CHAPTER 10 Graphs

1. Representing Graphs

Methods for representing g

- **Graphs**
- **Adjacency lists**

  -- lists that specify
  to each vertex
- **Adjacency matrice**
- **Incidence matrices**

An adjacency list for a directed graph	
Initial vertex	terminal vertices
a	b,c,d,e
b	b,d
c	a,c,e
d	
e	b,c,d

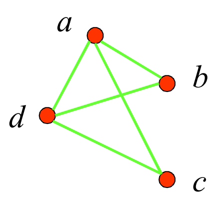
2. Adjacency Matrices

A simple graph $G = (V, E)$ with n vertices ($v_1, v_2, \cdots, v_n$) can be represented by its adjacency matrix, A, with respect to this listing of the vertices, where

$a_{ij} = 1$ if $\{v_i, v_j\}$ is an edge of G,
$a_{ij} = 0$ otherwise.

Note: An adjacency matrix of a graph is based on the ordering chosen for the vertices.

〖Example 1〗 What is the adjacency matrix A_G for the following graph G based on the order of vertices a, b, c, d ?



Solution:

$$A_G = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Note: Adjacency matrices of undirected graphs are always symmetric.

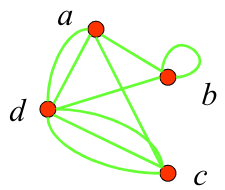
◆ The adjacency matrix of a multigraph or pseudograph

For the representation of graphs with multiple edges, we can no longer use zero-one matrices.

Instead, we use matrices of nonnegative integers.

The (i, j)th entry of such a matrix equals the number of edges that are associated to $\{v_i, v_j\}$.

〖Example 2〗 What is the adjacency matrix A_G for the following graph G based on the order of vertices a, b, c, d ?



Note: For undirected multigraph or pseudograph , adjacency matrices are symmetric.

Solution:

$$A_G = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 3 \\ 2 & 1 & 3 & 0 \end{bmatrix}$$

◆ The adjacency matrix of a directed graph

Let $G = (V, E)$ be a **directed graph** with $|V| = n$. Suppose that the vertices of G are listed in an arbitrary order as v_1, v_2, …, v_n.

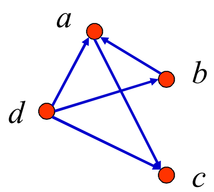
The adjacency matrix A (or A_G) of G, with respect to this listing of the vertices, is the $n\times n$ **zero-one matrix** with 1 as its (i, j)th entry when there is an edge from v_i to v_j, and 0 otherwise.

In other words, for an adjacency matrix $A = [a_{ij}]$,

$a_{ij} = 1$ if (v_i, v_j) is an edge of G,

$a_{ij} = 0$ otherwise.

〖Example 3〗 What is the adjacency matrix A_G for the following graph G based on the order of vertices a, b, c, d ?

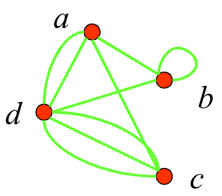


Solution:

$$A_G = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Question:

1. **What is the sum of the entries in a row of the adjacency matrix for an undirected graph?**



$$A_G = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 3 \\ 2 & 1 & 3 & 0 \end{bmatrix}$$

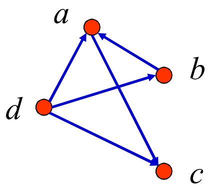
10.3 Representing Graphs and Graph Isomorphism

Question:

1. **What is the sum of the entries in a row of the adjacency matrix for an undirected graph?**

The number of edges incident to the vertex *i*, which is the same as degree of *i* minus the number of loops at *i*.

For a directed graph?



$$A_G = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Question:

1. What is the sum of the entries in a row of the adjacency matrix for an undirected graph?

   The number of edges incident to the vertex *i*, which is the same as degree of *i* minus the number of loops at *i*.

   For a directed graph?

   $\deg^{+}(v_i)$

2. What is the sum of the entries in a column of the adjacency matrix for an undirected graph?

   The number of edges incident to the vertex *i*, which is the same as degree of *i* minus the number of loops at *i*.

   For a directed graph?

