Coursework (5) for *Introductory Lectures on Optimization*

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Excercise 1. Prove the following theorem:
for any $\boldsymbol{x}_0 \in \mathrm{dom}\ f$, all vectors $\boldsymbol{g} \in \partial f(\boldsymbol{x}_0)$ are supporting to the level set $\mathcal{L}_f(f(\boldsymbol{x}_0))$:

$$\langle \boldsymbol{g},\ \boldsymbol{x}_0 - \boldsymbol{x} \rangle \geq 0, \quad \forall \boldsymbol{x} \in \mathcal{L}_f(f(\boldsymbol{x}_0)) \equiv \{\boldsymbol{x} \in \mathrm{dom}\ f : f(\boldsymbol{x}) \leq f(\boldsymbol{x}_0)\} \tag{1}$$

Proof of Excercise 1: Since $\boldsymbol{g} \in \partial f(\boldsymbol{x}_0)$, we have:

$$\forall \boldsymbol{x} \in \mathrm{dom}\ f, f(\boldsymbol{x}) \geq f(\boldsymbol{x}_0) + \langle \boldsymbol{g},\ \boldsymbol{x} - \boldsymbol{x}_0 \rangle \tag{2}$$

Also, we have $\boldsymbol{x} \in \mathcal{L}_f(f(\boldsymbol{x}_0))$, which means $f(\boldsymbol{x}) \leq f(\boldsymbol{x}_0)$. Thus, we have:

$$f(\boldsymbol{x}_0) \geq f(\boldsymbol{x}) \geq f(\boldsymbol{x}_0) + \langle \boldsymbol{g},\ \boldsymbol{x} - \boldsymbol{x}_0 \rangle \tag{3}$$

Thus we have $\langle \boldsymbol{g},\ \boldsymbol{x} - \boldsymbol{x}_0 \rangle \leq 0$. That is :

$$\langle \boldsymbol{g}, \boldsymbol{x} \rangle \leq \langle \boldsymbol{g}, \boldsymbol{x}_0 \rangle, \quad \forall \boldsymbol{x} \in \mathcal{L}_f(f(\boldsymbol{x}_0)) \tag{4}$$

So by the definition of the supporting hyperplane, we have $\boldsymbol{g}$ is supporting to the level set $\mathcal{L}_f(f(\boldsymbol{x}_0))$.

□

Excercise 2. Prove the following theorem:
let $Q \subseteq \mathrm{dom}\ f$ be a closed convex set, $\boldsymbol{x}_0 \in Q$ and

$$\boldsymbol{x}^* = \mathrm{argmin}\{f(\boldsymbol{x}) | \boldsymbol{x} \in Q\} \tag{5}$$

Then for any $g \in \partial f(\boldsymbol{x}_0)$ we have $\langle \boldsymbol{g},\ \boldsymbol{x}_0 - \boldsymbol{x}^* \rangle \geq 0$.

Proof of Excercise 2: Since $\boldsymbol{x}^* = \mathrm{argmin}\{f(\boldsymbol{x}) | \boldsymbol{x} \in Q\}$, we have:

$$\forall \boldsymbol{x} \in Q, f(\boldsymbol{x}^*) \leq f(\boldsymbol{x}) \tag{6}$$

Also, we have $\boldsymbol{x}_0 \in Q$, which means $f(\boldsymbol{x}_0) \geq f(\boldsymbol{x}^*)$. Thus, we have:

$$f(\boldsymbol{x}_0) \geq f(\boldsymbol{x}^*) \geq f(\boldsymbol{x}_0) + \langle \boldsymbol{g},\ \boldsymbol{x}^* - \boldsymbol{x}_0 \rangle \tag{7}$$

Thus we have $\langle \boldsymbol{g},\ \boldsymbol{x}^* - \boldsymbol{x}_0 \rangle \leq 0$. That is: For any $\boldsymbol{g} \in \partial f(\boldsymbol{x}_0)$, we have $\langle \boldsymbol{g},\ \boldsymbol{x}_0 - \boldsymbol{x}^* \rangle \geq 0$.

□

Excercise 3. Prove the following theorem:
let f be closed and convex. Assume that it is differentiable on its domain. Then $\partial f(\boldsymbol{x}) = \{\nabla f(\boldsymbol{x})\}$ for any $\boldsymbol{x} \in \mathrm{int}(\mathrm{dom}\ f)$.

Proof of Excercise 3: Notice that $\forall \boldsymbol{p} \in \mathbb{R}^n$, we have:

$$\langle \nabla f(\boldsymbol{x}), \boldsymbol{p} \rangle = \lim_{t \to 0} \frac{f(\boldsymbol{x} + t\boldsymbol{p}) - f(\boldsymbol{x})}{t} \tag{8}$$

Becausen $f^{'}(\boldsymbol{x}_0; \boldsymbol{p}) = max\left\{ \langle \boldsymbol{g}, \boldsymbol{p} \rangle | \boldsymbol{g} \in \partial f(\boldsymbol{x}_0) \right\}$ for all $\boldsymbol{x} \in int(\text{dom } f)$ So we have:

$$\langle \nabla f(\boldsymbol{x}), \boldsymbol{p} \rangle = f^{'}(\boldsymbol{x}; \boldsymbol{p}) \geq \langle \boldsymbol{g}, \boldsymbol{p} \rangle \tag{9}$$

That is:$\forall \boldsymbol{p} \in \mathbb{R}^n, \langle \nabla f(\boldsymbol{x}) - \boldsymbol{g}, \boldsymbol{p} \rangle \geq 0$. Let $\boldsymbol{p} = \boldsymbol{g} - \nabla f(\boldsymbol{x})$,we have:

$$\|\boldsymbol{g} - \nabla f(\boldsymbol{x})\|^2 \leq 0 \tag{10}$$

So we have $\boldsymbol{g} = \nabla f(\boldsymbol{x})$. Because of the arbitrariness of $\boldsymbol{p}$ thus the arbitrariness of $\boldsymbol{g}$, we can reach the conclution that $\partial f(\boldsymbol{x}) = \{\nabla f(\boldsymbol{x})\}$ for any $\boldsymbol{x} \in \text{int}(\text{dom } f)$.

$\square$