1 (10%) Question 1

Let $C \subseteq \mathbf{R}^n$ be a convex set, with $x_1, \dots, x_k \in C$, and let $\theta_1, \dots, \theta_k \in \mathbf{R}$ satisfy $\theta_i \geq 0$, $\theta_1 + \cdots + \theta_k = 1$. Show that $\theta_1 x_1 + \cdots + \theta_k x_k \in C$. (The definition of convexity is that this holds for $k = 2$; you must show it for arbitrary k.) Hint. Use induction on k.

Solution. This is readily shown by induction from the definition of convex set. We illustrate the idea for $k = 3$, leaving the general case to the reader. Suppose that $x_1, x_2, x_3 \in C$, and $\theta_1 + \theta_2 + \theta_3 = 1$ with $\theta_1, \theta_2, \theta_3 \geq 0$. We will show that $y = \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 \in C$. At least one of the θ_i is not equal to one; without loss of generality we can assume that $\theta_1 \neq 1$. Then we can write

$$y = \theta_1 x_1 + (1 - \theta_1)(\mu_2 x_2 + \mu_3 x_3)$$

where $\mu_2 = \theta_2/(1 - \theta_1)$ and $\mu_2 = \theta_3/(1 - \theta_1)$. Note that $\mu_2, \mu_3 \geq 0$ and

$$\mu_1 + \mu_2 = \frac{\theta_2 + \theta_3}{1 - \theta_1} = \frac{1 - \theta_1}{1 - \theta_1} = 1.$$

Since C is convex and $x_2, x_3 \in C$, we conclude that $\mu_2 x_2 + \mu_3 x_3 \in C$. Since this point and x_1 are in C, $y \in C$.

2 (10%) Question 2

Show that a set is convex if and only if its intersection with any line is convex. Show that a set is affine if and only if its intersection with any line is affine.

Solution. We prove the first part. The intersection of two convex sets is convex. Therefore if S is a convex set, the intersection of S with a line is convex.
Conversely, suppose the intersection of S with any line is convex. Take any two distinct points x_1 and $x_2 \in S$. The intersection of S with the line through x_1 and x_2 is convex. Therefore convex combinations of x_1 and x_2 belong to the intersection, hence also to S.

3 (20%) Question 3

Which of the following sets S are polyhedra? If possible, express S in the form $S = \{x \mid Ax \preceq b, Fx = g\}$.

(a) $S = \{y_1 a_1 + y_2 a_2 \mid -1 \leq y_1 \leq 1,\ -1 \leq y_2 \leq 1\}$, where $a_1, a_2 \in \mathbf{R}^n$

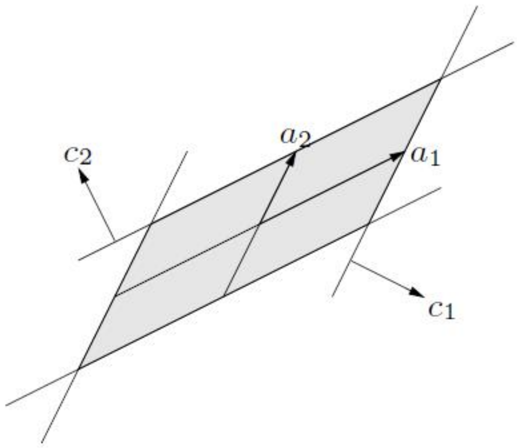
(b) $S = \{x \in \mathbf{R}^n \mid x \succeq 0, \mathbf{1}^T x = 1, \sum_{i=1}^n x_i a_i = b_1, \sum_{i=1}^n x_1 a_i^2 = b_2\}$, where $a_1, \dots, a_n \in \mathbf{R}$ and $b_1, b_2 \in \mathbf{R}$.

(c) $S = \{x \in \mathbf{R}^n \mid x \succeq 0, x^T y \leq 1 \text{ for all } y \text{ with } \|y\|_2 = 1\}$

(d) $S = \{x \in \mathbf{R}^n \mid x \succeq 0, x^T y \leq 1 \text{ for all } y \text{ with } \sum_{i=1}^n |y_i| = 1\}$

Solution.

(a) S is a polyhedron. It is the parallelogram with corners $a_1 + a_2$, $a_1 - a_2$, $-a_1 + a_2$, $-a_1 - a_2$, as shown below for an example in $\mathbf{R}^2$.



