CHAPTER 10 Graphs

【Definition】 A graph is called *planar* if it can be drawn in the plane without any edges crossing .

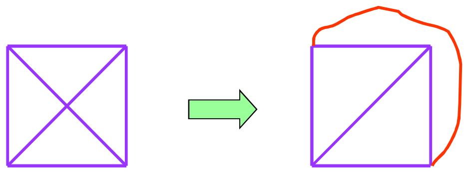
Such a drawing is called a *planar representation* of the graph.

Understanding planar graph is important:

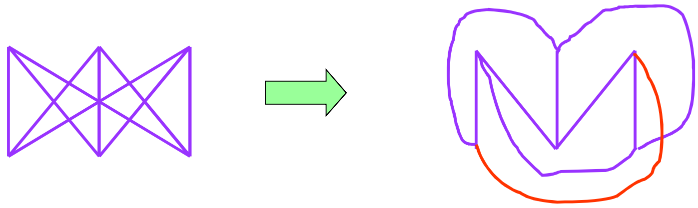
- **Any graph representation of maps is planar.**
- **Electronic circuits usually represented by planar graphs.**

〖Example 1〗 Are the following graphs planar?

(1)



(2)



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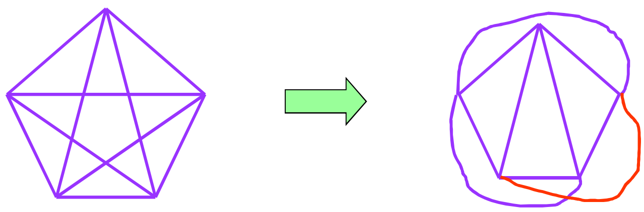


Note:

To prove that a graph is planar amounts to redrawing the edges in a way that no edges will cross. May need to move vertices around and the edges may have to be drawn in a very indirect fashion.

1. Euler’s Formula

Region

-- A region is a part of the plane completely disconnected off from other parts of the plane by the edges of the graph.

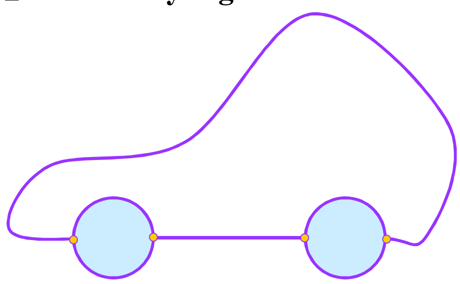
- Bounded region
- Unbounded region

Note:

There is one unbounded region in a planar graph.

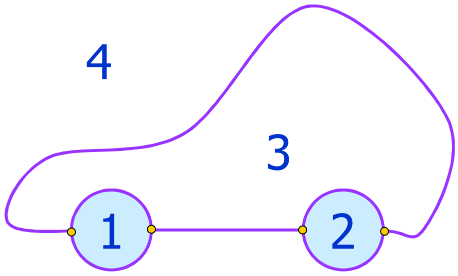
〖Example 2〗 How many regions does the following graph have?

(1)

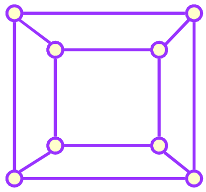


〖Example 2〗 How many regions does the following graph have?

(1)

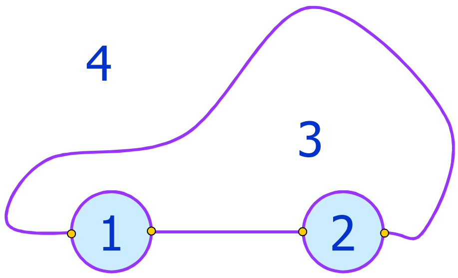


(2)

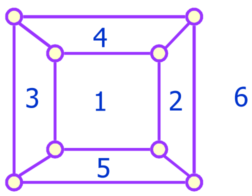


〖Example 2〗 How many regions does the following graph have?

(1)



(2)



【 Theorem 1】 Euler's formula
Let G be a *connected planar simple* graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r=e-v+2$.

Proof:

First, we specify a planar representation of G. We will prove the theorem by constructing a sequence of subgraphs G_1, G_2, $\cdots$, $G_e = G$, successively adding an edge at each stage.

