CHAPTER 10 Graphs

10.1 Graphs and Graph Models

10.2 Graph Terminology and Special Types of Graphs

10.3 Representing Graphs and Graph Isomorphism

10.4 Connectivity

10.5 Euler and Hamilton Paths

10.6 Shortest Path Problems

10.7 Planar Graphs

10.8 Graph Coloring

Graph Theory

Graph theory is an old subject with many modern applications .

For example, graphs can be used to

- study the structure of the World Wide Web.

- determine whether a circuit can be implemented on a planar circuit board.

- solve problems such as finding the shortest path between two cities in a transportation network.

- to schedules exams, and so on.

10.1 Graphs and Graph Models

【Definition 1】 **A *graph* G=(V,E) consists of V, a nonempty set of *vertices* (or *nodes*) and E, a set of *edges*. Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to *connect* its endpoints.**

Remark:

- ✓ ***Unrelated to graphs of functions studied in Chapter 2***
- ✓ ***All that matters is the connections made by the edges, not the particular geometry depicted.***
- ✓ ***Infinite Graph, finite Graph***

Types of ***Undirected Graphs***

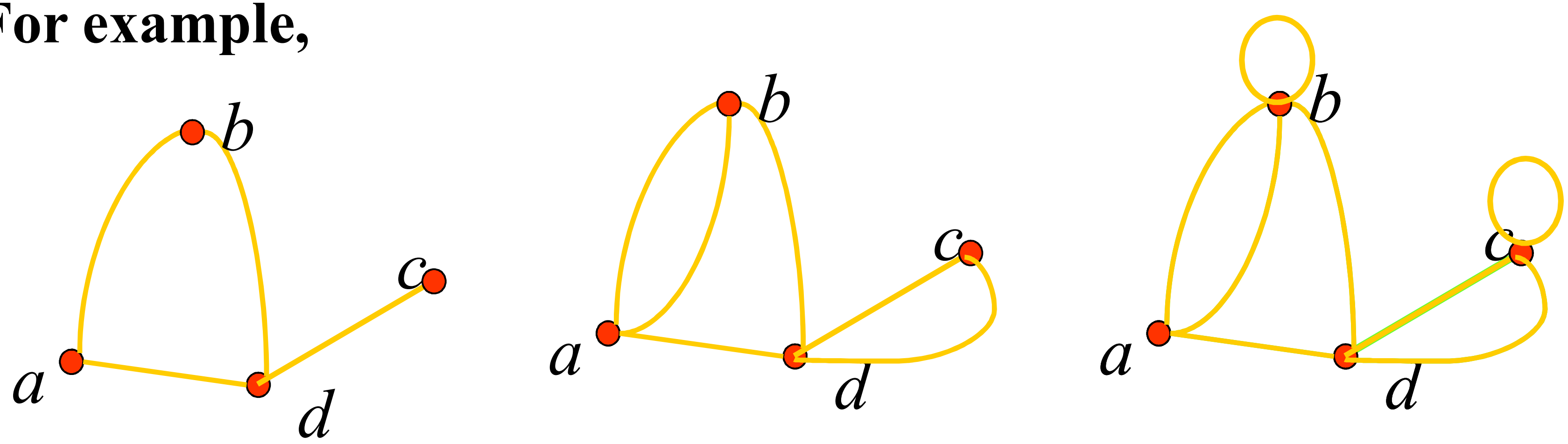
- Simple graph
- Multigraph
- Pseudograph

Simple **graph: A graph in which each edge connects two different vertices *and* where no two edges connect the same pair of vertices.**

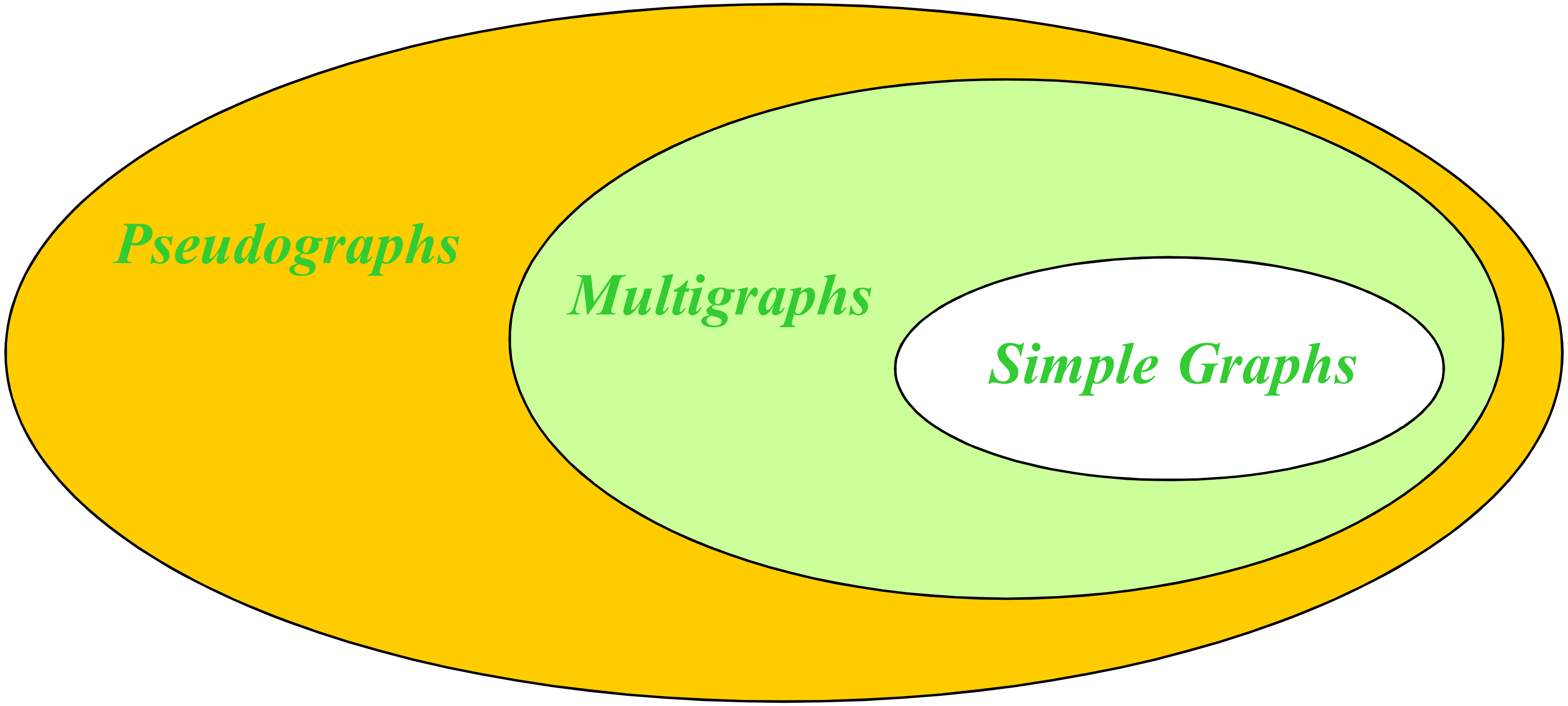
Multigraph: **Graphs that may have multiple edges connecting the same vertices.**

Pseudograph: **Graphs that may include loops, and possibly multiple edges connecting the same pair of vertices.**

