证明：$L_x = yP_z - zP_y$

$$[L_x, p_y] = [yP_z - zP_y, P_y]$$

$$= [yP_z, P_y] - [zP_y, P_y]$$

$$= y[P_z, P_y] + [y, P_y]P_z - z[P_y, P_y] - [z, p_y]P_y$$

$$= -(-i\hbar \frac{\partial y}{\partial y})P_z = i\hbar P_z$$

$$[L_z, P_z] = [xP_y - yP_x, P_z]$$

$$= x[P_y, P_z] + [x, P_z]P_y - y[P_x, P_z] - [y, P_z]P_x$$

$$= 0$$

1-27. $\hat{L}^2 = \hat{L}_x^2+\hat{L}_y^2+\hat{L}_z^2$

$$[\hat L^2,\hat L_x] = [\hat L_x^2+\hat L_y^2+\hat L_z^2, \hat L_x]$$

$$= [\hat L_x^2,\hat L_x]+[\hat L_y^2,\hat L_x]+[\hat L_z^2,\hat L_x]$$

$$= \hat L_x[\hat L_x,\hat L_x]+[\hat L_x,\hat L_x]\hat L_x+\hat L_y[\hat L_y,\hat L_x]+[\hat L_y,\hat L_x]\hat L_y+\hat L_z[\hat L_z,\hat L_x]+[\hat L_z,\hat L_x]\hat L_z$$

$$=0+0+\hat L_y(-i\hbar\hat L_z)+(-i\hbar \hat L_z)\hat L_y + i\hbar \hat L_z\hat L_y + i\hbar\hat L_y\hat L_z$$

$$= -i\hbar \hat L_y\hat L_z + i\hbar\hat L_y\hat L_z - i\hbar\hat L_z\hat L_y + i\hbar \hat L_z\hat L_y$$

$$=0$$

同理，有 $[\hat L^2,\hat L_y]=[\hat L^2,\hat L_z]=[\hat L^2,\hat L_x]=0$

1-28. • $\hat L = \hat r\times\hat p$，$[\hat L, \frac1r] = [\hat r\times\hat p, \frac1r]$

$$=\hat r\times\hat p\cdot\frac1r - \frac1r\cdot\hat r\times\hat p$$

$$=\hat r\times(\hat p\cdot\frac1r) - \hat r\times(\frac1r\hat p)$$

$$=\hat r\times(\hat p\cdot\frac1r-\frac1r\hat p)$$

$$=\hat r\times[\hat p,\frac1r]$$

$$=\hat r\times(i\hbar\frac{r}{r^3})=0$$

• $[\hat L,\hat p^2]=[\hat L,\hat p_x^2+\hat p_y^2+\hat p_z^2]$

$$=[\hat L,\hat p_x^2]+[\hat L,\hat p_y^2]+[\hat L,\hat p_z^2]$$

$$=\hat p_x[\hat L,\hat p_x]+[\hat L,\hat p_x]\hat p_x+\hat p_y[\hat L,\hat p_y]+[\hat L,\hat p_y]\hat p_y+\hat p_z[\hat L,\hat p_z]+[\hat L,\hat p_z]\hat p_z$$

令 $\hat L\Rightarrow\hat L_x$，上式 $=0+0+i\hbar\hat p_y\hat p_z+i\hbar\hat p_z\hat p_y+\hat p_z[\hat L_x,\hat p_z]+[\hat L_x,\hat p_z]\hat p_z$

下求 $[\hat L_x,\hat p_z]$： $[\hat L_x,\hat p_z]=[y\hat p_z - z\hat p_y, \hat p_z]$

$$=y[\hat p_z,\hat p_z]+[y,\hat p_z]\hat p_z - z[\hat p_y,\hat p_z]-[z,\hat p_z]\hat p_y = -i\hbar\hat p_y$$

∴ 上式 $= i\hbar\hat p_y\hat p_z+i\hbar\hat p_z\hat p_y - i\hbar\hat p_z\hat p_y - i\hbar\hat p_y\hat p_z=0$

同理 $[\hat L_y,\hat p^2]=[\hat L_z,\hat p^2]=0$ ∴ $[\hat L,\hat p^2]=0$

$2i \quad -4 \quad 4 \quad 2ie^{2i\phi}$

1-32. $[\hat{L}_i,\hat{P}_j]=[j\hat{P}_k-k\hat{P}_j,\hat{P}_j]\quad(i\neq j)$

$=j[\hat{P}_k,\hat{P}_j]+[j,\hat{P}_j]\hat{P}_k-k[\hat{P}_j,\hat{P}_j]-[k,\hat{P}_j]\hat{P}_j$