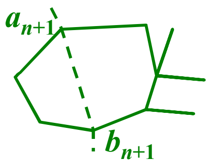
The constructing method: Arbitrarily pick one edge of G to obtain G_1. Obtain G_n from G_{n-1} by arbitrarily adding an edge that is incident with a vertex already in G_{n-1}, adding the other vertex incident with this edge if it is not already in G_{n-1}

Let r_n, e_n, and v_n represent the number of regions, edges, and vertices of the planar representation of G_n induced by the planar representation of G, respectively.

(1) The relationship $r_1= e_1 - v_1 +2$ is true for G_1, since e_1=1, v_1 =2,and r_1= 1.

(2) Now assume that $r_n= e_n - v_n +2$ .Let $\{a_{n+1}, b_{n+1}\}$be the edge that is added to G_n to obtain G_{n+1}.

◆ Both a_{n+1} and b_{n+1} are already in G_n.



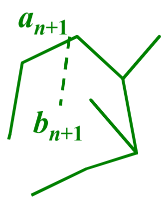
These two vertices must be on the boundary of a common region R, or else it would be impossible to add the edge $\{a_{n+1}, b_{n+1}\}$ to G_n without two edges crossing (and G_{n+1} is planar).

The addition of this new edge splits R into two regions.

Consequently, $r_{n+1}= r_n +1$, $e_{n+1}= e_n +1$,and $v_{n+1}= v_n$ .Thus, $r_{n+1}= e_{n+1}- v_{n+1} +2$.

- **One of the two vertices of the new edge is not already in G_n.**

Suppose that a_{n+1} is in G_n but that b_{n+1} is not.



Adding this new edge does not produce any new regions, since b_{n+1} must be in a region that has a_{n+1} on its boundary.

Consequently, $r_{n+1} = r_n$. Moreover, $e_{n+1} = e_n + 1$ and $v_{n+1} = v_n + 1$.

Hence, $r_{n+1} = e_{n+1} + 1 - v_{n+1} - 1 + 2$.

Note:

1) The Euler's formula is a *necessary condition*.

2) How about unconnected simple planar graph?

〖Example 3〗 Suppose that a connected planar simple graph has 20 vertices, each of degree 3. How many regions does this planar graph have?

Solution:

By handshaking theorem,

$$3v = 2e \Rightarrow e = 30$$

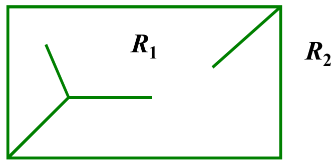
From Euler's formula, the number of regions is

$$r = e - v + 2 = 30 - 20 + 2 = 12$$

【Definition】 Suppose R is a region of a connected planar simple graph, the number of the edges on the boundary of R is called the *Degree of R* .

Notation: Deg(R)

For example,



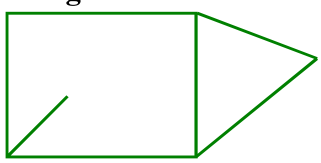
Deg(R_1) = 12
Deg(R_2) = 4

【 Corollary 1 】 If G is a connected planar simple graph with e edges and v vertices where $v \geq 3$, then $e \leq 3v-6$

Proof:

Suppose that a connected planar simple graph divides the plane into r regions, the degree of each region is at least 3.

Since $2e = \sum \deg(R_i) \geq 3r$,



it imply $r \leq (2/3)e$

Using Euler's formula $e-v+2=r$, we obtain

$e-v+2 \leq (2/3)e$, this shows that

$e \leq 3v-6$

Note:

- **The equality holds if and only if every region has exactly three edges.**
- **For unconnected planar simple graph,** $e \le 3v - 6$ **also holds.**

Since for a component, $e_i \le 3v_i - 6$

$$e = \sum e_i \le \sum (3v_i - 6) < 3\sum v_i - 6 = 3v - 6$$

【 Corollary 2 】 If *G* is a connected planar simple graph, then *G* has *a* vertex of degree not exceeding five.

Proof:

(1) *G* has one or two vertices

(2) G has at least three vertices

By Corollary 1 , we know that $e \le 3v-6$, so $2e \le 6v-12$

If the degree of every vertex were at least six, then

$2e \ge 6v$

【 Corollary 3 】 If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length 3,then $e \leq 2v-4$.

