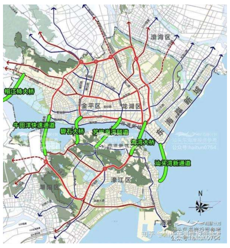
Network Flow
网络流

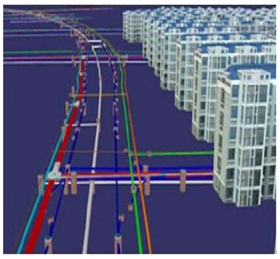
Max flow-Min Cut
最大流最小割定理

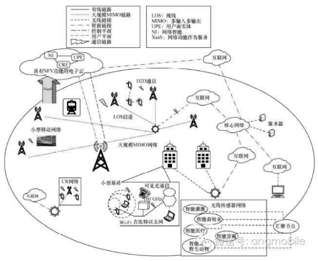
Outline

- Network flow 网络流
- Flows and Cuts 流和割
- Residual Graph 残差图
- Augmenting Paths 增广路径
- Ford Fulkerson Algorithm 福特-福克森算法
- Max flow-Min Cut Theorem 最大流最小割
- Application – Bipartite Matching

Network Flow

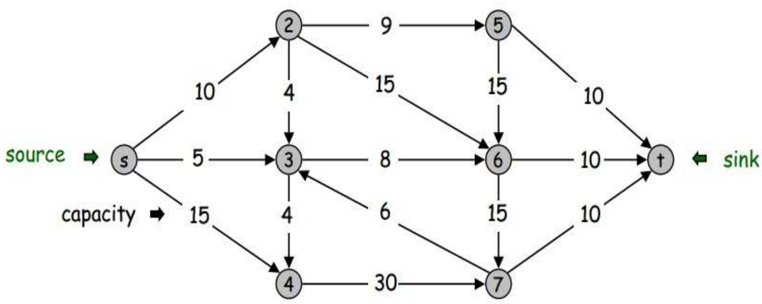






Network Flow Definitions

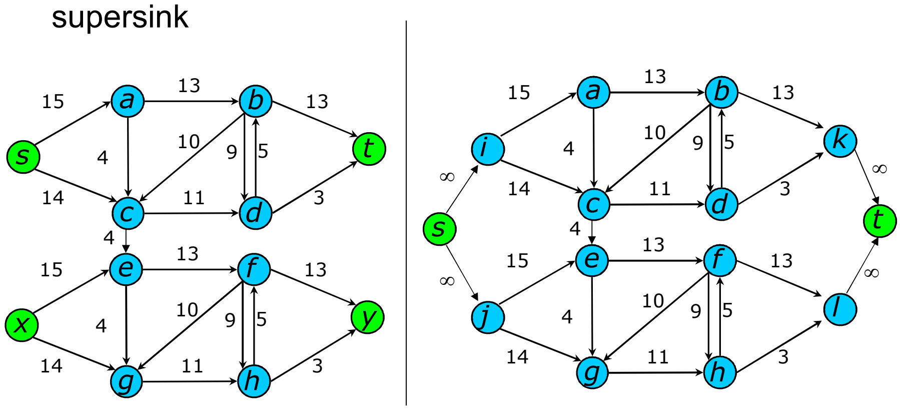
- Flowgraph: Directed graph with distinguished vertices s (source源点) and t (sink收点/汇点)
- Capacities（容量） on the edges: c(e) >= 0



边上标注的是容量值

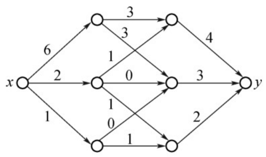
Multiple Sources or Sinks

- What if you have a problem with more than one source and more than one sink?
- Modify the graph to create a single supersource and supersink



Network Flow

- Problem, assign flows f(e) to the edges such that:
  - 0 <= f(e) <= c(e) （容量约束）
  - Flow is conserved at vertices other than s and t
    - Flow conservation: flow going into a vertex equals the flow going out （守恒约束）



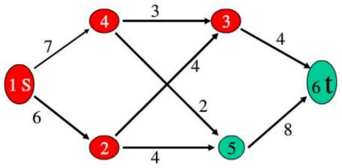
一个可行流，边上标注的是流量

  - 优化目标： The flow leaving the source is as large as possible
    - Denoted by |f|

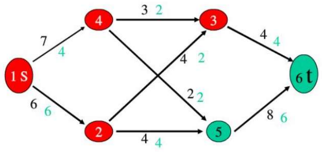
Network Flow

实例：

有一自来水管道输送系统，起点是S，目标是T，途中经过的管道都有一个最大的容量。



- 问题：问从S到T的最大水流量是多少？



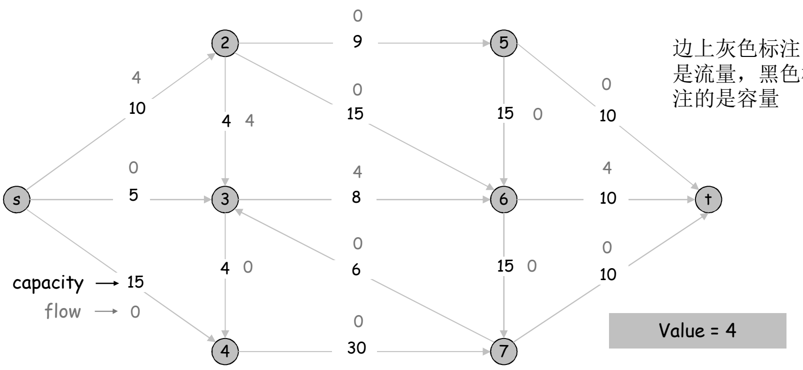
最大水流量是10

Flows

Def. An s-t flow （可行流）is a function that satisfies:

- For each e ∈ E: $0 \le f(e) \le c(e)$ [capacity]
- For each v ∈ V – {s, t}: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ [conservation]
- 特例：零流

Def. The value of a flow f is: $v(f) = \sum_{e \text{ out of } s} f(e)$.



