Lecture 0 - Review

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Sets - Notation

A set of elements a, b, and c is written as

$$\mathcal{S} = \{a, b, c\}.$$

To indicate that a is in set $\mathcal{S}$, we saw 'a is an element of set $\mathcal{S}$', and write

$$a \in \mathcal{S}.$$

Sets - Named Sets

- Natural numbers: $\mathbb{N} = \{1, 2, 3, 4, \dots\}$
- Integers: $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- Rational numbers: $\mathbb{Q} = \{x | \; x = a/b \text{ where } a, b \in \mathbb{Z}\}$
  - Notation: the $|$ means 'such that'
  - So, the above says:
    '$\mathbb{Q}$ is the set of all numbers x such that x is equal to the quotient a/b where a and b are both integers.'
- Real numbers: $\mathbb{R}$
- Empty (Null) Set: $\varnothing = \{\}$

Sets - Operations and Subsets

Common Set Operations:

- Union:
  $A \cup B$ is the set of all things that are in either A or B.
- Intersection:
  $A \cap B$ is the set of all things that are in both A and B.
- Complement:
  $\overline{A}$ is the set of all things that are not in A.

Subsets:

- Subset:
  $A \subseteq B$, every element in set A is also in set B.
- Proper Subset:
  $A \subset B$, set A is a subset of set B and $A \neq B$.

Boolean Logic

Consider the literals P and Q that can have the boolean values True ($\top$) or False ($\perp$).

- Think statements like ‘all pigs can fly,’ or ‘my shirt is blue.’

Boolean Logic - Operations

- Disjunction (OR): $P \vee Q$ is true if either P or Q is true.
- Conjunction (AND): $P \wedge Q$ is true if both P and Q are true.
- Negation (NOT): $\neg P$ is true if P is not true.
- Exclusive-Or (XOR): $P \oplus Q$, P or Q but not both
- Implies: $P \Rightarrow Q$ means 'if P is true, then Q is true'
- Equality: $P \Leftrightarrow Q$ means 'P is true if and only if Q is true'
  - We write 'iff' as shorthand for 'if and only if':
    'P is true iff Q is true'
  - To prove equality, must prove both directions:
    Forward direction $P \Rightarrow Q$ and reverse direction $Q \Rightarrow P$.

Sequences and Tuples

A sequence is a list of objects in some order.

- Ex: (3,65,2) or (65,3,2). Note: these are not the same sequence because order matters.
- Note: a sequence can be infinite.

A tuple is a sequence with a finite length, and a tuple of length k is called a k-tuple.

- Ex: (3,65,2) is a 3-tuple and (5,4) is a 2-tuple.
- 2-tuples are also called ordered pairs.

Sets, Sequences, and Tuples

Sets and sequences can be elements of other sets and sequences.

- Ex: the power set of set A is the set of all subsets of A.

  If $A = \{0, 1\}$, then the power set of A is the set:

  $$\{\ \emptyset, \{0\}, \{1\}, \{0, 1\}\ \}.$$

- Ex: The set of all ordered pairs with elements 0 and 1:

  $$\{\ (0, 0),\ (0, 1),\ (1, 0),\ (1, 1)\ \}.$$

The Cartesian Product (aka. Cross Product) of sets A and B:

$$A \times B = \{(a, b) | a \in A, b \in B\}.$$

Functions

A function is a mapping of inputs to outputs.

- The set of all possible inputs to a function is its domain.
- The outputs of a function are all contained in a set called its range.
- We write $f : D \to R$.
- Note: a function does not necessarily use all of the elements of the specified range. If it does, we say f maps D onto R.

Relations

A predicate or property is a function whose range is:

$$\{\text{True}, \text{False}\}.$$

A relation is a predicate whose domain is a set of k-tuples, and a binary relation is a relation whose input is an ordered pair.

- Infix notation: xRy means $R(x, y) = True$.
- Ex: 'less than', with infix notation using the symbol $<$ i.e. $x < y$.

