RLC Circuits

It doesn't matter how beautiful your theory is, it doesn't matter how smart you are. If it doesn't agree with experiment, it's wrong.

Richard Feynman (1918-1988)

OBJECTIVES

To observe free and driven oscillations of an RLC circuit.

THEORY

The circuit of interest is shown in Fig. 1, including sine-wave sources. We start with the series connection, writing Kirchoff's law for the loop in terms of the charge q_C on the capacitor and the current $i = dq_C/dt$ in the loop. The sum of the voltages around the loop must be zero, so we obtain

$$V_L + V_R + V_C = V_D \sin(\omega t) \tag{1}$$

$$L\frac{d^2 q_C}{dt^2} + R\frac{dq_C}{dt} + \frac{q_C}{C} = V_D \sin(\omega t) \tag{2}$$

For reasons that will become clear shortly, we rewrite this as

$$\frac{d^2 q_C}{dt^2} + \frac{2}{\tau}\frac{dq_C}{dt} + \omega_0^2 q_C = \frac{V_D}{L}\sin(\omega t) \tag{3}$$

with $\boldsymbol{\tau = 2L/R}$ and $\omega_0^2 = 1/LC$. Two situations will be of interest. First, we will examine the "free oscillations case" when the sine-wave generator is replaced with a short circuit, and a charged capacitor provides the initial energy to the circuit. Then we will connect the sine wave generator and calculate the response as a function of frequency.

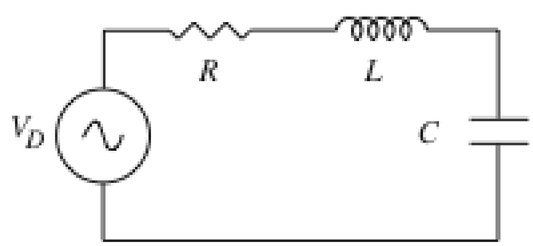

Fig. 1 Idealized series RLC circuit driven by a sine-wave voltage source.

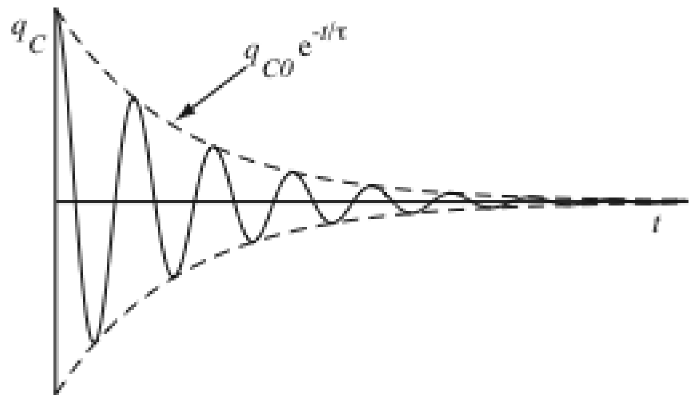

Fig. 2 Damped oscillation, showing decay envelope

If the resistance in the circuit is small, the solution to the "free oscillations case" is of the form

$$q_c = q_{c0} e^{-t/\tau} sin(\omega_1 t + \phi) \tag{4}$$

Where q_{C0} and ϕ are determined by initial conditions, and

$$\omega_1 = \omega_0 \left[1 - (\omega_0 \tau)^{-2}\right]^{1/2} \tag{5}$$

This solution is plotted in Fig. 2 for a case where the capacitor is initially fully charged. (For a mass on a spring the equivalent situation would be to pull the mass aside and release it from rest.) The charges on the capacitor will oscillation at ω_1, approximately equal to ω_0, within an exponential envelope. Note that the amplitude falls to 1/e of the initial value when $t = \tau$. **A system that exhibit damped oscillation like in Fig. 2 is said to be underdamped.**

As τ gets smaller (**larger resistance *R***), ω_1 becomes smaller and finally imaginary. When ω_1 becomes imaginary, the corresponding solutions do not oscillate at all. For $\omega_0 < 1/\tau$, the solution is the sum of two exponentials

$$q_C = A_1 e^{-t/\tau_1} + A_2 e^{-t/\tau_2} \tag{6}$$

where τ_1 and τ_2 differ somewhat from τ. When $\boldsymbol{\omega_0 = 1/\tau}$ the solution is slightly simpler:

$$q_C = (A_1 + A_2 t) e^{-t/\tau} \tag{7}$$

If the capacitor is initially charged, the results from the previous page tell us that we will get a monotonic decay for sufficiently large R. The case $\omega_0 = 1/\tau$ or equivalently $R = R_{Critical} = 2L\omega_o = 2\sqrt{\frac{L}{C}}$ is referred to as "critically damped" because the charge reaches zero in the shortest time without any charge oscillation. For $R > R_{Critical}$, the system is called "overdamped", since it takes a longer time than for the critically damped case for the charge to reach zero.

