# Generating Functions

Section 8.4

## Section Summary

- Generating Functions
- Useful Generating Functions
- Counting Problems and Generating Functions
- Solving Recurrence Relations Using Generating Functions
- Proving Identities Using Generating Functions

# Generating Functions

**Definition**: The *generating function for the sequence* $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k.$$

***A generating function is a clothesline on which we hang up a sequence of numbers for display.***

— **Herbert Wilf, *Generating functionology* (1994)**

- *Questions about the convergence of these series are ignored.*
- *The fact that a function has a unique power series around x = 0 will also be important*

# Generating Functions

**Examples**:

- The sequence $\{a_k\}$ with $a_k = 3$ has the generating function $\sum_{k=0}^{\infty} 3x^k = \frac{3}{1-x}$ $for |x| < 1$
- The sequence $\{a_k\}$ with $a_k = k + 1$ has the generating function has the generating function $\sum_{k=0}^{\infty} (k+1)x^k.$
- The sequence $\{a_k\}$ with $a_k = 2^k$ has the generating function has the generating function $\sum_{k=0}^{\infty} 2^k x^k.$

## Generating Functions for Finite Sequences

- Generating functions for finite sequences of real numbers can be defined by extending a finite sequence $a_0, a_1, \ldots, a_n$ into an infinite sequence by setting $a_{n+1} = 0$, $a_{n+2} = 0$, and so on.
- The generating function $G(x)$ of this infinite sequence $\{a_n\}$ is a polynomial of degree n because no terms of the form $a_j x^j$ with $j > n$ occur, that is,

$$G(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

## Generating Functions for Finite Sequences (continued)

**Example**: What is the generating function for the sequence 1,1,1,1,1,1?

**Solution**: The generating function of 1,1,1,1,1,1 is

$$1 + x + x^2 + x^3 + x^4 + x^5.$$

By Theorem 1 of Section 2.4, we have

$$(x^6 - 1)/(x - 1) = 1 + x + x^2 + x^3 + x^4 + x^5$$

when $x \neq 1$.

Consequently $G(x) = (x^6 - 1)/(x - 1)$ is the generating function of the sequence.

# Generating Functions for Finite Sequences (continued)

**Example**:

**Let** $a_k = C(m,k), k = 0,1,2,\cdots,m$**. The generating function for this sequence is**

$$G(x) = C(m,0) + C(m,1)x + C(m,2)x^2 + \cdots + C(m,m)x^m = (1+x)^m$$

# Counting Problems and Generating Functions

***Generating functions can be used to solve a wide variety of counting problems, such as***

- ✓ **Count the number of combinations from a set when repetition is allowed and additional constraints exist.**
- ✓ **Count the number of permutations**

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# Counting Problems and Generating Functions

**Example**: Find the number of solutions of

$e_1 + e_2 + e_3 = 17,$

where $e_1$, $e_2$, and $e_3$ are nonnegative integers with $2 \le e_1 \le 5$, $3 \le e_2 \le 6$, and $4 \le e_3 \le 7$.

**Solution**: The number of solutions is the coefficient of $x^{17}$ in the expansion of

$(x^2 + x^3 + x^4 + x^5)\ (x^3 + x^4 + x^5 + x^6)\ (x^4 + x^5 + x^6 + x^7).$

This follows because a term equal to is obtained in the product by picking a term in the first sum $x^{e_1}$, a term in the second sum $x^{e_2}$, and a term in the third sum $x^{e_3}$, where $e_1 + e_2 + e_3 = 17$.

There are three solutions since the coefficient of $x^{17}$ in the product is 3.