Generally, if every region of a connected planar simple graph has at least k edges, then

$$e \leq \frac{(v-2)k}{k-2}$$

r=e−v+2

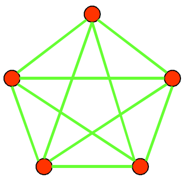
Kr<=2e

r<=2e/k

〖Example 4〗 Show that k_5, $k_{3,3}$ are nonplanar.

Proof:

(1)

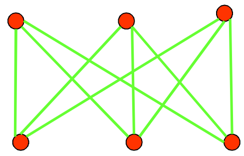


The graph k_5 has 5 vertices and 10 edges.

However, the inequality $e \leqq 3v-6$ is not satisfied for this graph since e=10 and $3v$-6=9.

Therefore, k_5 is not planar.

(2)



$k_{3,3}$ has 6 vertices and 9 edges.

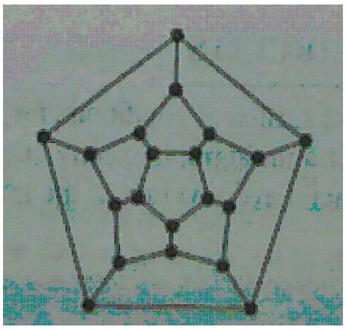
Since $k_{3,3}$ has no circuits of length 3 (this is easy to see since it is bipartite), Corollary 3 can be used .

Since e=9 and 2v-4=8, corollary 3 shows that $k_{3,3}$ is nonplanar.

〖Example 5〗 The construction of Dodecahedron .

Solution:

Since the degree of every vertex is 3 and the degree of every region is 5. Then

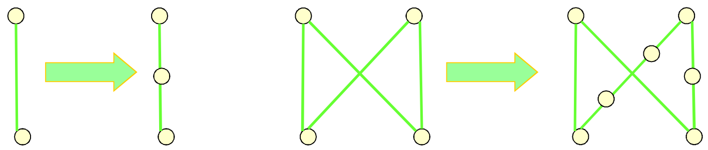


$$\begin{cases} 2e = 3v \\ 2e = 5r \\ r = e - v + 2 \end{cases}$$

It follows that v=20, e=30 and r=12.

2. KURATOWSKI'S THEOREM

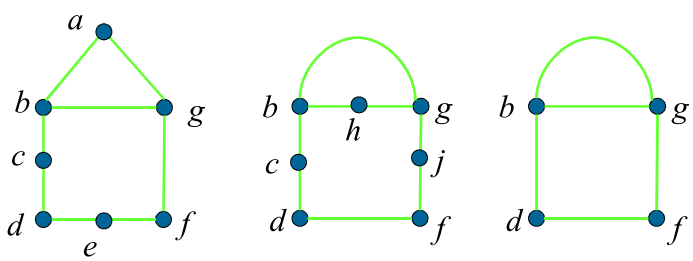
Elementary subdivision



Homeomorphic

-- **The graph $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivision.**

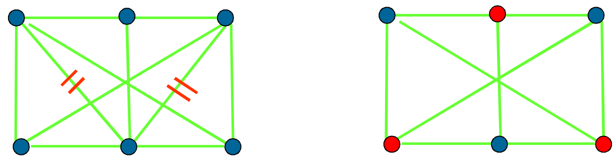
For example,



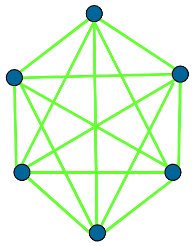
【Theorem 2】 A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

〖Example 6〗 Determine whether the following graphs are planar.