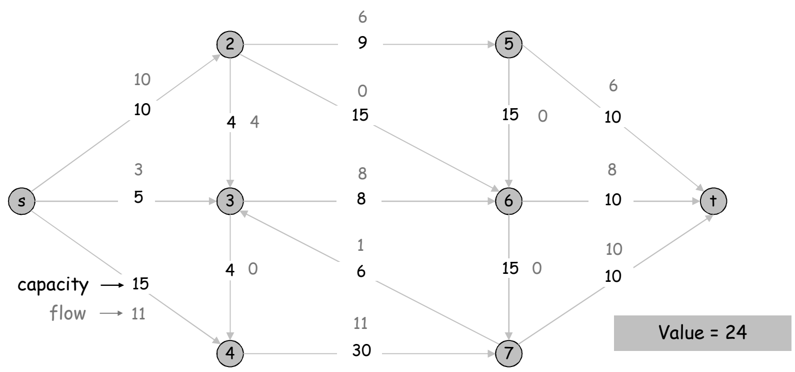
边上灰色标注的是流量，黑色标注的是容量

Flows

Def. An s-t flow is a function that satisfies:

- For each $e \in E$: $0 \le f(e) \le c(e)$ [capacity]
- For each $v \in V - \{s, t\}$: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ [conservation]

Def. The value of a flow f is: $v(f) = \sum_{e \text{ out of } s} f(e)$.

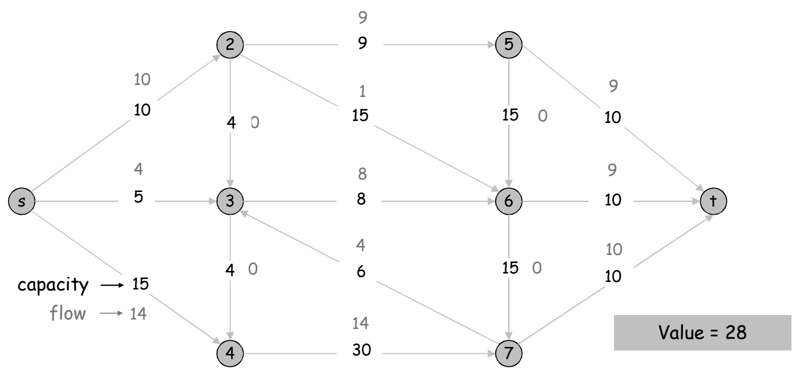


9

Maximum Flow Problem

Max flow problem. Find s-t flow of maximum value.

所有可行流中，流量最大的流的流量



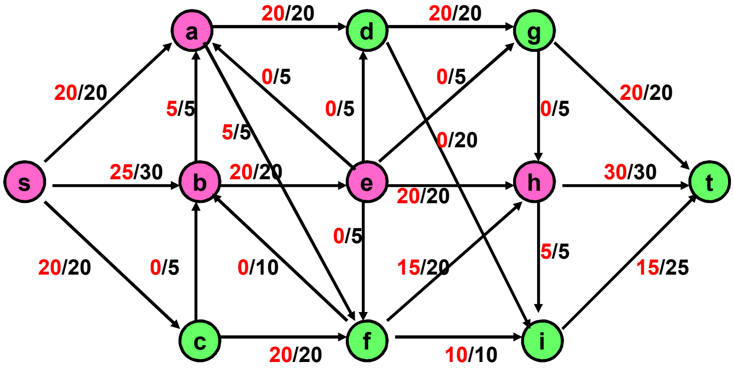
10

Cuts in a graph

- Cut: Partition of V into disjoint sets S, T with s in S and t in T.
- Cap(S,T): sum of the capacities of edges from S to T
- Flow(S,T): net flow out of S
  - Sum of flows out of S minus sum of flows into S
- Flow(S,T) <= Cap(S,T)

What is Cap(S,T) and Flow(S,T)

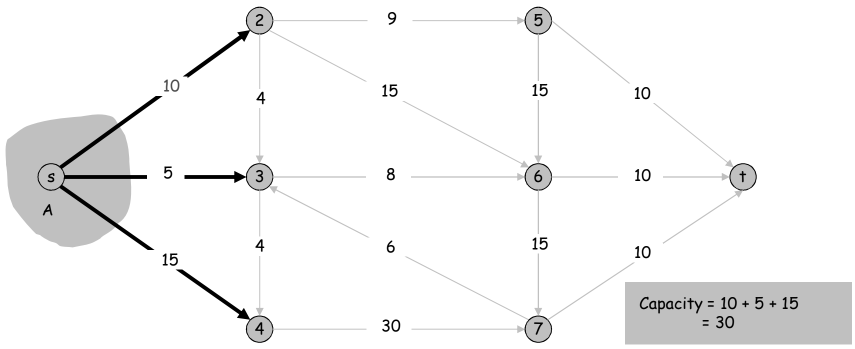
S={s, a, b, e, h}, T = {c, f, i, d, g, t}



Cuts

Def. An s-t cut is a partition (A, B) of V with s ∈ A and t ∈ B.

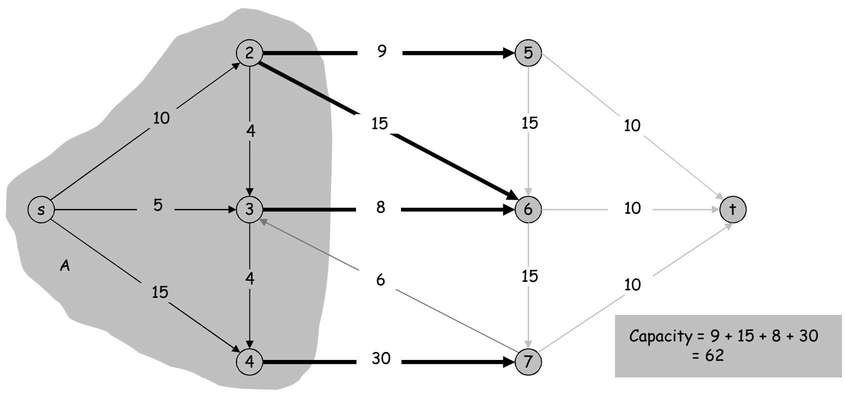
Def. The capacity of a cut (A, B) is: $cap(A, B) = \sum_{e \text{ out of } A} c(e)$



Cuts

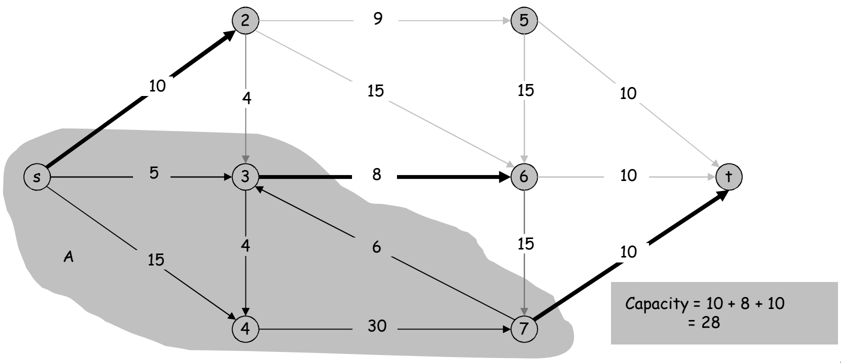
Def. An s-t cut is a partition (A, B) of V with s ∈ A and t ∈ B.