〖**Example** 〗**Determine the number of ways to insert tokens worth \$1,\$2 and \$5 into a vending machine to pay for an item that costs *r* dollars in both the cases when the order in which the tokens are inserted does not matter and when the order does matter.**

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*Solution:*

**(1) The order in which the tokens are inserted does not matter**

$$G(x) = (1 + x + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots)(1 + x^5 + x^{10} + x^{15} + \cdots)$$

**The coefficient of $x^r$ in the expansion of $G(x)$ is the solution of this problem.**

**(2) The order in which the tokens are inserted does matter**

- **The number of ways to insert exactly $n$ tokens to produce a total of $r$\$ is the coefficient of $x^r$ in** $(x + x^2 + x^5)^n$
- **Since any number of tokens may be inserted, the number of ways to produce $r$\$ using \$1,\$2 and \$5 tokens, is the coefficient of $x^r$ in**

$$1 + (x + x^2 + x^5) + (x + x^2 + x^5)^2 + \ldots = \frac{1}{1 - (x + x^2 + x^5)}$$

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# Counting Problems and Generating Functions (*continued*)

**Example**: Use generating functions to find the number of $k$-combinations of a set with $n$ elements, i.e., $C(n,k)$.

**Solution**: Each of the n elements in the set contributes the term $(1 + x)$ to the generating function

$$f(x) = \sum_{k=0}^{n} a^k x^k.$$

Hence $f(x) = (1 + x)^n$ where $f(x)$ is the generating function for $\{a^k\}$, where $a^k$ represents the number of $k$-combinations of a set with $n$ elements.

By the binomial theorem, we have

$$f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k,$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Hence,

$$C(n,k) = \frac{n!}{k!(n-k)!}.$$

Useful Facts About Power Series

【Theorem 1】Let $f(x)=\sum_{k=0}^{\infty}$

(1) $f(x)+g(x)=\sum_{k=0}^{\infty}(a_k+b_k)x^k$

(2) $\alpha\cdot f(x)=\sum_{k=0}^{\infty}\alpha\cdot a_k x^k$ $\quad\alpha$

(3) $x\cdot f'(x)=\sum_{k=0}^{\infty}k\cdot a_k x^k$

(4) $f(\alpha x)=\sum_{k=0}^{\infty}\alpha^k\cdot a_k x^k$

(5) $f(x)g(x)=\sum_{k=0}^{\infty}(\sum_{j=0}^{k}a_j b_{k-j})x^k$

*Proof:*

$$\sum_{k=0}^{\infty}ka_k x^k=\sum_{k=0}^{\infty}a_k\cdot x\cdot kx^{k-1}$$

$$=x\sum_{k=0}^{\infty}a_k(x^k)'$$

$$=x(\sum_{k=0}^{\infty}a_k x^k)'$$

$$=xf'(x)$$



Useful Facts About Power Series

**Let** $f(x)=\sum_{k=0}^{\infty}a_k$

(1) $f(x)+g(x)=$

(2) $\alpha\cdot f(x)=\sum$

(3) $x\cdot f'(x)=$

(4) $f(\alpha x)=\sum_{k=0}^{\infty}$

(5) $f(x)g(x)=$

*Proof:*

$$\sum_{k=0}^{\infty}(\sum_{j=0}^{k}a_j b_{k-j})x^k$$

$$=a_0b_0+(a_0b_1+a_1b_0)x+(a_0b_2+a_1b_1+a_2b_0)x^2$$

$$+\ldots+(\sum_{j=1}^{k}a_j b_{k-j})x^k+\ldots$$

$$=(a_0+a_1x+a_2x^2+\ldots)(b_0+b_1x+b_2x^2+\ldots)$$

$$=f(x)\cdot g(x)$$

*Using the above properties, the generating functions of some sequences can be obtained easily.*

**〖Example 〗 What is the generating function for the sequence 0,1,2,3,4,5,…?**

*Solution:*

$$b_k = k$$

$$G(x) = \sum_{k=0}^{\infty} kx^k$$

$$= x\left(\frac{1}{1-x}\right)'$$

$$= \frac{x}{(1-x)^2}$$



**〖Example 〗 Suppose that the generating function of the sequence: $a_0, a_1, a_2, \cdots, a_n, \cdots$ is $G(x)$. What is the generating function for the sequence $b_k = \sum_{i=0}^{k} a_i$ ?**

*Solution:*

$a_k \leftrightarrow G(x), \quad b_k$

$$c_k = 1$$

$$b_k = \sum_{i=0}^{k} a_i$$

$$= \sum_{i=0}^{k} a_i \times c_{k-i}$$

$$F(x) = G(x) \cdot \frac{1}{1-x}$$

For example:

$1,1,1,\cdots \leftrightarrow \frac{1}{1-x}$

$1,2,3,4,\cdots,k+1,\cdots \leftrightarrow \frac{1}{(1-x)^2}$

〖Example 〗 Let $f(x) = \frac{1}{1-4x^2}$. Find the coefficient $a_0, a_1, a_2, \cdots, a_n, \cdots$ in the expansion $f(x) = \sum_{k=0}^{\infty} a_k x^k$ .