(1)



(2)



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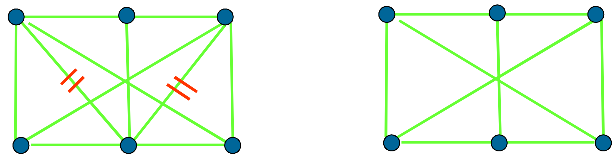
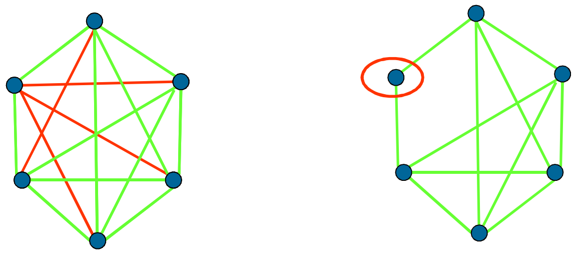
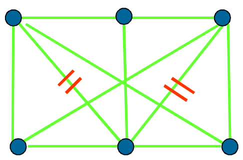


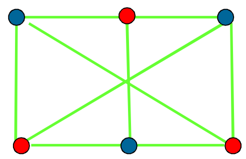
(2)



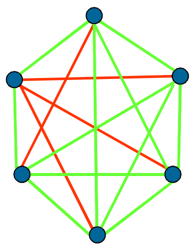
〖Example 6〗 Determine whether the following graphs are planar.

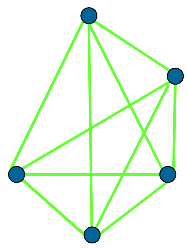
(1)





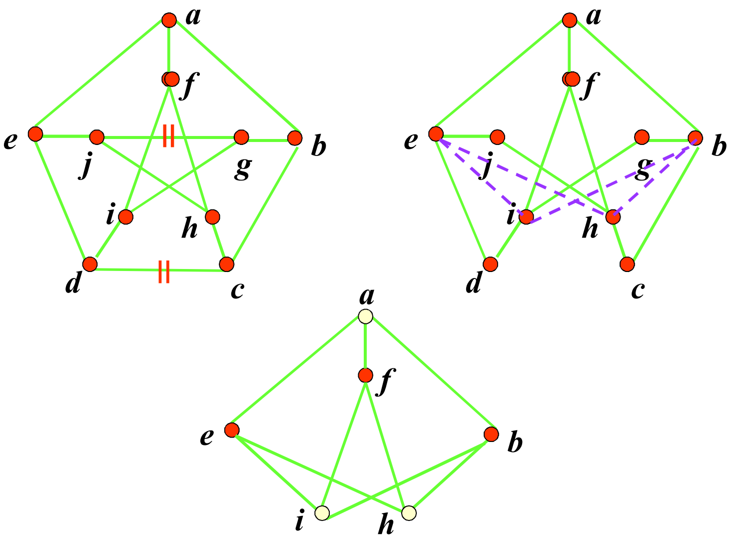
(2)





〖Example 6〗 Determine whether the following graphs are planar.

(3)



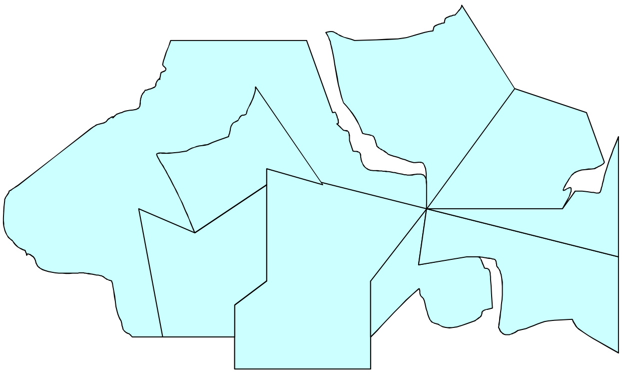
Homework:

Sec. 10.7 7, 20, 22, 23, 25

CHAPTER 10 Graphs

The problem related to the coloring of maps:

Determining the least number of colors that can be used to color a map so that adjacent regions never have the same color.