For example,



The relations of different undirected graphs



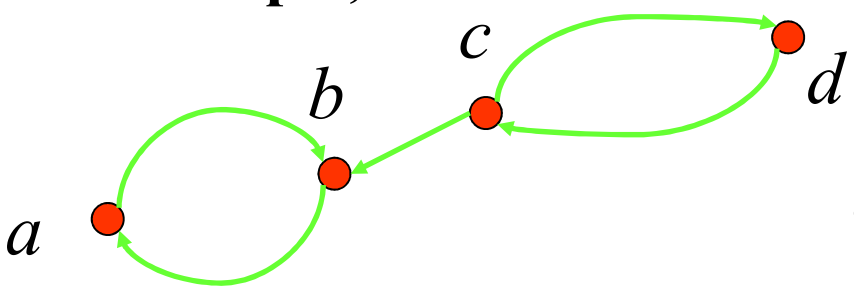
【Definition 2】A *directed graph* (or *digraph*) *(V, E)* consists of a nonempty set of vertices *V* and a set of *directed edges* (or *arcs*) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (*u*,*v*) is said to *start* at *u* and *end* at *v*.

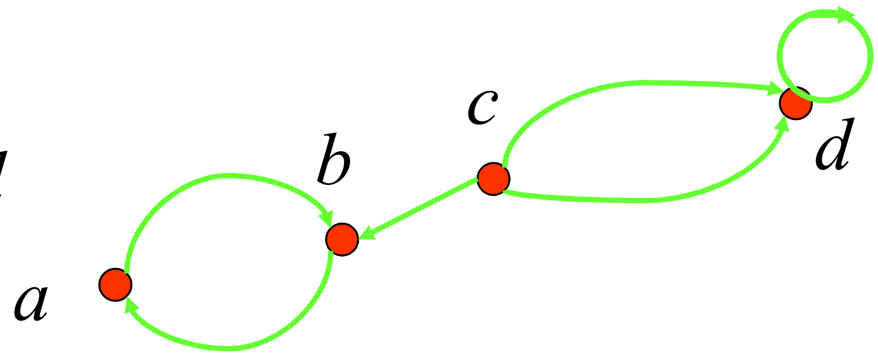
Types of digraphs:

Simple directed graph: **a directed graph has no loops and has no multiple directed edges.**

directed multigraph: **a directed graphs that may have multiple directed edges from a vertex to a second (possibly the same) vertex.**

For example,





Graph Models

Problems in almost every conceivable discipline can be solved using graph models.

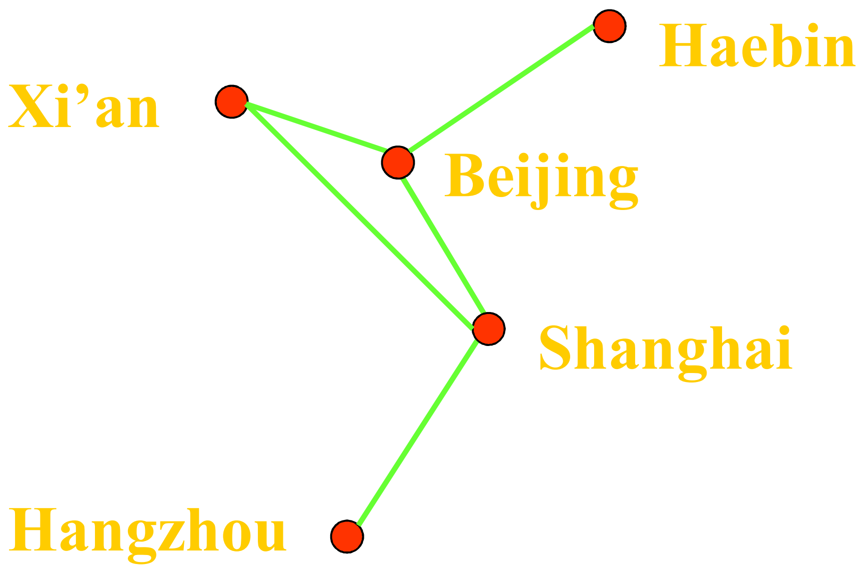
For example,

- ✓ **Niche overlap Graphs in Ecology**
- ✓ **Influence Graphs**
- ✓ **The Hollywood Graph**
- ✓ **Round-Robin Tournament**
- ✓ **The Web Graph**
- ✓ **……**

〖Example 1〗 How can we represent a network of (bi-directional) railways connecting a set of cities?

Solution:

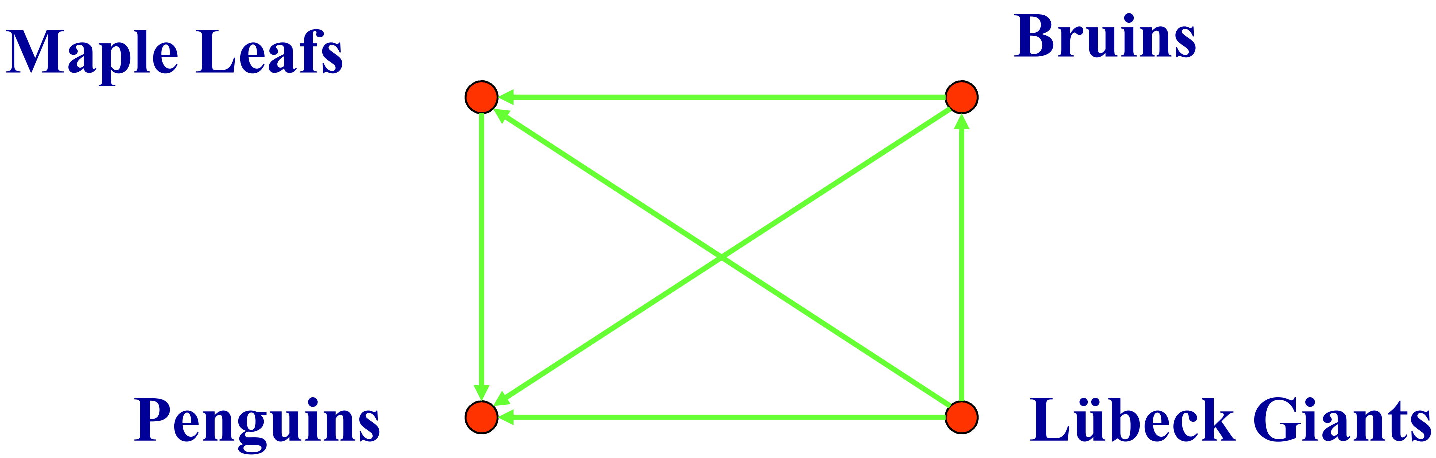
We should use a simple graph with an edge {*a*, *b*} indicating a direct train connection between cities *a* and *b*.



〖Example 2〗 In a round-robin tournament, each team plays against each other team exactly once. How can we represent the results of the tournament (which team beats which other team)?

Solution:

We should use a *directed graph* with an edge (a, b) indicating that team a beats team b.



