Lecture 8 Conic Programming

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Why conic programming?

- Conic programming including Linear, Conic Quadratic and Semidefinite Programming can be seen as a special class of convex optimization problems that are "well-structured".
- Wide range of applications in fields like control theory, finance, signal processing, engineering, and machine learning

Cone, convex cone, conic combination

Cone: $C \subseteq \mathbb{R}^n$ such that

$$x \in C \implies tx \in C \text{ for all } t \geq 0$$

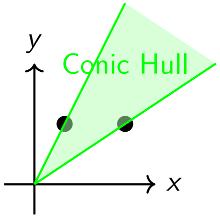
Convex cone: cone that is also convex, i.e.,

$$x_1, x_2 \in C \implies t_1x_1 + t_2x_2 \in C \text{ for all } t_1, t_2 \geq 0$$

Conic combination of $x_1, \ldots, x_k \in \mathbb{R}^n$:

$$\lambda_1 x_1 + \cdots + \lambda_k x_k$$

with $\lambda_i \geq 0, i = 1, \ldots, k$.



Conic hull collects all conic combinations

Proper convex cones

We consider proper convex cones $\mathcal{K}$ in $\mathbb{R}^n$:

- Closed
- Pointed: $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$.
- Non-empty interior.

Dual-cone:

$$\mathcal{K}^* = \left\{ v \in \mathbb{R}^n \mid u^T v \geq 0, \forall u \in \mathcal{K} \right\}$$

If $\mathcal{K}$ is a proper cone, then $\mathcal{K}^\star$ is also proper.
We use the notation:

$$\begin{aligned} x \succeq_{\mathcal{K}} y \quad &\Longleftrightarrow (x-y) \in \mathcal{K} \\ x \succ_{\mathcal{K}} y \quad &\Longleftrightarrow (x-y) \in \operatorname{int} \mathcal{K} \end{aligned}$$

What is conic programming?

Let $\mathcal{K}$ be a proper convex cone in $\mathbb{R}^m$ (pointed, closed, and with a nonempty interior). Given an objective $c \in \mathbb{R}^n$, an $m \times n$ constraint matrix A, and a right-hand side $b \in \mathbb{R}^m$, consider the optimization problem

$$\min_x \left\{ c^T x \mid Ax \succeq_{\mathcal{K}} b \right\}$$

where $Ax \succeq_{\mathcal{K}} b \Leftrightarrow Ax - b \in \mathcal{K}$

- Linear program is a special case of conic program (why?). For LP, $\mathcal{K} = \{y \in \mathbb{R}^m \mid y \geq 0\}$.

Dual of conic program

The dual of the conic program is

$$\max \left\{ b^T \lambda \mid A^T \lambda = c, \lambda \succeq_{\mathcal{K}^*} 0 \right\}$$

where $\mathcal{K}^*$ represents the dual cone of $\mathcal{K}$.

- Note that for LP, $\mathcal{K}$ is self-dual. $\mathcal{K} = \mathcal{K}^* = \{y \in \mathbb{R}^m \mid y \geq 0\}$. We reproduce LP duality!
- The duality is **symmetric**: the dual problem is conic, and the problem dual to dual is (equivalent to) the primal.
- **Weak duality**: value of the dual objective at every dual feasible solution λ is the value of the primal objective at every primal feasible solution x, so that the duality gap

  $$c^T x - b^T \lambda \geq 0$$

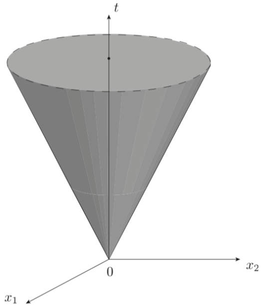
  at every primal-dual feasible pair (x, λ).

Strong duality

- If the primal is bounded below and strictly feasible, i.e., $Ax \succ_{\mathcal{K}}$ for some x, then the dual is solvable and the optimal values in the problems are equal to each other.
- If the dual is bounded above and strictly feasible (i.e., exists $\lambda \succ_{\mathcal{K}_*} 0$ such that $A^T\lambda = c$), then the primal is solvable and we have strong duality.
- Assume that at least one of the problems is bounded and strictly feasible. Then a primal-dual feasible pair (x, λ) is a pair of optimal solutions to the respective problems if and only if the following condition hold
  - $b^T\lambda = c^T x$ [zero duality gap]
  - $\lambda^T[Ax - b] = 0$ [complementary slackness].

Second-order cone

- The second-order cone (SOC) in $\mathbb{R}^3$ is the set of vectors (x_1, x_2, t) such that $\sqrt{x_1^2 + x_2^2} \leq t$. Horizontal sections of this set at level $\alpha \geq 0$ are disks of radius α.



- In arbitrary dimension: an $(n+1)$-dimensional SOC is the following set:

$$\mathcal{K}_n = \{(x, t), x \in \mathbb{R}^n, t \in \mathbb{R} : \|x\|_2 \leq t\}.$$

- SOC is self-dual: $\mathcal{K}_n^* = \mathcal{K}_n$

Rotated second order cone

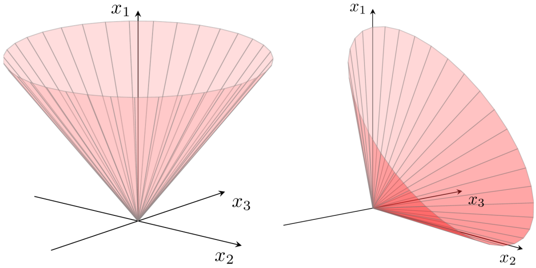


Figure: Boundary of second order cone $x_1 \geq \sqrt{x_2^2 + x_3^2}$ and rotated second cone $2x_1x_2 \geq x_3^2, x_1, x_2 \geq 0$.

