CS2601 Linear and Convex Optimization: Homework 3

SJTU 2022 Fall

Nov. 06, 2023

Submission Guideline

Deadline: 23:59pm, Sunday, Dec. 03, 2023

Submissions later than the deadline will be discounted:

(a) within 0-24 hours, 20% off;

(b) within 24-48 hours, 50% off;

(c) larger than 48 hours, not acceptable.

Acceptable submission formats:

(1) You are encouraged to submit the electronic version of your homework to the Canvas. You may write your answers in a paper by hand, and then take photos of the answer sheet to get the electronic version.

1 (15%) Question 1

Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}$$

Make a sketch of the feasible set. For each of the following objective functions, give the optimal set and the optimal value.

(a) $f_0(x_1, x_2) = x_1 + x_2$.

(b) $f_0(x_1, x_2) = -x_1 - x_2$.

(c) $f_0(x_1, x_2) = x_1$.

2 (15%) Question 2

Prove that $x^* = (1, 1/2, -1)$ is optimal for the optimization problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T P x + q^T x + r \\ \text{subject to} & -1 \leq x_i \leq 1, \quad i = 1, 2, 3, \end{array}$$

where

$$P = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}, \quad q = \begin{bmatrix} -22.0 \\ -14.5 \\ 13.0 \end{bmatrix}, \quad r = 1.$$

3 (15%) Question 4

Give an explicit solution of each of the following LPs.

(a) Minimizing a linear function over an affine set.

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b. \end{array}$$

(b) Minimizing a linear function over a halfspace.

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a^T x \leq b, \quad a \neq 0. \end{array}$$

(c) Minimizing a linear function over a rectangle.

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \ell \preceq x \preceq u, \end{array}$$

where ℓ and u satisfy $\ell \preceq u$.

4 (10%) Question 5

Square LP. Consider the LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b, \end{array}$$

with A square and nonsingular. Show that the optimal value is given by

$$p^* = \begin{cases} c^T A^{-1} b & A^{-T} c \preceq 0 \\ -\infty & \text{otherwise.} \end{cases}$$

5 (10%) Question 5

Problems with one inequality constraint. Express the dual problem of

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & f(x) \leq 0, \end{array}$$

with $c \neq 0$, in terms of the conjugate f. Explain why the problem you give is convex. We do not assume f is convex.

6 (15%) Question 6

Consider the inequality form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b, \end{array}$$

with $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$. Lagrange dual of inequality form LP

$$\begin{array}{ll} \text{maximize} & -b^T \lambda \\ \text{subject to} & A^T \lambda + c = 0, \\ & \lambda \succeq 0. \end{array}$$

In this exercise we develop a simple geometric interpretation of the dual LP. Let $w \in \mathbf{R}^m_+$. If x is feasible for the LP, i.e., satisfies $Ax \preceq b$, then it also satisfies the inequality

$$w^T A x \leq w^T b$$

Geometrically, for any $w \succeq 0$, the halfspace $H_w = \{x \mid w^T A x \leq w^T b\}$ contains the feasible set for the LP. Therefore if we minimize the objective $c^T x$ over the halfspace H_w we get a lower bound on p^*.

(a) Derive an expression for the minimum value of $c^T x$ over the halfspace H_w (which will depend on the choice of $w \succeq 0$).

(b) Formulate the problem of finding the best such bound, by maximizing the lower bound over $w \succeq 0$.

(c) Relate the results of (a) and (b) to the Lagrange dual of the inequality form LP.