Other Applications of Graphs

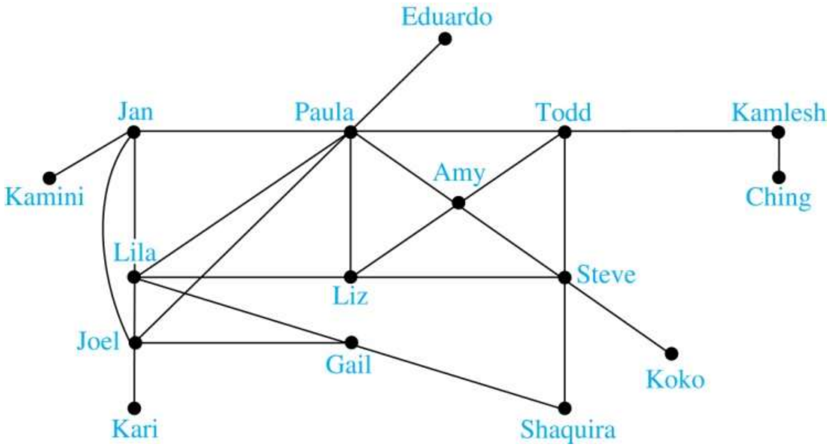
- We will illustrate how graph theory can be used in models of:
  - Social networks
  - Communications networks
  - Information networks
  - Software design
  - Transportation networks
  - Biological networks
- It’s a challenge to find a subject to which graph theory has not yet been applied. Can you find an area without applications of graph theory?

Graph Models: Social Networks

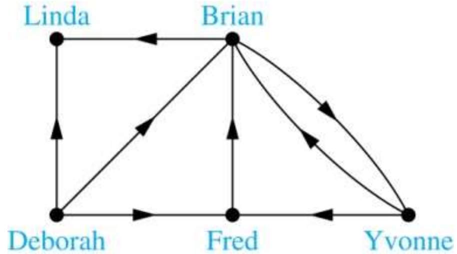
- Graphs can be used to model social structures based on different kinds of relationships between people or groups.
- In a *social network*, vertices represent individuals or organizations and edges represent relationships between them.
- Useful graph models of social networks include:
  - *friendship graphs* - undirected graphs where two people are connected if they are friends (in the real world, on Facebook, or in a particular virtual world, and so on.)
  - *collaboration graphs* - undirected graphs where two people are connected if they collaborate in a specific way
  - *influence graphs* - directed graphs where there is an edge from one person to another if the first person can influence the second person

Graph Models: Social Networks (*continued*)

➤Example: A friendship graph where two people are connected if they are Facebook friends.



➤Example: An influence graph



➤*Next Slide: Collaboration Graphs*

Examples of Collaboration Graphs

- The *Hollywood graph* models the collaboration of actors in films.
  - We represent actors by vertices and we connect two vertices if the actors they represent have appeared in the same movie.
  - We will study the Hollywood Graph in Section 10.4 when we discuss Kevin Bacon numbers.
- An *academic collaboration graph* models the collaboration of researchers who have jointly written a paper in a particular subject.
  - We represent researchers in a particular academic discipline using vertices.
  - We connect the vertices representing two researchers in this discipline if they are coauthors of a paper.
  - We will study the academic collaboration graph for mathematicians when we discuss *Erdős numbers* in Section 10.4.

Applications to Information Networks

- Graphs can be used to model different types of networks that link different types of information.
- In a *web graph*, web pages are represented by vertices and links are represented by directed edges.
  - A web graph models the web at a particular time.
  - We will explain how the web graph is used by search engines in Section 11.4.
- In a *citation network*:
  - Research papers in a particular discipline are represented by vertices.
  - When a paper cites a second paper as a reference, there is an edge from the vertex representing this paper to the vertex representing the second paper.

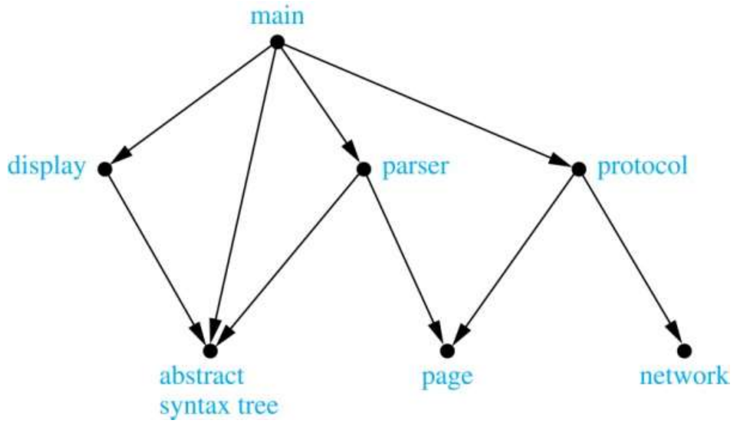
Transportation Graphs

- Graph models are extensively used in the study of transportation networks.
- Airline networks can be modeled using directed multigraphs where
  - airports are represented by vertices
  - each flight is represented by a directed edge from the vertex representing the departure airport to the vertex representing the destination airport
- Road networks can be modeled using graphs where
  - vertices represent intersections and edges represent roads.
  - undirected edges represent two-way roads and directed edges represent one-way roads.

Software Design Applications

- Graph models are extensively used in software design. We will introduce two such models here; one representing the dependency between the modules of a software application and the other representing restrictions in the execution of statements in computer programs.
- When a top-down approach is used to design software, the system is divided into modules, each performing a specific task.
- We use a *module dependency graph* to represent the dependency between these modules. These dependencies need to be understood before coding can be done.
  - In a module dependency graph vertices represent software modules and there is an edge from one module to another if the second module depends on the first.

➢Example: The dependencies between the seven modules in the design of a web browser are represented by this module dependency graph.



Software Design Applications (*continued*)

- We can use a directed graph called a *precedence graph* to represent which statements must have already been executed before we execute each statement.
  - Vertices represent statements in a computer program
  - There is a directed edge from a vertex to a second vertex if the second vertex cannot be executed before the first

➢Example: This precedence graph shows which statements must already have been executed before we can execute each of the six statements in the program.

S_1 $a := 0$

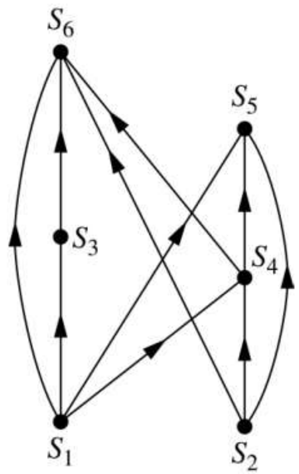
S_2 $b := 1$

S_3 $c := a + 1$

S_4 $d := b + a$

S_5 $e := d + 1$

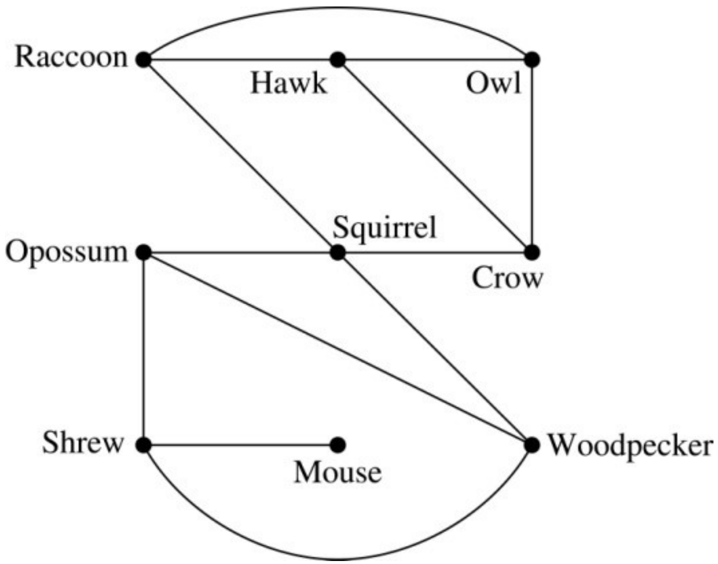
S_6 $e := c + d$



Biological Applications

- Graph models are used extensively in many areas of the biological science. We will describe two such models, one to ecology and the other to molecular biology.
- *Niche overlap graphs* model competition between species in an ecosystem
  - Vertices represent species and an edge connects two vertices when they represent species who compete for food resources.