Def. The capacity of a cut (A, B) is: $cap(A, B) = \sum_{e \text{ out of } A} c(e)$



Minimum Cut Problem

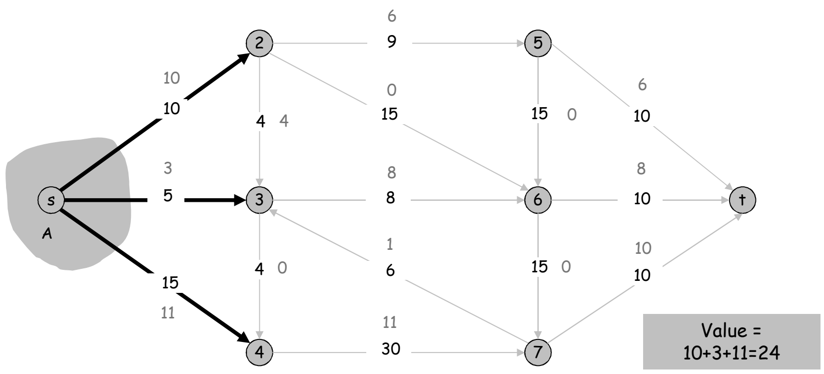
Min s-t cut problem. Find an s-t cut of minimum capacity.



Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then, the net flow sent across the cut is equal to the amount leaving s.

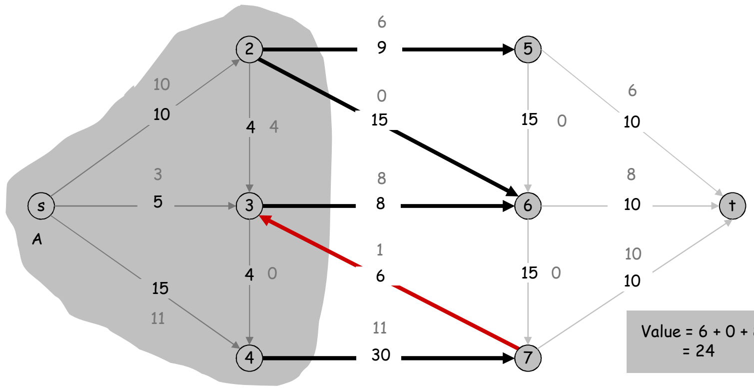
$$\sum_{e \text{ out of } A} f(e) \; - \sum_{e \text{ in to } A} f(e) \; = \; v(f)$$



Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then, the net flow sent across the cut is equal to the amount leaving s.

$$\sum_{e \text{ out of } A} f(e) \; - \sum_{e \text{ in to } A} f(e) \; = \; v(f)$$

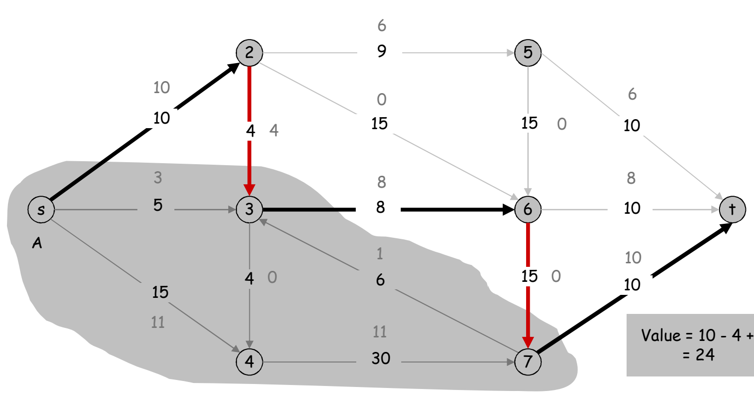


Value = 6 + 0 + 8 - 1 + 11
= 24

Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then, the net flow sent across the cut is equal to the amount leaving s.

$$\sum_{e \text{ out of } A} f(e) \; - \sum_{e \text{ in to } A} f(e) \; = \; v(f)$$



Value = 10 - 4 + 8 - 0 + 10
= 24

Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f).$$

Pf.

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

by flow conservation, all terms except v = s are 0 $\longrightarrow$

$$= \sum_{v \in A} \left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ in to } v} f(e) \right)$$

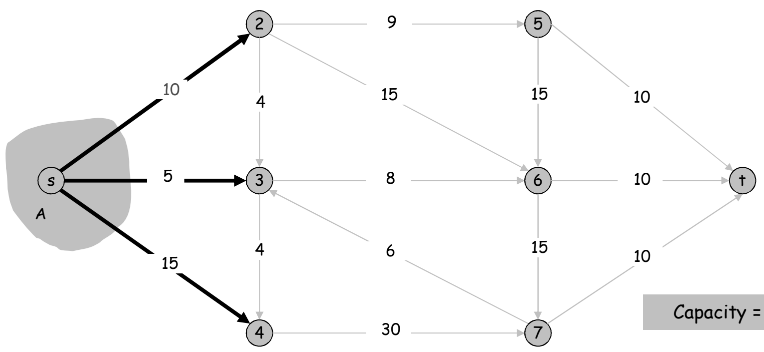
$$= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to A}} f(e).$$

19

Flows and Cuts

Weak duality. Let f be any flow, and let (A, B) be any s-t cut. Then the value of the flow is at most the capacity of the cut.

