Course Name: Algorithm Design and Analysis

Dynamic Programming

For problems 1-4, you should do at least the following things:

1. Modeling: how you analyse the problem;
2. Algorithm description: describe your algorithm in **natural language**;
3. Time complexity: provide the time complexity and explain the reasoning behind it;
4. Space complexity: provide the space complexity and explain the reasoning behind it.

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1. Step Problem

   A frog can jump either 1 step or 2 steps at a time. How many different ways can the frog jump to reach the top of an n-step staircase?

2. Buy!

   The teaching assistant has a budget of n yuan to decorate his workspace with items.

   The teaching assistant wants to buy many items, which will definitely exceed his budget. Therefore, he assigned an importance level to each item, ranging from 1 to 5, with 5 being the most important.

   Let the price of the j_{th} item be v_j, and the importance level be w_j. Suppose k items are selected, labeled as $j_1, j_2, \ldots, j_k$. The total value is given by:

   $$v_{j_1} \times w_{j_1} + v_{j_2} \times w_{j_2} + \cdots + v_{j_k} \times w_{j_k}$$

   Please help the teaching assistant design a shopping list that meets the requirements: given the prices and importance levels, the goal is to maximize the total value of selected items without exceeding the budget.

3. Counting

   An array $a[1...n]$ of length n is considered *nice* if there exists a way to partition it into several intervals, where the minimum value of each interval equals the length of that interval. Given a set of m positive integers $S[1...m]$, we need to determine **how many** *nice* arrays a exist such that each $a[i] \in S$. The integers n and m are provided, with S satisfying $1 \leq S[1] < S[2] < \ldots < S[m] \leq 10^6$.

   For instance, when $n = 2$ and $S = \{1, 2\}$, there are exactly two *nice* arrays: $[1, 1]$ and $[2, 2]$. Additionally, when $n = 5$ and $S = \{1, 4, 5\}$, your answer should be 32.

4. Ex. Buy! Buy! Buy!

   The teaching assistant has a budget of n yuan to decorate her workspace with main items and accessories (accessories are subordinate to a certain main item). The table below shows some examples of main items and their accessories.

   If she wants to buy accessories, she must first purchase the corresponding main item. Each main item can have 0, 1, or 2 accessories, and each accessory corresponds to one main item, with no further accessories for the accessories themselves.

main item	accessories
laptop	mouse, keyboard
bookshelf	book
desk	reading lamp, stationery
chair	-

The teaching assistant wants to buy many items, which will definitely exceed her budget. Therefore, she assigned an importance level to each item, ranging from 1 to 5, with 5 being the most important.

Let the price of the j_{th} item be v_j, and the importance level be w_j. Suppose k items are selected, labeled as $j_1, j_2, \ldots, j_k$. The total value is given by:

$$v_{j_1} \times w_{j_1} + v_{j_2} \times w_{j_2} + \cdots + v_{j_k} \times w_{j_k}$$

Please help the teaching assistant design a shopping list that meets the requirements: given the prices, importance levels, and their relationships, the goal is to maximize the total value of selected items without exceeding the budget.