1 Question 1

Base Step: for k = 1, it holds that $\theta x_1 = x_1 \in C$.
for k = 2, it holds by definition of convexity.

Induction Hypothesis: for arbitrary k > 2, assume that $\theta_1 x_1 + ... + \theta_k x_k \in C$, with $x_1, ..., x_k \in C, \theta_i \geq 0$ and $\sum_{i=1}^{k} \theta_i = 1$.

Inductive Step:let $x_1, x_2, ..., x_k, x_{k+1} \in C, \theta_i \geq 0, \sum_{i=1}^{k+1} \theta_i = 1$, then we have $\theta_1 x_1 + ... + \theta_{k+1} x_{k+1} = \theta_1 x_1 + ... + \theta_{k-1} x_{k-1} + (\theta_k + \theta_{k+1})(\frac{\theta_k}{\theta_k + \theta_{k+1}} x_k + \frac{\theta_{k+1}}{\theta_k + \theta_{k+1}} x_{k+1}) \in C$,because $x_k, x_{k+1} \in C$, $\frac{\theta_k}{\theta_k+\theta_{k+1}}, \frac{\theta_{k+1}}{\theta_k+\theta_{k+1}} \geq 0$ and $\frac{\theta_k}{\theta_k+\theta_{k+1}} + \frac{\theta_{k+1}}{\theta_k+\theta_{k+1}} = 1$. By definition of convexity, $\frac{\theta_k}{\theta_k+\theta_{k+1}} x_k + \frac{\theta_{k+1}}{\theta_k+\theta_{k+1}} x_{k+1} \in C$, so with $\sum_{i=1}^{k+1} \theta_i = 1$, it holds $\theta_1 x_1 + ... + \theta_{k+1} x_{k+1} \in C$.
Therefore, it holds that $\theta_1 x_1 + ... + \theta_k x_k \in C$.

2 Question 2

(a)First, show that a set is convex $\Rightarrow$ its intersection with any line is convex:
If $C = \emptyset$, or C only contains a single point, then its intersection with any line is itself which is still convex. If not, for any two points $x_1, x_2 \in C, C$ is a convex set, its intersection with any line can be expressed as: $\theta x_1 + (1-\theta)x_2, \theta \in \{\theta_i\}$, defined as S. Because C is a convex set and $x_1, x_2 \in C$, $\theta x_1 + (1-\theta)x_2 \in C$ for $0 \leq \theta \leq 1$. Meanwhile, by definition of a line, $\theta x_1 + (1-\theta)x_2$ is in this line for $\forall \theta \in R$. So $\theta x_1 + (1-\theta)x_2 \in C$'s intersection with any line, S.

Then, show that a set's intersection with any line is convex $\Rightarrow$ this set is convex:
Let c be a set, and S be its intersection with any line. If $S = \emptyset$ or only contains a point, it's obvious that C is convex. If not, for any two points $x_1, x_2 \in C$, S can be expressed as $\theta x_1 + (1-\theta)x_2, \theta \in \{\theta_i\}$. Because S is convex, $\theta x_1 + (1-\theta)x_2 \in S, 0 \leq \theta \leq 1$. And because $S \subseteq C, \theta x_1 + (1-\theta)x_2 \in C, 0 \leq \theta \leq 1$. So C is convex.

(b)First, show that a set is affine $\Rightarrow$ its intersection with any line is affine:
If $C = \emptyset$, or C only contains a single point, then its intersection with any line is itself which is still affine. If not, for any two points $x_1, x_2 \in C, C$ is an affine set, then we have $\theta x_1 + (1-\theta)x_2 \in C, for\ \ \forall \theta \in R$. C's intersection with any line can be expressed as: $\theta x_1 + (1-\theta)x_2, \theta \in \{\theta_i\}$, defined as S. For any θ, $\theta x_1 + (1-\theta)x_2$ is in any line, too. It's both in C and any line, so it's in S. Therefore, S is affine.

Then, show that a set's intersection with any line is affine $\Rightarrow$ this set is affine:
Let C be a set, and S be its intersection with any line. If $S = \emptyset$ or only contains a point, it's obvious that C is still affine. If not, for any two points $x_1, x_2 \in C$, S can be expressed as $\theta x_1 + (1-\theta)x_2, \theta \in \{\theta_i\}$. Because S is affine, $\theta x_1 + (1-\theta)x_2 \in S$ for $\forall \theta$. And because $S \subseteq C$, then $\theta x_1 + (1-\theta)x_2 \in C$ for $\forall \theta$. Therefore, C is affine.

