CS2601 Linear and Convex Optimization: Homework 4

SJTU 2023 Fall

Dec. 04, 2023

Submission Guideline

Deadline: 23:59pm, Sunday, Dec. 31, 2023

Submissions later than the deadline will be discounted:

(a) within 0-24 hours, 20% off;

(b) within 24-48 hours, 50% off;

(c) larger than 48 hours, not acceptable.

Acceptable submission formats:

(1) You are encouraged to submit the electronic version of your homework to the Canvas. You may write your answers in a paper by hand, and then take photos of the answer sheet to get the electronic version.

(2) You may also submit your answer sheet in paper version in Monday class.

1 (10%) Question 1

Derive a dual problem for

$$\text{minimize} \quad \sum_{i=1}^{N} \|A_i x + b_i\|_2 + (1/2)\|x - x_0\|_2^2.$$

The problem data are $A_i \in \mathbf{R}^{m_i \times n}$, $b_i \in \mathbf{R}^{m_i}$, and $x_0 \in \mathbf{R}^n$. First introduce new variables $y_i \in \mathbf{R}^{m_i}$ and equality constraints $y_i = A_i x + b_i$.

2 (15%) Question 2

Consider the following problem:

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & (x_1 - 1)^2 + (x_2 - 1)^2 \le 1 \\ & (x_1 - 1)^2 + (x_2 + 1)^2 \le 1 \end{array}$$

with variable $x \in \mathbf{R}^2$.

(a) Sketch the feasible set and level sets of the objective. Find the optimal point x^* and optimal value p^*.

(b) Give the KKT conditions. Do there exist Lagrange multipliers λ_1^* and λ_2^* that prove that x^* is optimal?

(c) Derive and solve the Lagrange dual problem. Does strong duality hold?

3 (10%) Question 3

Consider the equality constrained least-squares problem

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 \\ \text{subject to} & Gx = h \end{array}$$

where $A \in \mathbf{R}^{m \times n}$ with $\mathbf{rank}(A) = n$, and $G \in \mathbf{R}^{p \times n}$ with $\mathbf{rank}(G) = p$. Give the KKT conditions, and derive expressions for the primal solution x^* and the dual solution ν^*.

4 (15%) Question 4

ℓ_1-, ℓ_2-, and ℓ_∞-norm approximation by a constant vector. What is the solution of the norm approximation problem with one scalar variable $x \in \mathbf{R}$,

$$\text{minimize} \quad \|x\mathbf{1} - b\|,$$

for the ℓ_1-, ℓ_2-, and ℓ_∞-norms?

5 (10%) Question 5

Derive a Lagrange dual for the problem of Deadzone-linear penalty (with deadzone width $a = 1$):

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^m \phi(r_i) \\ \text{subject to} & r = Ax - b, \end{array}$$

where the penalty function $\phi : \mathbf{R} \to \mathbf{R}$ is given by

$$\phi(u) = \begin{cases} 0 & |u| \leq 1 \\ |u| - 1 & |u| > 1 \end{cases}$$

and $x \in \mathbf{R}^n$, $r \in \mathbf{R}^m$ are variables of the problem.

6 (20%) Question 6

The pure Newton method. Newton's method with fixed step size $t = 1$ can diverge if the initial point is not close to x^*. In this problem we consider two examples.

(a) $f(x) = \log(e^x + e^{-x})$ has a unique minimizer $x^* = 0$. Run Newton's method with fixed step size $t = 1$, starting at $x^{(0)} = 1$ and at $x^{(0)} = 1.1$.

(b) $f(x) = -\log x + x$ has a unique minimizer $x^* = 1$. Run Newton's method with fixed step size $t = 1$, starting at $x^{(0)} = 3$.

Plot f and f', and show the first few iterates.

7 (20%) Question 7

Suppose $Q \succeq 0$. The problem

$$\begin{array}{ll} \text{minimize} & f(x) + (Ax - b)^T Q (Ax - b) \\ \text{subject to} & Ax = b, \end{array}$$

is equivalent to the original equality constrained optimization problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b. \end{array}$$

Is the Newton step for this problem the same as the Newton step for the original problem?