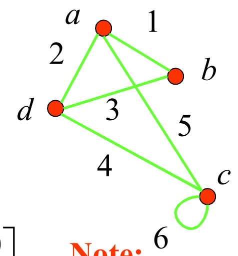
   $\deg^{-}(v_i)$

3. Incidence matrices

Let $G = (V, E)$ be an undirected graph. Suppose that $v_1, v_2, \cdots, v_n$ are the vertices and $e_1, e_2, \cdots, e_m$ are the edges of G. Then the *incidence matrix* with respect to this ordering of V and E is $n \times m$ matrix $\mathrm{M} = [m_{ij}]_{n\times m}$, where

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise} \end{cases}$$

〖Example 4〗 What is the incidence matrix M for the following graph G based on the order of vertices a, b, c, d and edges 1, 2, 3, 4, 5, 6?



Solution:

$$M = \begin{array}{c} a \\ b \\ c \\ d \end{array} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Note:

Incidence matrices of undirected graphs contain two 1s per column for edges connecting two distinct vertices and one 1 per column for loops.

4. Isomorphism of Graphs

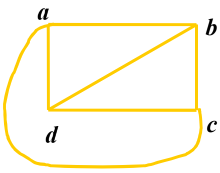
Graphs with the same structure are said to be *isomorphic*.

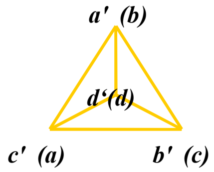
Formally, two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a 1-1 and onto function f from V_1 to V_2 such that for all a and b in V_1, a and b are adjacent in G_1 iff $f(a)$ and $f(b)$ are adjacent in G_2.

Such a function f is called an *isomorphism*.

In other words, when two simple graphs are isomorphic, there is a one-to-one correspondence between vertices of the two graphs that preserves the adjacency relationship.

For example,





Question:

How to determine whether two simple graphs are isomorphic?

It is usually difficult since there are $n!$ possible 1-1 correspondence between the two vertex sets with n vertices. However, some properties (called *invariants*) in the graphs may be used *to show that they are not isomorphic*.

invariants

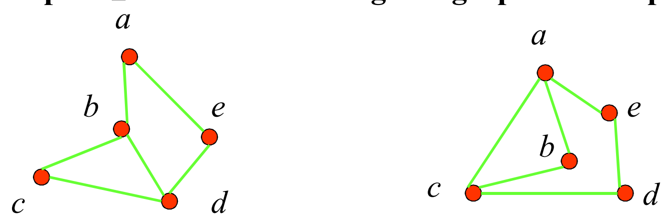
-- things that G_1 and G_2 must have in common to be isomorphic.

Important *invariants in isomorphic graphs*:

- the number of vertices
- the number of edges
- the degrees of corresponding vertices
- if one is bipartite, the other must be
- if one is complete, the other must be
- if one is a wheel, the other must be

etc.

〖Example 5〗 Are the following two graphs isomorphic?



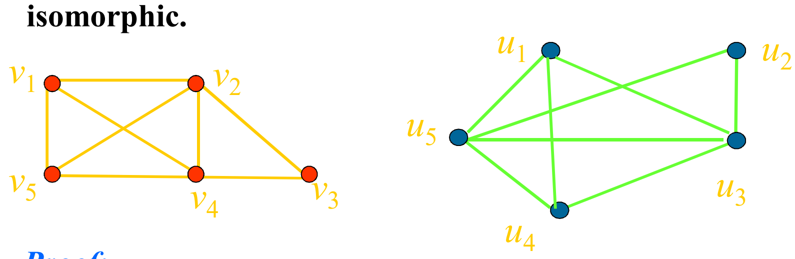
Solution:

They are isomorphic, because they can be arranged to look identical.

You can see this if in the right graph you move vertex *b* to the left of the edge {*a*, *c*}. Then the isomorphism *f* from the left to the right graph is:

$f(a) = e$, $f(b) = a$,
$f(c) = b$, $f(d) = c$, $f(e) = d$.

〖Example 6〗 Show that the following two graphs are isomorphic.