➢Example: This is the niche overlap graph for a forest ecosystem with nine species.



Homework:

Sec. 10.1 1, 3-9

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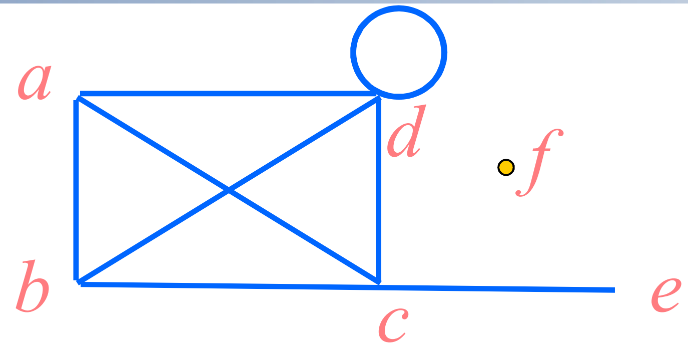
1. Basic Terminology

Undirected Graphs $G=(V, E)$



- ***Vertex, edge***
- **Two vertices, *u* and *v* in an undirected graph *G* are called *adjacent* (or *neighbors*) in *G*, if {*u*, *v*} is an edge of *G*.**
- **An edge *e* connecting *u* and *v* is called *incident with vertices u and v*, or is said to connect *u* and *v*.**
- **The vertices *u* and *v* are called *endpoints* of edge {*u*, *v*}.**
- ***loop***
- **The *degree of a vertex* in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex**

Notation: deg(*v*)

- **If deg(*v*) = 0, *v* is called *isolated.***
- **If deg(*v*) = 1, *v* is called *pendant*.**

【Theorem 1】 The Handshaking Theorem

Let $G = (V, E)$ be an undirected graph with e edges. Then

$$\sum_{v \in V} \deg(v) = 2e$$

The sum of the degrees over all the vertices is twice the number of edges.

Proof:

Each edge contributes twice to the degree count of all vertices.

Note:

This applies even if multiple edges and loops are present.

Questions:

If a graph has 5 vertices, can each vertex have degree 3? 4?

- **The sum is 3•5 = 15 which is an odd number.**

  Not possible.

- **The sum is 20 = 2 | *E* | and 20/2 = 10.**

  Possible.

【Theorem 2】 An undirected graph has an even number of vertices of odd degree.

Proof:

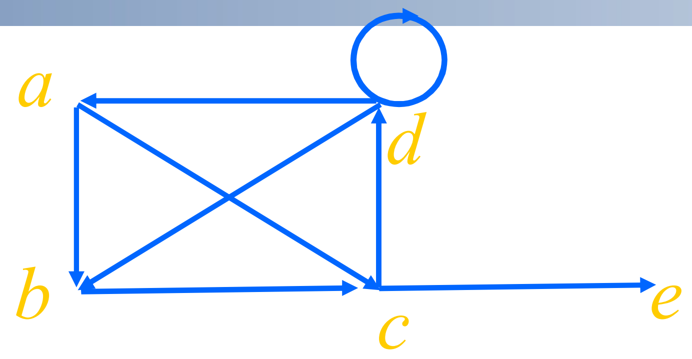
Let V_1, V_2 be the set of vertices of even degree and the set of vertices of odd degree, respectively.

$$\sum_{v \in V_1} d(v) + \sum_{v \in V_2} d(v) = 2m$$

Question:

Is it possible to have a graph with 3 vertices each of which has degree 3?

Directed Graphs *G*=(*V*, *E*)



Let (*u*, *v*) be an edge in *G*. Then *u* is an *initial vertex* and is *adjacent to* *v* and *v* is a *terminal vertex* and is *adjacent from* *u*.

The *in degree* of a vertex *v*, denoted $\deg^-(v)$ is the number of edges which terminate at *v*.

Similarly, the *out degree* of *v*, denoted $\deg^+(v)$, is the number of edges which initiate at *v*.

underlying undirected graph

【 Theorem 3】 Let *G* = *(V, E)* be a graph with direct edges. Then

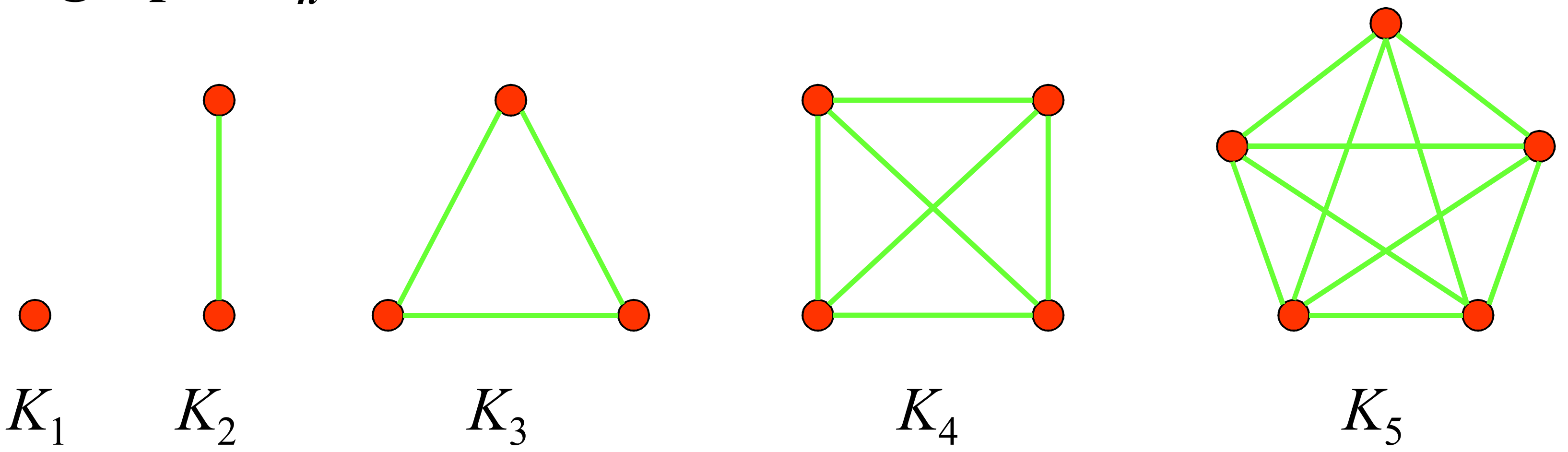
$$\sum_{v \in V} d^{+}(v) = \sum_{v \in V} d^{-}(v) \quad = |E|$$

2. Some Special Simple Graphs

(1) Complete Graphs - K_n: the simple graph with

- n vertices
- exactly one edge between every pair of distinct vertices.

