1-38. $\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ 本征矢 $|\uparrow\rangle_z = \begin{pmatrix}1\\0\end{pmatrix}$, $|\downarrow\rangle_z = \begin{pmatrix}0\\1\end{pmatrix}$

$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ 本征矢 $|+\rangle_x = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$, $|-\rangle_x = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}$

$\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$ 本征矢 $|+\rangle_y = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{pmatrix}$, $|-\rangle_y = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}}\end{pmatrix}$

$\sigma_y\begin{pmatrix}a\\b\end{pmatrix} = \lambda\begin{pmatrix}a\\b\end{pmatrix} \Rightarrow \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \lambda\begin{pmatrix}a\\b\end{pmatrix}$

$\lambda = \pm 1$

$\lambda=1$ 时：$a=-ib$　　$\lambda=-1$ 时 $a=ib$

$\therefore |+\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$　　$|-\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$

$|+\rangle_x = \alpha|+\rangle_y + \beta|-\rangle_y$，本征值 $\frac{\hbar}{2}$，$-\frac{\hbar}{2}$

$\frac{\hbar}{2}$：$\alpha = {}_y\langle+|+\rangle_x = \frac{1}{\sqrt{2}}(1,-i)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1-i}{2}$

$-\frac{\hbar}{2}$：$\beta = {}_y\langle-|+\rangle_x = \frac{1}{\sqrt{2}}(1,i)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1+i}{2}$

本征值 $\frac{\hbar}{2}$ 概率 $\left|\frac{1-i}{2}\right|^2 = \frac{1}{2}$，$-\frac{\hbar}{2}$：$\left|\frac{1+i}{2}\right|^2 = \frac{1}{2}$

同理 $|-\rangle_x = \alpha|+\rangle_y + \beta|-\rangle_y$，本征值 $\frac{\hbar}{2}$、$-\frac{\hbar}{2}$，概率同上

${}_y\langle+|\hat{S}_x|+\rangle_y = 0$　　${}_y\langle-|\hat{S}_x|-\rangle_y = 0$

1-39. (1) $(\hat{\sigma}\cdot\hat{A})^2 = (\hat{\sigma}_x\hat{A}_x + \hat{\sigma}_y\hat{A}_y + \hat{\sigma}_z\hat{A}_z)^2$

$= \hat{\sigma}_x^2\hat{A}_x^2 + \hat{\sigma}_y^2\hat{A}_y^2 + \hat{\sigma}_z^2\hat{A}_z^2 + \hat{\sigma}_x\hat{\sigma}_y\hat{A}_x\hat{A}_y + \hat{\sigma}_y\hat{\sigma}_x\hat{A}_y\hat{A}_x$

$+ \hat{\sigma}_y\hat{\sigma}_z\hat{A}_y\hat{A}_z + \hat{\sigma}_z\hat{\sigma}_y\hat{A}_z\hat{A}_y + \hat{\sigma}_z\hat{\sigma}_x\hat{A}_z\hat{A}_x + \hat{\sigma}_x\hat{\sigma}_z\hat{A}_x\hat{A}_z$

由 $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$ 以及 $\hat{\sigma}_x\hat{\sigma}_y = i\hat{\sigma}_z$，$\hat{\sigma}_y\hat{\sigma}_z = i\hat{\sigma}_x$，$\hat{\sigma}_z\hat{\sigma}_x = i\hat{\sigma}_y$

上式 $= \hat{A}_x^2 + \hat{A}_y^2 + \hat{A}_z^2 + i\hat{\sigma}_z(\hat{A}_x\hat{A}_y - \hat{A}_y\hat{A}_x) + i\hat{\sigma}_x(\hat{A}_y\hat{A}_z - \hat{A}_z\hat{A}_y)$

$+ i\hat{\sigma}_y(\hat{A}_z\hat{A}_x - \hat{A}_x\hat{A}_z) = \hat{A}^2 + i\hat{\sigma}\cdot(\hat{A}\times\hat{A})$

2. $e^{i\hat{\sigma}\cdot\hat{n}\phi}=\sum_{n=0}^{\infty}\frac{(i\hat{\sigma}\cdot\hat{n}\cdot\phi)^n}{n!}$

对于 $(\hat{\sigma}\cdot\hat{n})^2=(\alpha\hat{\sigma}_x+\beta\hat{\sigma}_x+\gamma\hat{\sigma}_z)^2=\alpha^2+\beta^2+\gamma^2=1$

$$\therefore\begin{cases}(\hat{\sigma}\cdot\hat{n})^n=1, & n\text{为偶数}\\ (\hat{\sigma}\cdot\hat{n})^n=\hat{\sigma}\cdot\hat{n}, & n\text{为奇数}\end{cases}$$

$$\text{故上式}=\sum_{n=\text{偶}}^{\infty}\frac{1}{n!}(i\phi)^n+(\hat{\sigma}\cdot\hat{n})\cdot\sum_{n=\text{奇}}^{\infty}\frac{1}{n!}(i\phi)^n$$

$$=\sum_{k=0}^{\infty}\frac{(i\phi)^{2k}}{(2k)!}+(\hat{\sigma}\cdot\hat{n})\sum_{k=0}^{\infty}\frac{(i\phi)^{2k+1}}{(2k+1)!}$$

$$=\cos\phi+i\hat{\sigma}\cdot\hat{n}\cdot\sin\phi$$

附：$\hat{S}_x=\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$，$\hat{S}_y=\frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$

$$\langle\hat{S}_x\rangle=\frac{\hbar}{2}(1,0)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=0$$

$$\langle\hat{S}_y\rangle=\frac{\hbar}{2}(1,0)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=0$$

$$\hat{S}_x^2=\frac{\hbar^2}{4}\hat{\sigma}_x^2=\frac{\hbar^2}{4}$$

$$\hat{S}_y^2=\frac{\hbar^2}{4}\hat{\sigma}_y^2=\frac{\hbar^2}{4}$$

$$\Delta S_x=\left(\overline{\hat{S}_x^2}-\overline{\hat{S}_x}^2\right)^{\frac{1}{2}}=\frac{\hbar}{2}$$

$$\Delta S_y=\left(\overline{\hat{S}_y^2}-\overline{\hat{S}_y}^2\right)^{\frac{1}{2}}=\frac{\hbar}{2}$$