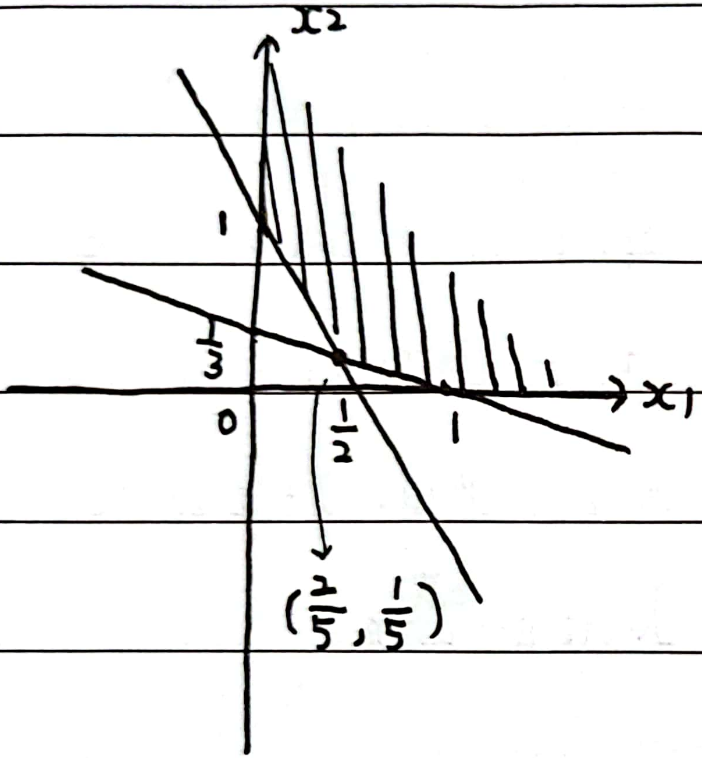
1.



(a) $(\frac{2}{5}, \frac{1}{5})$, $\frac{3}{5}$

(b) unbounded below

(c) $(0, x_2)$, $x_2 \geq 1$, 0

2. $f_0: \frac{1}{2}x^TPx + q^Tx + r$

$f_1: x_1 - 1 \le 0$, $f_2: -x_1 - 1 \le 0$, $f_3: x_2 - 1 \le 0$, $f_4: -x_2 - 1 \le 0$, $f_5: x_3 - 1 \le 0$
$f_6: -x_3 - 1 \le 0$

Largrangian: $L(x, \lambda) = f_0 + \sum_{i=1}^{6} \lambda_i f_i$

$\nabla f_0(x) = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, $\nabla f_1(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\nabla f_2(x) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$, $\nabla f_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\nabla f_4 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$,

$\nabla f_5 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\nabla f_6 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

$\nabla f_0(x) + \sum_{i=1}^{6} \lambda_i \nabla f_i = \begin{pmatrix} \lambda_1 - \lambda_2 - 1 \\ \lambda_3 - \lambda_4 \\ \lambda_5 - \lambda_6 + 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ①

$\lambda_i f_i(x) = 0$: $\lambda_1 \cdot (1-1) = 0$, $\lambda_2(-1-1) = 0 \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 0 \end{cases}$ a feasible (λ_1, λ_2)

$\lambda_3(\frac{1}{2} - 1) = 0$, $\lambda_4(-\frac{1}{2} - 1) = 0 \Rightarrow \begin{cases} \lambda_3 = 0 \\ \lambda_4 = 0 \end{cases}$

$\lambda_5(-1-1) = 0$, $\lambda_6(1-1) = 0 \Rightarrow \begin{cases} \lambda_5 = 0 \\ \lambda_6 = 2 \end{cases}$ a feasible (λ_1, λ_2)

and $\lambda_i \ge 0$ is satisfied, $-1 \le 1 \le 1$, $-1 \le \frac{1}{2} \le 1$, $-1 \le -1 \le 1$