Cut capacity = 30 $\Rightarrow$ Flow value $\leq$ 30



Capacity = 30

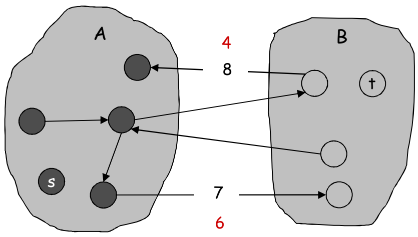
20

Flows and Cuts

Weak duality. Let f be any flow. Then, for any s-t cut (A, B) we have $v(f) \leq cap(A, B)$.

Pf.

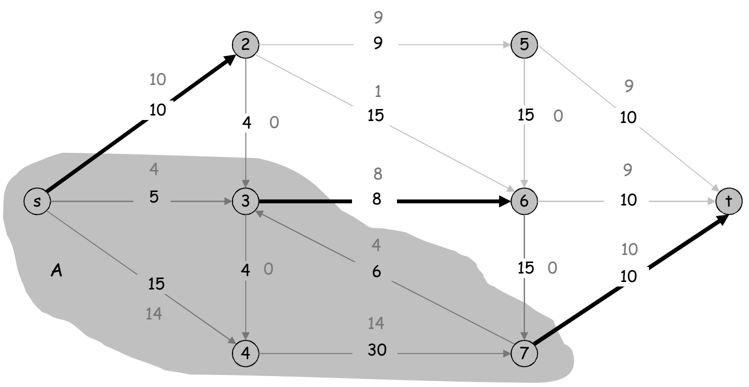
$$\begin{aligned} v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\ &\leq \sum_{e \text{ out of } A} f(e) \\ &\leq \sum_{e \text{ out of } A} c(e) \\ &= \text{cap}(A,B) \quad \blacksquare \end{aligned}$$



Certificate of Optimality

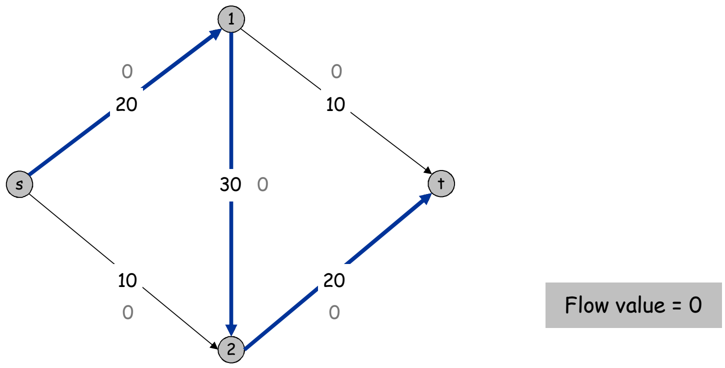
Corollary. Let f be any flow, and let (A, B) be any cut.
If v(f) = cap(A, B), then f is a max flow and (A, B) is a min cut.

Value of flow = 28
Cut capacity = 28 $\Rightarrow$ Flow value $\leq$ 28



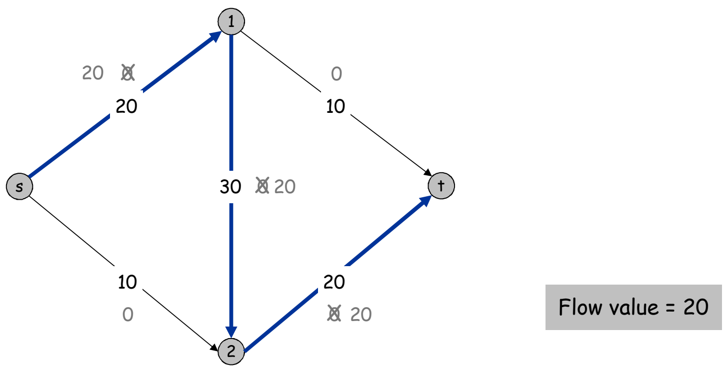
Towards a Max Flow Algorithm

Greedy algorithm.
- Start with f(e) = 0 for all edge e ∈ E.
- Find an s-t path P where each edge has f(e) < c(e).
- Augment flow along path P.
- Repeat until you get stuck.



Towards a Max Flow Algorithm

Greedy algorithm.
- Start with f(e) = 0 for all edge e ∈ E.
- Find an s-t path P where each edge has f(e) < c(e).
- Augment flow along path P.
- Repeat until you get stuck.

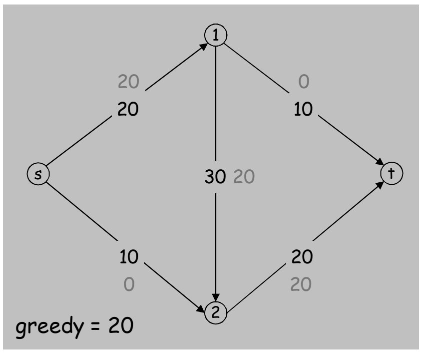


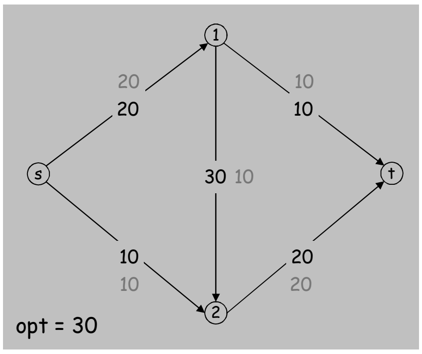
Towards a Max Flow Algorithm

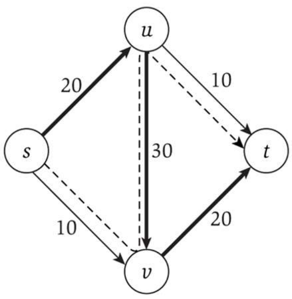
Greedy algorithm.

- Start with f(e) = 0 for all edge e ∈ E.
- Find an s-t path P where each edge has f(e) < c(e).
- Augment flow along path P.
- Repeat until you get stuck.

