1. $Pr[X_{t^*} = Y_{t^*}]$

There exists two conditions: 1. $X_{t^*} = Y_{t^*}$ at exactly t^* step

2. $X_{t^*} = Y_{t^*}$ at previous step.

For the first condition:

$Pr[X_{t^*} = Y_{t^*}] \geq Pr[X_{t^*} = Y_{t^*} = 1]$

Construct a coupling of Markov Chains: $(X'_t, Y'_t) \sim \omega_t$

as in the note: $Y_0 \sim \pi$ while $X_0 \sim \mu_0$

$$\overset{\mu_0}{\underset{}{X_0}} \to \overset{\mu_1}{X_1} \to \dots \to \overset{\mu_t}{X_t} \to X_{t+1} \to \dots$$

$$\underset{\pi}{Y_0} \to \underset{\pi}{Y_1} \to \dots \to \underset{\pi}{Y_t} \to Y_{t+1} \to \dots$$

So $Pr[X_{t^*} = Y_{t^*} = 1] \geq Pr[X'_{t^*} = Y'_{t^*} = 1]$ in the 1st condition

the above $\Rightarrow Pr[X'_{t^*} = 1] \cdot Pr[Y'_{t^*} = 1] - Pr[X'_t = Y'_t = a$, for all $t < t^*] \cdot Pr[X'_{t^*} = 1 \mid X'_t = a]\, Pr[Y'_{t^*} = 1 \mid Y'_t = a]$

(t: the first time X'_t and Y'_t meet, a: all possible states)

For the second condition:

$Pr[X_{t^*} = Y_{t^*}] \geq Pr[X_{t^*} = Y_{t^*} = 1]$

$\geq Pr[X_{t^*}' = Y_{t^*}' = a$, for all $t < t^*] \cdot Pr[X'_{t^*} = 1 \mid X'_t = a]$

$Pr[Y'_{t^*} = 1 \mid Y_{t^*} = a]$ since Probility ≤ 1

While X'_t, Y'_t independent. $Pr[X'_{t^*} = 1], Pr[Y'_{t^*} = 1] \geq \alpha$

Combine the two conditions: the overall $Pr = Pr(1st) + Pr(2st)$

Since two conditions are opposite. we have $Pr[X_{t^*} = Y_{t^*}] \geq \alpha \cdot \alpha = \alpha^2$

2. • $X_t \in \Omega$, $\Omega = \{1,2,3,4\}$

Since at each vertex, $\frac{1}{2}$ stay, $\frac{1}{2}$ move at random one direction, which means the move will only be infected by the previous states

suppose $X_{t-1} = a_{t-1}$ then X_t will either be a_{t-1} or vertex nearby.

So we have $Pr[X_t = a_t \mid X_{t-1} = a_{t-1}, \dots, X_1 = a_1, X_0 = a_0] = Pr[X_t = a_t \mid X_{t-1} = a_{t-1}]$

So it's a markov chain

• $P = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 2 & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ 3 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 4 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \end{array}$, generally. we have $P(i,j) = \begin{cases} 0 & (i,j \text{ 不连}) \\ \frac{1}{2d_i}, & i \neq j \\ \frac{1}{2}, & i = j \end{cases}$

(for the following G)

• By definition, A finite Markov chain is irreducible if its transition graph is strongly connected

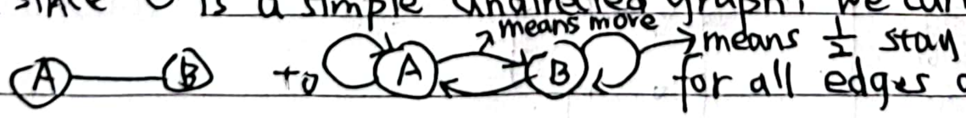
irreducible $\longrightarrow$ connected:

if G is not connected, instead, it's divided into n parts $P_1, P_2, \dots$, then, $\forall i \in P_i$, $j \in P_j$ (i, j are vertex), i, j is disconnected. so i can't transfer to j, vice versa, which means the Markov Chain is reducible.

connected $\longrightarrow$ irreducible:

$\forall i, j$ in G. i can go to j in finite step.

since G is a simple undirected graph, we can transfer



for all edges and vertices to form a transition graph. Since G is connected, $\forall i \in G$ can always transfer to $j \in G$ alongside a simple path in G.

So it's irreducible

• $\pi^T P = \pi^T$, we have aperiodic, finite.

① If irreducible, we have unique π, $\pi(i) = \frac{1}{E_i[T_i]}$ as learned in class.