For simplicity we assume that a_1 and a_2 are independent. We can express S as the intersection of three sets:

- S_1: the plane defined by a_1 and a_2
- $S_2 = \{z + y_1 a_1 + y_2 a_2 \mid a_1^T z = a_2^T z = 0, -1 \leq y_1 \leq 1\}$. This is a slab parallel to a_2 and orthogonal to S_1
- $S_3 = \{z + y_1 a_1 + y_2 a_2 \mid a_1^T z = a_2^T z = 0, -1 \leq y_2 \leq 1\}$. This is a slab parallel to a_1 and orthogonal to S_1

Each of these sets can be described with linear inequalities.

- S_1 can be described as

$$v_k^T x = 0, \quad k = 1, \ldots, n-2$$

where v_k are $n-2$ independent vectors that are orthogonal to a_1 and a_2 (which form a basis for the nullspace of the matrix $[a_1 \; a_2]^T$).

- Let c_1 be a vector in the plane defined by a_1 and a_2, and orthogonal to a_2. For example, we can take

$$c_1 = a_1 - \frac{a_1^T a_2}{\|a_2\|_2^2} a_2.$$

Then $x \in S_2$ if and only if

$$-|c_1^T a_1| \le c_1^T x \le |c_1^T a_1|.$$

- Similarly, let c_2 be a vector in the plane defined by a_1 and a_2, and orthogonal to a_1, *e.g.*,

$$c_2 = a_2 - \frac{a_2^T a_1}{\|a_1\|_2^2} a_1.$$

Then $x \in S_3$ if and only if

$$-|c_2^T a_2| \le c_2^T x \le |c_2^T a_2|.$$

Putting it all together, we can describe S as the solution set of $2n$ linear inequalities

$$\begin{array}{rcl} v_k^T x & \le & 0, \quad k = 1, \ldots, n-2 \\ -v_k^T x & \le & 0, \quad k = 1, \ldots, n-2 \\ c_1^T x & \le & |c_1^T a_1| \\ -c_1^T x & \le & |c_1^T a_1| \\ c_2^T x & \le & |c_2^T a_2| \\ -c_2^T x & \le & |c_2^T a_2|. \end{array}$$

(b) S is a polyhedron, defined by linear inequalities $x_k \geq 0$ and three equality constraints.

(c) S is not a polyhedron. It is the intersection of the unit ball $\{x \mid \|x\|_2 \leq 1\}$ and the nonnegative orthant $\mathbf{R}^n_+$. This follows from the following fact, which follows from the Cauchy-Schwarz inequality:

$$x^T y \leq 1 \text{ for all } y \text{ with } \|y\|_2 = 1 \iff \|x\|_2 \leq 1.$$

Although in this example we define S as an intersection of halfspaces, it is not a polyhedron, because the definition requires infinitely many halfspaces.

(d) S is a polyhedron. S is the intersection of the set $\{x \mid |x_k| \leq 1, \ \ k = 1, \ldots, n\}$ and the nonnegative orthant $\mathbf{R}^n_+$. This follows from the following fact:

$$x^T y \leq 1 \text{ for all } y \text{ with } \sum_{i=1}^{n} |y_i| = 1 \iff |x_i| \leq 1, \quad i = 1, \ldots, n.$$

We can prove this as follows. First suppose that $|x_i| \leq 1$ for all i. Then

$$x^T y = \sum_i x_i y_i \leq \sum_i |x_i||y_i| \leq \sum_i |y_i| = 1$$

if $\sum_i |y_i| = 1$.

Conversely, suppose that x is a nonzero vector that satisfies $x^T y \leq 1$ for all y with $\sum_i |y_i| = 1$. In particular we can make the following choice for y: let k be an index for which $|x_k| = \max_i |x_i|$, and take $y_k = 1$ if $x_k > 0$, $y_k = -1$ if $x_k < 0$, and $y_i = 0$ for $i \neq k$. With this choice of y we have

$$x^T y = \sum_i x_i y_i = y_k x_k = |x_k| = \max_i |x_i|.$$

Therefore we must have $\max_i |x_i| \leq 1$.

All this implies that we can describe S by a finite number of linear inequalities: it is the intersection of the nonnegative orthant with the set $\{x \mid -\mathbf{1} \preceq x \preceq \mathbf{1}\}$, *i.e.*, the solution of $2n$ linear inequalities

$$\begin{array}{rcl} -x_i & \leq & 0, \ \ i = 1, \ldots, n \\ x_i & \leq & 1, \ \ i = 1, \ldots, n. \end{array}$$

Note that as in part (c) the set S was given as an intersection of an infinite number of halfspaces. The difference is that here most of the linear inequalities are redundant, and only a finite number are needed to characterize S.

None of these sets are affine sets or subspaces, except in some trivial cases. For example, the set defined in part (a) is a subspace (hence an affine set), if $a_1 = a_2 = 0$; the set defined in part (b) is an affine set if $n = 1$ and $S = \{1\}$; etc.