$=i\hbar\hat{P}_k$

$i=j$时，$[\hat{L}_i,\hat{P}_j]=[\hat{L}_i,\hat{P}_i]$

$=j[\hat{P}_k,\hat{P}_i]+[j,\hat{P}_i]\hat{P}_k-k[\hat{P}_j,\hat{P}_i]-[k,\hat{P}_i]\hat{P}_j=0$

$\therefore [\hat{L}_i,\hat{P}_j]=i\hbar\varepsilon_{ijk}\hat{P}_k$

1-33. $\hat{L}_z Y_{2,2}(\theta,\phi)=-i\hbar\frac{\partial}{\partial\phi}\sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{2i\phi}=2\hbar Y_{2,2}(\theta,\phi)$

$\therefore Y_{2,2}(\theta,\phi)$是$\hat{L}_z$的本征函数，本征值为$2\hbar$

$\hat{L}^2 Y_{2,2}(\theta,\phi)=-\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{2i\phi}$

$=-\hbar^2\sqrt{\frac{15}{32\pi}}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(2\sin^2\theta\cos\theta)-4\right)e^{2i\phi}$

$=-\hbar^2\sqrt{\frac{15}{32\pi}}\left(\frac{1}{\sin\theta}(4\sin\theta\cos^2\theta-2\sin^3\theta)-4\right)e^{2i\phi}$

$=-\hbar^2\sqrt{\frac{15}{32\pi}}\left(4\cos^2\theta-2\sin^2\theta-4\right)e^{2i\phi}$

$=-\hbar^2\sqrt{\frac{15}{32\pi}}\left(-6\sin^2\theta\right)e^{2i\phi}$

$=6\hbar^2 Y_{2,2}(\theta,\phi)$

$\therefore Y_{2,2}(\theta,\phi)$是$\hat{L}^2$的本征函数，本征值为$6\hbar^2$

故$Y_{2,2}(\theta,\phi)$是$\hat{L}^2$、$\hat{L}_z$的共同本征函数

1-34. $\int_0^{2\pi}|\psi(\phi)|^2 d\phi=\int_0^{2\pi}\psi^*(\phi)\psi(\phi)d\phi$

$=\int_0^{2\pi}\left(\frac{1}{2\pi}+\sqrt{\frac{1}{18\pi^2}}(e^{i\phi}+e^{-i\phi})\right)d\phi=\int_0^{2\pi}\frac{1}{2\pi}d\phi+0$

$=2\pi\cdot\frac{1}{2\pi}=1$

$\therefore$ 满足归一化条件

$\hat{L}_z$的本征值为$m\hbar$，本征函数为$\frac{1}{\sqrt{2\pi}}e^{im\phi}$

$\therefore \psi(\phi)=\sqrt{\frac{2}{3}}\cdot\frac{1}{\sqrt{2\pi}}e^{i\cdot 0\cdot\phi}+\sqrt{\frac{1}{3}}\cdot\frac{1}{\sqrt{2\pi}}e^{i\cdot(-1)\cdot\phi}$

对应m分别为0和-1

故可能得到0和$-\hbar$，概率分别是$\left(\sqrt{\frac{2}{3}}\right)^2=\frac{2}{3}$和$\left(\sqrt{\frac{1}{3}}\right)^2=\frac{1}{3}$

$\langle\hat{L}_z\rangle=\frac{2}{3}\times 0-\frac{1}{3}\times\hbar=-\frac{\hbar}{3}$

1-35 (1) $H\phi=\lambda\phi$

$$\begin{bmatrix}E_1 & V\\ V & E_2\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\lambda\begin{bmatrix}a\\ b\end{bmatrix}$$

$$\begin{vmatrix}E_1-\lambda & V\\ V & E_2-\lambda\end{vmatrix}=0 \Rightarrow \begin{cases}\lambda_1=\frac{E_1+E_2-\sqrt{(E_1-E_2)^2+4V^2}}{2}\\ \lambda_2=\frac{E_1+E_2+\sqrt{(E_1-E_2)^2+4V^2}}{2}\end{cases}$$

$\lambda=\lambda_1$时：可解得$\phi_1=\left[-\frac{1}{2\lambda}\left(-E_1+E_2+\sqrt{(E_1-E_2)^2+4\lambda^2}\right),1\right]^T$

$\lambda=\lambda_2$时：可解得$\phi_2=\left[-\frac{1}{2\lambda}\left(-E_1+E_2-\sqrt{(E_1-E_2)^2+4\lambda^2}\right),1\right]^T$

(2) $t>0$时，令量子态为ψ

则$\psi=C_1\phi_1e^{-\frac{i\lambda_1t}{\hbar}}+C_2\phi_2e^{-\frac{i\lambda_2t}{\hbar}}$

由$t=0$时处于$\begin{bmatrix}0\\1\end{bmatrix}$态：$C_1\phi_1+C_2\phi_2=\begin{bmatrix}V\\E_2\end{bmatrix}$

