Name______ Student Number______ Class______ Tutor______

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Discrete Mathematics Quiz II (Chapters 5-8)
Zhejiang University 2020

Number	1	2	3	4	5	6	7	Total
Records								

1. (32%) Filling in the blanks.
(1) Select 3 numbers from $1, 2, \ldots, 100$ such that the sum of these 3 numbers must be divided by 4. Then the total number of different selection ways is _____ $(C_{25}^3 + 3C_{25}^1 C_{25}^2 + 25^3)$.
(2) Assume that you have 20 balls and three boxes (labeled A, B and C).
a) Assuming that the balls are distinguishable, in how many ways can you put the balls in the boxes _____ (3^{20}).
b) In how many ways can you put the balls in the boxes, assuming that the balls are identical and each box must have at least two balls put into it? _____ $(C(3-1+14, 14))$.
(3) If $G(x)$ is the generating function for $a_0, a_1, a_2, a_3, \ldots$, the generating function for $0, a_1, 2a_2, 3a_3, \ldots$ in terms of $G(x)$ is _____ $(xG'(x))$.
(5) For the Fibonacci sequence f_n, it is divisible by 3 if and only if n is divisible by 4. This statement is _____. (True or false) (True)
(6) Let $a_0 = 3, a_1 = 1$, and $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 2$. Then $a_n =$_____ $(a_n = 3^n + 2(-1)^n)$.
(7) The number of 10-combinations from three apples, four oranges, and five pears is _____ (6).
(8) The coefficient of $x_1^3 x_2^3 x_3 x_4^2$ in the expansion of $(x_1 - x_2 + 2x_3 - 2x_4)^9$ is _____ $(-\frac{9!}{3!3!2!}2^3 = -40320)$.

2. (12%)How many different ways can you put 9 coins in 9 boxes $B_1, \ldots, B_9$ if
a)the coins are all different and no box is empty? 9!
b)the coins are all different and only two boxes B_1 and B_9 are empty? $7!S(9,7)$
c)the coins are all different and exactly four boxes are not empty? $C_9^4 4! S(9,4)$
d) the coins are all different and each box is either empty or contains exactly three coins? $C_9^3 C_9^{3,3,3}$
e) the coins are identical. C_{17}^9
f) the coins are identical and exactly six boxes are empty. $C_9^6 C_8^6$

3. (12%)How many ways are there to distribute six objects to five boxes if
a) both the objects and boxes are labeled? 15625, 5^6
b) the objects are labeled, but the boxes are unlabeled? 202, Stirling numbers
c) the objects are unlabeled, but the boxes are labeled? 210, C_{10}^4
d) both the objects and the boxes are unlabeled? 10

4. (11%) Find the number of solutions of $x_1 + x_2 + x_3 = 20$, where x_1, x_2, x_3 are integers. a) $x_i > i$, for $i = 1, 2, 3$. b) $x_1 \geq -6$, $x_2 \geq -7$ and $x_3 \geq 8$.
Anser: a) 20-2-3-4=11; C(3+11-1,11)=C(13,11)=13*12/2=78;
b) 20+6+7-8=25; C(3+25-1,25)=C(27,25)=27*26/2=351;

5. (11%) Compute the ways that the digits 0,1,2,3,4,5,6,7,8,9 are arranged so that the first 4 digits are not in their original positions.

   Answer:

   Let A0 be the set 0 is arranged in its original position. Let A1 be the set 1 is arranged in its original position. Let A2 be the set 2 is arranged in its original position. Let A3 be the set 3 is arranged in its original position. Then what we want to obtain is: $10! - 4 * 9! + 6 * 8! - 4 * 7! + 6!$.

6. (10%) Find the sequence with each of these functions as its exponential generating function.

$$\begin{array}{ll} a) f(x) = e^{-x} & b) f(x) = e^{3x} - 3e^{2x} \\ c) f(x) = e^{-3x} - (1 + x) + \frac{1}{1-2x} & d) f(x) = e^{x^2} \end{array}$$

7. (12%)

   A coding system encodes messages using strings of base 3 digits (that is, digits from the set $\{0, 1, 2\}$). A codeword is valid if and only if it contains an even number of 0s and an even number of 1s. Let a_n equal the number of valid codewords of length n. Furthermore, let b_n, c_n, and d_n equal the number of strings of base 3 digits of length n with an even number of 0s and an odd number of 1s, with an odd number of 0s and an even number of 1s, and with an odd number of 0s and an odd number of 1s, respectively.

   a) Show that $d_n = 3^n - a_n - b_n - c_n$. Use this to derive the recurrence relation between a_{n+1} and a_n, b_n, c_n, d_n, b_{n+1} and a_n, b_n, c_n, d_n, and c_{n+1} and a_n, b_n, c_n, d_n.

   b)What are a_1, b_1, c_1, and d_1?

   c) Use parts (a) and (b) to find a_3, b_3, c_3, and d_3.

   d) Use the recurrence relations in part (a), together with the initial conditions in part (b), to set up three equations relating the generating functions $A(x), B(x)$, and $C(x)$ for the sequences $\{a_n\}, \{b_n\}$, and $\{c_n\}$, respectively.

   e) Solve the system of equations from part (d) to get explicit formulae for $A(x), B(x)$, and $C(x)$ and use these to get explicit formulae for a_n, b_n, c_n, and d_n.