac{Z_T}{e^{\lambda s}}|Z_T\ge e^{\lambda s}]$

$=e^{-\lambda s}\cdot E[Z_T|Z_T\ge e^{\lambda s}]$

Since $Z_T\ge e^{\lambda s}\subseteq \mathcal{F}_N$, by the property of $\{Z_n\}$, we have:

$e^{-\lambda s}\cdot E[Z_T|Z_T\ge e^{\lambda s}]\le e^{-\lambda s}E[Z_N|Z_T\ge e^{\lambda s}]=e^{-\lambda s}E[Z_N|\max Z_n\ge e^{\lambda s}]\le e^{-\lambda s}\cdot E[Z_N]$ (Since there is another condition makes $Z_N=\max Z_n$)

Therefore, $Pr[\max_{1\le n\le N} Z_n\ge e^{\lambda s}]\le\frac{E[Z_n']}{e^{\lambda s}}$

Proof: $E[Z_T|Z_T\ge e^{\lambda s}]\le E[Z_N|Z_T\ge e^{\lambda s}]$. (Property)

$E[Z_N|Z_T\ge e^{\lambda s}]=\sum_{n=0}^{N}E[Z_N|Z_T\ge e^{\lambda s}\cap\{T=n\}]\ge\sum_{n=0}E[Z_n|Z_T\ge e^{\lambda s}\cap\{T=n\}]=\sum_{n=0}^{N}E[Z_T|Z_T\ge e^{\lambda s}\cap\{T=n\}]=E[Z_T|Z_T\ge e^{\lambda s}]$

3. Let $n = n_0$ when $\left|\sum_{j=1}^{n} X_j\right|$ is maximal

$\Rightarrow Pr\left[\max_{1\le n\le N}\left|\sum_{j=1}^{n} X_j\right| \ge s\right] = Pr\left[\left|\sum_{j=1}^{n_0} X_j\right| \ge s\right]$

Since $E[X_n]=0$ as assumed, we have $E\left[\sum_{j=1}^{n_0} X_j\right] = \sum_{j=1}^{n_0} E[X_j]$

$=0$

$\therefore Pr\left[\left|\sum_{j=1}^{n_0} X_j\right| \ge s\right] = Pr\left[\left|\sum_{j=1}^{n_0} X_j - E\left[\sum_{j=1}^{n_0} X_j\right]\right| \ge s\right]$

By Hoeffding's Inequality, we have:

$Pr\left[\left|\sum_{j=1}^{n_0} X_j - E\left[\sum_{j=1}^{n_0} X_j\right]\right| \ge s\right] \le 2\exp\left(-\frac{2s^2}{\sum_{i=1}^{n_0}(b-a)^2}\right) = 2e^{-\frac{2s^2}{n_0(b-a)^2}}$

$\le 2e^{-\frac{2s^2}{N(b-a)^2}}$

Problem 2.

1. $E[Z_{n+1} \mid X_1, \dots, X_n] = E\left[Z_n \cdot \frac{g(X_{n+1})}{f(X_{n+1})} \,\middle|\, X_1, \dots, X_n\right]$

$= Z_n \cdot E\left[\frac{g(X_{n+1})}{f(X_{n+1})} \,\middle|\, X_1, \dots, X_n\right] = Z_n \cdot E\left[\frac{g(X_{n+1})}{f(X_{n+1})}\right]$

if (H_1) holds, then X_{n+1}'s density is $f(x)$

$\therefore E\left[\frac{g(X_{n+1})}{f(X_{n+1})}\right] = \int_{-\infty}^{\infty} \frac{g(x)}{f(x)} \cdot f(x) \cdot dx = \int_{-\infty}^{\infty} g(x)\,dx$

$\because g(x)$ is a density $\therefore \int_{-\infty}^{\infty} g(x)\,dx = 1$

$\therefore E[Z_{n+1} \mid X_1, \dots, X_n] = Z_n$

Therefore, $\{Z_n\}_{n\ge 0}$ is a martingale with respect to $\{X_n\}_{n\ge 1}$

$b-a$

2. Assuming (H_1), $\{Z_n\}_{n\geq 0}$ is a martingale.

T_{ab} is a stopping time since $Pr[T_{ab}<\infty]=1$ and $a\leq Z_t\leq b$, $|Z_n|\leq b$ for $\forall t\leq T_{ab}$

Therefore, $E[Z_{T_{ab}}]=E[Z_0]=1$

Since at T_{ab}, $Z_{T_{ab}}$ is either a or b.

We have: $1=Pr[Z_{T_{ab}}=a]\cdot a+Pr[Z_{T_{ab}}=b]\cdot b$

$\Rightarrow 1=Pr[Z_{T_{ab}}=a]\cdot a+(1-Pr[Z_{T_{ab}}=a])\cdot b$

$\Rightarrow Pr[Z_{T_{ab}}=a]=\frac{1-b}{a-b}=\frac{b-1}{b-a}$

Assuming (H_2), $\{\frac{1}{Z_n}\}_{n\geq 0}$ is a martingale, the proof is similar to 1: $E[\frac{1}{Z_{n+1}}|X_1,\dots X_n]=E[\prod_{i=1}^{n}\frac{f(X_i)}{g(X_i)}\cdot\frac{f(X_{n+1})}{g(X_{n+1})}|X_1,\dots,X_n]$

$=\frac{1}{Z_n}\cdot E[\frac{f(X_{n+1})}{g(X_{n+1})}]=\frac{1}{Z_n}$.

