Stochastic Process Solution #5

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Problem 1

(1)

$x \mapsto e^{\lambda x}$ is a measurable function and $\sum_{i=1}^{n} X_i$ is $\mathcal{F}_n$-measurable, thus we conclude that Z_n is $\mathcal{F}_n$-measurable. We have that for any $n \in \{0\} \cup [N-1]$,

$$\begin{aligned}
\mathbf{E}[Z_{n+1}|\mathcal{F}_n] &= Z_n \cdot \mathbf{E}[e^{\lambda X_{n+1}}|\mathcal{F}_n] \\
&= Z_n \cdot \mathbf{E}[e^{\lambda X_{n+1}}] && \text{(independence)} \\
&\geq Z_n \cdot e^{\lambda \mathbf{E}[X_{n+1}]} && \text{(convexity of } x \mapsto e^{\lambda x}\text{)} \\
&= Z_n
\end{aligned}$$

Thus $\mathbf{E}[Z_{n+1}|\mathcal{F}_n] \geq Z_n$ and $\{Z_n\}_{n\in\{0\}\cup[N]}$ is a submartingale with respect to $\{\mathcal{F}_k\}_{k\geq 0}$.

(2)

Let $\tau := N \wedge \min\{n \leq N : Z_n \geq e^{\lambda s}\}$. The event $[\tau = k]$ only depends on the values of $X_1, ..., X_k$, thus $[\tau = k] \in \mathcal{F}_k$ for $k \in \{0\} \cup [N]$. For any $k \in \{0\} \cup [N]$,

$$Z_k \leq \mathbf{E}[Z_{k+1}|\mathcal{F}_k] \leq \mathbf{E}[\mathbf{E}[Z_{k+2}|\mathcal{F}_{k+1}]|\mathcal{F}_k] = \mathbf{E}[Z_{k+2}|\mathcal{F}_k] \leq ... \leq \mathbf{E}[Z_N|\mathcal{F}_k]$$

Then we have that

$$\begin{aligned}
\mathbf{E}[Z_\tau] &= \sum_{n=1}^{N} \mathbf{E}[Z_n|\tau = n]\mathbf{Pr}[\tau = n] \\
&\leq \sum_{n=1}^{N} \mathbf{E}[\mathbf{E}[Z_N|\mathcal{F}_n]|\tau = n]\mathbf{Pr}[\tau = n] \\
&= \sum_{n=1}^{N} \mathbf{E}[Z_N|\tau = n]\mathbf{Pr}[\tau = n] && ([\tau = n] \in \mathcal{F}_n) \\
&= \mathbf{E}[Z_N]
\end{aligned}$$

Therefore, by Markov's inequality, we have that

$$\mathbf{Pr}[\max_{1\leq n\leq N} Z_n \geq e^{\lambda s}] = \mathbf{Pr}[Z_\tau \geq e^{\lambda s}] \leq \frac{\mathbf{E}[Z_\tau]}{e^{\lambda s}} \leq \frac{\mathbf{E}[Z_N]}{e^{\lambda s}}$$

(3)

By Problem 1.2, we have that

$$\begin{aligned}
\mathbf{Pr}[\max_{1\le n\le N}\sum_{j=1}^{n} X_j \ge s] &= \mathbf{Pr}[\max_{1\le n\le N} Z_n \ge e^{\lambda s}] \\
&\le \frac{\mathbf{E}[Z_N]}{e^{\lambda s}} && \text{(Problem 1.2)} \\
&= \frac{\prod_{i\in[N]}\mathbf{E}[e^{\lambda X_i}]}{e^{\lambda s}} && \text{(independence)} \\
&\le \exp(\frac{\lambda^2(b-a)^2N}{8} - \lambda s) && \text{(Hoeffding's lemma)}
\end{aligned}$$

Let $\lambda = \frac{4s}{N(b-a)^2}$, we conclude that

$$\mathbf{Pr}[\max_{1\le n\le N}\sum_{j=1}^{n} X_j \ge s] \le e^{\frac{-2s^2}{N(b-a)^2}}$$

Observe that $\{-X_k\}_{k\in[N]}$ are independent random variables taking values in $[-b,-a]$ and $\mathbf{E}[-X_n] = 0$ for $n \in [N]$. Thus following the same way, we have that

$$\mathbf{Pr}[\max_{1\le n\le N}\sum_{j=1}^{n} -X_j \ge s] \le e^{\frac{-2s^2}{N(-a-(-b))^2}}$$

By union bound,

$$\begin{aligned}
\mathbf{Pr}[\max_{1\le n\le N}|\sum_{j=1}^{n} X_j| \ge s] &= \mathbf{Pr}[\max_{1\le n\le N}\sum_{j=1}^{n} X_j \ge s \vee \min_{1\le n\le N}\sum_{j=1}^{n} X_j \le -s] \\
&\le \mathbf{Pr}[\max_{1\le n\le N}\sum_{j=1}^{n} X_j \ge s] + \mathbf{Pr}[\max_{1\le n\le N}\sum_{j=1}^{n} -X_j \ge s] \\
&\le 2e^{\frac{-2s^2}{N(b-a)^2}}
\end{aligned}$$