After studying the "free oscillations case", we will connect the circuit to a sine wave generator. Equation 3 describes how the charges on the capacitor changes with time in this case. The solution to equation 3 is a sine function that describes charge oscillation at the frequency of the driving voltage:

$$q_C = A\sin(\omega t + \phi) \tag{8}$$

We can find A and ϕ by substituting into the differential equation (eq. 3) and solving:

$$A = \frac{V_D / L}{\left[\left(\omega^2 - \omega_0^2\right)^2 + \left(2\omega/\tau\right)^2\right]^{1/2}} \tag{9}$$

$$\tan\phi = \frac{2\omega/\tau}{\left(\omega^2 - \omega_0^2\right)} \tag{10}$$

These two equations are plotted in Fig. 3, where we use the fact that $V_C = q_C/C$ to plot the maximum voltage across the capacitor relative to the maximum driving voltage. (Our apparatus

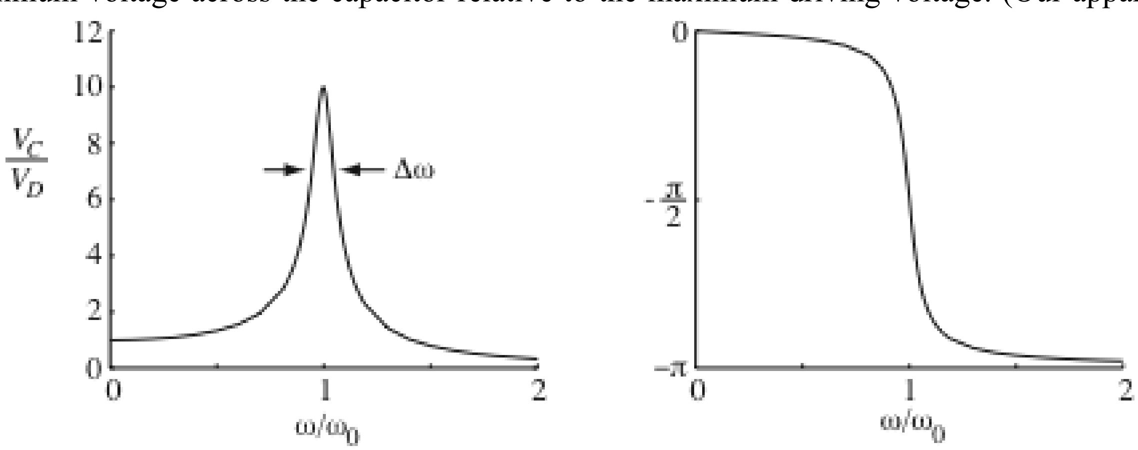

Fig. 3 Amplitude and phase of capacitor voltage as a function of frequency.

does not allow us to observe the phase of the response, so we won't consider that further.)
The angular frequency for maximum amplitude is given by

$$\omega_{peak} = \omega_0 \left[1 - 2/(\omega_0 \tau)^2\right]^{1/2} \tag{11}$$

At ω_{peak}, the oscillation amplitude of V_C is considerably greater than the driving amplitude of V_D. In fact, if τ were infinite (no damping), the maximum amplitude of V_C would be infinite at $\omega_{peak} \approx \omega_0$.

Since the shape of the peak in V_C characterizes the resonance, it is convenient to have some parameter to specify the sharpness of the peak. Traditionally, this is taken to be the full width $\Delta\omega$, shown in Fig. 3, at which the voltage or current has fallen to $1/\sqrt{2}$ of the peak value. The reason for this choice is that the power dissipation is proportional to I^2, so these frequencies correspond to the points at which the power dissipation is half of the maximum. Using Eq. 9 we find that the width is related to τ by

$$\Delta\omega = \frac{2}{\tau} \tag{12}$$

demonstrating the intimate relation between time and frequency response parameters.

EXPERIMENTAL PROCEDURE

The RLC circuit is assembled from a variable resistor and a 0.1 μF capacitor on the circuit panel, and a separate large solenoid coil that is NOT on the circuit panel. The large solenoid coil will act as the inductor for your RLC circuit. The circuit can be charged up with a DC power supply to study the free oscillations, or driven with a sine wave source for forced oscillations.