T_{ab} is a stopping time since $Pr[T_{ab}<\infty]=1$ and $\frac{1}{b}\leq\frac{1}{Z_t}\leq\frac{1}{a}$

$|\frac{1}{Z_n}|\leq\frac{1}{a}$ for $\forall t\leq T_{ab}$

Therefore, $E[\frac{1}{Z_{T_{ab}}}]=E[\frac{1}{Z_0}]=1$

Similarly, $1=Pr[\frac{1}{Z_{T_{ab}}}=\frac{1}{b}]\cdot\frac{1}{b}+Pr[\frac{1}{Z_{T_{ab}}}=\frac{1}{a}]\cdot\frac{1}{a}$

$\Rightarrow Pr[\frac{1}{Z_{T_{ab}}}=\frac{1}{b}]=\frac{ab-b}{a-b}=\frac{b(1-a)}{b-a}$

If we choose b to be a large number and a be a small number, $Pr[Z_{T_{ab}}=a]$, $Pr[\frac{1}{Z_{T_{ab}}}=\frac{1}{b}]$ will converge to 1. high probability.

$b \log x^{b}$: $\log x \le x-1$

3. $E[\log Z_{n+1} \mid X_1, \dots, X_n] = E[\log Z_n + \log \frac{g(X_{n+1})}{f(X_{n+1})} \mid X_1, \dots, X_n]$
$= \log Z_n + E[\log \frac{g(X_{n+1})}{f(X_{n+1})}]$
for $E[\log \frac{g(X_{n+1})}{f(X_{n+1})}]$, $LHS = \int_{-\infty}^{\infty} \log \frac{g(x)}{f(x)} \cdot f(x)\, dx$
$\le \int_{-\infty}^{\infty} (\frac{g(x)}{f(x)} - 1) \cdot f(x)\, dx = \int_{-\infty}^{\infty} [g(x) - f(x)]\, dx$
$= \int_{-\infty}^{\infty} g(x)\, dx - \int_{-\infty}^{\infty} f(x)\, dx = 0$
$\therefore E[\log Z_{n+1} \mid X_1, \dots, X_n] \le \log Z_n + 0$
Therefore, $\{\log Z_n\}_{n \ge 0}$ is a supermartingale with respect to $\{X_n\}_{n \ge 1}$

4. To make M_n a martingale, we should let
$E[M_{n+1} \mid X_1, \dots, X_n] = M_n = \log Z_n + A_n$
$E[\log Z_n + \log \frac{g(X_{n+1})}{f(X_{n+1})} + A_{n+1} \mid X_1, \dots, X_n] = \log Z_n + E[\log \frac{g(X_{n+1})}{f(X_{n+1})}]$
$+ E[A_{n+1} \mid X_1, \dots, X_n] = \log Z_n + A_n$
$\therefore E[A_{n+1} \mid X_1, \dots, X_n] + E[\log \frac{g(X_{n+1})}{f(X_{n+1})}] = A_n$
When $A_{n+1} = -(n+1) \cdot E[\log \frac{g(X_{n+1})}{f(X_{n+1})}]$
$E[A_{n+1} \mid X_1, \dots, X_n] + E[\log \frac{g(X_{n+1})}{f(X_{n+1})}] = -(n+1) \cdot E[\log \frac{g(X_{n+1})}{f(X_{n+1})}] +$
$E[\log \frac{g(X_{n+1})}{f(X_{n+1})}] = -n E[\log \frac{g(X_{n+1})}{f(X_{n+1})}] = A_n$ by tower rule
Since H_1 holds $E[\log \frac{g(X_{n+1})}{f(X_{n+1})}] \le 0$, thus A_n is an increasing sequence.

5. T_{ab} is a stopping time of M_n since $E[T_{ab}] < \infty$

and $\exists c.\ \forall t \le T_{ab},\ E[\,|M_{t+1} - M_t|\,|\,X_1, \dots, X_t] \le C$

Therefore, we have $E[M_{T_{ab}}] = E[M_0] = E[\log Z_0] = 0$

$$\therefore E[\log Z_{T_{ab}} - T_{ab} E[\log \frac{g(X_{n+1})}{f(X_{n+1})}]] = 0$$

$$\Rightarrow E[\log Z_{T_{ab}}] = E[T_{ab}] \cdot E[\log \frac{g(X)}{f(X)}]$$

$$\Rightarrow E[T_{ab}] = \frac{E[\log Z_{T_{ab}}]}{E[\log \frac{g(X)}{f(X)}]}$$

$$E[\log Z_{T_{ab}}] = \frac{b-1}{b-a} \log a + \frac{1-a}{b-a} \log b$$

$$E[\log \frac{g(X)}{f(X)}] = \int_{-\infty}^{\infty} [\log g(X) - \log f(X)] f(X)\, dx \quad (H_1)$$

$$\therefore E[T_{ab}] = \frac{(b-1)\log a + (1-a)\log b}{(b-a) \cdot \int_{\mathbb{R}} (\log g(X) - \log f(X)) f(X)\, dX}$$