1. First, prove it's recurrent:

As mentioned in class, if the chain is irreducible, then if a state is recurrent $\longrightarrow$ all the states are recurrent.

Suppose there is a state that is transient, then all the states are transient, so we have $\forall i, j \in S$, $p_{ij}^{(n)} \to 0$

$\Rightarrow \sum_{j=1}^{N} p_{ij}^{(n)} \to 0$, since the states' number N is finite,

which contradicts to the property that $\sum_{j=1}^{N} p_{ij}^{(n)} = 1$. Therefore, all the states are recurrent.

Then, prove [PR]

Similarly, suppose state i is null recurrent, then all the states are recurrent, futher suppose state j is positive recurrent.

since $E_i[T_i] = \infty$

$\Rightarrow \lim_{n\to\infty} p_{ii}^{(n)} = 0$, Since irreducible, $\exists t_1, t_2$, $p_{ij}^{(t_1)} > 0$, $p_{ji}^{(t_2)} > 0$

then $p_{ii}^{(t_1+t_2)} \geq p_{ij}^{(t_1)} \cdot p_{ji}^{(t_2)} > 0 \Rightarrow p_{ii}^{(t_1+n+t_2)} \geq p_{ij}^{(t_1)} \cdot p_{jj}^{(n)} \cdot p_{ji}^{(t_2)} \geq 0$

then $\lim_{n\to\infty} p_{ii}^{(t_1+n+t_2)} \to 0 \geq p_{ij}^{(t_1)} \cdot p_{jj}^{(n)} \cdot p_{ji}^{(t_2)} \geq 0$ (Since it's the sum o[f] p·p

since $p_{ij}^{(t_1)} \cdot p_{ji}^{(t_2)} > 0$, we have $\lim_{n\to\infty} p_{jj}^{(n)} = 0$, which means state j is null recurrent. Contradiction! So if i is null recurrent, then all the states are null recurrent.

Therefore, we have $\lim_{n\to\infty} p_{ij}^{(n)} = 0 \Rightarrow \sum_{j=1}^{N} p_{ij}^{(n)} = 0$, Contradiction!

So all the states are positive recurrent.

[F] + [I] $\Rightarrow$ [PR]

2. the proof is similar.

Since irreducible, then [R] is a class property as mentioned in class.

Suppose i is null recurrent, $\forall j$

$\Rightarrow \lim_{n\to\infty} p_{ii}^{(n)}=0$, irreducible, $\exists t_1, t_2$, $p_{ij}^{(t_1)}>0$, $p_{ji}^{(t_2)}>0$

then $p_{ii}^{(t_1+t_2)} \geq p_{ij}^{(t_1)} \cdot p_{ji}^{(t_2)} > 0$ (since $p_{ii}^{(t_1+t_2)}$ is the sum of $p_{ij}^{(t)} p_{ji}^{(t)}$)

$\Rightarrow \lim_{n\to\infty} p_{ii}^{(t_1+n+t_2)} \geq p_{ij}^{(t_1)} \cdot \lim_{n\to\infty} p_{jj}^{(n)} \cdot p_{ji}^{(t_2)} \geq 0$

Since $\lim_{n\to\infty} p_{ii}^{(t_1+n+t_2)} = 0$, we have $\lim_{n\to\infty} p_{jj}^{(n)} = 0$, which suggests that j is null recurrent. Therefore, if one state is null recurrent, then all states are null recurrent. Therefore, if one state if [PR], then all states all recurrent. Combined with the property that there is no null recurrent state, all states must be [PR] $\Rightarrow$ [PR] is a class property

3. Let ϕ be the 2-CNF formula and $V=\{v_1, v_2, \dots, v_n\}$ be its set of variables. Let $\sigma_1, \sigma_2, \dots, \sigma_T$ be the assignment, $T=2n^2$

$X_t \triangleq \{v \in V \mid \sigma^*(v) = \sigma_t(v)\}$. the number of varibles that are the same with σ^*. we have $Pr[X_{t+1}=X_t+1 \mid \sigma_t] \geq \frac{1}{2}$, $Pr[X_{t+1}=X_t-1 \mid \sigma_t] \leq \frac{1}{2}$. It's a markov chain with $\geq \frac{1}{2}$ walk right and $\leq \frac{1}{2}$ walk left.