Problem 2

(1)

Since f,g are probability density functions, f and g are measurable. Thus g/f is measurable on $\mathrm{supp}(f)$ and $g(X_i)/f(X_i)$ is $\sigma(X_i)$-measurable for any $i \in \mathbb{N}_+$. Since $\{g(X_n)/f(X_n)\}_{n\in\mathbb{N}_+}$ are independent nonnegative random variables, Z_n is $\sigma(X_1,...,X_n)$-measurable.
We have that for any $n \ge 0$,

$$\begin{aligned}
\mathbf{E}[Z_{n+1}|X_1,...,X_n] &= \prod_{i=1}^{n}\frac{g(X_i)}{f(X_i)}\mathbf{E}[\frac{g(X_{n+1})}{f(X_{n+1})}|X_1,...,X_n] \\
&= Z_n\mathbf{E}[\frac{g(X_{n+1})}{f(X_{n+1})}] && \text{(independence)} \\
&= Z_n\int_{\mathbb{R}}\frac{g(x)}{f(x)}f(x)\mathrm{d}x \\
&= Z_n && (g \text{ is a density})
\end{aligned}$$

Thus $\mathbf{E}[Z_{n+1}|X_1,...,X_n] = Z_n$ and $\{Z_n\}_{n\ge 0}$ is a martingale with respect to $\{X_n\}_{n\ge 1}$.

(2)

Assume **(H1)**. The event $[T_{ab} = k]$ only depends on the values of $X_1, ..., X_k$, thus $[T_{ab} = k] \in \sigma(X_1, ..., X_k)$ for $k \in \mathbb{N}$ and T_{ab} is a stopping time with respect to $\{X_n\}_{n\geq 1}$. The measure of $x \in \mathbb{R}$ satisfying $f(x) \neq g(x)$ is larger than zero implies that there exists constants $c_1, c_2 > 0$ such that

$$\mathbf{Pr}[\frac{f(X_i)}{g(X_i)} > 1 + c_1] \geq c_2$$

for any $i \in \mathbb{N}_+$. By the independence of $\{X_i\}_{i\in\mathbb{N}_+}$, The probability of translating from a to b in $\log_{1+c_1}(b/a)$ steps in at least $(c_2)^{\log_{1+c_1}(b/a)}$. That is, for any $n \in \mathbb{N}_+$.

$$\mathbf{Pr}[\prod_{j=n}^{n+\log_{1+c_1}(b/a)} \frac{f(X_j)}{g(X_j)} > \frac{b}{a}] > (c_2)^{\log_{1+c_1}(b/a)}$$

Therefore,

$$\mathbf{Pr}[T_{ab} = \infty] \leq \lim_{k\to\infty} (1 - (c_2)^{\log_{1+c_1}(b/a)})^k = 0$$

And $|Z_t| < b$ for all $t < T_{ab}$. Thus by Optional Stopping Theorem, $\mathbf{E}[Z_{T_{ab}}] = \mathbf{E}[Z_0] = 1$. $Z_{T_{ab}}$ equals either a or b, then

$$1 = \mathbf{E}[Z_{T_{ab}}] = a\mathbf{Pr}[Z_{T_{ab}} = a] + b(1 - \mathbf{Pr}[Z_{T_{ab}} = a]) \implies \mathbf{Pr}[Z_{T_{ab}} = a] = \frac{b-1}{b-a}$$

Assume **(H2)**. In the same way, $\{\frac{1}{Z_n}\}_{n\geq 0}$ is a martingale with respect to $\{X_n\}_{n\geq 1}$. $|\frac{1}{Z_t}| < \frac{1}{a}$ for all $t < T_{ab}$. Thus by Optional Stopping Theorem, $\mathbf{E}[\frac{1}{Z_{T_{ab}}}] = \mathbf{E}[\frac{1}{Z_0}] = 1$ and

$$1 = \mathbf{E}[\frac{1}{Z_{T_{ab}}}] = \frac{1}{a}(1 - \mathbf{Pr}[Z_{T_{ab}} = b]) + \frac{1}{b}\mathbf{Pr}[Z_{T_{ab}} = b] \implies \mathbf{Pr}[Z_{T_{ab}} = b] = \frac{b-ab}{b-a}$$

In conclusion, if **(H1)** holds, the probabilty that the algorithm provides a correct answer is $\frac{b-1}{b-a}$. If **(H2)** holds, the probabilty that the algorithm provides a correct answer is $\frac{b-ab}{b-a}$.

