Algorithm Design and Analysis (Fall 2023)
Assignment 4
Deadline: Dec 26, 2023

1. (30 points) Consider that you are in a stock market and you would like to maximize your profit. Suppose the prices of the stock for the n days, $p_1, p_2, \ldots, p_n$, are given to you. On the i-th day, you are allowed to do exactly one of the following operations:

   - Buy one unit of the stock and pay the price p_i. Your stock will increase by 1.
   - Sell one unit of stock and get the reward p_i if your stock is at least 1. Your stock will decrease by one.
   - Do nothing.

   Design an $O(n^2)$ time dynamic programming algorithm.

   Remark: [Not for credits] There exits a clever greedy algorithm that runs in $O(n \log n)$ time. Can you figure it out?

2. (30 points) Given two strings $x = x_1 x_2 \cdots x_n$ and $y = y_1 y_2 \cdots y_n$, we wish to find the length of their *longest common subsequence*, that is, the largest k for which there are indices $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$ with $x_{i_1} x_{i_2} \cdots x_{i_k} = y_{j_1} y_{j_2} \cdots y_{j_k}$. Design an $O(n^2)$ dynamic programming algorithm for this problem.

3. (40 points) In the *Traveling Salesman Problem* (TSP), we are given an undirected weighted complete graph $G = (V, E, w)$ (where $(i, j) \in E$ for any $i \neq j \in V$). The objective is to find a cycle of length $|V|$ with minimum total weight, i.e., to find a tour that visit each vertex exactly once such that the total distance traveled in the tour is minimized. Obviously, the naïve exhaustive search algorithm requires $O((n-1)!)$ time. In this question, you are to design a dynamic programming algorithm for the TSP problem with time complexity $O(n^2 \cdot 2^n)$.

   (a) (10 points) Show that $n^2 \cdot 2^n = o((n-1)!)$, so that the above-mentioned algorithm is indeed faster than the naïve exhaustive search algorithm.

   (b) (30 points) Design this algorithm. Hint: label all vertices as $1, 2, \ldots, n$; given $i \in V$ and $S \subseteq V \setminus \{1, i\}$, let $d(S, i)$ be the length of the shortest path from 1 to i where the intermediate vertices are *exactly* those in S; show that the minimum weight cycle/tour is $\min_{i=2,3,\ldots,n}\{d(V \setminus \{1, i\}, i) + w(i, 1)\}$.

4. How long does it take you to finish the assignment (including thinking and discussion)? Give a score (1,2,3,4,5) to the difficulty. Do you have any collaborators? Please write down their names here.