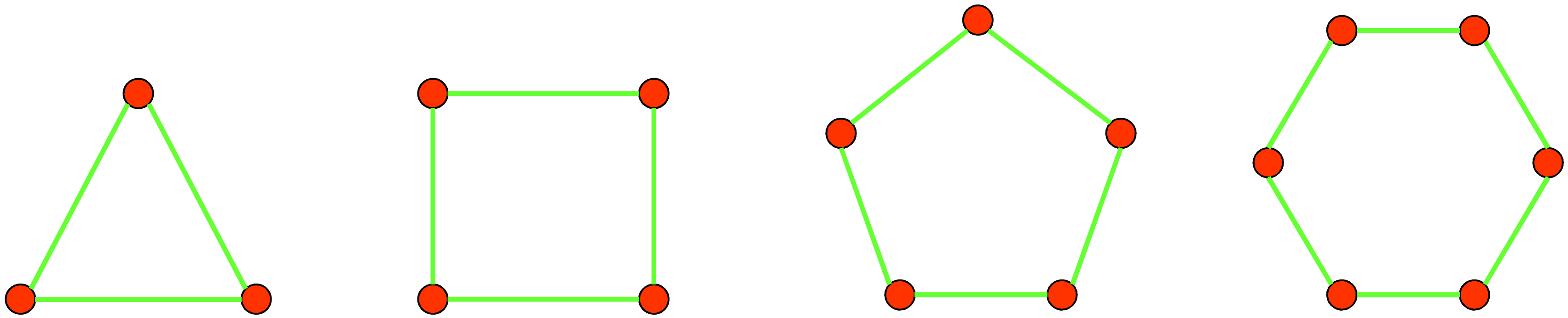
The graphs K_n for n=1,2,3,4,5.



Question:

The number of edges in K_n ?

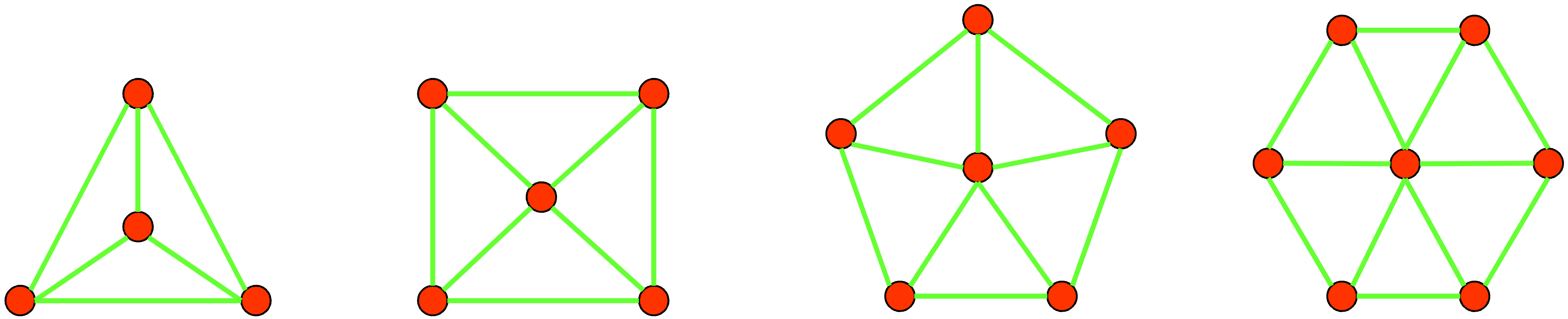
(2) Cycles C_n (n>2)



C_3 C_4 C_5 C_6

(3) Wheels W_n (n>2)

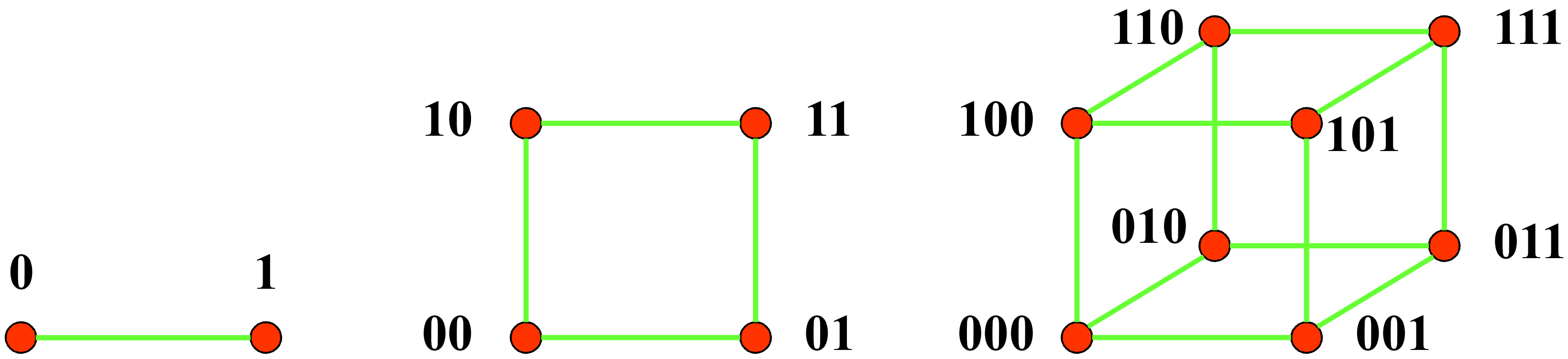
Add one additional vertex to the cycle C_n and add an edge from each vertex in C_n to the new vertex to produce W_n.



W_3 W_4 W_5 W_6

(4) *n-Cubes* Q_n (n>0)

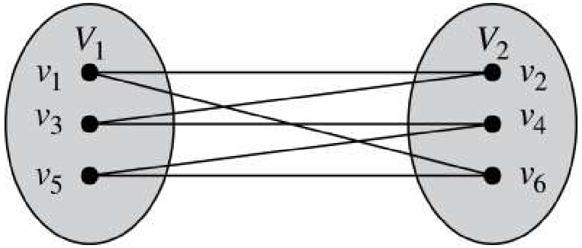
Q_n is the graph with 2^n vertices representing bit strings of length n. An edge exists between two vertices that differ in exactly one bit position.



Q_1 Q_2 Q_3

3. Bipartite Graphs

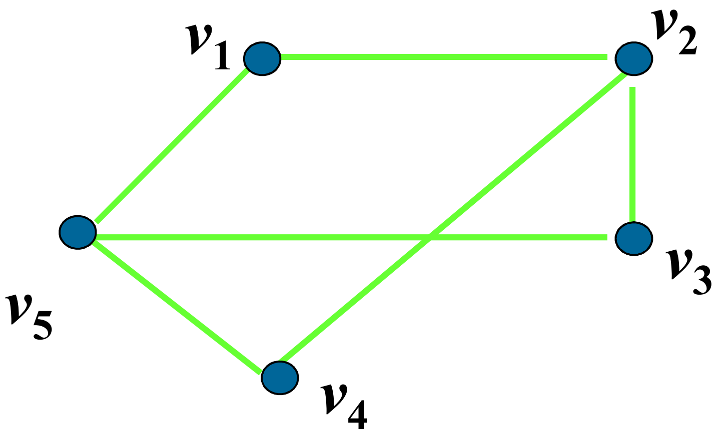
A simple graph G is *bipartite* if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2.



