Algorithm Design and Analysis (Fall 2023)

Assignment 2

Deadline: Nov 27, 2023

1. (25 points) Design a polynomial time algorithm that, given a directed unweighted graph $G = (V, E)$ and $s \in V$, outputs "yes" if there is a vertex $u \in V$ such that there are at least two different simple paths from s to u and outputs "no" otherwise. A simple path is a path that does not visit a vertex more than once. Two paths are different if they differ in at least one edge. Prove the correctness of your algorithm and analyze its time complexity.

2. (25 points) Given an undirected connected graph $G = (V, E)$ and a vertex s, if the DFS tree and the BFS tree rooted at s are identical, prove that G is a tree.

3. (25 points) Consider a directed *acyclic* edge-weighted graph $G = (V, E, w)$ where edge weights can be negative and a vertex $s \in V$.

   (a) (20 points) Adapt the Bellman-Ford algorithm to find the distances of all vertices from s. Your algorithm must run in $O(|V| + |E|)$ time. Prove the correctness of your algorithm and analyze its time complexity.

   (b) (5 points) Suppose now we want to find the length of the longest path from s to each vertex $u \in V$ (the length is measured by the sum of the weights of the edges on the path, not by the number of the edges). Can we negate the weight of every edge and use the algorithm from the first part? If so, prove it; if not, provide a counterexample.

4. (25 points) It is possible that the shortest path from s to t in an edge-weighted graph is not unique. Given a directed edge-weighted graph $G = (V, E, w)$ with positive edge weights and $s \in V$, design an $O((|V| + |E|) \log |V|)$ time algorithm to output a Boolean array B, with vertices being the array indices, such that $B[u] =$ `true` if the shortest path from s to u is unique and $B[u] =$ `false` otherwise. You can assume that every vertex $u \in V$ is reachable from s. Prove the correctness of your algorithm.

5. How long does it take you to finish the assignment (including thinking and discussion)? Give a score (1,2,3,4,5) to the difficulty. Do you have any collaborators? Please write down their names here.