Problem 1.

1. let X be vehicles in the last minute, we have:

$$Pr(X=k \mid X_{bus}=5) = \frac{Pr(X=k \wedge X_{bus}=5)}{P(X_{bus}=5)} \quad (1)$$

Since $P_{bus}=\frac{1}{10}$, $P_{car}=\frac{9}{10}$, we have: $N(t)$ and $N_{bus}(t)$ are independent. $\because Pr(X_{bus}=5)=Pr(N_{bus}(1)=5)$

$$\therefore (1) = \frac{Pr[N(1)=k \wedge N_{bus}(1)=5]}{Pr[N_{bus}(1)=5]}$$

$$= \frac{Pr[N(1)=k]\cdot Pr[N_{bus}(1)=5]}{Pr[N_{bus}(1)=5]}$$

$$= Pr[N(1)=k] = Pr[X=k]$$

Therefore, $E[X \mid X_{bus}=5]=E[X]$

Since $X \sim Pos(10)$, $E[X]=10$, so the average number of vehicles is 10.

2. • $N(1) \sim Pos(\lambda)$, $N(T) \sim Pos(\lambda T)$, $N(T-s) \sim Pos(\lambda(T-s))$

$N(T)-N(T-s) \sim Pos(\lambda s)$

$Pr[\text{achieve}] = Pr[N(T)-N(T-s)=1] = \frac{(\lambda s)^1}{1!}e^{-\lambda s} = \lambda s e^{-\lambda s}$

• $Pr[\text{achieve}] = \lambda s e^{-\lambda s}$, $\frac{\partial(\lambda s e^{-\lambda s})}{\partial s} = \lambda \frac{\partial(s e^{-\lambda s})}{\partial s}$

$= \lambda(1-\lambda s)e^{-\lambda s}$, $s=\frac{1}{\lambda}$ when $Pr[\text{achieve}]$ reaches its maximal value, the corresponding success probility is $\frac{1}{e}$.

Problem 2.

1. $Pr[X=\lambda+k]=\frac{\lambda^{(\lambda+k)}}{(\lambda+k)!}e^{-\lambda}$, $Pr[X=\lambda-k-1]=\frac{\lambda^{(\lambda-k-1)}}{(\lambda-k-1)!}e^{-\lambda}$ $\therefore$

$$\frac{Pr[X=\lambda+k]}{Pr[X=\lambda-k-1]}=\frac{\lambda^{(\lambda+k)}\cdot(\lambda-k-1)!}{\lambda^{(\lambda-k-1)}\cdot(\lambda+k)!}=\frac{\lambda^{2k+1}}{(\lambda+k)(\lambda+k-1)\ldots(\lambda-k)}$$

$\because (\lambda+k)(\lambda-k)=\lambda^2-k^2<\lambda^2$ and $(\lambda+k)(\lambda+k-1)\ldots(\lambda-k)=(\lambda+k)(\lambda-k)\cdot(\lambda+k-1)\cdot(\lambda-k+1)\cdot\ldots\cdot\lambda<\lambda^2\cdot\lambda^2\cdot\ldots\cdot\lambda=\lambda^{2k+1}$

$\therefore \frac{\lambda^{2k+1}}{(\lambda+k)(\lambda+k-1)\ldots(\lambda-k)}>\frac{\lambda^{2k+1}}{\lambda^{2k+1}}=1$. Therefore, $Pr[X=\lambda+k]\geq Pr[X=\lambda-k-1]$.

$$\begin{cases} Pr[X\geq\lambda]=\sum_{k=0}^{\infty}Pr[X=\lambda+k] \\ Pr[X\leq\lambda]=\sum_{k=0}^{\lambda+1}Pr[X=\lambda-k+1] \end{cases}$$

$\because \sum_{k=0}^{\infty}Pr[X=\lambda+k]\geq\sum_{k=0}^{\lambda+1}Pr[X=\lambda+k]\geq\sum_{k=0}^{\lambda+1}Pr[X=\lambda-k+1]$

and $\sum_{k=0}^{\infty}Pr[X=\lambda+k]+\sum_{k=0}^{\lambda+1}Pr[X=\lambda-k+1]=1$

$\therefore Pr[X\geq\lambda]=\sum_{k=0}^{\infty}Pr[X=\lambda+k]\geq\frac{1}{2}$

2. $E[f(Y_1,Y_2,\ldots,Y_n)]=\sum_{k=0}^{\infty}E[f(Y_1,Y_2,\ldots,Y_n)\mid\sum_{i=1}^{n}Y_i=k]\,Pr[\sum_{i=1}^{n}Y_i=k]$