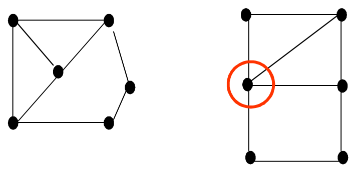
Proof:

- Try to find an 1-1 and onto function *f*
- Show that *f* isomorphism (preserves adjacency relation)

  - The adjacency matrix of a graph *G* is the same as the adjacency matrix of another graph *H*, when rows and columns are labeled to correspond to the images under *f* of the vertices in *G* that are the labels of these rows and columns in the M_G

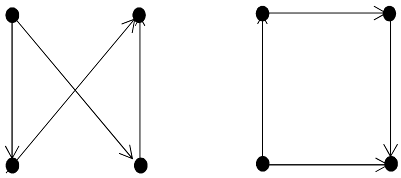
〖Example 7〗 Determine whether the given pair of graphs is isomorphic?

(1)



(2)

(3)



Homework:

第8版 Sec. 10.3 **8, 15, 17, 38-41**

CHAPTER 10 Graphs

10.1 Graphs and Graph Models

10.2 Graph Terminology and Special Types of Graphs

10.3 Representing Graphs and Graph Isomorphism

10.4 Connectivity

10.5 Euler and Hamilton Paths

10.6 Shortest Path Problems

10.7 Planar Graphs

10.8 Graph Coloring

1. Paths

A ***path of length n*** in a simple graph is a sequence of vertices $v_0, v_1, \ldots, v_n$ such that $\{v_0, v_1\}$, $\{v_1, v_2\}$, ..., $\{v_{n-1}, v_n\}$ are n edges in the graph.

The path is a ***circuit*** if it begins and ends at the same vertex (length greater than 0).

A path is ***simple*** if it does not contain the same edge more than once.

Note:

1. There is nothing to prevent traversing an edge back and forth to produce arbitrarily long paths. This is usually not interesting which is why we define a simple path.
2. The notation of a path: vertex sequence
3. A path of length zero consists of a single vertex.

Path in directed graph

A *path of length n* in a directed graph is a sequence of vertices $v_0, v_1, . . . v_n$ such that (v_0, v_1) , (v_1, v_2) , ..., (v_{n-1}, v_n) are n directed edges in the graph.

Circuit or cycle : the path begins and ends with the same vertex.

Simple path: the path does not contain the same edge more than once.

Paths represent useful information in many graph models.

〖Example 1〗 Path in Acquaintanceship Graphs

In an acquaintanceship graph there is a path between two people if there is a chain of people linking these people, where two people adjacent in the chain know one other.

Many social scientists have conjectured that almost every pair of people in the world are linked by a small chain of people, perhaps containing just five or fewer people.

The play *Six Degrees of Separation* by John Guare is based on this notion.

2. Counting paths between vertices

The number of paths between two vertices in a graph can be determined using its adjacency matrix.

【Theorem 2】 The *number of different paths of length r* from v_i to v_j is equal to the $(i, j)th$ entry of A^r, where A is the adjacency matrix representing the graph consisting of vertices $v_1, v_2, \ldots v_n$.

Note: This is the standard power of A, not the Boolean product.

Proof:

Let $A = [a_{ij}]_{n\times n}$

(1) A

$a_{ij} = 1$

$a_{ij} = 0$

(3)

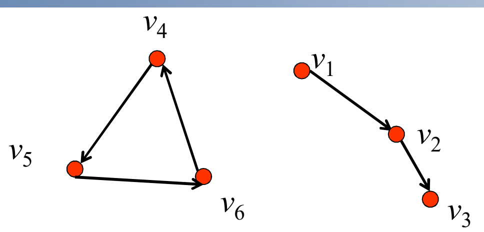
$$A^r = (c_{ij})_{n\times n}$$

$$A^{r+1} = A^r \cdot A = (d_{ij})_{n\times n}$$

$$d_{ij} = c_{i1}a_{1j} + c_{i2}a_{2j} + \cdots + c_{in}a_{nj} = \sum_{k=1}^{n} c_{ik}a_{kj}$$

from v_i to v_j

〖Example 2〗



(1) **How many paths of length 2 are there from v_5 to v_4?**

a_{54} in A^2: 1

(2) **The number of paths not exceeding 6 are there from v_4 to v_5?**

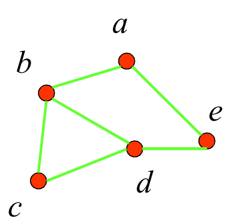
a_{45} in $A+A^2+A^3+A^4+A^5+A^6$: 2

(3) **The number of circuits starting at vertex v_5 whose length is not exceeding 6?**

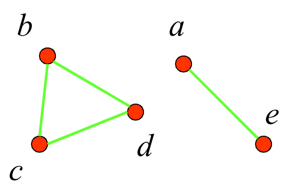
3. Connectedness in undirected graphs

An undirected graph is called *connected* if there is a path between every pair of distinct vertices of the graph.