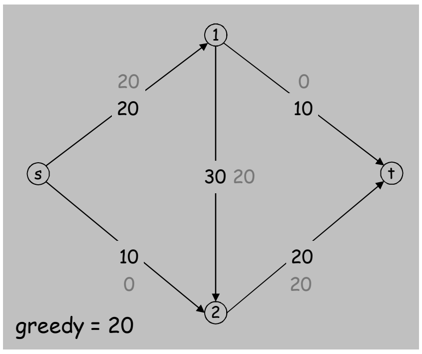
locally optimality ⇏ global optimality

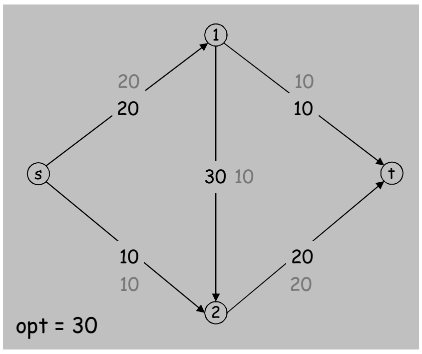






观察虚线：利用逆向边。

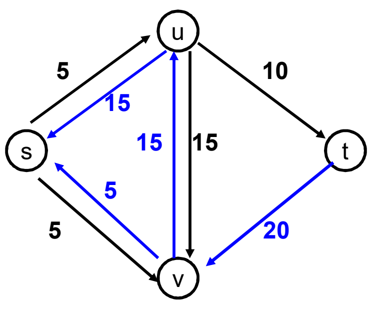




Residual Graph（残差图）

- Flow graph showing the remaining capacity
- Flow graph G, Residual Graph G_R
  - G: edge e from u to v with capacity c and flow f
  - G_R: edge e' from u to v with capacity c – f
  - G_R: edge e'' from v to u with capacity f

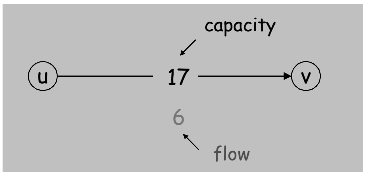
The residual graph



the residual graph

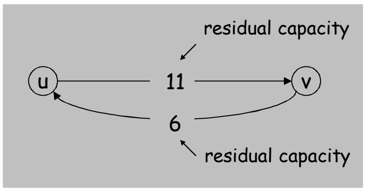
Residual Graph

Original edge: $e = (u, v) \in E$.
- Flow $f(e)$, capacity $c(e)$.



Residual edge.
- "Undo" flow sent.
- $e = (u, v)$ and $e^R = (v, u)$.
- Residual capacity:

$$c_f(e) = \begin{cases} c(e) - f(e) & \text{if } e \in E \\ f(e) & \text{if } e^R \in E \end{cases}$$

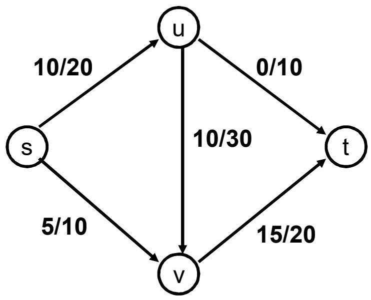


Residual graph: $G_f = (V, E_f)$.
- Residual edges with positive residual capacity.
- $E_f = \{e : f(e) < c(e)\} \cup \{e^R : f(e) > 0\}$.

29

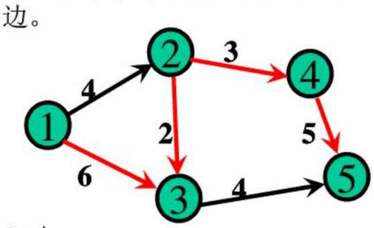
Augmenting Path Algorithm

- Augmenting path（增广路径）
  - Vertices $v_1, v_2, \ldots, v_k$
    - $v_1 = s$, $v_k = t$
    - Possible to add b units of flow between v_j and v_{j+1} for j = 1 … k-1



Augmenting Path Algorithm

- 从 s 到 t 的一条简单路径，若边 (u, v) 的方向与该路径的方向一致，称 (u, v) 为正向边，方向不一致时称为逆向边。



简单路：1→3 → 2→4→5中。

（1，3）（2，4）（4，5）是正向边。（3，2）是逆向边。

Augmenting Path Algorithm

增广路径：

若路径上所有的边满足：

①所有正向边有：f （ u, v ）< c （ u, v）

②所有逆向边有：f （ u, v ）> 0

则称该路径为一条**增广路径(可增加流量)**

```
Augment(f, c, P) {
   b ← bottleneck(P)                    b > 0
   foreach e ∈ P {
      if (e ∈ E) f(e) ← f(e) + b        forward edge
      else       f(e^R)← f(e^R) - b     reverse edge
   }
   return f
}
```

找到这样一条路径，其流量未达到容量上限。增广后，总流量增加了b。

Ford-Fulkerson Algorithm (1956)

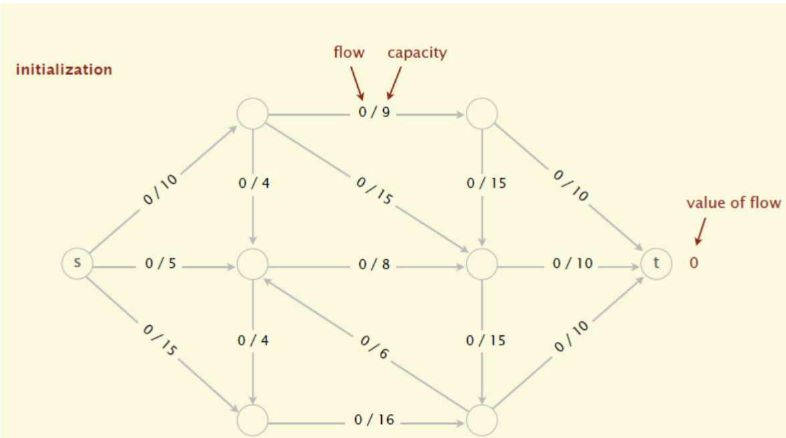
```
Ford-Fulkerson(G, s, t, c) {
   foreach e ∈ E  f(e) ← 0
   G_f ← residual graph

   while (there exists augmenting path P) {
      f ← Augment(f, c, P)
      update G_f
   }
   return f
}
```

