Algorithm Design and Analysis (Fall 2023)

Assignment 6

Deadline: Jan 9, 2023

Choose **two** of the first four questions to submit. Question 5 is the bonus question.

1. Prove that the following problem is NP-complete. Given an undirected graph G and an undirected graph H, decide if H is a subgraph of G.

2. Prove that the following problem is NP-complete. Given an undirected graph G and a positive integer $k \geq 2$, decide if G contains a spanning tree with maximum degree at most k.

   First, prove that the problem is in NP. The certificate is edges and vertices that forms the spanning tree.
   Next, we show that HamiltonianPath $\leq_k$ the problem. Let G be the instance of HamiltonianPath and G' be the instance of this problem with maximum degree k where $G' = G$ and $k = 2$ there.
   If G is a yes instance, the graph has a HamiltonianPath. Therefore, there is a maximum spanning tree with maximum degree 2 which makes G' a yes instance.
   If G' is a yes instance, there is a spanning tree with maximum degree 2. So form u to v, we can find a HamiltonianPath where u and v are the two vertices with degree 1.
   Since HamiltonianPath is NP-complete, the problem is NP-complete.

3. Given an undirected graph $G = (V, E)$, prove that it is NP-complete to decide if G contains an independent set with size *exactly* $|V|/3$.

   First, prove that the problem is in NP. The certificate is the set of vertices in the independent set.
   Next, we show that VertexCover $\leq_k$ the problem. Let G be the instance of VertexCover and G' be the instance of this problem. Let $G' = G$ and the independent set $|S| = |V|/3$ and the vertex cover $|S'| = 2|V|/3$.
   If G is a yes instance, the graph has a vertex cover with $2|V|/3$. Since $G' = G$, we can find the independent set $V \setminus S'$ with size $|V|/3$ which makes G' a yes instance.
   If G' is a yes instance, there is an independent set with size $|V|/3$. So we can find the vertex cover $V \setminus S$ with size $2|V|/3$, making G a yes instance.
   Since VertexCover is NP-complete, the problem is NP-complete.

4. Consider the decision version of *Knapsack*. Given a set of n items with weights $w_1, \ldots, w_n \in \mathbb{Z}^+$ and values $v_1, \ldots, v_n \in \mathbb{Z}^+$, a capacity constraint $C \in \mathbb{Z}^+$, and a positive integer $V \in \mathbb{Z}^+$, decide if there exists a subset of items with total weight at most C and total value at least V. Prove that this decision version of Knapsack is NP-complete.

5. (**Bonus**) In the class, we have seen that 3SAT is NP-complete. In this question, we investigate the 2SAT problem and its variants. Similar to the 3SAT problem, in the 2SAT problem, we are given a 2-CNF Boolean formula (where each clause contains two literals) and we are to decide if this formula is satisfiable.

   (a) Prove that 2SAT is in P. (Hint: a clause $(a_i \vee a_j)$ with two literals a_i and a_j can be represented as two logical implications: $\neg a_i \implies a_j$ and $\neg a_j \implies a_i$; you may want to construct a directed graph with $2n$ vertices corresponding to $x_1, \neg x_1, x_2, \neg x_2, \ldots, x_n, \neg x_n$.)

   First, construct a directed graph G with $2n$ vertices, where n is the number of variables in the formula. For each variable x_i, create two vertices: x_i and $\neg\, x_i$. For each clause $(a_i \vee a_j)$, add two directed edges to the graph: $(\neg a_i, a_j)$ and $(\neg a_j, a_i)$. Next, find all the strongly connected components and check them to find if there exists a variable x_i and its negation $\neg x_i$ in the same strongly connected component. If there is no such x_i, then this formula is satisfiable, otherwise not satisfiable. The time complexity of finding all the strongly connected components takes $O(|V|^2 + |E||V|)$ while constructing and checking them takes $O(|V|)$. The algorithm can solve the problem in polynomial time, so 2SAT is in P.

   (b) Consider this variant of the 2SAT problem: given a 2-CNF Boolean formula ϕ and a positive integer k, decide if there is a Boolean assignment to the variables such that at least k clauses of ϕ are satisfied. Notice that 2SAT is the special case of this problem with k equals to the number of the clauses. Prove that this problem is NP-complete.
   We will prove that 3SAT $\leq_k$ the problem.
   First, for a clause in 3SAT: $(a \vee b \vee c)$, consider the following 10 clauses in 2SAT: $(a \vee a), (b \vee b), (c \vee c), (d \vee d), (\neg a \vee \neg b), (\neg b \vee \neg c), (\neg a \vee \neg c), (a \vee \neg d), (b \vee \neg d), (c \vee \neg d)$.
   By listing all the condition, we can prove that a value assignment that satisfies $(a \vee b \vee c)$ can be extended to satisfy 7 above clauses and no more, whereas a value assignment that does not satisfy $(a \vee b \vee c)$ can be extended to satisfy 6 above clauses and no more.
   The certificate of the problem is a value assignment of each variable.
   Let S be the instance of 3SAT and S' be the instance of the problem. Meanwhile, let S has n clauses, S' has $10n$ clauses where each clause in 3SAT corresponds to 10 clauses as mentioned above. Let $k = 7n$.
   If S is a yes instance, then there is a value assignment that satisfies all the n clauses in 3SAT. Therefore, we can find $7n$ clauses in S' true by assigning value to each

variable the same as in 3SAT. So S' is a yes instance

If S' is a yes instance, then there is a value assignment that satisfies at least $7n$ clauses are satisfied. Hence 7 clauses of each group of 10 clauses must be satisfied because each group can have at most 7 clauses satisfied. Therefore, each clause in 3SAT is satisfied. S is a yes instance.

Since 3SAT is NP-complete, this problem is NP-complete.

6. How long does it take you to finish the assignment (including thinking and discussion)? Give a score (1,2,3,4,5) to the difficulty. Do you have any collaborators? Please write down their names here.

   3,3,5

   2 hours.

   Reference: https://www.csie.ntu.edu.tw/ lyuu/complexity/2017/20171107.pdf