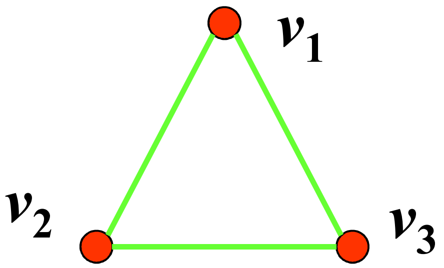
Note:

There are no edges which connect vertices in V_1 or in V_2.

For example,

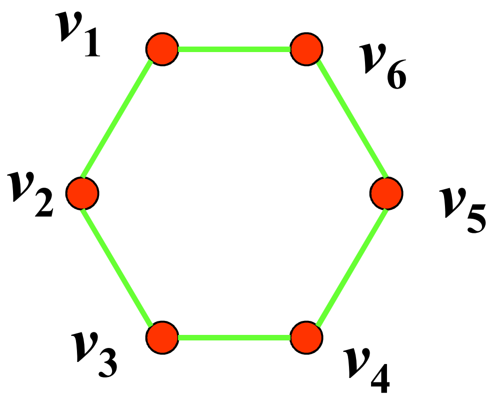


〖Example 1〗 Is C_3 bipartite?

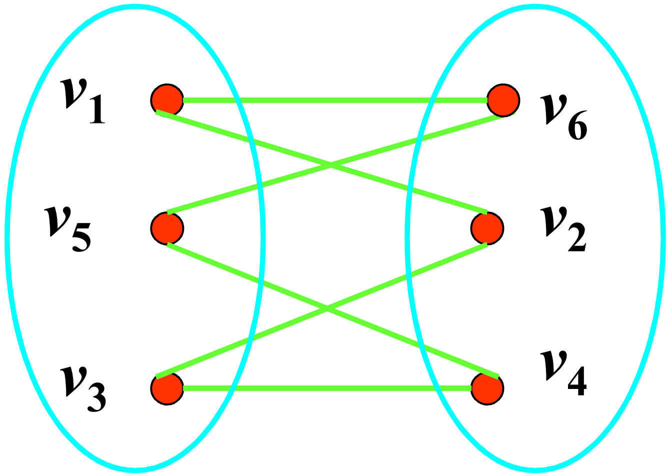


No.

〖Example 2〗 Is C_6 bipartite?



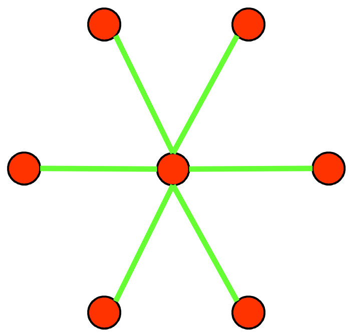
Yes. Because we can display C_6 like this:



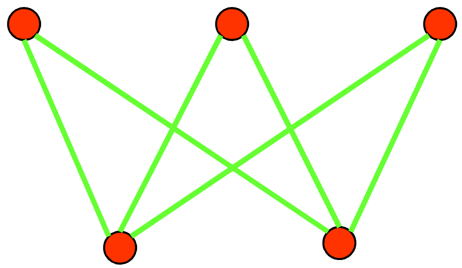
The *complete bipartite graph* is the simple graph that has its vertex set partitioned into two subsets V_1 and V_2 with m and n vertices, respectively, and *every vertex* in V_1 is connected to *every vertex* in V_2, denoted by $K_{m,n}$, where $m = |V_1|$ and $n = |V_2|$.

For example,

(1) A Star network is a $K_{1,n}$



(2) $K_{3,2}$



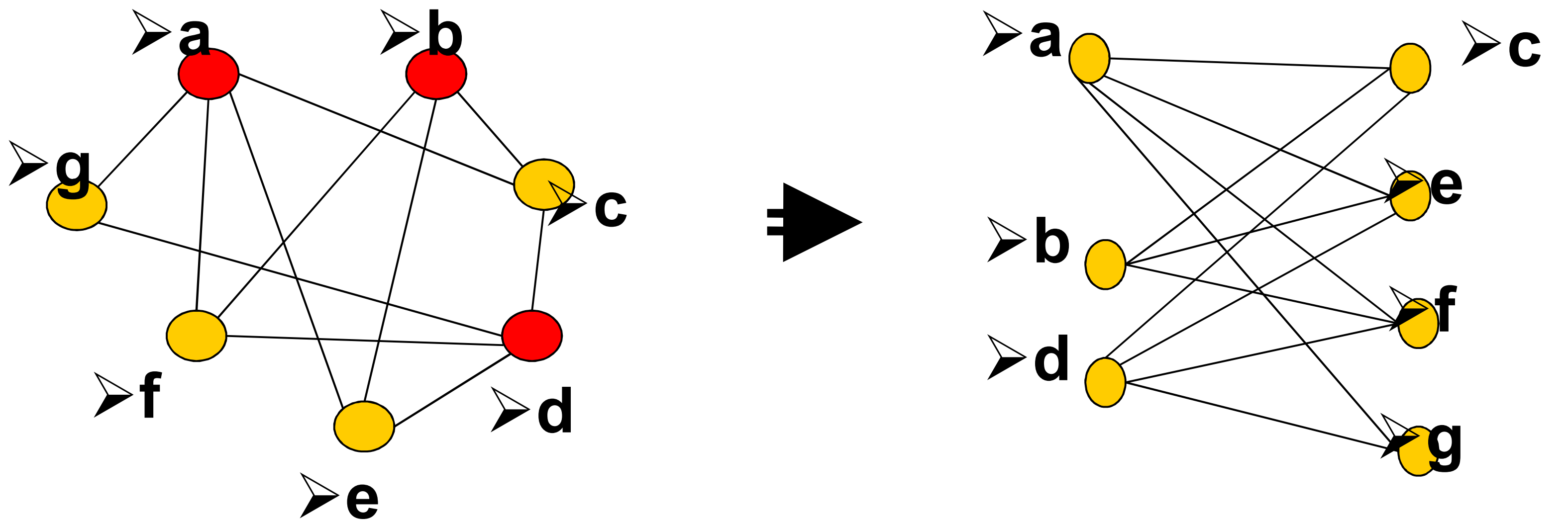
【Theorem 4】 A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Proof:

(1) Suppose that $G=(V, E)$ is a bipartite simple graph. Then $V=V_1 \cup V_2$, where V_1, V_2 are disjoint sets and every edge in E connects a vertex in V_1 and a vertex in V_2.

(2) Suppose that it is possible to assign colors to the vertices of the graph using just two colors so that no two adjacent vertices are assigned the same color.

【 Theorem 4】 A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.



Regular graph

A simply graph is called *regular* if every vertex of this graph has the same degree.

A *regular graph* is called *n*-regular if every vertex in this graph has degree *n*.