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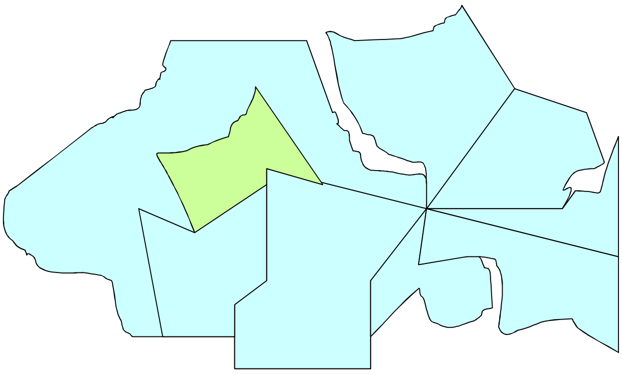

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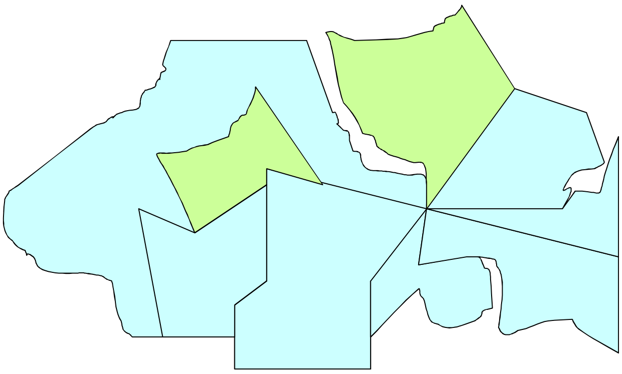

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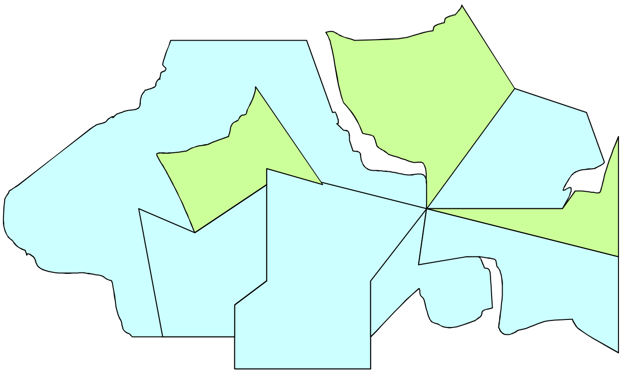

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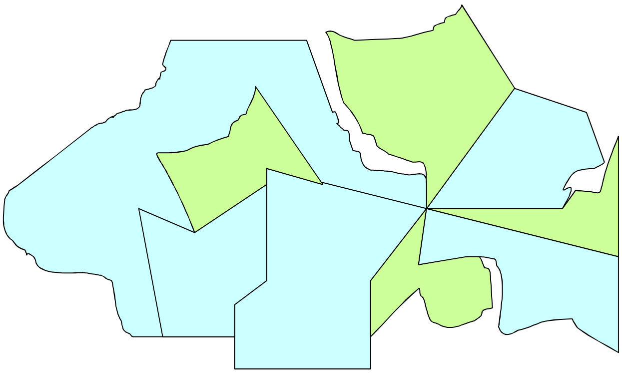

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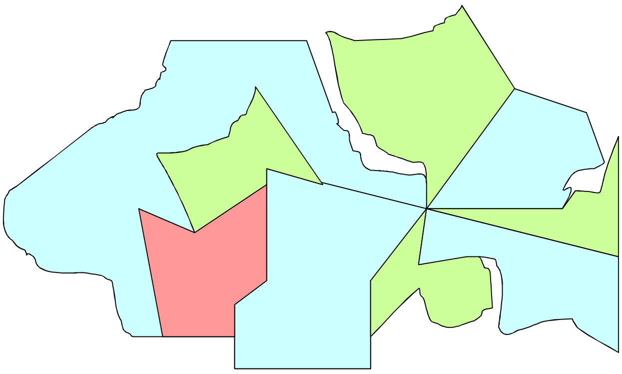

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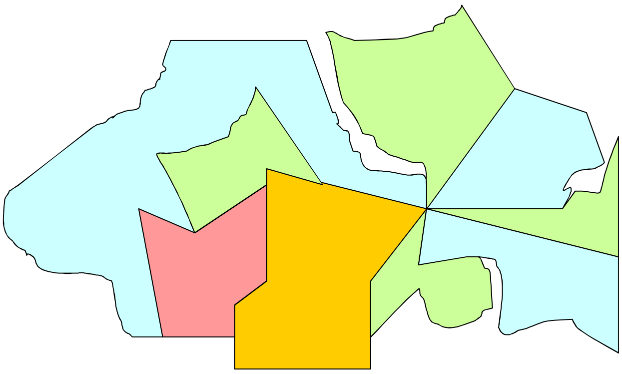
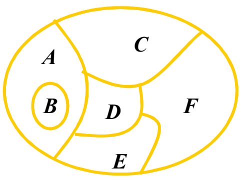


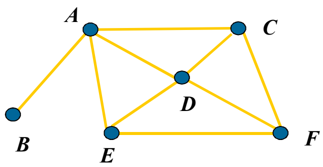
The problem of coloring a map, can be reduced to a graph-theoretic problem.