以及归一化条件

可解得：
$$\begin{cases}C_1=\frac{E_1-E_2+\sqrt{(E_1-E_2)^2+4\lambda^2}}{2\sqrt{(E_1-E_2)^2+4\lambda^2}}\\ C_2=\frac{-E_1+E_2+\sqrt{(E_1-E_2)^2+4\lambda^2}}{2\sqrt{(E_1-E_2)^2+4\lambda^2}}\end{cases}$$

$-\lambda^3+2\lambda=0$ $\quad$ $-\lambda^2+2$

1-36. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda^3+2\lambda=0 \Rightarrow \begin{cases} \lambda_1=0 \\ \lambda_2=\sqrt{2} \\ \lambda_3=-\sqrt{2} \end{cases}$$

$\lambda_1=0:\ \phi_1=\frac{1}{\sqrt{2}}[1,0,-1]^T$

$\lambda_2=\sqrt{2}:\ \phi_2=\frac{1}{2}[1,\sqrt{2},1]^T$

$\lambda_3=-\sqrt{2}:\ \phi_3=\frac{1}{2}[1,-\sqrt{2},1]^T$

幺正矩阵：$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

1-40. $\langle \hat{L}_x \rangle = \langle Y_{lm} | \hat{L}_x | Y_{lm} \rangle$

$= \int Y_{lm}^* \hat{L}_x Y_{lm} \, d\Omega$

由 $[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x$

上式 $= \frac{1}{i\hbar}\int Y_{lm}^* \hat{L}_y \hat{L}_z Y_{lm} \, d\Omega - \frac{1}{i\hbar}\int Y_{lm}^* \hat{L}_z \hat{L}_y Y_{lm} \, d\Omega$

由角动量算符的厄密性，有：

上式 $= \frac{1}{i\hbar}\left[\int Y_{lm}^* \hat{L}_y (\hat{L}_z Y_{lm}) d\Omega - \int (\hat{L}_z Y_{lm})^* \hat{L}_y Y_{lm} \, d\Omega\right]$

$= \frac{1}{i\hbar}\left[m\hbar \int Y_{lm}^* \hat{L}_y Y_{lm} \, d\Omega - m\hbar \int Y_{lm}^* \hat{L}_y Y_{lm} \, d\Omega\right]$

$= 0$

同理 $\langle \hat{L}_y \rangle = 0$

1-41. $\phi(x)$为偶宇称实函数

$\bar{x}=\int_{-\infty}^{\infty}x\phi^2dx=0$

$\overline{x^2}=\int_{-\infty}^{\infty}x^2\cdot\left(\frac{\pi}{a}\right)^{-\frac{1}{2}}\cdot e^{-ax^2}dx$

$=2\left(\frac{\pi}{a}\right)^{-\frac{1}{2}}\int_0^{\infty}x^2e^{-ax^2}dx$

$=2\left(\frac{\pi}{a}\right)^{-\frac{1}{2}}\frac{1}{4a}\cdot\left(\frac{\pi}{a}\right)^{\frac{1}{2}}=\frac{1}{2a}$

$\Delta x=\left(\overline{x^2}-\bar{x}^2\right)^{\frac{1}{2}}=\frac{1}{\sqrt{2a}}$

$\bar{p}=-i\hbar\int_{-\infty}^{\infty}\phi^*\phi'dx=0$ (奇函数)

$\overline{p^2}=(-i\hbar)^2\int_{-\infty}^{\infty}\phi^*\phi''dx$

$=-\hbar^2\cdot a\cdot\left(\frac{\pi}{a}\right)^{-\frac{1}{2}}\cdot 2\cdot\left(-\frac{\sqrt{\pi}}{4\sqrt{a}}\right)=\frac{\hbar^2a}{2}$

$\Delta p=\left(\overline{p^2}-\bar{p}^2\right)^{\frac{1}{2}}=\frac{\hbar\sqrt{a}}{\sqrt{2}}$

$\Delta x\cdot\Delta p=\frac{\hbar}{2}\geq\frac{\hbar}{2}$，验证成功

1-42. $\Delta S_x\Delta S_y\geq\frac{1}{2}|\langle[\hat{S}_x,\hat{S}_y]\rangle|$

由$[\hat{S}_x,\hat{S}_y]=i\hbar\hat{S}_z$，得$\Delta S_x\cdot\Delta S_y\geq\frac{\hbar}{2}|\langle\hat{S}_z\rangle|$

$\Delta L_x\cdot\Delta L_y\geq\frac{1}{2}|\langle[\hat{L}_x,\hat{L}_y]\rangle|$ 由$[\hat{L}_x,\hat{L}_y]=i\hbar\hat{L}_z$

得$\Delta L_x\cdot\Delta L_y\geq\frac{\hbar}{2}|\langle\hat{L}_z\rangle|$

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