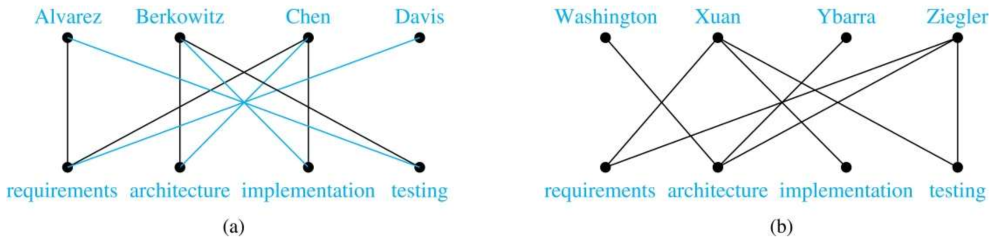
For example,

(1) k_n **is a (*n*-1)-regular.**

(2) For which values of *m* and *n* is $K_{m,n}$ regular?

Bipartite Graphs and Matchings

- Bipartite graphs are used to model applications that involve matching the elements of one set to elements in another, for example:
- *Job assignments* - vertices represent the jobs and the employees, edges link employees with those jobs they have been trained to do. A common goal is to match jobs to employees so that the most jobs are done.



- *Marriage* - vertices represent the men and the women and edges link a a man and a woman if they are an acceptable spouse. We may wish to find the largest number of possible marriages.

See the text for more about matchings in bipartite graphs.

Bipartite Graphs and Matchings

- A **matching** M in a simple graph $G = (V, E)$ is a subset of the set E of edges of the graph such that no two edges are incident with the same vertex.
- A vertex that is the endpoint of an edge of a matching M is said to be **matched** in M
- A **maximum matching** is a matching with the largest number of edges
- We say that a matching M in a bipartite graph $G = (V, E)$ with bipartition $(V1, V2)$ is a **complete matching from** $V1$ **to** $V2$ if every vertex in $V1$ is the endpoint of an edge in the matching

Theorem 5 (Hall 1935)

- **HALL'S MARRIAGE THEOREM** **The bipartite graph *G = (V ,E) with bipartition*** *(V1, V2) has a complete matching from V1 to V2 if and only if* $|N(A)| \geq |A|$ *for all* $A \subseteq V1$.
- ($\Rightarrow$) ***Proof: We first prove the only if part of the theorem***
  - $a \in V1$ has at least one neighbor, i.e., $|N(\{a\})| \geq |\{a\}| = 1$.
    - For any $a \in V1$, there is an edge *ab* in *M* for some $b \in V2$
  - Neighbors of *a* and *a'* in *M* can be distinct if $a \neq a'$.
    - We can choose the other ends of *a*, *a'* in *M* as the neighbors.
  - For any *A*, at least |*A*| distinct neighbors can be found.
    - That is, $|N(A)| \geq |A|$ for all $A \subseteq V1$.

Theorem 5 (Hall 1935) (2/6)

- ($\Leftarrow$) To prove the *if part of the theorem, the more difficult part, we need to show that if* $|N(A)| \geq |A|$ *for all* $A \subseteq V_1$*, then there is a complete matching M from* V_1 *to* V_2*. We will* use strong induction on $|V_1|$ *to prove this.*
- *Basis step: If* $|V_1| = 1$*, then* V_1 *contains a single vertex* v_0*. Because* $|N(\{v_0\})| \geq |\{v_0\}| = 1$*,*
- there is at least one edge connecting v_0 *and a vertex* $w_0 \in V_2$*. Any such edge forms a complete* matching from V_1 *to* V_2*.*

Theorem 5 (Hall 1935) (3/6)

- *Inductive step: We first state the inductive hypothesis.*
- *Inductive hypothesis: Let k be a positive integer. If $G = (V, E)$ is a bipartite graph with bipartition $(V1, V2)$, and $|V1| = j \le k$, then there is a complete matching M from $V1$ to $V2$ whenever* the condition that *$|N(A)| \ge |A|$ for all $A \subseteq V1$ is met.*
- Now suppose that *$H = (W, F)$ is a bipartite graph with bipartition $(W1, W2)$ and $|W1| = k + 1$. We will prove that the inductive holds using a proof by cases, using two case.*
- *Case (i)* applies when for all integers *j with $1 \le j \le k$, the vertices in every set of j elements from$W1$ are* adjacent to at least *$j + 1$ elements of $W2$. Case (ii) applies when for some j with $1 \le j \le k$ there* is a subset *$W1$* of *j vertices such that there are exactly j neighbors of these vertices in$W2$. Because* either Case *(i) or Case (ii) holds, we need only consider these cases to complete the inductive step.*

Theorem 5 (Hall 1935) (4/6)

- *Case (i): Suppose that for all integers j with $1 \le j \le k$, the vertices in every subset of j elements* from *$W1$ are adjacent to at least $j + 1$ elements of $W2$. Then, we select a vertex $v \in W1$ and an* element *$w \in N(\{v\})$, which must exist by our assumption that $|N(\{v\}| \ge |\{v\}| = 1$. We delete v and w and all edges incident to them from H. This produces a bipartite graph H* with bipartition *$(W1 - \{v\}, W2 - \{w\})$. Because $|W1 - \{v\}| = k$, the inductive hypothesis tells us* there is a complete matching from *$W1 - \{v\}$ to $W2 - \{w\}$. Adding the edge from v to w to this*complete matching produces a complete matching from *$W1$ to $W2$.*

Theorem 5 (Hall 1935) (5/6)

- *Case (ii): Suppose that for some j with 1 ≤ j ≤ k, there is a subset W’*1 of *j vertices such that* there are exactly *j neighbors of these vertices inW2. LetW’*2 be the set of these neighbors. Then, by the inductive hypothesis there is a complete matching from *W’*1 to *W’*2. Remove these 2*j* vertices from *W1 and W2 and all incident edges to produce a bipartite graph K with bipartition(W1 − W’*1*,W2 − W’*2*).*

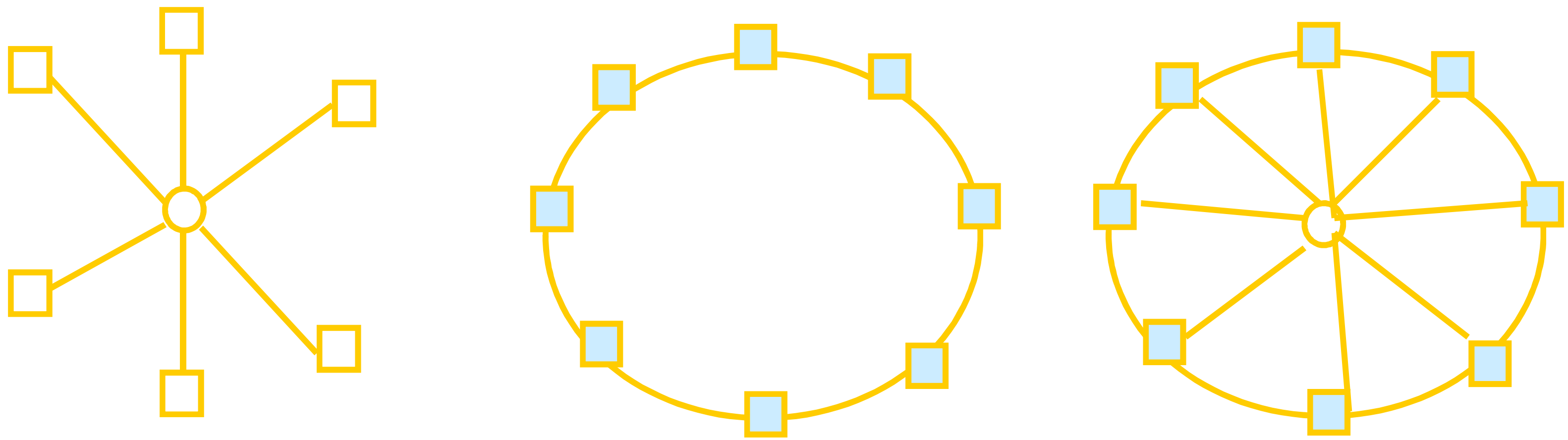
Theorem 5 (Hall 1935) (6/6)