*Solution:*

$$f(x) = \frac{1}{1-4x^2} = \frac{1}{(1-2x)(1+2x)} = \frac{1}{2}\left(\frac{1}{1-2x} + \frac{1}{1+2x}\right)$$

$$2^k \qquad (-2)^k$$

$$a_k = \frac{1}{2}(2^k + (-2)^k) = \begin{cases} 2^k & k \text{ is even} \\ 0 & k \text{ is odd} \end{cases}$$

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❋ The extended binomial coefficient

**Recall:**

$$\binom{m}{k} = C(m,k) = \frac{m!}{k!(m-k)!}$$

**Where $m$, $k$ are nonnegative integers, $k \le m$**

**【Definition 2】 Let $u$ be a real number and $k$ a nonnegative integer. Then the** ***extended binomial coefficient*** **is defined by**

$$\binom{u}{k} = \begin{cases} u(u-1)\cdots(u-k+1)/k! & \text{if } \ k > 0 \\ 1 & \text{if } \ k = 0 \end{cases}$$

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〖**Example** 〗 (1) $\binom{1/2}{3} = ?$ (2) $\binom{-n}{k} = ?$

*Solution:*

$$(1)\binom{1/2}{3} = \frac{(1/2)(1/2-1)(1/2-2)}{3!} = 1/16$$

$$(2)\binom{-n}{k} = \frac{(-n)(-n-1)...(-n-k+1)}{k!}$$

$$= \frac{(-1)^k n(n+1)...(n+k-1)}{k!}$$

$$= (-1)^k C(n+k-1,k)$$



## ❈ The extended Binomial Theorem

【Theorem 2】Let $x$ be a real number with $|x| < 1$ and let $u$ be a real number. Then

$$(1+x)^u = \sum_{k=0}^{\infty}\binom{u}{k}x^k$$

〖Example 〗 **Find the generating functions for**

$$(1+x)^{-n} \; and \; (1-x)^{-n}$$

**where *n* is a positive integer, using the extended Binomial Theorem.**

*Solution:*

**By the extended Binomial Theorem , it follows that**

$$(1+x)^{-n} = \sum_{k=0}^{\infty}\binom{-n}{k}x^k = \sum_{k=0}^{\infty}(-1)^k C(n+k-1,k)x^k$$

$$(1-x)^{-n} = \sum_{k=0}^{\infty}\binom{-n}{k}(-x)^k = \sum_{k=0}^{\infty}(-1)^k C(n+k-1,k)(-1)^k x^k = \sum_{k=0}^{\infty}C(n+k-1,k)x^k$$

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# Useful Generating Functions

**TABLE 1 Useful Generating Functions.**

| $G(x)$ | $a_k$ |
|---|---|
| $(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k = 1 + C(n,1)x + C(n,2)x^2 + \cdots + x^n$ | $C(n,k)$ |
| $(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k = 1 + C(n,1)ax + C(n,2)a^2x^2 + \cdots + a^n x^n$ | $C(n,k)a^k$ |
| $(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk} = 1 + C(n,1)x^r + C(n,2)x^{2r} + \cdots + x^{rn}$ | $C(n,k/r)$ if $r \mid k$; 0 otherwise |
| $\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$ | 1 if $k \le n$; 0 otherwise |
| $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$ | 1 |
| $\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$ | $a^k$ |
| $\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$ | 1 if $r \mid k$; 0 otherwise |
| $\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$ | $k+1$ |
| $\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k = 1 + C(n,1)x + C(n+1,2)x^2 + \cdots$ | $C(n+k-1,k) = C(n+k-1,n-1)$ |
| $\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k = 1 - C(n,1)x + C(n+1,2)x^2 - \cdots$ | $(-1)^k C(n+k-1,k) = (-1)^k C(n+k-1,n-1)$ |
| $\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k = 1 + C(n,1)ax + C(n+1,2)a^2x^2 + \cdots$ | $C(n+k-1,k)a^k = C(n+k-1,n-1)a^k$ |
| $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ | $1/k!$ |
| $\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ | $(-1)^{k+1}/k$ |