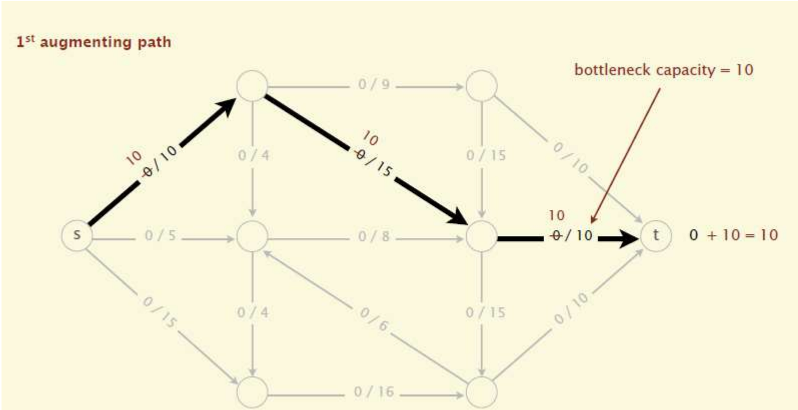
先假定容量为非负**整数**：

If the sum of the capacities of edges leaving S is at most C, then the algorithm takes at most C iterations.（每次都增加，但总量有限，一定会结束）

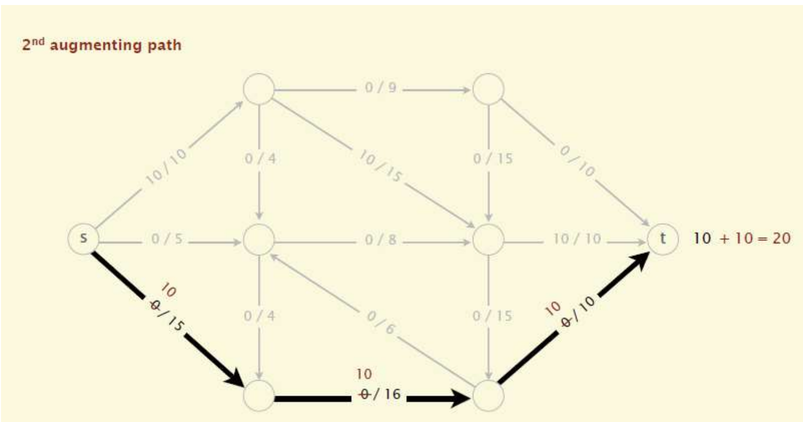
Ford-Fulkerson Algorithm (1956)



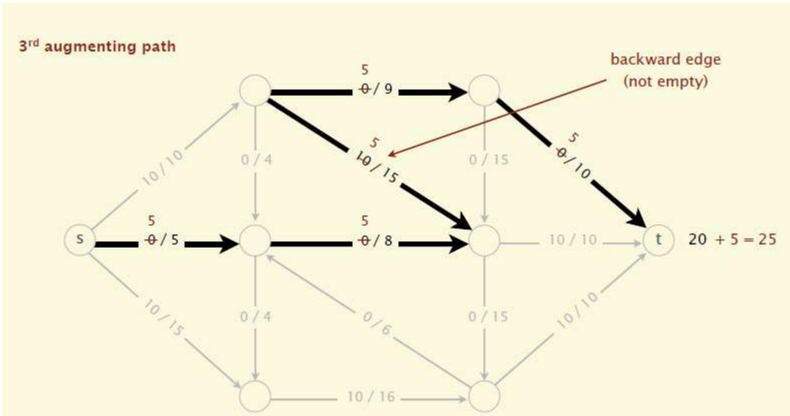
Ford-Fulkerson Algorithm (1956)



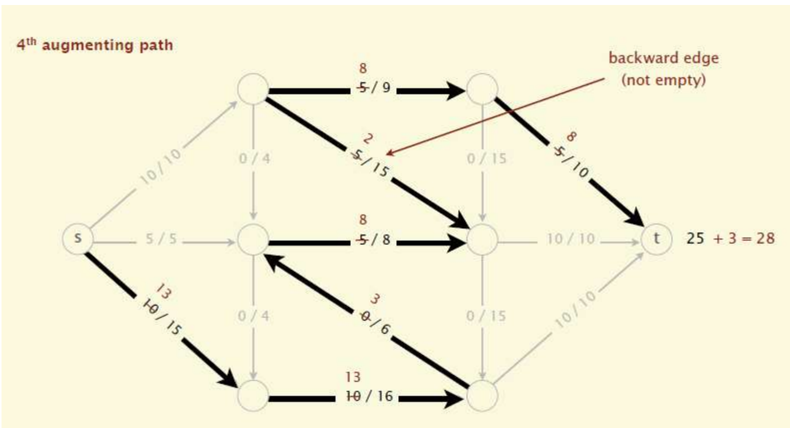
Ford-Fulkerson Algorithm (1956)



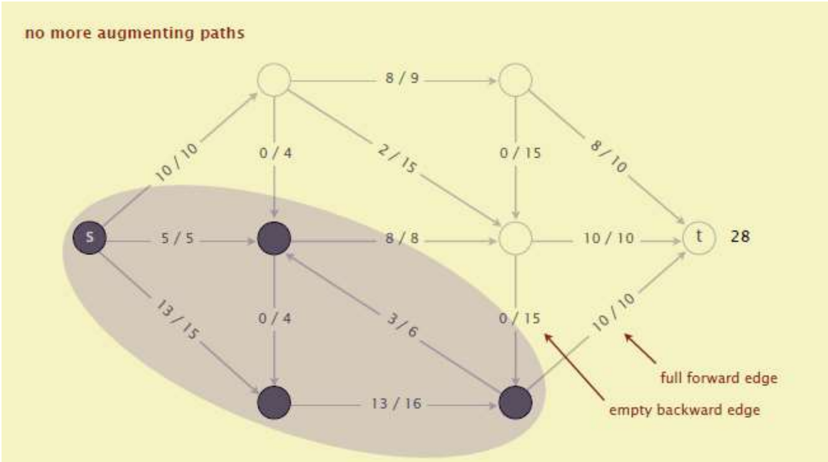
Ford-Fulkerson Algorithm (1956)



Ford-Fulkerson Algorithm (1956)



Ford-Fulkerson Algorithm (1956)



Max-Flow Min-Cut Theorem

Augmenting path theorem. Flow f is a max flow iff there are no augmenting paths.

