CS2601 Linear and Convex Optimization: Homework 2

SJTU 2023 Fall

Oct. 31, 2023

Submission Guideline

Deadline: 23:59pm, Sunday, Nov. 12, 2023

Submissions later than the deadline will be discounted:

(a) within 0-24 hours, 20% off;

(b) within 24-48 hours, 50% off;

(c) larger than 48 hours, not acceptable.

Acceptable submission formats:

(1) You are encouraged to submit the electronic version of your homework to the Canvas. You may write your answers in a paper by hand, and then take photos of the answer sheet to get the electronic version.

1 (10%) Question 1

Define $f(x_1, x_2) = 10 - 2(x_2 - x_1^2)^2$, and

$$S = \{(x_1, x_2) \mid -11 \le x_1 \le 1,\ -1 \le x_2 \le 1\}.$$

Is $f(x_1, x_2)$ a convex function over the set S?

2 (20%) Question 2

Suppose $f : \mathbf{R} \to \mathbf{R}$ is convex, and $a, b \in \mathbf{dom}(f)$ with $a < b$.

(a) Show that

$$f(x) \le \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b), \quad \forall x \in [a, b].$$

(b) Show that

$$\frac{f(x) - f(a)}{x-a} \le \frac{f(b) - f(a)}{b-a} \le \frac{f(b) - f(x)}{b-x}, \quad \forall x \in (a, b).$$

Draw a sketch that illustrate this inequality.

(c) Suppose f is differentiable. Use the result in (b) to show that

$$f'(a) \le \frac{f(b) - f(a)}{b-a} \le f'(b).$$

(d) Suppose f is twice differentiable. Use the result in (c) to show that $f''(a) \ge 0$ and $f''(b) \ge 0$.

3 (10%) Question 3

Inverse of an increasing convex function. Suppose $f : \mathbf{R} \to \mathbf{R}$ is increasing and convex on its domain (a, b). Let g denote its inverse, i.e., the function with domain $(f(a), f(b))$ and $g(f(x)) = x$ for $a < x < b$. What can you say about convexity or concavity of g?

4 (10%) Question 4

Kullback-Leibler (KL) divergence between two positive vectors u,$v \in \mathbf{R}^n_{++}$ is given by

$$D_{kl} = \sum_{i=1}^{n} (u_i \log(u_i/v_i) - u_i + v_i).$$

Prove the *information inequality*: $D_{kl}(u, v) \geq 0$ for all u, $v \in \mathbf{R}^n_{++}$. Also, show that $D_{kl}(u, v) = 0$ if and only if $u = v$. Hint: The Kullback-Leibler divergence can be expressed as

$$D_{kl}(u, v) = f(u) - f(v) - \nabla f(v)^T (u - v),$$

where $f(v) = \sum_{i=1}^{n} v_i \log v_i$ is the negative entropy of v.

5 (20%) Question 5

For each of the following functions determine whether it is convex, concave.

(a) $f(x) = e^x - 1$ on $\mathbf{R}$.

(b) $f(x_1, x_2) = x_1 x_2$ on $\mathbf{R}^2_{++}$.

(c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbf{R}^2_{++}$.

(d) $f(x_1, x_2) = x_1/x_2$ on $\mathbf{R}^2_{++}$.

(e) $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, where $0 \leq \alpha \leq 1$, on $\mathbf{R}^2_{++}$.