Each map in the plane can be represented by a graph, namely *the dual graph of the map*.

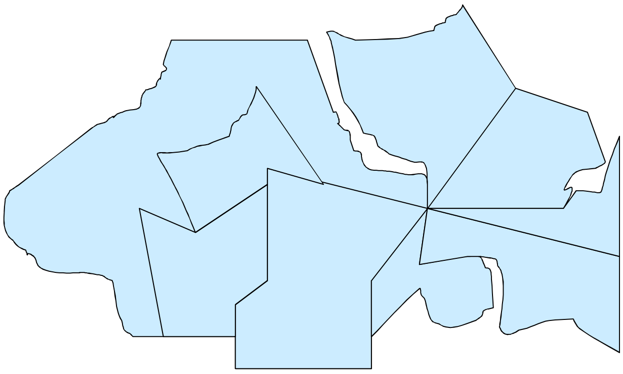
- **Each region of the map is represented by a vertex.**
- **An edge connect two vertices if the regions represented by these vertices have a common border.**
- **Two regions that touch at only one point are not considered adjacent.**

For example,

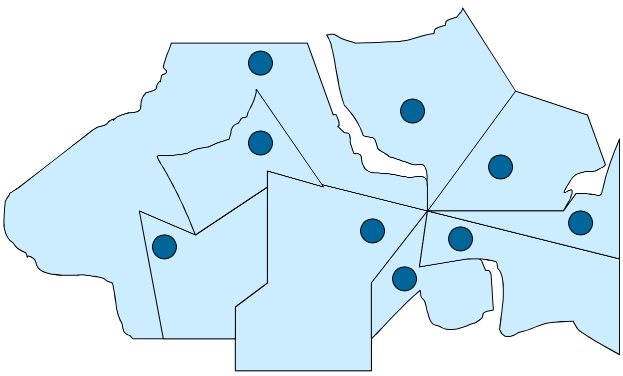




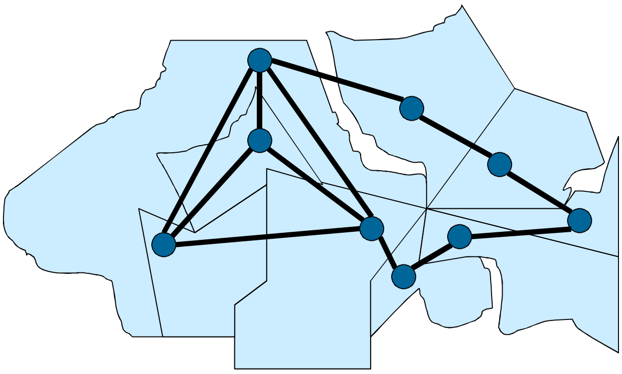
Coloring a map is equivalent to coloring the vertices of its dual graph so that no two adjacent vertices in this graph have the same color.



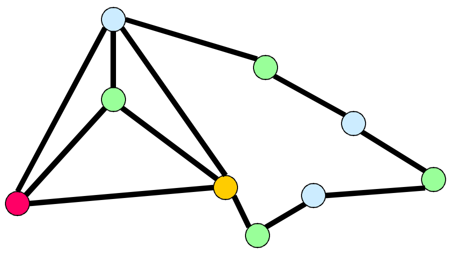
For each region introduce a vertex.



For each pair of regions with a positive-length common border introduce an edge.



Coloring regions of a map is equivalent to coloring vertices of its dual graph.