*Note:* The series for the last two generating functions can be found in most calculus books when power series are discussed.

| Sequence | Generating function |
|---|---|
| (1) $C(n,k)$ | $\sum_{k=0}^{\infty} C(n,k)x^k = (1+x)^n$ |
| (2) $C(n,k)a^k$ | $(1+ax)^n$ |
| (3) 1,1,…,1 | $1+x+x^2+\cdots+x^n = \frac{1-x^{n+1}}{1-x}$ |
| (4) $1,1,1,\cdots$ | $\frac{1}{1-x}$ |
| (5) $a^k$ | $\frac{1}{1-ax}$ |
| (6) $k+1$ | $\frac{1}{(1-x)^2}$ |

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| Sequence | Generating function |
|---|---|
| (7) $C(n+k-1,k)$ | $(1-x)^{-n}$ |
| (8) $(-1)^k C(n+k-1,k)$ | $(1+x)^{-n}$ |
| (9) $C(n+k-1,k)a^k$ | $(1-ax)^{-n}$ |
| (10) $\frac{1}{k!}$ | $e^x$ |
| (11) $\frac{(-1)^{k+1}}{k}$ | $\ln(1+\mathrm{x})$ |

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# Counting Problems and Generating Functions (*continued*)

**Example**: Use generating functions to find the number of $k$-combinations of a set with $n$ elements, i.e., $C(n,k)$.

**Solution**: Each of the n elements in the set contributes the term $(1 + x)$ to the generating function

$$f(x) = \sum_{k=0}^{n} a^k x^k.$$

Hence $f(x) = (1 + x)^n$ where $f(x)$ is the generating function for $\{a^k\}$, where $a^k$ represents the number of $k$-combinations of a set with $n$ elements.

By the binomial theorem, we have

$$f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k,$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Hence,

$$C(n,k) = \frac{n!}{k!(n-k)!}.$$

〖Example 〗Use generating functions to find the number of $r$-combinations from a

*Solution:*

**Since there are $n$ e**

**times, one times a**

$$G(x) = (1+$$

$$(1-x)^{-n} = (1+(-x))^{-n} = \sum_{r=0}^{\infty} \binom{-n}{r} (-x)^r.$$

$$\binom{-n}{r}(-1)^r = (-1)^r C(n+r-1,r) \cdot (-1)^r$$
$$= C(n+r-1,r).$$

**the number of $r$-co**

**repetitions allowed, is the coefficient $a_r$ of $x^r$ in the expansion of $G(x)$ . Since**

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k$$

**Then the coefficient $a_r$ equals $C\,(n+r-1,r)$**

## 5. Using Generating Functions to Solve Recurrence Relations

**The Methods of Solving Recurrence Relations**

- ❑ **Iterative approach**
- ❑ **Use a systematic way to solve an important class of recurrence relations**
- ❑ **Generating functions**

*Method:*

**(1) Use the recurrence relation to find the generating function of this sequence;**

**(2)** $G(x) \leftrightarrow a_n$



**〖Example 〗 Use generating functions to solve the recurrence relation** $a_n = 2a_{n-1} + 3a_{n-2} + 4^n + 6$ **with initial conditions** $a_0 = 20, a_1 = 60$ **.**