② If reducible, we have sereval π, and ① is a special case

• 求 $\mu_{10}^T = \mu_0^T P^{10}$ and P is in previous picture, $\mu_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\mu_{10} = \begin{bmatrix} \frac{219}{583} \\ \frac{394}{1581} \\ \frac{385}{3065} \\ \frac{394}{1581} \end{bmatrix} \sim \begin{bmatrix} 0.376 \\ 0.249 \\ 0.126 \\ 0.249 \end{bmatrix}$$

3. for $\forall i$ and its neighbour j, $\frac{g(j)}{g(i)} = \frac{f(j)d(i)}{d(j)\cdot f(i)}$; $\pi(i) = \frac{f(i)}{\sum f(j)} \Rightarrow \frac{g(j)}{g(i)} = \frac{\pi(j)d(i)}{\pi(i)d(j)}$

Its transition matrix P has its $p(i,j)$ decided by $Ber(\frac{g(j)}{g(i)} \wedge 1)$

Then, there are two conditions:

1. $g(j) < g(i)$, $Ber(\frac{g(j)}{g(i)} \wedge 1) = \frac{g(j)}{g(i)} = \frac{\pi(j)d(i)}{\pi(i)d(j)}$

so we have $p(i,j) = \frac{1}{2}\cdot\frac{1}{d(i)}\cdot\frac{\pi(j)\cdot d(i)}{\pi(i)d(j)} = \frac{1}{2}\cdot\frac{\pi(j)}{\pi(i)d(j)}$

while $Ber(\frac{g(i)}{g(j)} \wedge 1) = 1$, we have $p(j,i) = \frac{1}{2}\cdot\frac{1}{d(j)}$

$\Rightarrow \pi(i)\cdot p(i,j) = \pi(j)\cdot p(j,i)$

2. $g(j) \geq g(i)$, $Ber(\frac{g(j)}{g(i)} \wedge 1) = 1$, $p(i,j) = \frac{1}{2d(i)}$

$p(j,i) = \frac{1}{2}\cdot\frac{1}{d(j)}\frac{\pi(i)d(j)}{\pi(j)d(i)} = \frac{1}{2}\frac{\pi(i)}{\pi(j)d(i)}$

$\Rightarrow \pi(j)\cdot p(j,i) = \pi(i)p(i,j)$

If π is a stationary distribution, we have:

$\sum_{i=1}^{n}\pi(i)p(i,j) = \pi(j) \Rightarrow \sum_{i=1}^{n}\pi(i)p(i,j) = \sum_{g(j)\geq g(i)}\pi(i)p(i,j) + \sum_{g(j)<g(i)}\pi(i)p(i,j) + \pi(j)p(j,j)$

$= \sum_{g(j)\geq g(i)}\pi(j)p(j,i) + \sum_{g(j)<g(i)}\pi(j)p(j,i) + p(j,j)\cdot\pi(j) = \pi(j)\sum_{i=1}^{n}p(j,i) = \pi(j)$. Prove!

4. To prove it's irreducible, we need to prove that for any two states S_1, S_2, $S_1 \to S_2$ S_1 can transfer to S_2.

The process: By shuffling, we can iteratively pick i-th in S_1 and compare with S_2, if i-th cards are the same, then, let $i=j$, otherwise, switch the card in S_1 to make the i-th card the same in two states S_1, S_2. In this way, S_1 can transfer to S_2, so irreducible.

Since there is a chance that $i=j$ (self loop). the chain is aperiodic.

The chain has finite state, so it has a unique stationary distribution, and it's the same as other shuffling so uniform distribution is π

• Since the two picks in the following is independent, the position i can't infect c. Therefore, it's equal to pick two i, j (i can equal to j) and switch (since the stationary distribution is the uniform distribution, random card c could be randomly in the deck)

J. Construct a coupling of Markov Chain X_t, Y_t

$X_t \xrightarrow{\text{Switch}} X_{t+1}$, $Y_t \xrightarrow{\text{Switch}} Y_{t+1}$

We apply the following simple coupling rule:

the two decks pick the same i, c

There are different situations:

1 i-th cards are the same in X_t, Y_t, then whether or not the card c are in the same position, it remains the same the amount of cards to be shuffled to let $X=Y$

2. i-th cards are different in X_t, Y_t. Then if card c are in the same position, it also remains the same the amount. If c are in different position, then the amount of cards to be shuffled decrease by 1.

Combined with the above 2(4) situation, the selection to decrease the amount has $Pr(\text{decrease}) = \frac{C_t}{n} \cdot \frac{C_t}{n} = \frac{C_t^2}{n^2}$ (C_t refers to the amount of cards in the same position but differs)

So the mixing time $T = \sum_{i=1}^{n} T_i$, $T_i = \frac{n^2}{C_t^2} = \frac{n^2}{(n-i+1)^2}$

$$T = \sum_{i=1}^{n} \frac{n^2}{(n-i+1)^2} \le \sum_{i=1}^{n-1} n^2 \cdot \left(\frac{1}{n-i} - \frac{1}{n-i+1}\right) + n^2 \le n^2 \cdot \sum_{i=1}^{n-1}\left(\frac{1}{n-i} - \frac{1}{n-i+1}\right) + n^2$$

$$= n^2 \cdot \left(\frac{1}{n-1} - \frac{1}{n} + \frac{1}{n-2} - \frac{1}{n-1} + \ldots + 1 - \frac{1}{2}\right) + n^2$$

$$= 2n^2 - n$$

Therefore, the mixing time is $O(n^2)$

Reference: Your previous note & 刘书瑜