Terminologies:

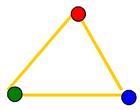
coloring

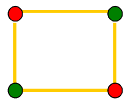
-- A *coloring* of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

The chromatic number of a graph

-- is the least number of colors needed for a coloring of this graph, denoted by $x(G)$

For example,





【 Theorem 1】 The Four Color Theorem

The chromatic number of a planar graph is no greater than four.

- Any planar map of regions can be depicted using 4 colors so that no two regions with a common border have the same color.
- The four color theorem was originally proposed as a conjecture in 1850s.
- Proof by Haken and Appel used exhaustive computer search in 1976.
- The four color theorem applies only to planar graphs. Nonplanar graphs can have arbitrarily large chromatic numbers.

1. The chromatic numbers of some simple graphs

Show that the chromatic numbers of a graph is *n*.

- **Show that the graph can be colored with *n* colors.**

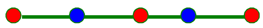
  Method: constructing such a coloring.
- **Show that the graph cannot be colored using fewer than *n* colors.**

The chromatic numbers of some simple graphs:

(1) The graph *G* contains only some isolated vertices.

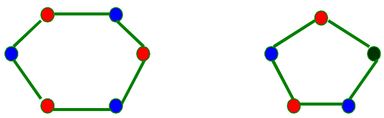
$$x(G) = 1$$

(2) The graph *G* is a path containing no circuit.



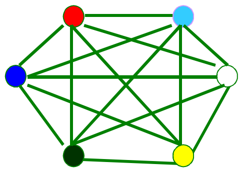
$$x(G) = 2$$

(3) C_n



$$\begin{cases} x(G) = 2 & \text{if } n \text{ is even} \\ x(G) = 3 & \text{if } n \text{ is odd} \end{cases}$$

(4) K_n



$x(G) = n$

(5) A simple graph with a chromatic number of 2 is bipartite.

A connected bipartite graph has a chromatic number of 2.

2. Applications of graph colorings

(1) Scheduling Exams

How can the exams at a university be scheduled so that no student has two exams at the same time?

Solution:

This scheduling problem can be solved using a graph model, with vertices representing courses and with an edge between two vertices if there is a common student in the courses they represent.

Each time slot for a final exam is represented by a different color.

A scheduling of the exams corresponds to a coloring of the associated graph.

For example, Suppose we want to schedule some final exams for CS courses with following code numbers:

1007, 3137, 3157, 3203, 3261, 4115, 4118, 4156

Suppose also that there are no common students in the following pairs of courses because of prerequisites:

1007-3137

1007-3157, 3137-3157

1007-3203

1007-3261, 3137-3261, 3203-3261

1007-4115, 3137-4115, 3203-4115, 3261-4115

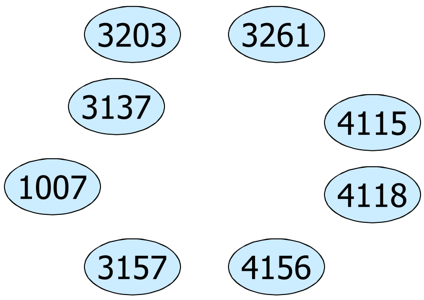
1007-4118, 3137-4118

1007-4156, 3137-4156, 3157-4156

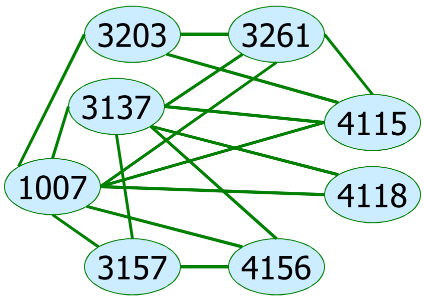
How many exam slots are necessary to schedule exams?

Turn this into a graph coloring problem.