*Solution:*

$$a_n = 2a_{n-1} + 3a_{n-2} + 4^n + 6 \qquad \times x^n$$

$$a_n x^n = 2a_{n-1}x^n + 3a_{n-2}x^n + 4^n x^n + 6x^n$$

$$\sum_{n=2}^{\infty} a_n x^n = 2\sum_{n=2}^{\infty} a_{n-1}x^n + 3\sum_{n=2}^{\infty} a_{n-2}x^n + \sum_{n=2}^{\infty} 4^n x^n + 6\sum_{n=2}^{\infty} x^n$$

$$G(x) - a_0 - a_1 x \qquad 2x\sum_{n=1}^{\infty} a_n x^n \qquad 3x^2\sum_{n=0}^{\infty} a_n x^n \qquad \frac{1}{1-4x} - 1 - 4x \qquad 6\left(\frac{1}{1-x} - 1 - x\right)$$

$$2x(G(x) - a_0) \qquad 3x^2 G(x)$$

$$(1-2x-3x^2)G(x)=\frac{20-80x+2x^2+40x^3}{(1-4x)(1-x)}$$

$$G(x)=\frac{20-80x+2x^2+40x^3}{(1-4x)(1-x)(1+x)(1-3x)}$$

$$=\frac{16/5}{1-4x}+\frac{-3/2}{1-x}+\frac{31/20}{1+x}+\frac{67/4}{1-3x}$$

$$\frac{16}{5}\times 4^n \quad -\frac{2}{3}\times 1^n \quad \frac{31}{20}\times(-1)^n \quad \frac{67}{4}\times 3^n$$

$$a_n=\frac{16}{5}\times 4^n-\frac{2}{3}+\frac{31}{20}\times(-1)^n+\frac{67}{4}\times 3^n$$

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6. Proving Identities via Generating Functions

**The method of proving combinatorial identities:**

- **Use combinatorial proofs**
- **Use generating functions**

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〖Example 〗 **Use generating functions to prove Pascal's identity** $C(n,r)=C(n-1,r)+C(n-1,r-1)$ **when** $n$ **and** $r$ **are positive integers with** $r<n$ **.**



*Proof:* $G(x)=(1+x)^n=\sum_{r=0}^{n}C(n,r)x^r$

$$(1+x)^n=(1+x)(1+x)^{n-1}=(1+x)^{n-1}+x(1+x)^{n-1}$$

$$\sum_{r=0}^{n}C(n,r)x^r=\sum_{r=0}^{n-1}C(n-1,r)x^r+\sum_{r=0}^{n-1}C(n-1,r)x^{r+1}$$

$$=\sum_{r=0}^{n-1}C(n-1,r)x^r+\sum_{r=1}^{n}C(n-1,r-1)x^r$$

$$1+\sum_{r=1}^{n-1}C(n,r)x^r+x^n$$

$$=1+\sum_{r=1}^{n-1}C(n-1,r)x^r+\sum_{r=1}^{n-1}C(n-1,r-1)x^r+x^n$$

$$\sum_{r=1}^{n-1}\underline{C(n,r)}x^r=\sum_{r=1}^{n-1}[\underline{C(n-1,r)+C(n-1,r-1)}]x^r$$

## Homework

第8版 Sec. 8.4 6(d,f), 10 (c, d, e), 16, 24(a), 32, 36, 45

# Inclusion-Exclusion

Section 8.5

## Section Summary

- The Principle of Inclusion-Exclusion
- Examples

## Principle of Inclusion-Exclusion

- In Section 2.2, we developed the following formula for the number of elements in the union of two finite sets:

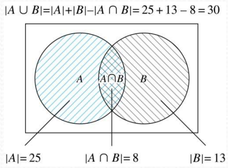
$$|A \cup B| = |A| + |B| - |A \cap B|$$

- We will generalize this formula to finite sets of any size.

# Two Finite Sets

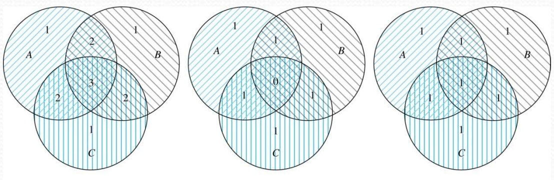
**Example**: In a discrete mathematics class every student is a major in computer science or mathematics or both. The number of students having computer science as a major (possibly along with mathematics) is 25; the number of students having mathematics as a major (possibly along with computer science) is 13; and the number of students majoring in both computer science and mathematics is 8. How many students are in the class?