Max-flow min-cut theorem. [Elias-Feinstein-Shannon 1956, Ford-Fulkerson 1956]
The value of the max flow is equal to the value of the min cut.
(There exists a flow which has the same value of the minimum cut)

Pf. We prove both simultaneously by showing TFAE (the following are equivalent):

- (i) There exists a cut (A, B) such that v(f) = cap(A, B).
  存在一个割的容量等于flow f的值
- (ii) Flow f is a max flow.
  f是最大流
- (iii) There is no augmenting path relative to f.
  对于f没有增广路径

40

Max-Flow Min-Cut Theorem

(i) $\Rightarrow$ (ii) This was the corollary to weak duality lemma.

假设我们有一个割（A,B）的容量等于f的值，那么所有流的值<=(A,B)的容量，从而（2）成立

(ii) $\Rightarrow$ (iii) We show contrapositive.

- Let f be a flow. If there exists an augmenting path, then we can improve f by sending flow along path.
  我们来证明它的逆否命题。对于f如果还有还有增广路径，那f不是最大流。这很显然。如果按照FF算法的话，我们还可以增加flow f的值，因此f就不会是最大流，因此逆否命题成立，也就代表（2）->(3)成立。

41

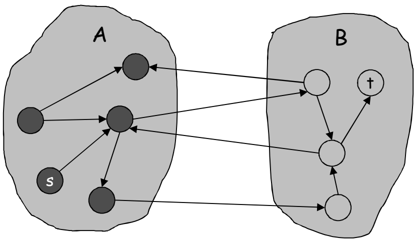
Proof of Max-Flow Min-Cut Theorem

(iii) $\Rightarrow$ (i)

- Let f be a flow with no augmenting paths.
- Let A be set of vertices reachable from s in residual graph.
  - 通过这些边可达：要么是不是满的前向边，要么是非空的反向边。
- By definition of A, s $\in$ A.
- By definition of f, t $\notin$ A. （因为没有增广路径）

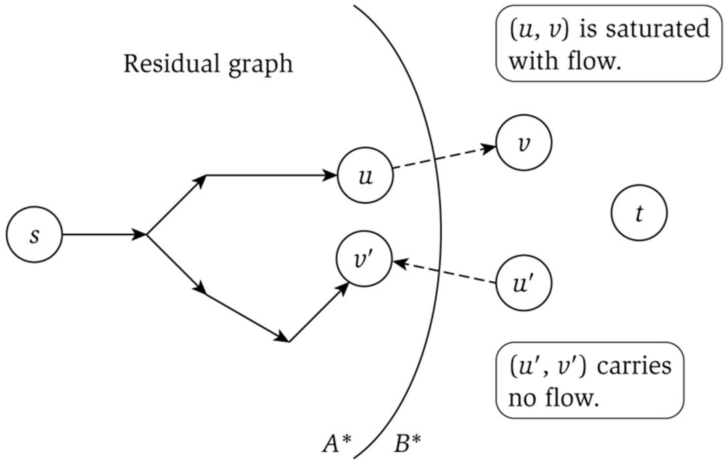
$$\begin{aligned} v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\ &= \sum_{e \text{ out of } A} c(e) \\ &= cap(A, B) \end{aligned}$$

Explained next page

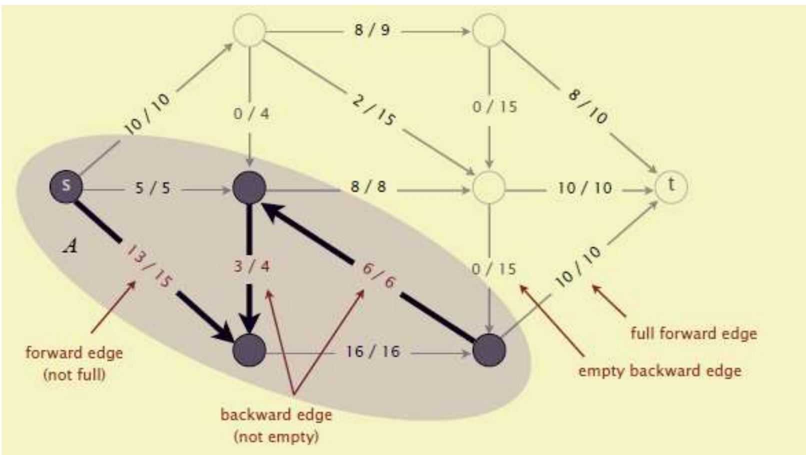


original network

Proof of Max-Flow Min-Cut Theorem



Proof of Max-Flow Min-Cut Theorem

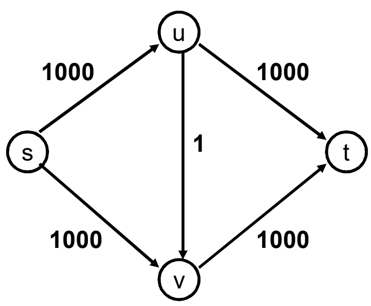


Max Flow - Min Cut Theorem

- Ford-Fulkerson algorithm finds a flow where the residual graph is disconnected, hence FF finds a maximum flow.
- If we want to find a minimum cut, we begin by looking for a maximum flow.

Performance

- The worst case performance of the Ford-Fulkerson algorithm is horrible

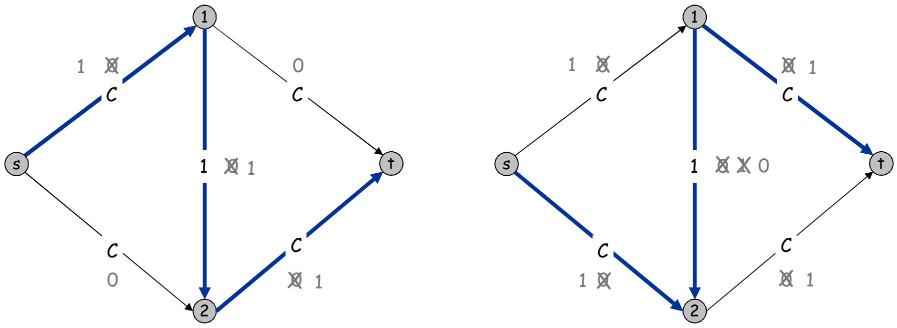


Ford-Fulkerson: Exponential Number of Augmentations

Q. Is generic Ford-Fulkerson algorithm polynomial in input size?

m, n, and log C

A. No. If max capacity is C, then algorithm can take C iterations.



Running Time

Assumption. All capacities are integers between 1 and C.

Invariant. Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains an integer throughout the algorithm.