$\geq\sum_{k=m}^{\infty}E[f(Y_1,Y_2,\ldots,Y_n)\mid\sum_{i=1}^{n}Y_i=k]\,Pr[\sum_{i=1}^{n}Y_i=k]$

$\geq\sum_{k=m}^{\infty}E[f(Y_1,Y_2,\ldots,Y_n)\mid\sum_{i=1}^{n}Y_i=m]\,Pr[\sum_{i=1}^{n}Y_i=k]$ (monotonical)

$=E[f(Y_1,Y_2,\ldots,Y_n)\mid\sum_{i=1}^{n}Y_i=m]\sum_{k=m}^{\infty}Pr[\sum_{i=1}^{n}Y_i=k]$

$\because \sum_{i=1}^{n}Y_i\sim Pois(m)$, $\sum_{k=m}^{\infty}Pr[\sum_{i=1}^{n}Y_i=k]=Pr[\sum_{i=1}^{n}Y_i\geq m]\geq\frac{1}{2}$ (by 1.)

$\therefore E[f(Y_1,Y_2,\ldots,Y_n)]\geq E[f(X_1,\ldots,X_n)]\cdot\frac{1}{2}$

Therefore, $E[f(X_1,\ldots,X_n)]\leq 2E[f(Y_1,Y_2,\ldots,Y_n)]$

3. Let X_i be the number of students who share the same birthday on i-th day

Further let $X = \max_{i \in [n]} X_i$, we need to prove that:

$$\Pr[X \geq 5] \leq 0.01$$

$$\Rightarrow \Pr[X \geq 5] = \Pr[\exists i \in [n], X_i \geq 5] \leq \sum_{i=1}^{n} \Pr[X_i \geq 5]$$

$$= n \cdot \Pr[X_1 \geq 5] \leq n \cdot C_m^5 \cdot \frac{1}{n^5} \approx 0.005 < 0.01$$ Proved

Problem 3.

1. Let $H \in \mathcal{F}_1$, we have: $\int_H E[X|\mathcal{F}_1]\,dP = \int_H X\,dP$

Let $G \in \mathcal{F}_2$, $\int_G E[X|\mathcal{F}_2]\,dP = \int_G X\,dP$

Since $\mathcal{F}_1 \subseteq \mathcal{F}_2$, we have $\int_H E[X|\mathcal{F}_2]\,dP = \int_H X\,dP$

Therefore, we have: $\int_H E[X|\mathcal{F}_1]\,dP = \int_H E[X|\mathcal{F}_2]\,dP$,

which indicates $E[X|\mathcal{F}_1] = E[E[X|\mathcal{F}_2]|\mathcal{F}_1]$

$E[X|\mathcal{F}_1]$ is $\mathcal{F}_1$-measurable, hence its also $\mathcal{F}_2$-measurable ($\mathcal{F}_1 \subseteq \mathcal{F}_2$). Therefore, we have $E[X|\mathcal{F}_1] = E[E[X|\mathcal{F}_1]|\mathcal{F}_2]$.

So, $E[E[X|\mathcal{F}_1]|\mathcal{F}_2] = E[E[X|\mathcal{F}_2]|\mathcal{F}_1] = E[X|\mathcal{F}_1]$

2. $X \sim \mathrm{Exp}(\lambda)$, $Y \sim \mathrm{Exp}(\lambda)$, $E[X^3 \mid X+Y = x+y]$

$p(x, x+y)$ is the joint density of X and $X+Y$

$p(X=x, X+Y=y) = p(X=x, Y=y-x)$

since X, Y are independent, $p(X=x, X+Y=y)$

$= P(X=x) \cdot P(Y=y-x) = \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda(y-x)} = \lambda^2 e^{-\lambda y}$

$$E[X^3 \mid X+Y=y] = \frac{\int_0^y x^3 \lambda^2 e^{-\lambda y} dx}{\int_0^y \lambda^2 e^{-\lambda y} dx} = \frac{\lambda^2 e^{-\lambda y} \int_0^y x^3 dx}{\lambda^2 e^{-\lambda y}} = \frac{1}{4} y^3$$

Ref: 刘禹宏 for last conditional Expectation

Tower Rule's wiki for 1.

(proofwiki.org/wiki/Tower_Property_of_Conditional_Expectation)