Free oscillations

To study the "free oscillations case" we will use LoggerPro to record the voltage across the capacitor as a function of time. The required circuit is shown in Fig. 4. Since the coil is not made with superconducting wire, we account for the wire resistance with R_L. **The variable resistor (use the one with $20\ k\Omega, 10\ turns, on\ the\ circuit\ panel$), shown on Fig. 4 by a resistor symbol with an arrow in the middle on the circuit box, should be set initially to the minimum possible resistance.** (You can use the DMM in ohmmeter mode to check that the variable resistor is set to the minimum value before connecting *the variable resistor* into the circuit.) **The power supply should be set to its maximum output.** The toggle switch should be placed between the power supply and the variable resistor.

Start *LoggerPro* with the file RLC.cmbl located on the desktop of the computer. This will configure the program to collect data at 10,000 samples per second, **triggered when the voltage decreases across 9.5 V**. Set the toggle switch to position *b* to charge the capacitor, and then start data acquisition by pushing the "play" button in the LoggerPro program. **After pushing the "play" button in the LoggerPro program, you should wait eight seconds**, and then flip the toggle switch to position *a,* which will start discharging the capacitor, and you should see a plot resembling Fig. 2. There may be some irregularity at the beginning due to bouncing when the switch first makes contact, but you can ignore that in your analysis.

When you have a plot that looks similar to Fig. 2, select the portion of the data after any effects of switch bounce have ended, and fit that portion of the data to a damped sine wave, like Eq. 4. **By matching the fit parameters output by *LoggerPro* with Eq. 4, you should be able to obtain estimates of the time constant τ and the angular frequency ω_1 for your circuit.** **(Delete the fit or restart RLC.cmbl before taking more data. Otherwise the computer might freeze up.)**

Sketch on your lab report by hands the shapes of the voltage vs time curves that you observe on the computer screen as the variable resistor *R* is increased to show underdamped, critically damped and overdamped responses. **Include a time scale** so that changes in τ can be recognized. Critical damping occurs when *R* is just big enough that the voltage no longer crosses zero during the decay. Physically adjusts the variable resistor until the circuit exhibits critical damping, and measure the value of the variable resistor using a DMM. The exact value is not very clear, but makes a reasonable attempt. **Reminder: You must disconnect *R* from the circuit to get an accurate measurement with the DMM.** Describe what happens to the time constant as *R* becomes progressively larger while the system is underdamped.

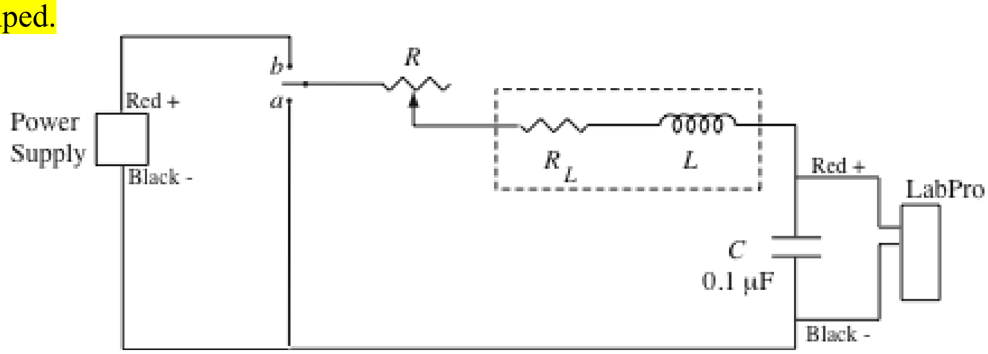

Figure 4: circuit to study free-oscillations.

Driven oscillations

To study the driven solution we need to use a sine-wave signal source whose operation is explained below in Fig. 6. The real function generator is equivalent to an ideal sine-wave generator in series with a R_g = **50 Ω resistor**. R_g is labeled in Fig. 5. See Page 7 for details for operating the sine-wave generator. Since R_g is large enough to affect the damping of the circuit, we add the 3.3 Ω resistor in parallel to reduce the effective resistance of the generator. The other change from Fig. 4 is to replace the LabPro (The box that is connected to the computer you used in the free-oscillations portion of the lab) with **a DMM (The orange box), set to read AC voltage**. For this part of the experiment, you will have to manually read out and record the voltage across the capacitor. (The DMM can read AC voltages between about 30 Hz and 1000 Hz. It is unreliable outside of that range.) Also, there is no toggle switch in this circuit.