(3)

$x \mapsto \log x$ is a measurable function and Z_n is $\sigma(X_1, ..., X_n)$-measurable. Thus $\log Z_n$ is $\sigma(X_1, ..., X_n)$-measurable. We have that

$$\begin{aligned}
\mathbf{E}[\log Z_{n+1}|X_1, ..., X_n] &= \log \prod_{i=1}^{n} \frac{g(X_i)}{f(X_i)} + \mathbf{E}[\log \frac{g(X_{n+1})}{f(X_{n+1})}|X_1, ..., X_n] \\
&= \log Z_n + \mathbf{E}[\log \frac{g(X_{n+1})}{f(X_{n+1})}] && \text{(independence)} \\
&\leq \log Z_n + \log \mathbf{E}[\frac{g(X_{n+1})}{f(X_{n+1})}] && \text{(concavity of } x \mapsto \log x\text{)} \\
&= \log Z_n + \log 1 = \log Z_n && \text{((\textbf{H1}) holds)}
\end{aligned}$$

Thus $\mathbf{E}[\log Z_{n+1}|X_1, ..., X_n] \leq \log Z_n$ and $\{Z_n\}_{n\geq 0}$ is a supermartingale with respect to $\{X_n\}_{n\geq 1}$.

(4)

Let $A_n = n\mathbf{E}[\log \frac{f(X)}{g(X)}]$ and the density of X is f. Observe that $\mathbf{E}[\log \frac{f(X)}{g(X)}] = D_{KL}(f \parallel g) > 0$, thus $\{A_n\}_{n\geq 0}$ is an increasing sequence. We have that

$$\begin{aligned}
\mathbf{E}[M_{n+1}|X_1, ..., X_n] &= \log Z_n + (n+1)\mathbf{E}[\log \frac{f(X)}{g(X)}] + \mathbf{E}[\log \frac{g(X_{n+1})}{f(X_{n+1})}|X_1, ..., X_n] \\
&= M_n + \mathbf{E}[\log \frac{f(X)}{g(X)}] + \mathbf{E}[\log \frac{g(X_{n+1})}{f(X_{n+1})}] && \text{(independence)} \\
&= M_n + \mathbf{E}[\log \frac{f(X_{n+1})}{g(X_{n+1})}] + \mathbf{E}[\log \frac{g(X_{n+1})}{f(X_{n+1})}] \\
&= M_n + \mathbf{E}[\log 1] = M_n
\end{aligned}$$

The third equation holds since X and X_{n+1} share the same distribution f and they are independent. Thus $\{M_n\}_{n\geq 0}$ is a martingale with respect to $\{X_n\}_{n\geq 1}$.

(5)

We follow the notation in Problem 2.2, let $C = \log_{1+c_1}(b/a)$. We conclude that

$$\mathbf{E}[T_{ab}] = \sum_{n=1}^{\infty} \mathbf{Pr}[T_{ab} \geq n] \leq \sum_{n=1}^{\infty} (1 - c_2^C)^{\lfloor \frac{n}{C} \rfloor} \leq \frac{1}{1 - c_2^C} \sum_{n=1}^{\infty} ((1 - c_2^C)^{\frac{1}{C}})^n < \infty$$

And we have that

$$\begin{aligned}
\mathbf{E}[|M_{n+1} - M_n| \mid X_1, ..., X_n] &= \mathbf{E}[|\mathbf{E}[\log \frac{f(X)}{g(X)}] + \log \frac{g(X_{n+1})}{f(X_{n+1})}| \mid X_1, ..., X_n] \\
&\leq \mathbf{E}[\log \frac{f(X)}{g(X)}] + \mathbf{E}[|\log \frac{g(X)}{f(X)}|] \leq 2\mathbf{E}[|\log \frac{g(X)}{f(X)}|]
\end{aligned}$$

Since $\mathbf{E}[|\log \frac{g(X)}{f(X)}|] < \infty$, $\mathbf{E}[|M_{n+1} - M_n| \mid X_1, ..., X_n]$ is bounded by some constant. By Optional Stopping Theorem, We have that

$$0 = \mathbf{E}[M_0] = \mathbf{E}[M_{T_{ab}}] = \mathbf{E}[\log Z_{T_{ab}}] + \mathbf{E}[A_{T_{ab}}] = \mathbf{E}[\log Z_{T_{ab}}] + \mathbf{E}[T_{ab}]\mathbf{E}[\log \frac{f(X)}{g(X)}]$$

By Problem 2.2, $\mathbf{E}[\log Z_{T_{ab}}] = \frac{(b-1)\log a + (1-a)\log b}{b-a}$. Therefore,

$$\mathbf{E}[T_{ab}] = \frac{-\mathbf{E}[\log Z_{T_{ab}}]}{\mathbf{E}[\log \frac{f(X)}{g(X)}]} = \frac{(b-1)\log a + (1-a)\log b}{(b-a) \cdot \int_{\mathbb{R}} (\log g(x) - \log f(x)) f(x) \mathrm{d}x}$$