4 (20%) Question 4

Voronoi sets and polyhedral decomposition. Let $x_0, \ldots, x_K \in \mathbf{R}^n$. Consider the set of points that are closer (in Euclidean norm) to x_0 than the other x_i, i.e.,

$$V = \{x \in \mathbf{R}^n \mid \|x - x_0\|_2 \leq \|x - x_i\|_2, i = 1, \ldots, K\}.$$

V is called the *Voronoi region* around x_0 with respect to $x_1, \ldots, x_K$.

(a) Show that V is a polyhedron. Express V in the form $V = \{x \mid Ax \preceq b\}$.

(b) Conversely, given a polyhedron P with nonempty interior, show how to find $x_0, \ldots, x_K$ so that the polyhedron is the Voronoi region of x_0 with respect to $x_0, \ldots, x_K$.

Solution.

(a) x is closer to x_0 than to x_i if and only if

$$\begin{aligned} \|x - x_0\|_2 \leq \|x - x_i\|_2 &\iff (x - x_0)^T(x - x_0) \leq (x - x_i)^T(x - x_i) \\ &\iff x^T x - 2x_0^T x + x_0^T x_0 \leq x^T x - 2x_i^T x + x_i^T x_i \\ &\iff 2(x_i - x_0)^T x \leq x_i^T x_i - x_0^T x_0, \end{aligned}$$

which defines a halfspace. We can express V as $V = \{x \mid Ax \preceq b\}$ with

$$A = 2\begin{bmatrix} x_1 - x_0 \\ x_2 - x_0 \\ \vdots \\ x_K - x_0 \end{bmatrix}, \qquad b = \begin{bmatrix} x_1^T x_1 - x_0^T x_0 \\ x_2^T x_2 - x_0^T x_0 \\ \vdots \\ x_K^T x_K - x_0^T x_0 \end{bmatrix}.$$

(b) Conversely, suppose $V = \{x \mid Ax \preceq b\}$ with $A \in \mathbf{R}^{K \times n}$ and $b \in \mathbf{R}^K$. We can pick any $x_0 \in \{x \mid Ax \prec b\}$, and then construct K points x_i by taking the mirror image of x_0 with respect to the hyperplanes $\{x \mid a_i^T x = b_i\}$. In other words, we choose x_i of the form $x_i = x_0 + \lambda a_i$, where λ is chosen in such a way that the distance of x_i to the hyperplane defined by $a_i^T x = b_i$ is equal to the distance of x_0 to the hyperplane:

$$b_i - a_i^T x_0 = a_i^T x_i - b_i.$$

Solving for λ, we obtain $\lambda = 2(b_i - a_i^T x_0)/\|a_i\|_2^2$, and

$$x_i = x_0 + \frac{2(b_i - a_i^T x_0)}{\|a_i\|^2} a_i.$$

5 (10%) Question 5

Show that if S_1 and S_2 are convex sets in $\mathbf{R}^{m+n}$, then so is their partial sum

$$S = \{(x, y_1 + y_2) \mid x \in \mathbf{R}^m, y_1, y_2 \in \mathbf{R}^n, (x, y_1) \in S_1, (x, y_2) \in S_2\}.$$

Solution. We consider two points $(\bar{x}, \bar{y}_1 + \bar{y}_2)$, $(\tilde{x}, \tilde{y}_1 + \tilde{y}_2) \in S$, *i.e.*, with

$$(\bar{x}, \bar{y}_1) \in S_1, \qquad (\bar{x}, \bar{y}_2) \in S_2, \qquad (\tilde{x}, \tilde{y}_1) \in S_1, \qquad (\tilde{x}, \tilde{y}_2) \in S_2.$$

For $0 \leq \theta \leq 1$,

$$\theta(\bar{x}, \bar{y}_1 + \bar{y}_2) + (1-\theta)(\tilde{x}, \tilde{y}_1 + \tilde{y}_2) = (\theta\bar{x} + (1-\theta)\tilde{x}, (\theta\bar{y}_1 + (1-\theta)\tilde{y}_1) + (\theta\bar{y}_2 + (1-\theta)\tilde{y}_2))$$

is in S because, by convexity of S_1 and S_2,

$$(\theta\bar{x} + (1-\theta)\tilde{x}, \theta\bar{y}_1 + (1-\theta)\tilde{y}_1) \in S_1, \qquad (\theta\bar{x} + (1-\theta)\tilde{x}, \theta\bar{y}_2 + (1-\theta)\tilde{y}_2) \in S_2.$$