Define Y_t, $Y_{t+1} = Y_t + \begin{cases} 1, & p=\frac{1}{2} \\ -1, & p=\frac{1}{2} \end{cases}$.

Then Couple (X_t, Y_t), then $X_t \geq Y_t$

$Pr[\text{algo fail}] = (1 - Pr[\exists t \in \{0, 2n^2\}, X_t = n])^{\sqrt{0}} \geq$

$(1 - Pr[\exists t \in \{0, 100n^2\}, Y_t = n])^{\sqrt{0}}$

$Pr[\text{algo fail}] = (Pr[\max_{t \in [2n^2]} X_t < n])^{\sqrt{0}} \leq (Pr[\max_{t \in [2n^2]} Y_t < n])^{\sqrt{0}}$

Then define $T_{i \to n}$: the step to take from i to n. $(i \geq 1)$

$T_{i \to n} = \sum_{k=i}^{n-1} T_{k \to k+1}$, as mentioned in class, $E[T_{i \to n}] = n^2 - i^2 \leq n^2$

Then, by markov inequility, $Pr[T_{Y_0 \to n} > 2n^2] \leq \frac{E[T_{Y_0 \to n}]}{2n^2} \leq \frac{1}{2}$

Therefore $Pr[\text{algo fail}] = (Pr[T_{Y_0 \to n} > 2n^2])^{\sqrt{0}} \leq (\frac{1}{2})^{\sqrt{0}} = \frac{1}{2^{\sqrt{0}}}$

4. Suppose a clause $V_{j1} \vee V_{j2} \vee V_{j3}$ (the same if $v_j \to \bar{v}_j$)

Since it's unsatisfied, the current is 0, 0, 0

there's 7 conditions, from the table below, the $Pr[X_{t+1}=X_t+1]$ all $\geq \frac{1}{3}$

V_{j1}	V_{j2}	V_{j3}		$X_{t+1}=X_t+1$	$X_{t+1}=X_t-1$
0	0	1		$\frac{1}{3}$	$\frac{2}{3}$
0	1	0		$\frac{1}{3}$	$\frac{2}{3}$
0	1	1	random	$\frac{2}{3}$	$\frac{1}{3}$
1	0	0	flip	$\frac{1}{3}$	$\frac{2}{3}$
1	0	1		$\frac{2}{3}$	$\frac{1}{3}$
1	1	0		$\frac{2}{3}$	$\frac{1}{3}$
1	1	1		1	0

and $Pr[X_{t+1}=X_{t-1}]$ all $\leq \frac{2}{3}$. So the conclusion holds

$\frac{1}{n}\left(\frac{1}{2^1\cdot 1}\right)+\frac{1}{n} \quad \frac{1}{n}\left(\frac{1}{2^1\cdot 1}+\frac{1}{2^2\cdot 2} \geq \frac{\quad}{2^i\sqrt{i}} \geq \frac{0}{i2^i}. \quad \frac{\quad}{n2^n}\right.$

5. Suppose repeating for $c\cdot 2^n$ times

$Pr[\text{algo fail}] = 1 - Pr[\exists t \in \{0, c\cdot 2^n\}, X_t = n]$

$= Pr[\max_{x\in[c\cdot 2^n]} X_t < n] \leq Pr[\max_{t\in[c\cdot 2^n]} Y_t < n]$, Define Y_t: $Y_{t+1} = Y_t + \begin{cases} 1, & p=\frac{1}{3} \\ -1, & p=\frac{2}{3} \end{cases}$