〖Example 3〗 Are the following graphs connected?



Yes.



No.

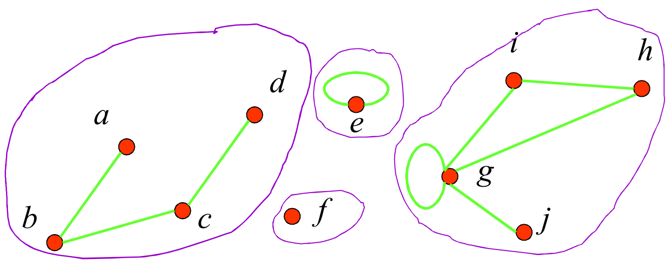
【 Theorem 1】 There is a simple path between every pair of distinct vertices of a connected undirected graph.

Proof:

Because the graph is connected there is a path between *u* and *v*. Throw out all redundant circuits to make the path simple.

The maximally connected subgraphs of *G* are called the *connected components* or just the *components*.

For example,

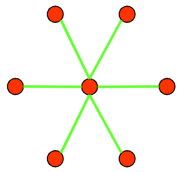


A vertex is a *cut vertex* (or *articulation point*), if removing it and all edges incident with it results in more connected components than in the original graph.

Similarly if removal of an edge creates more components the edge is called a *cut edge* or *bridge*.

For example,

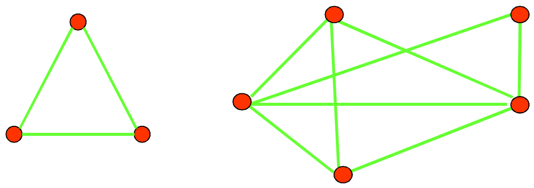
(1)



In the star network the center vertex is a cut vertex.

All edges are cut edges.

(2)



There are no cut edges or vertices in the graph *G* above.

Removal of any vertex or edge does not create additional components.

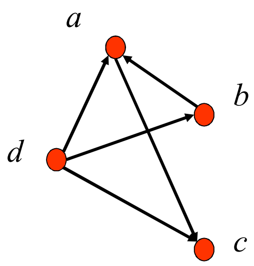
4. Connectedness in directed graphs

A directed graph is *strongly connected* if there is a path from *a* to *b* and from *b* to *a* for all vertices *a* and *b* in the graph.

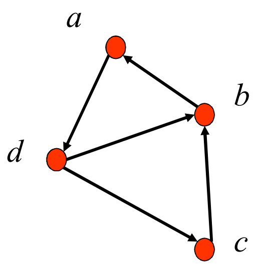
The graph is *weakly connected* if the underlying undirected graph is connected.

Note: By the definition, any strongly connected directed graph is also weakly connected.

For example,



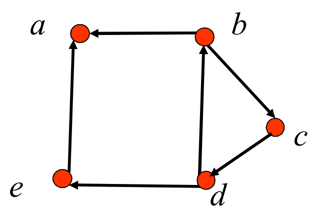
Weakly connected



Strongly connected

For directed graph, the maximal strongly connected subgraphs are called the *strongly connected components* or just the *strong components.*

For example,



Problems:

1. How to determine whether a given directed graph is strongly connected or weakly connected ?
2. How to find the strongly connected components in a directed graph ?

    Kosararu

    Tarjan

5. Paths and Isomorphism

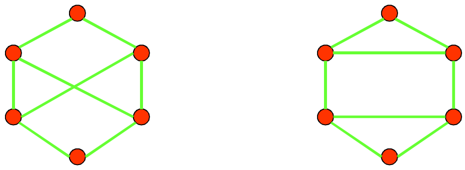
Idea:

(1) Some other invariants

- The number and size of connected components
- Path
- ✓ Two graphs are isomorphic only if they have simple circuits of the same length.
- ✓ Two graphs are isomorphic only if they contain paths that go through vertices so that the corresponding vertices in the two graphs have the same degree.

(2) We can also use paths to find mapping that are potential isomorphism.

〖Example 4〗 Are these two graphs isomorphic?



Solution:

No.

Because the right graph contains circuits of length 3, while the left graph does not.