**Solution**: $|A \cup B| = |A| + |B| - |A \cap B|$
$= 25 + 13 - 8 = 30$




# Three Finite Sets

$$|A \cup B \cup C| =$$
$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



(a) Count of elements by $|A|+|B|+|C|$

(b) Count of elements by $|A|+|B|+|C|-|A \cap B|-|A \cap C|-|B \cap C|$

(c) Count of elements by $|A|+|B|+|C|-|A \cap B|-|A \cap C|-|B \cap C|+|A \cap B \cap C|$

# Three Finite Sets Continued

**Example**: A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken a course in at least one of Spanish French and Russian, how many students have taken a course in all 3 languages.

**Solution**: Let $S$ be the set of students who have taken a course in Spanish, $F$ the set of students who have taken a course in French, and $R$ the set of students who have taken a course in Russian. Then, we have $|S| = 1232$, $|F| = 879$, $|R| = 114$, $|S \cap F| = 103$, $|S \cap R| = 23$, $|F \cap R| = 14$, and $|S \cup F \cup R| = 2092$.

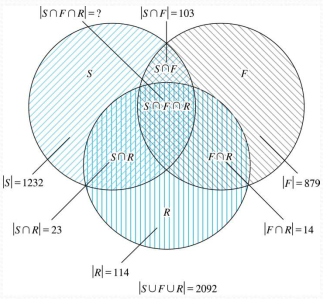
Using the equation

$$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|,$$

we obtain $2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|$.

Solving for $|S \cap F \cap R|$ yields 7.

# Illustration of Three Finite Set Example



# The Principle of Inclusion-Exclusion

**Theorem 1. The Principle of Inclusion-Exclusion**: Let $A_1, A_2, \ldots, A_n$ be finite sets. Then:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| =$$

$$\sum_{1 \le i \le n} |A_i| - \sum_{1 \le i \le j \le n} |A_i \cap A_j| +$$

$$\sum_{1 \le i \le j \le k \le n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

# The Principle of Inclusion-Exclusion (*continued*)

**Proof:** An element in the union is counted exactly once in the right-hand side of the equation. Consider an element $a$ that is a member of $r$ of the sets $A_1, \ldots, A_n$ where $1 \le r \le n$.

- It is counted $C(r,1)$ times by $\sum |A_i|$
- It is counted $C(r,2)$ times by $\sum |A_i \cap A_j|$
- In general, it is counted $C(r,m)$ times by the summation of $m$ of the sets $A_i$.

# The Principle of Inclusion-Exclusion (cont)

- Thus the element is counted exactly
  $C(r,1) - C(r,2) + C(r,3) - \cdots + (-1)^{r+1} C(r,r)$
  times by the right hand side of the equation.
- By Corollary 2 of Section 6.4, we have
  $C(r,0) - C(r,1) + C(r,2) - \cdots + (-1)^{r} C(r,r) = 0.$
- Hence,
  $1 = C(r,0) = C(r,1) - C(r,2) + \cdots + (-1)^{r+1} C(r,r).$

**〖Example 〗How many positive integers not exceeding 1000 that are not divisible by 5, 6 or 8?**

*Solution:*

***U*: the set of positive integers not exceeding 1000**

***A*: the set of positive integers not exceeding 1000 that are divisible by 5,**

***B*: the set of positive integers not exceeding 1000 that are divisible by 6,**

***C*: the set of positive integers not exceeding 1000 that are divisible by 8.**

$$\left|\overline{A}\cap\overline{B}\cap\overline{C}\right| = |U| - |A\cup B\cup C|$$

$$= |U| - (|A|+|B|+|C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$$

$$= 1000 - (\left\lfloor\frac{1000}{5}\right\rfloor + \left\lfloor\frac{1000}{6}\right\rfloor + \left\lfloor\frac{1000}{8}\right\rfloor - \left\lfloor\frac{1000}{5\times 6}\right\rfloor - \left\lfloor\frac{1000}{2\times 3\times 4}\right\rfloor - \left\lfloor\frac{1000}{5\times 8}\right\rfloor + \left\lfloor\frac{1000}{5\times 2\times 3\times 4}\right\rfloor)$$

$$= 600$$

## Homework

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# Applications of Inclusion-Exclusion