Properties:

- R is reflexive if for every x: xRx.
- R is symmetric if for every x and y: $xRy \Rightarrow yRx$.
- R is transitive if for every x, y, and z: xRy and $yRz \Rightarrow xRz$.

A relation is an equivalence relation if it is reflexive, symmetric, and transitive.

Strings

We define an alphabet as a nonempty finite set of symbols. We typically use the letters Σ and Γ to denote alphabets.

- $\Sigma = \{0, 1\}$
- $\Gamma = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$

A string over an alphabet is a finite sequence of symbols from that alphabet.

- $\Sigma = \{0, 1\}$,
  Ex: 0011, 01001, 0, 10
- The empty string is a string of length 0, denoted ε.

Properties:

- The length of a string w is written $|w|$.
- The reverse of a string w is written $w^{\mathcal{R}}$.
- A string z is a substring of w if z appears within w.
  Ex: 'abba' is a substring of 'bbabba'

Strings

- Let x be the string $x_1x_2\ldots x_n$ of length n, where $x_1, \ldots, x_n$ are all symbols of the given alphabet.

- Let y be the string $y_1y_2\ldots y_m$ of length m, where $y_1, \ldots, y_m$ are all symbols of the given alphabet.

- The concatenation of x and y is written xy and is the string $x_1x_2\ldots x_ny_1y_2\ldots y_m$ of length $n+m$.

- We can concatenate a string to itself. If we do this k times, we denote the resulting string as x^k.

Languages

A language is a set of strings.

Proof Techniques

Direct Proof (Proof by Construction)

Say we want to prove that $P \Rightarrow Q$. A direct proof has 2 steps:

1. Assume that P is true.
2. Use P to show that Q is true.

Ex: Prove that if a and b are consecutive integers, then their sum $a + b$ is odd.

Proof: Assume that a and b are consecutive integers. This means that $b = a + 1$. So we can see that $a + b = 2a + 1$, which is clearly odd.

Proof Techniques

Proof by Contradiction

Say we want to prove that $P \Rightarrow Q$. A proof by contradiction has 2 steps:

1. Assume that P is true but that Q is not true.
2. Use this to demonstrate a contradiction, i.e. an obviously false conclusion.

Ex: Prove that if a and b are consecutive integers, then their sum $a + b$ is odd.

Proof: Assume that a and b are consecutive integers, but that their sum $a + b$ is not odd. Since the sum $a + b$ is not odd, there exists no integer k such that $a + b = 2k + 1$. However, since a and b are consecutive integers, $b = a + 1$ and so $a + b = 2a + 1$. This contradicts the previous sentence, and therefore if a and b are consecutive integers, their sum $a + b$ must be odd.

Proof Techniques

Proof by Induction

Say we want to prove that all of the elements of an ordered infinite set have a given property. Let's call this property $P(x)$, and so our goal is to prove that $P(x)$ is true for all x.

- Base Case:
  We start with the base case, the first (and often simplest) subproblem: prove that $P(1)$ is true.
- Inductive Hypothesis:
  Assume that $P(k)$ is true for some k.
- Inductive Step:
  Using the assumption that $P(k)$ is true, prove that $P(k+1)$ is true.

Why does this work?

Proof Techniques

Ex: Prove that if a and b are consecutive positive integers, then their sum $a+b$ is odd.

Proof:

- Base Case:
  Let $a=1$ and $b=2$. Then $a+b=3$, which is clearly odd.
- Inductive Hypothesis (IH):
  Consider the integer k. Let $a=k$ and $b=k+1$. Assume that their sum $a+b=k+(k+1)$ is odd.
- Inductive Step:
  Consider $k+1$. Let $a=k+1$ and $b=k+2$ so

  $$a+b=(k+1)+(k+2)=(k+1)+k+2=k+(k+1)+2.$$

  From the IH, we have assumed that $k+(k+1)$ is odd.
  Adding 2 to a number does not change whether it is odd.
  Hence the sum $a+b=k+(k+1)+2$ is also odd.