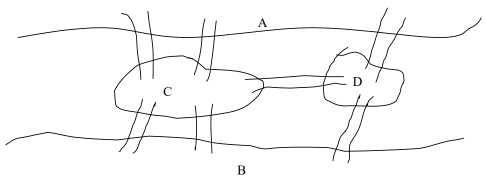
Homework:

Sec. 10.4 27(e), 28, 29, 62

CHAPTER 10 Graphs

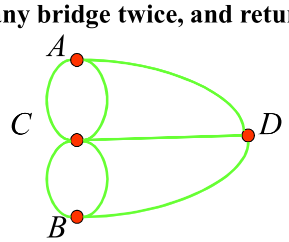
9.5.1 Euler Paths

1. Konigsberg Seven Bridge Problem



Problem: Is it possible to start at some location in the town, travel across all the bridges without crossing any bridge twice, and return the starting point?

The question becomes: Is there a simple circuit in this multigraph that contains every edge?



Terminologies:

- ***Euler Path***

  An Euler path is a simple path containing every edge of *G*.

- ***Euler Circuit***

  An Euler circuit is a simple circuit containing every edge of *G*.

- ***Euler Graph***

  A graph contains an Euler circuit.

2. Necessary and sufficient conditions for Euler circuit and paths

【 Theorem 1】 A connected multigraph has an Euler circuit if and only if each of its vertices has even degree.

Proof:

(1) Necessary condition

G has an Euler circuit $\Rightarrow$ Every vertex in V has even degree

Consider the Euler circuit.

- **the vertex a which the Euler circuit begins with**
- **intermeidate vertices**

(2) sufficient condition

We will form a simple circuit that begins at an arbitrary vertex *a* of *G*.

- **Build a simple circuit $x_0=a, x_1, x_2,\ldots,x_n=a$.**
- **An Euler circuit has been constructed if all the edges have been used. otherwise,**
- **Consider the subgraph *H* from *G* by deleting the edges already uses and vertices that are not incident with any remaining edges.**

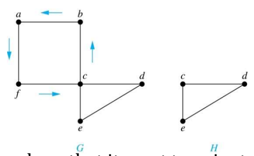
  Let *w* be a vertex which is the common vertex of the circuit and *H*.

  Beginning at *w*, construct a simple path in *H*.

Sufficient Conditions for Euler Circuits and Paths

Suppose that *G* is a connected multigraph with ≥ 2 vertices, all of even degree. Let $x_0 = a$ be a vertex of even degree. Choose an edge $\{x_0, x_1\}$ incident with *a* and proceed to build a simple path $\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{n-1}, x_n\}$ by adding edges one by one until another edge can not be added.

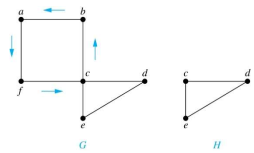
We illustrate this idea in the graph G here. We begin at *a* and choose the edges $\{a, f\}, \{f, c\}, \{c, b\}$, and $\{b, a\}$ in succession.



- The path begins at *a* with an edge of the form $\{a, x\}$; we show that it must terminate at *a* with an edge of the form $\{y, a\}$. Since each vertex has an even degree, there must be an even number of edges incident with this vertex. Hence, every time we enter a vertex other than *a*, we can leave it. Therefore, the path can only end at *a*.
- If all of the edges have been used, an Euler circuit has been constructed. Otherwise, consider the subgraph *H* obtained from *G* by deleting the edges already used.

➢In the example *H* consists of the vertices *c*, *d*, *e*.

Sufficient Conditions for Euler Circuits and Paths (*continued*)



- Because G is connected, H must have at least one vertex in common with the circuit that has been deleted.

  - In the example, the vertex is c.

- Every vertex in H must have even degree because all the vertices in G have even degree and for each vertex, pairs of edges incident with this vertex have been deleted. Beginning with the shared vertex construct a path ending in the same vertex (as was done before). Then splice this new circuit into the original circuit.

  - In the example, we end up with the circuit a, f, c, d, e, c, b, a.