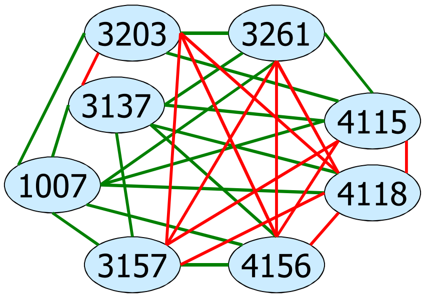
-- Vertices are courses, and edges connect courses which *cannot* be scheduled simultaneously because of possible common students in both courses



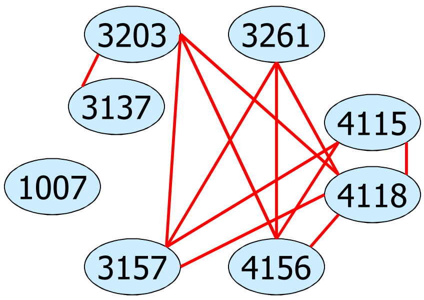
One way to do this is to put edges down where students mutually excluded…



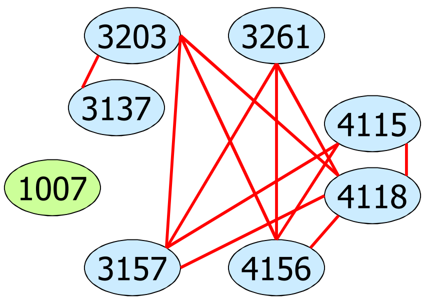
…and then compute the complementary graph:



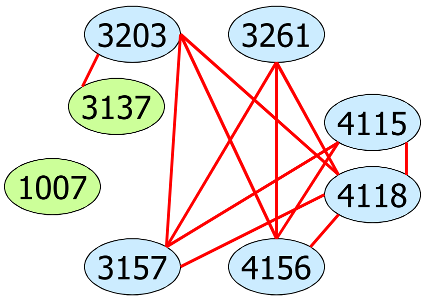
…and then compute the complementary graph:



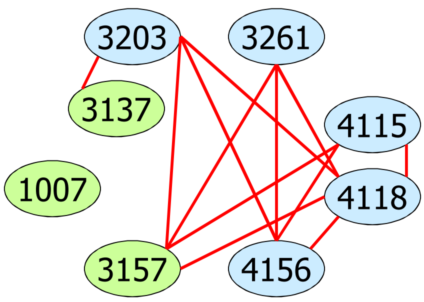
Coloring …



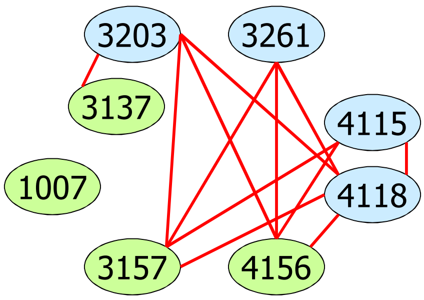
Coloring …



Coloring …

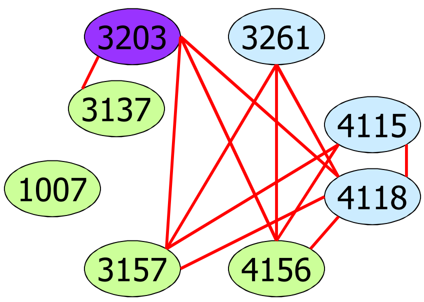


Coloring …

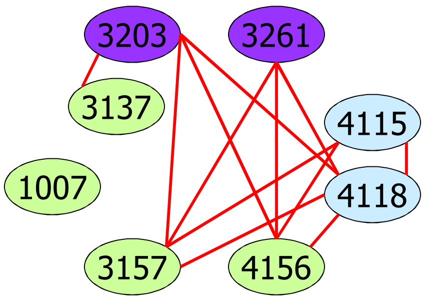


10.8 Graph Coloring

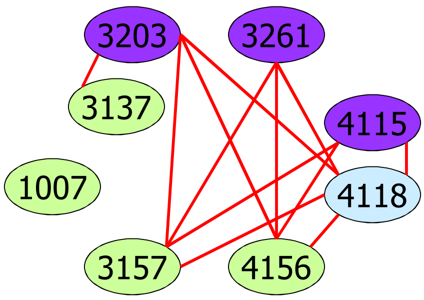
Coloring …



Coloring …



Coloring …



2. Applications of graph colorings

(2) Set up natural habitats of animals in a zoo

Solution:

Let the vertices of a graph represent the animals.

Draw an edge between two vertices if the animals they represent cannot be in the same habitat because of their eating habits.

A coloring of this graph gives an assignment of habitats.

Homework:

Sec. 10.8 3, 8, 9, 10, 17