Section 8.6

# Section Summary

- An alternative form of inclusion-exclusion
- Counting Onto-Functions
- Derangements



1. An alternative form of inclusion-exclusion

**Problems that ask for the number of elements in a set that have none of *n* properties**

$$P_1, P_2, \cdots, P_n \text{.}$$

**Let** $A_i$ **be the subset containing the elements that have property** $P_i$ **.**

$N(P_1'P_2'\cdots P_n')$

**----The number of elements with none of the properties** $P_1, P_2, \cdots, P_n$**.**

**From the inclusion-exclusion principle, we see that**

$$N(P_1'P_2'\cdots P_n') = N - |A_1 \cup A_2 \cdots \cup A_n| = N - \sum_{1\le i\le n} N(P_i) + \sum_{1\le i<j\le n} N(P_iP_j) + \cdots + (-1)^n N(P_1P_2\cdots P_n)$$

〖Example 1〗 How many solutions does $x_1 + x_2 + x_3 = 13$ have, where $x_i$ are nonnegative integers with $x_i < 6, i = 1,2,3$ ?

*Solution:*

Let a solution have property $P_1$ is $x_1 \ge 6$, property $P_2$ is $x_2 \ge 6$ , property $P_3$ is $x_3 \ge 6$ .

The number of solutions is

$$N(P_1'P_2'P_3') = N - N(P_1) - N(P_2) - N(P_3) + N(P_1P_2) + N(P_1P_3) + N(P_2P_3) - N(P_1P_2P_3)$$

$C(3-1+13,13)$

$N(P_i) = C(3-1+7,7)$

$N(P_iP_j) = C(3-1+1,1)$

$N(P_1P_2P_3) = 0$





2. The sieve of Eratosthenes

〖Example 2〗 Find the number of primes not exceeding a specified positive integer.

Take 100 for example.

*Solution:*

- ✧ A composite integer is divisible by a prime not exceeding its square root.
- ✧ Composite integer not exceeding 100 must have a prime factor not exceeding 10.
- ✧ Since the only primes less than 10 are 2,3,5,7, the primes not exceeding 100 are these four primes and the positive integers greater than 1 and not exceeding 100 that are divisible by none of 2,3,5,7.

**$P_1$: the property that an integer is divisible by 2**
**$P_2$: the property that an integer is divisible by 3**
**$P_3$: the property that an integer is divisible by 5**
**$P_4$: the property that an integer is divisible by 7**

**The number of primes not exceeding positive integer 100 is**

$$4 + N(P_1'P_2'P_3'P_4')$$
$$= 4 + N - N(P_1) - N(P_2) - N(P_3) - N(P_4) + N(P_1P_2) + N(P_1P_3) + N(P_1P_4)$$
$$+ N(P_2P_3) + N(P_2P_4) + N(P_3P_4) - N(P_1P_2P_3) - N(P_1P_2P_4) - N(P_1P_3P_4) - N(P_2P_3P_4) + N(P_1P_2P_3P_4)$$
$$= 25$$

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$\lfloor 100/2 \rfloor$

$\lfloor 100/(2\times 3) \rfloor$

$\lfloor 100/(2\times 3\times 5) \rfloor$

$\lfloor 100/(2\times 3\times 5\times 7) \rfloor$





## The sieve of Eratoshenes -1

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 |
| 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 |
| 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 |
| 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 |
| 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |

## The sieve of Eratoshenes -1

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 |
| 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 |
| 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 |
| 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 |
| 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |



8.6 Applications of Inclusion-Exclusion

## The sieve of Eratoshenes -1

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 |
| 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 |
| 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 |
| 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 |
| 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |

8.6 Applications of Inclusion-Exclusion

## The sieve of Eratoshenes -1

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 |
| 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 |
| 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 |
| 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 |
| 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |

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8.6 Applications of Inclusion-Exclusion

## The sieve of Eratoshenes -1

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 |
| 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 |
| 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 |
| 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 |
| 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |

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## The Number of Onto Functions

**Example**: How many onto functions are there from a set with six elements to a set with three elements?

**Solution**: Suppose that the elements in the codomain are $b_1$, $b_2$, and $b_3$. Let $P_1$, $P_2$, and $P_3$ be the properties that $b_1$, $b_2$, and $b_3$ are not in the range of the function, respectively. The function is onto if none of the properties $P_1$, $P_2$, and $P_3$ hold.

By the inclusion-exclusion principle the number of onto functions from a set with six elements to a set with three elements is

$$N - [N(P_1) + N(P_2) + N(P_3)] + [N(P_1P_2) + N(P_1P_3) + N(P_2P_3)] - N(P_1P_2P_3)$$

- Here the total number of functions from a set with six elements to one with three elements is $N = 3^6$.
- The number of functions that do not have in the range is $N(P_1) = 2^6$. Similarly, $N(P_2) = N(P_1) = 2^6$.
- Note that $N(P_1P_2) = N(P_1P_3) = N(P_2P_3) = 1$ and $N(P_1P_2P_3)= 0$.

Hence, the number of onto functions from a set with six elements to a set with three elements is:

$$3^6 - 3 \cdot 2^6 + 3 = 729 - 192 + 3 = 540$$

## The Number of Onto Functions (continued)

**Theorem 1**: Let m and n be positive integers with $m \geq n$. Then there are

$$n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \cdots + (-1)^{n-1}C(n,n-1)\cdot 1^m$$

onto functions from a set with $m$ elements to a set with $n$ elements.

Proof follows from the principle of inclusion-exclusion (*see Exercise* 27).

# Derangements

**Definition**: A *derangement* is a permutation of objects that leaves no object in the original position.

**Example**: The permutation of 21453 is a derangement of 12345 because no number is left in its original position. But 21543 is not a derangement of 12345, because 4 is in its original position.

# Derangements (continued)

**Theorem 2**: The number of derangements of a set with $n$ elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right].$$

Proof follows from the principle of inclusion-exclusion (*see text*).

# Derangements (continued)

**The Hatcheck Problem**: A new employee checks the hats of *n* people at restaurant, forgetting to put claim check numbers on the hats. When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats. What is the probability that no one receives the correct hat.

**Solution**: The answer is the number of ways the hats can be arranged so that there is no hat in its original position divided by $n!$, the number of permutations of $n$ hats.

**Remark**: It can be shown that the probability of a derangement approaches $1/e$ as $n$ grows without bound.

$$\frac{D_n}{n!} = \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right]$$

TABLE 1 The Probability of a Derangement.

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $D_n/n!$ | 0.50000 | 0.33333 | 0.37500 | 0.36667 | 0.36806 | 0.36786 |

*Proof:*

**Let a permutation have property $P_i$ if it fixes element $i$.**

**The number of derangements is the number of permutations having none of the properties $P_i$ for $i$=1, 2, …, $n$.**

$$\begin{aligned}
D_n &= N(P_1'P_2'\cdots P_n') \\
&= N - \sum_{1\le i\le n} N(P_i) + \sum_{1\le i<j\le n} N(P_iP_j) + \cdots + (-1)^n N(P_1P_2\cdots P_n) \\
&= n! - C(n,1)(n-1)! + C(n,2)(n-2)! - C(n,3)(n-3)! + \ldots + (-1)^n \times C(n,n)(n-n)! \\
&= n! - \frac{n!}{1!(n-1)!}\times(n-1)! + \frac{n!}{2!(n-2)!}\times(n-2)! - \frac{n!}{3!(n-3)!}\times(n-3)! + \ldots + (-1)^n \frac{n!}{n!(n-n)!}\times(n-n)! \\
&= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right)
\end{aligned}$$

# Homework

Sec. 8.6 6, 11, 16