Theorem. The algorithm terminates in at most $v(f^*) \le nC$ iterations.
Pf. Each augmentation increase value by at least 1. ▪

Corollary. If $C = 1$, Ford-Fulkerson runs in $O(mn)$ time.

Integrality theorem. If all capacities are integers, then there exists a max flow f for which every flow value $f(e)$ is an integer.
Pf. Since algorithm terminates, theorem follows from invariant. ▪

Choosing Good Augmenting Paths

Use care when selecting augmenting paths.

- Some choices lead to exponential algorithms.
- Clever choices lead to polynomial algorithms.
- If capacities are irrational, algorithm not guaranteed to terminate!

Goal: choose augmenting paths so that:

- Can find augmenting paths efficiently.
- Few iterations.

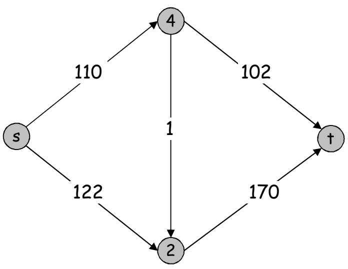
Choose augmenting paths with: [Edmonds-Karp 1972, Dinitz 1970]

- Max bottleneck capacity.
- Sufficiently large bottleneck capacity.
- Fewest number of edges.

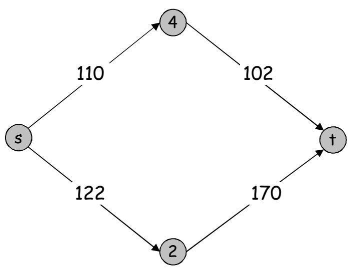
Capacity Scaling

Intuition. Choosing path with highest bottleneck capacity increases flow by max possible amount.

- Don't worry about finding exact highest bottleneck path.
- Maintain scaling parameter Δ.
- Let $G_f(\Delta)$ be the subgraph of the residual graph consisting of only arcs with capacity at least Δ.



G_f



G_f (100)

Capacity Scaling

```
Scaling-Max-Flow(G, s, t, c) {
   foreach e ∈ E  f(e) ← 0
   Δ ← smallest power of 2 greater than or equal to C
   Gf ← residual graph

   while (Δ ≥ 1) {
      Gf(Δ) ← Δ-residual graph
      while (there exists augmenting path P in Gf(Δ)) {
         f ← augment(f, c, P)
         update Gf(Δ)
      }
      Δ ← Δ / 2
   }
   return f
}
```


Capacity Scaling: Correctness

Assumption. All edge capacities are integers between 1 and C.

Integrality invariant. All flow and residual capacity values are integral.

Correctness. If the algorithm terminates, then f is a max flow.
Pf.

- By integrality invariant, when $\Delta = 1 \Rightarrow G_f(\Delta) = G_f$.
- Upon termination of $\Delta = 1$ phase, there are no augmenting paths. ▪

Theorem. The scaling max-flow algorithm finds a max flow in $O(m \log C)$ augmentations. It can be implemented to run in $O(m^2 \log C)$ time. ▪

Performance of finding augmenting paths

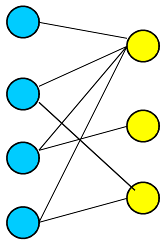
- Find the maximum capacity augmenting path
  - $O(m^2\log(C))$ time algorithm for network flow
- Find the shortest augmenting path
  - $O(m^2n)$ time algorithm for network flow
- Find a blocking flow in the residual graph
  - $O(mn\log n)$ time algorithm for network flow

Application – Bipartite Matching

- Example – given a community with *n* men and *m* women
- Assume we have a way to determine which couples (man/woman) are compatible for marriage
  - E.g. (Joe, Susan) or (Fred, Susan) but not (Frank, Susan)
- Problem: Maximize the number of marriages
  - No polygamy allowed
  - Can solve this problem by creating a flow network out of a bipartite graph

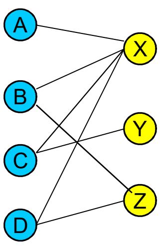
Bipartite Graph

- A bipartite graph is an undirected graph G=(V,E) in which V can be partitioned into two sets V_1 and V_2 such that $(u,v) \in E$ implies either $u \in V_1$ and $v \in V_2$ or vice versa.
- That is, all edges go between the two sets V_1 and V_2 and not within V_1 and V_2.

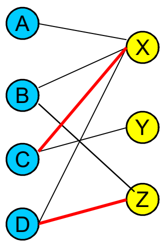


Model for Matching Problem

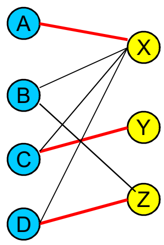
- Men on leftmost set, women on rightmost set, edges if they are compatible



Men Women



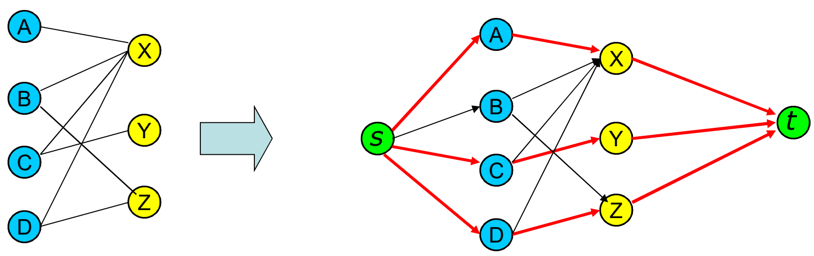
A matching



Optimal matching

Solution Using Max Flow

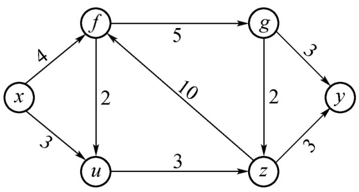
- Add a supersouce, supersink, make each undirected edge directed with a flow of 1



Since the input is 1, flow conservation prevents multiple matchings

Homework

1、In the network below, find a maximum flow from x to y , calculate its flow value, and prove that it is the maximum flow.



2、There are two classic algorithms for finding the strongly-connected components of a directed graph.

✓ *Tarjan algorithm*

✓ *Kosaraju-Sharir algorithm*

You just have to choose one of the two algorithms. Search for relevant documents and study by yourself, then use your own description to summarize the algorithm. Both Chinese and English are acceptable.