CS2601 Linear and Convex Optimization: Homework 1

SJTU 2023 Fall

Oct. 09, 2023

Submission Guideline

Deadline: 23:59pm, Sunday, Oct. 29, 2023

Submissions later than the deadline will be discounted:

(a) within 0-24 hours, 20% off;

(b) within 24-48 hours, 50% off;

(c) larger than 48 hours, not acceptable.

Acceptable submission formats:

(1) You are encouraged to submit the electronic version of your homework to the Canvas. You may write your answers in a paper by hand, and then take photos of the answer sheet to get the electronic version.

1 (10 points) Question 1

Let $C \subseteq \mathbf{R}^n$ be a convex set, with $x_1, \dots, x_k \in C$, and let $\theta_1, \dots, \theta_k \in \mathbf{R}$ satisfy $\theta_i \geq 0$, $\theta_1 + \cdots + \theta_k = 1$. Show that $\theta_1 x_1 + \cdots + \theta_k x_k \in C$. (The definition of convexity is that this holds for $k = 2$; you must show it for arbitrary k.) Hint. Use induction on k.

2 (20 points) Question 2

(a) Show that a set is convex if and only if its intersection with any line is convex.

(b) Show that a set is affine if and only if its intersection with any line is affine.

3 (20 points) Question 3

Which of the following sets S are polyhedra? If possible, express S in the form $S = \{x \mid Ax \preceq b, Fx = g\}$.

(a) $S = \{y_1 a_1 + y_2 a_2 \mid -1 \leq y_1 \leq 1,\ -1 \leq y_2 \leq 1\}$, where $a_1, a_2 \in \mathbf{R}^n$

(b) $S = \{x \in \mathbf{R}^n \mid x \succeq 0, \mathbf{1}^T x = 1, \sum_{i=1}^n x_i a_i = b_1, \sum_{i=1}^n x_i a_i^2 = b_2\}$, where $a_1, \dots, a_n \in \mathbf{R}$ and $b_1, b_2 \in \mathbf{R}$.

(c) $S = \{x \in \mathbf{R}^n \mid x \succeq 0, x^T y \leq 1 \text{ for all } y \text{ with } \|y\|_2 = 1\}$

(d) $S = \{x \in \mathbf{R}^n \mid x \succeq 0, x^T y \leq 1 \text{ for all } y \text{ with } \sum_{i=1}^n |y_i| = 1\}$

4 (20 points) Question 4

Voronoi sets and polyhedral decomposition. Let $x_0, \dots, x_K \in \mathbf{R}^n$. Consider the set of points that are closer (in Euclidean norm) to x_0 than the other x_i, i.e.,

$$V = \{x \in \mathbf{R}^n \mid \|x - x_0\|_2 \leq \|x - x_i\|_2, i = 1, \dots, K\}.$$

V is called the *Voronoi region* around x_0 with respect to $x_1, \dots, x_K$.

(a) Show that V is a polyhedron. Express V in the form $V = \{x \mid Ax \preceq b\}$.

(b) Conversely, given a polyhedron P with nonempty interior, show how to find $x_0, \dots, x_K$ so that the polyhedron is the Voronoi region of x_0 with respect to $x_1, \dots, x_K$.

5 (10 points) Question 5

Show that if S_1 and S_2 are convex sets in $\mathbf{R}^{m+n}$, then so is their partial sum

$$S = \{(x, y_1 + y_2) \mid x \in \mathbf{R}^m, y_1, y_2 \in \mathbf{R}^n, (x, y_1) \in S_1, (x, y_2) \in S_2\}.$$