- We will show that the graph *K satisfies the condition* $|N(A)| \geq |A|$ *for all subsets A of W1* − *W'*1. If not, there would be a subset of *t vertices of W1* − *W'*1 where 1 ≤ *t* ≤ *k* + *1* − *j* such that the vertices in this subset have fewer than *t vertices of W2* − *W'*2 as neighbors. Then, the set of *j* + *t vertices of W1 consisting of these t vertices together with the j vertices we* removed from *W1 has fewer than j* + *t neighbors in W2, contradicting the hypothesis that* $|N(A)| \geq |A|$ *for all* $A \subseteq W1$.

4. Some applications of special types of graphs

〖Example 3〗 Local Area Networks.

1. **Star topology**
2. **Ring topology**
3. **Hybrid topology**

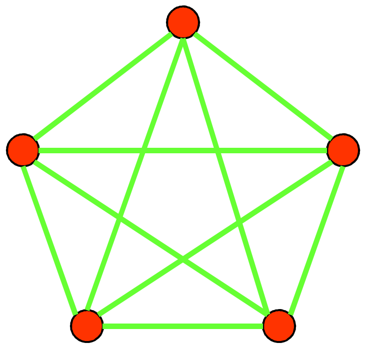


5. New Graphs from Old

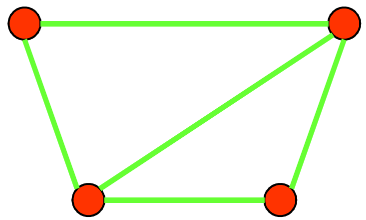
【Definition】 $G=(V, E)$, $H=(W, F)$

- H is a ***subgraph*** of G if $W \subseteq V, F \subseteq E$.
- subgraph H is a ***proper subgraph*** of G *if* $H \neq G$.
- H is a ***spanning(生成）subgraph*** of G if $W = V, F \subseteq E$.

For example,



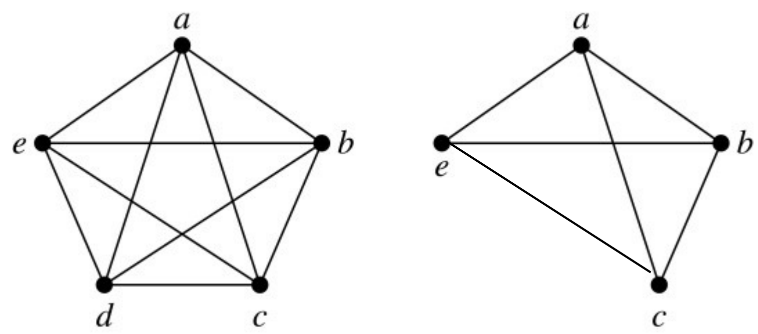
K_5



subgraph of K_5

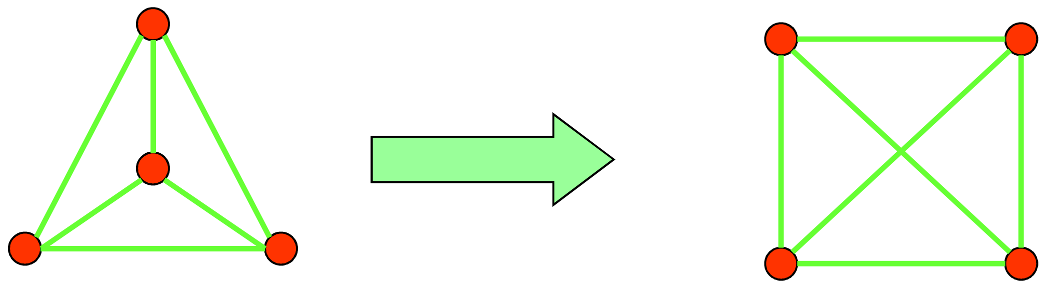
【Definition】 Let $G = (V, E)$ be a simple graph. The *subgraph induced*（诱导） by a subset W of the vertex set V is the graph (W,F), where the edge set F contains an edge in E if and only if both endpoints are in W.

Example: Here we show K_5 and the subgraph induced by $W = \{a,b,c,e\}$.



〖Example 4〗 How many subgraphs with at least one vertex does W_3 have?

Solution:



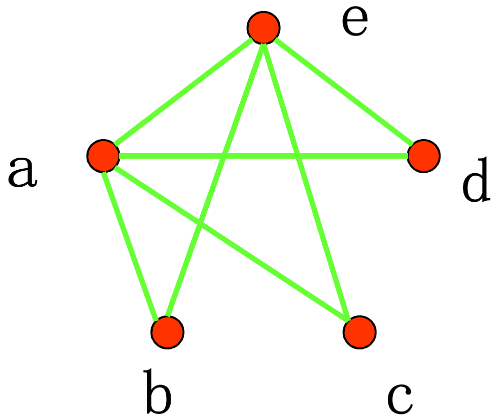
$$C(4,1)+C(4,2)\times 2+C(4,3)\times 2^{3}+C(4,4)\times 2^{6}$$

The union of G_1 and G_2

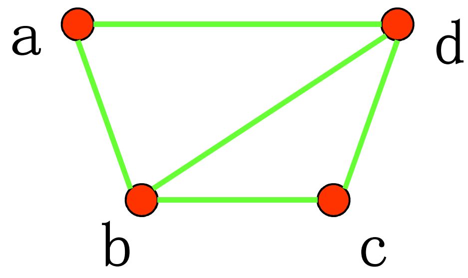
The *union* of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$.

Notation: $G_1 \cup G_2$

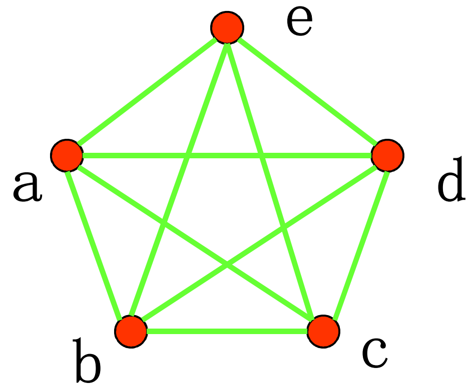
For example,



G_1



G_2



$G_1 \cup G_2 = K_5$

Homework:

第8版 Sec. 10.2 5, 24, 25, 44(b, f, h), 55, 62

注：允许**10.2**的作业有部分延迟到下次交。尽量本次一起交掉。