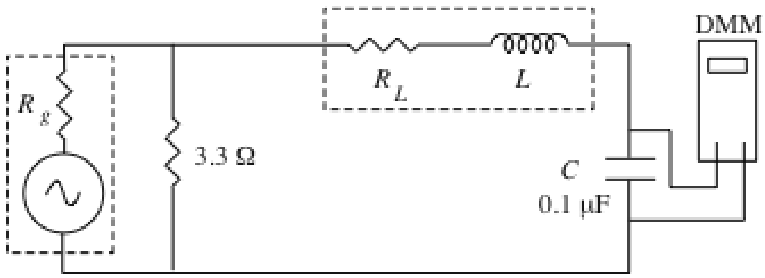

Fig. 5 Circuit for measuring driven oscillations in an RLC circuit.

Connect the circuit as shown in Fig 5, and then vary the driving frequency to find the frequency at which V_C is maximized. This identifies the resonance frequency, in Hertz. Next, you should plot V_C as a function of frequency, taking care to get enough data around the resonance frequency to clearly define the curve. The resonance frequency should be close to $f_{resonance} = \frac{\omega_1}{2\pi}$. This goes very quickly if you enter f and V_C directly into **Graph.cmbl** and then pick data points to trace the regions of interest. Does your plot look like Fig. 3?

You can get an accurate measure of the width by finding the frequencies just above and just below the resonance for which V_C is reduced to $1/\sqrt{2}$ of the peak value. The width $\Delta\omega$ is then $2\pi\Delta f$, where the 2π converts from frequency in Hertz to angular frequency. Do your values of $\Delta\omega$ extracted from the driven oscillations circuit and τ extracted from the free oscillations circuit satisfy Eq. 12? Describe one reason other than "experimental error", why your measured values of $\Delta\omega$ and τ might not satisfy Eq. 12 exactly.

Function generator operation

Figure 6 shows the control panel of the function generator most often used in this lab. The instrument is turned on with the power switch, item (1) in the figure. A sine, square or triangle waveform is selected with the buttons labeled FUNCTION (3). The frequency range is determined by pushbuttons (2), and the exact frequency is then set using the coarse and fine FREQUENCY knobs (12, 11, respectively). The output amplitude is determined by the AMPL knob (4). Pulling gently on the AMPL knob toward you decreases the output amplitude by a factor of 10. The display (13) shows the output frequency numerically in Hz or kHz, according to the indicator (14) and decimal point on the display. The selected output appears as a voltage at OUTPUT (5). The other controls and outputs are more specialized, and will not be used here.

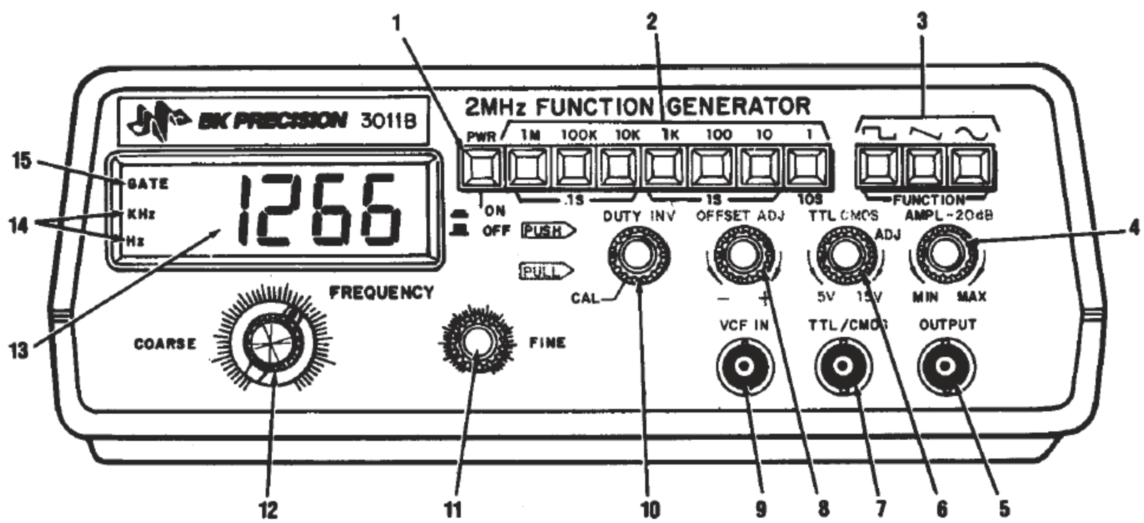

Fig. 6 The front panel of the function generator, with controls marked.