Coursework (2) for *Introductory Lectures on Optimization*

Hanxuan Li
3220106039

Oct. 16, 2023

Excercise 1. For the function $f(x) : \mathbb{R}^n \to \mathbb{R}^m$, please write down the zeroth-order Taylor expansion with an integral remainder term.

Solution of Excercise 1:

$$f(x) = f(a) + \int_0^1 \nabla f(a + t(x-a))(x-a)dt \tag{1}$$

□

Excercise 2. Please write down the definition of the p-norm for a n-dimensional real vector.

Solution of Excercise 2:

$$\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}} \tag{2}$$

□

Excercise 3. Please write down the definition of the matrix norms induced by vector p-norms.

Solution of Excercise 3:

$$\|A\|_p = \underset{x \neq 0}{sup} \frac{\|Ax\|_p}{\|x\|_p} \tag{3}$$

□

Excercise 4. Let A be an $n \times n$ symmetric matrix. Proof that A is positive semidefinite if and only if all eigenvalues of A are nonnegative. Moreover, A is positive definite if and only if all eigenvalues of A are positive.

Proof of Excercise 4:

1. A is positive semidefinite iff all eigenvalues of A are nonnegative
   (a) Sufficient condition: If A is semidefinite, then for any non-zero vector x, we have $x^T Ax \geq 0$. Let λ be an arbitary eigenvalue of A, and v be the corresponding eigenvector, thus we have $Av = \lambda v$.
   Then:
   $$v^T Av = v^T \lambda v = \lambda \|v\|^2 \geq 0 \tag{4}$$
   Since $\|x\| \geq 0$, it follows that λ must be nonnegative.
   Without loss of generality, all eigenvalues of A are nonnegative.

(b) Necessary condition: Since A is a symmetric matrix, so it must have n eigenvalues,let's say: $\lambda_1, \lambda_2, \cdots, \lambda_n$, and the corresponding eigenvectors(normalized,i.e.$\|v_i\| == 1$) $v_1, v_2, \cdots, v_n$ which are linearly independent and actually orthogonal to each other, that is:

$$v_i^T v_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{5}$$

So $\forall x$,we can express $x = c_1 v_1 + c_2 v_2 \cdots c_n v_n$, where $c_1, c_2, \cdots, c_n \in \mathbb{R}$.
Then we have:

$$x^T A x = (c_1 v_1 + c_2 v_2 \cdots c_n v_n)^T A (c_1 v_1 + c_2 v_2 \cdots c_n v_n) = c_1^2 \lambda_1 + c_2^2 \lambda_2 + \cdots + c_n^2 \lambda_n \geq 0 \tag{6}$$

So A is positive semidefinite.

2. A is positive definite iff all eigenvalues of A are positive.
The proof of this statement is almost the same to the previous one, with the only modification being changing "greater than or equal to" sign to "strict greater" sign.

$\square$