Algorithm Design and Analysis (Fall 2023)

Assignment 3

Deadline: Dec 12, 2023

1. (25 points) Given a constant $k \in \mathbb{Z}^+$, we say that a vertex u in an undirected graph *covers* a vertex v if the distance between u and v is at most k. In particular, a vertex u covers all those vertices that are within distance k from u, including u itself. Given an undirected *tree* $G = (V, E)$ and the parameter k, consider the problem of finding a minimum-size subset of vertices that covers all the vertices in G. Design a polynomial time algorithm for this problem. Prove the correctness of your algorithm and analyze its running time. You will receive 15 points if you can solve the problem for $k = 1$.

2. (25 points) Suppose you are a driver, and you plan to drive from A to B through a highway with distance D. Since your car's tank capacity C is limited, you need to refuel your car at the gas station on the way. We are given n gas stations on the highway with surplus supply. Let $d_i \in (0, D)$ be the distance between the starting point A and the i-th gas station. Let p_i be the price for each unit of gas at the i-th gas station. Suppose each unit of gas exactly supports one unit of distance. The car's tank is empty at the beginning, and the 1-st gas station is at A. Design efficient algorithms for the following tasks.

   (a) (5 points) Determine whether it is possible to reach B from A.

   (b) (20 points) Minimized the gas cost for reaching B.

   Please prove the correctness of your algorithms and analyze their running times.

3. (25 points) Given a ground set $U = \{1, \ldots, n\}$, a *set function on* U is a function $f : \{0,1\}^U \to \mathbb{R}$ that maps a subset of U to a real value. A set function f is *submodular* if

$$f(S \cup \{v\}) - f(S) \geq f(T \cup \{v\}) - f(T)$$

   holds for any $S, T \subseteq U$ with $S \subseteq T$ and any $v \in U \setminus T$. We make the following assumptions on a submodular set function f:

   - Nonnegative: $f(S) \geq 0$ for any $S \subseteq U$; you can assume $f(S)$ is always a rational number.
   - Monotone: $f(S) \leq f(T)$ for any $S, T \subseteq U$ with $S \subseteq T$.
   - $f(S)$ can be computed in a polynomial time with respect to $n = |U|$.

   Given a positive integer $k > 0$ as an input, the goal is to find $S \subseteq U$ that maximizes $f(S)$ subject to the cardinality constraint $|S| \leq k$. Design a polynomial time $(1 - 1/e)$-approximation algorithm for this maximization problem. Prove that the algorithm you design runs in a polynomial time and provides a $(1 - 1/e)$-approximation.

4. (25 points) Given an undirected graph $G = (V, E)$, a *matching* M is a subset of edges such that no two edges in M share an endpoint. The *maximum matching problem* takes the graph $G = (V, E)$ as the input and outputs a matching M with the maximum size $|M|$. Consider the following greedy algorithm.

   - initialize $M \leftarrow \emptyset$
   - while there exists $e \in E$ such that $M \cup \{e\}$ is a valid matching:
     - update $M \leftarrow M \cup \{e\}$
   - endwhile
   - return S

   (a) (20 points) Prove that this is a 0.5-approximation algorithm.

   (b) (5 points) Provide a tight example showing that this is not a $(0.5+\varepsilon)$-approximation algorithm for any $\varepsilon > 0$.

5. How long does it take you to finish the assignment (including thinking and discussion)? Give a score (1,2,3,4,5) to the difficulty. Do you have any collaborators? Please write down their names here.