$\Rightarrow \begin{pmatrix} \lambda_1 - \lambda_2 - 1 \\ \lambda_3 - \lambda_4 \\ \lambda_5 - \lambda_6 + 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, true

so $(1, \frac{1}{2}, -1)$ is optimal

3. (a) minimize c^Tx

s.t. $Ax = b$

case1: infeasible when $Ax = b$ has no feasible solution

case2: feasible, decompose c to $A^T\lambda + \hat{c}$, $A\hat{c} = 0$

suppose $\hat{c} \ne 0$, then let $x = x_0 - t\hat{c}$, $Ax_0 = b$

$c^Tx = c^T(x_0 - t\hat{c}) = (A^T\lambda + \hat{c})(x_0 - t\hat{c})$

$= \lambda^TAx_0 - (A^T\lambda)^T(t\hat{c}) + \hat{c}^Tx_0 - t\|\hat{c}\|_2 = \lambda^Tb + \hat{c}^Tx_0 - t\|\hat{c}\|_2$

since $\lambda^Tb + \hat{c}^Tx_0$ is a constant and $t\|\hat{c}\|_2$ can be $+\infty$,

$c^Tx \to -\infty$

case 3: $\hat{c}=0$, then according to case 2: the optimal value is $\lambda^T b$ for any feasible solution.

(b) minimize $c^T x$

subject to $a^T x \leq b$, $a \neq 0$

Similarly, let $c = a\lambda + \hat{c}$, $a^T \hat{c} = 0$

let $x = x_0 - t\hat{c}$, $a^T x_0 \leq b$

$c^T(x_0 - t\hat{c}) = (a\lambda + \hat{c})(x_0 - t\hat{c}) = \lambda^T(a^T x_0) - (a\lambda)^T(t\hat{c}) + \hat{c}^T x_0 - t\|\hat{c}\|_2$

$= \lambda^T(a^T x_0) + \hat{c}^T x_0 - t\|\hat{c}\|_2$

case 1: $\lambda > 0$: then $a^T x_0 \to -\infty \Rightarrow c^T x \Rightarrow -\infty$, unbounded below

case 2: $\hat{c} \neq 0$: when $t \to \infty$, $c^T x \Rightarrow -\infty$, unbounded below

case 3: $\hat{c} = 0$, $\lambda \leq 0$: the optimal value is $\lambda^T b$ for some x that satisfies $a^T x = b$

(c) minimize $c^T x$

subject to $l \leq x \leq u$

$c^T x = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

so if $c_i > 0$, let $x_i = l_i$

else if $c_i < 0$, let $x_i = u_i$

else, x_i can be in $[l_i, u_i]$

4. minimize $c^T x$

subject to $Ax \preceq b$

$f_0 = c^T x = c^T A^{-1} \cdot A \cdot x$ because A is squre and nonsingular.

therefore, if $A^{-T} c \preceq 0$, $(A^{-T}c)^T \cdot (Ax)$ is optimal when $Ax = b$.

So $p^* = c^T A^{-1} b$, when $A^{-T} c \preceq 0$

otherwise, if $A^{-T} c \succ 0$, then $(A^{-T}c)^T (Ax)$ is unbounded below since Ax can be $-\infty$

5. $L(x,\lambda) = c^T x + \lambda f(x)$

$g(\lambda) = \inf_x L(x,\lambda) = \inf_x [c^T x + \lambda f(x)]$, $\lambda \geq 0$

if $\lambda = 0$, $g(\lambda) = -\infty$

if $\lambda > 0$, $g(\lambda) = -\lambda \sup_x [(\frac{c}{\lambda})^T x - f(x)] = -\lambda f^*(\frac{c}{\lambda})$

So the dual problem is $\max[-\lambda f^*(\frac{c}{\lambda})]$

to show that the problem is convex, we only need to show that $f^*(y)$ is convex function, $f^*(y) = \sup_{x \in \mathrm{dom} f} (y^T x - f(x))$ is affine on y, so $f^*(y)$ is convex.

6. (a) $c = A^T \omega \cdot \lambda + \hat{c}$

only when $\hat{c} = 0$ and $\lambda \leq 0$ does the optimal value exist

according to 3(b), the optimal value is $\lambda \omega^T b$

(b) maximize $\lambda \omega^T b$

subject to $\lambda \leq 0$, $c = A^T \omega \lambda + \hat{c}$, $\hat{c} = 0$, $\omega \succeq 0$

(c) $\omega \succeq 0$, $\lambda \leq 0 \Rightarrow$ maximize $-y^T b$

subject to $y \succeq 0$, $A^T y + c = 0$

where $y = -\omega\lambda$