3 Question 3

(a)$\{x|a^Tx = 0, a$ is orthogonal to $b_1a_1 + b_2a_2,\ b_1, b_2 \in R, -|b^Ta_1| \leq b^Tx \leq |b^Ta_1|, b = a_1 - a_2\frac{a_1^Ta_2}{||a_2||_2^2}, -|c^Ta_2| \leq c^Tx \leq |c^Ta_2|, c = a_2 - a_1\frac{a_2^Ta_1}{||a_1||_2^2}\}$.

(b)$\{x|Ex \geq 0, \mathbf{1}^Tx = 1, a^Tx = b_1, a = (a_1, ..., a_n), a'^Tx = b_2, a' = (a_1^2, ..., a_n^2)\}$.

(c)S isn't a polyhedron.

(d)$\{x|Ex \geq 0, v_i^Tx \leq 1, vi :$ only a_{i1} is 1 and the rest of its elements are $0\}$.

4 Question 4

(a)By definition of V:

$$\begin{gathered}||x - x_0||_2 = ||x - x_i||_2 \\ ||x - x_0||_2^2 = ||x - x_i||_2^2 \\ (x - x_0)^T(x - x_0) \leq (x - x_i)^T(x - x_i) \\ x^Tx - 2x_0^Tx + x_0^Tx_0 \leq x^Tx - 2x_i^Tx + x_i^Tx_i \\ 2(x_i - x_0)^Tx \leq x_i^Tx_i - x_0^Tx_0\end{gathered}$$

So $V = \{Ax \leq b, A = \begin{bmatrix} 2(x_1 - x_0) \\ \vdots \\ 2(x_K - x_0) \end{bmatrix}, b = \begin{bmatrix} x_1^Tx_1 - x_0^Tx_0 \\ \vdots \\ x_K^Tx_K - x_0^Tx_0 \end{bmatrix}\}$.

(b)assume $P = \{x|Ax \leq b, A \in R^k \times n, b \in R^k\}$, then choose a x_0 in polyhedron P. So x_i can be expressed as the mirror point if x_0 with hyperplane $a_i^Tx = b_i$. The distance between x_0 and hyperplane $a_i^T = b_i$ is $\frac{|a_i^Tx_0 - b_i|}{||a_i||_2}$. So the distance between x_i and hyperplane $a_i^T = b_i$ is $\frac{|a_i^Tx_i - b_i|}{||a_i||_2}$. The two distances are equal, which means $b_i - a_i^Tx_0 = a_i^Tx_i - b_i$.
Since x_i is the mirror point of $x_0, x_i = x_0 + \frac{a_i}{||a_i||_2}\frac{|a_i^Tx_0 - b_i|}{||a_i||_2}$.

5 Question 5

For any two points $(x_1, y_1 + y_2), (x_2, y_3 + y_4)$ in $S, (x, y_1), (x, y_3) \in S_1, (x, y_2), (x, y_4) \in S_2$.
$\theta(x_1, y_1 + y_2) + (1 - \theta)(x_2, y_3 + y_4) = (\theta x_1 + (1 - \theta)x_2, \theta y_1 + (1 - \theta)y_3 + \theta y_2 + (1 - \theta)y_4), 0 \leq \theta \leq 1$.
Because $(x, y_1), (x, y_3) \in S_1$ and S_1 is a convex set, $\theta(x, y_1) + (1 - \theta)(x, y_3) = (\theta x + (1 - \theta)x, \theta y_1 + (1 - \theta)y_3) \in S1, 0 \leq \theta \leq 1$. Then we have $(\theta x_1 + (1 - \theta)x_2, \theta y_1 + (1 - \theta)y_3) \in S_1$. Similarly, we have $(\theta x_1 + (1 - \theta)x_2, \theta y_2 + (1 - \theta)y_4) \in S_2$.
So $(\theta x_1 + (1 - \theta)x_2, \theta y_1 + (1 - \theta)y_3 + \theta y_2 + (1 - \theta)y_4) \in S \Rightarrow \theta(x_1, y_1 + y_2) + (1 - \theta)(x_2, y_3 + y_4) \in S, 0 \leq \theta \leq 1$.