$T_{i\to n} = \sum_{k=i}^{n-1} T_{k\to k+1}$. A: 1st step toward right

$T_{k\to k+1} = 1[A] + 1[\bar{A}](1 + T_{k-1\to k+1})$

$E[T_{k\to k+1}] = Pr[A] + (1-Pr[A])\cdot(E[T_{k-1\to k}] + E[T_{k\to k+1}] + 1)$

$= \frac{1}{3} + \frac{2}{3}(E[T_{k-1\to k}] + E[T_{k\to k+1}] + 1)$

$\Rightarrow \frac{1}{3}E[T_{k\to k+1}] = \frac{1}{3} + \frac{2}{3}E[T_{k-1\to k}] + \frac{2}{3}$

$\Rightarrow E[T_{k\to k+1}] = 2E[T_{k-1\to k}] + 3$, $E[T_{0\to 1}] = 1$

$E[T_{k\to k+1}] = 2^{k+2} - 3$

Therefore $E[T_{i\to n}] = \sum_{k=i}^{n-1} E[T_{k\to k+1}] = 2^{n+2} - 2^{i+2} - 3n + 3j \leq 2^{n+2}$

$Pr[\text{algo fail}] \leq Pr[T_{Y_0\to n} > c\cdot 2^n] \leq \frac{E[T_{Y_0\to n}]}{c\cdot 2^n} \leq \frac{4}{c}$.

that is for suffieiently large enough c ($c>400$) the $Pr[\text{algo fail}] \leq \frac{1}{100}$. Proved

7. Starting at random.

$X_0 = n - i$ at first.

in first $3i$ steps, consider we have at most i steps toward left (in this condition, Algo successes)

$$\Pr[\text{the above statement}] \geq C_{3i}^{i} \left(\frac{1}{3}\right)^{2i} \left(\frac{2}{3}\right)^{i} = C_{3i}^{i} \frac{2^i}{3^{3i}} \geq \frac{1}{\sqrt{6i}} \cdot 2^{-i}$$

$$\Pr[\text{the Algo successes}] \geq \underbrace{\frac{1}{2^n}}_{i=0} + \sum_{i=1}^{n} \frac{1}{\sqrt{6i}} \cdot 2^{-i} \cdot C_n^i \cdot \frac{1}{2^n}$$

$$\geq \frac{1}{2^n\sqrt{6n}} \sum_{i=0}^{n} C_n^i \, 2^{-i} = \frac{1}{2^n\sqrt{6n}} \left(\frac{3}{2}\right)^n = \left(\frac{3}{4}\right)^n \cdot \frac{1}{\sqrt{6n}}$$

8. Algorithm: First, Start with a random distribution Y_0,
Then pick an unsatisfied clause and flip a varible at random, repeat it for $3n$ times. (at most). We call it a round.
Repeat it for $\frac{10}{p}\ln p$ ~~王~~ rounds.

Proof: why $\frac{10}{p}\ln n$: Suppose at each round, $Pr[Algo\ success] = p$, then after all rounds. $Pr[Algo\ fails] = (1-p)^t$ (t for t rounds),

we have $(1-p)^t \le e^{-pt} \xrightarrow{t=\frac{10}{p}\ln n} (1-p)^t \le e^{-pt} = n^{-10}$.

Therefore, we have Probility $\ge 99\%$

From 7. we know $Pr[Algo\ Success] \ge \frac{1}{\sqrt{6n}}\left(\frac{3}{4}\right)^n$

So in $\sqrt{6n}\left(\frac{4}{3}\right)^n \cdot 10 \cdot \ln n$ times

$$\downarrow$$

$$n^{O(1)}\ c^n \text{ time},\ c \in (1,2]$$

Reference: https://people.eecs.berkeley.edu/~venkatg/teaching/15252-sp21/notes/CMU-210-Schoning-3Sat.pdf ~~fo~~ for last 2 question

https://www.math.pku.edu.cn/teachers/lidf/course/stochproc/stochprocnotes/html/_book/markovc.html for Pro 1.