- Continue this process until all edges have been used. This produces an Euler circuit. Since every edge is included and no edge is included more than once.
- Similar reasoning can be used to show that a graph with exactly two vertices of odd degree must have an Euler path connecting these two vertices of odd degree

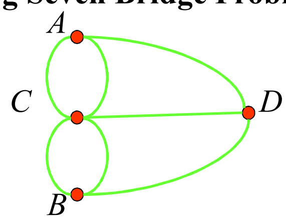
Algorithm for Constructing an Euler Circuits

In our proof we developed this algorithms for constructing a Euler circuit in a graph with no vertices of odd degree.

```
procedure Euler(G: connected multigraph with all vertices of even degree)
  circuit := a circuit in G beginning at an arbitrarily chosen vertex
  with edges
                  successively  added to form a path that returns to this
  vertex.
  H := G with the edges of this circuit removed
  while H  has edges
        subciruit  := a circuit in H beginning at a vertex in H that also is
                           an endpoint of an edge in circuit.
       H := H with edges of subciruit and all isolated vertices
  removed
       circuit := circuit with subcircuit inserted at the
  appropriate vertex.
return circuit{circuit is an Euler circuit}
```

【 Theorem 2】 **A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.**

〖Example 1〗 Konigsberg Seven Bridge Problem

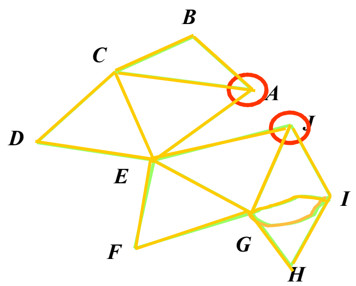


Solution:

1. The graph has four vertices of odd degree. Therefore, it does not have an Euler circuit.

2. It does not have an Euler path either.

〖Example 2〗 Determine whether the following graph has an Euler path. Construct such a path if it exists.



The Euler path:

A,C,E,F,G,I,J,E,A,B, C,D,E,G, H,I,G,J

Solution:

The graph has 2 vertices of odd degree, and all of other vertices have even degree . Therefore, this graph has an Euler path.

3. Euler circuit and paths in directed graphs

A directed multigraph having no isolated vertices has an Euler circuit if and only if

-- the graph is weakly connected

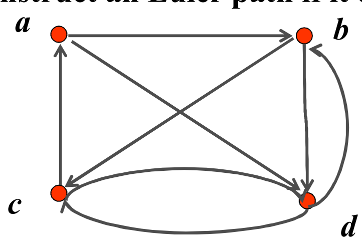
-- the in-degree and out-degree of each vertex are equal.

A directed multigraph having no isolated vertices has an Euler path but not an Euler circuit if and only if

-- the graph is weakly connected

-- the in-degree and out-degree of each vertex are equal for all but two vertices, one that has in-degree 1 larger than its out-degree and the other that has out-degree 1 larger than its in-degree.

〖Example 3〗 Determine whether the directed graph has an Euler path. Construct an Euler path if it exists.

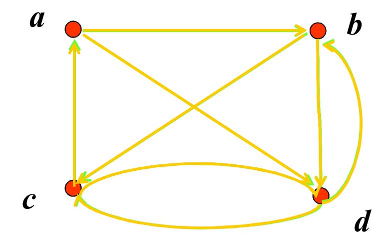


Solution:

	$\deg^-(v)$	$\deg^+(v)$
a	1	2
b	2	2
c	2	2
d	3	2

Hence, the directed graph has an Euler path.

〖Example 3〗 Determine whether the directed graph has an Euler circuit. Construct an Euler circuit if it exists.



Solution:

	$\deg^-(v)$	$\deg^+(v)$
a	1	2
b	2	2
c	2	2
d	3	2

Hence, the directed graph has an Euler path.

4. Applications

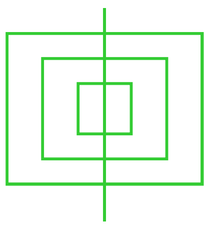
Euler path or circuit can be used to solve many practical problems.

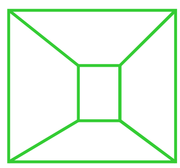
1) A type of puzzle

Draw a picture in a continuous motion without lifting a pencil so that no part of the picture is retraced.

The equivalent problem: Whether the graph exist an Euler path or circuit.

For example,





4. Applications

Euler path or circuit can be used to solve many practical problems.

2) The Chinese postman problem

- The problem is named in honor of Guan Meigu（管梅谷）, who posed it in 1962.

3) The other area, such as networking, molecular biology etc.

Homework:

下次本节讲完一起布置