Rotated second order cone

- The rotated second-order cone in $\mathbb{R}^{n+2}$ is the set

$$\mathcal{K}_n^r = \left\{ (x, y, z), x \in \mathbb{R}^n, y \in \mathbb{R}, z \in \mathbb{R} : x^\top x \leq 2yz, y \geq 0, z \geq 0 \right\}.$$

- The rotated second-order cone in $\mathbb{R}^{n+2}$ can be expressed as a linear transformation (actually, a rotation) of the (plain) second-order cone in $\mathbb{R}^{n+2}$, since

$$\|x\|_2^2 \leq 2yz, y \geq 0, z \geq 0 \Longleftrightarrow \left\| \begin{bmatrix} x \\ \frac{1}{\sqrt{2}}(y - z) \end{bmatrix} \right\|_2 \leq \frac{1}{\sqrt{2}}(y+z).$$

  That is, $(x, y, z) \in \mathcal{K}_n^r$ if and only if $(w, t) \in \mathcal{K}_n$, where

$$w = (x, (y - z)/\sqrt{2}), \quad t = (y + z)/\sqrt{2}.$$

Standard SOC constraint

- The standard format of a second-order cone constraint on a variable $x \in \mathbb{R}^n$ expresses the condition that $(y, t) \in \mathcal{K}_m$, with $y \in \mathbb{R}^m, t \in \mathbb{R}$, where y, t are some affine functions of x.
- These affine functions can be expressed as $y = Ax + b, t = c^\top x + d$, hence the condition $(y, t) \in \mathcal{K}_m$ becomes

$$\|Ax + b\|_2 \leq c^\top x + d,$$

  where $A \in \mathbb{R}^{m,n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$, and $d \in \mathbb{R}$.
- For example, the quadratic constraint

$$x^\top Q x + c^\top x \leq t, \quad Q \succeq 0$$

  can be expressed in conic form as

$$\left\| \begin{bmatrix} \sqrt{2} Q^{1/2} x \\ t - c^\top x - 1/2 \end{bmatrix} \right\|_2 \leq t - c^\top x + 1/2.$$

Second-order cone programs

- A second-order cone program is a convex optimization problem having linear objective and SOC constraints. When the SOC constraints have the standard form, we have a SOCP in standard inequality form:

$$\begin{array}{ll} \min_{x \in \mathbb{R}^n} & c^\top x \\ \text{s.t.:} & \|A_i x + b_i\|_2 \leq c_i^\top x + d_i, \quad i = 1, \dots, m, \end{array}$$

  where $A_i \in \mathbb{R}^{m_i,n}$ are given matrices, $b_i \in \mathbb{R}^{m_i}$, $c_i \in \mathbb{R}^n$ are vectors, and d_i are given scalars.

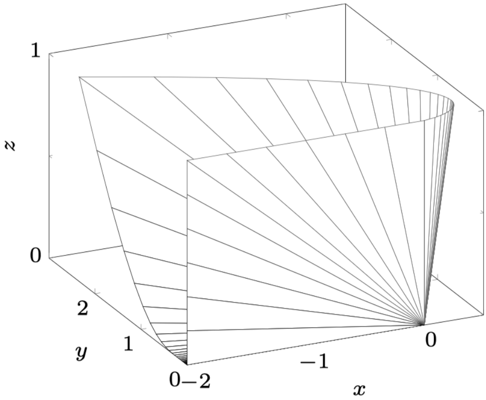
- SOCPs are representative of a quite large class of convex optimization problems. Indeed, LPs, convex QPs, and convex QCQPs can all be represented as SOCPs.

Exponential cone

the epigraph of the perspective of $\exp x$ is a non-proper cone

$$\mathcal{K} = \left\{ (x, y, z) \in \mathbb{R}^3 \mid y e^{x/y} \le z, y > 0 \right\}$$

the exponential cone is $\mathcal{K}_{\exp} = \mathbf{cl}\,\mathcal{K} = \mathcal{K} \cup \{(x, 0, z) \mid x \le 0, z \ge 0\}$



Exponential cone representable functions

- The epigraph $t \geq e^x$ is a section of $\mathcal{K}_{\exp}$:

$$t \geq e^x \iff (x, 1, t) \in \mathcal{K}_{\exp} \; .$$

- Similarly, we can express the hypograph $t \leq \log x, x \geq 0$:

$$t \leq \log x \iff (t, 1, x) \in \mathcal{K}_{\exp} \; .$$

- The entropy function $H(x) = -x \log x$ can be maximized using the following representation.:

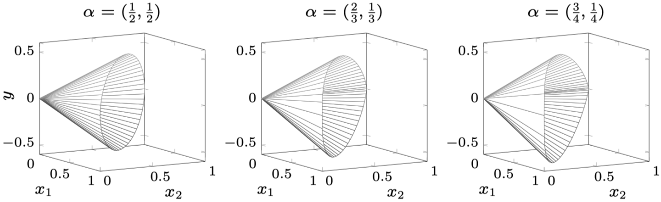
$$t \leq -x \log x \iff t \leq x \log(1/x) \iff (t, x, 1) \in \mathcal{K}_{\exp} \; .$$

Power cone

Definition: for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) > 0$ and $\sum_{i=1}^m \alpha_i = 1$

$$K_\alpha = \left\{(x, y) \in \mathbb{R}_+^m \times \mathbb{R} \,\middle|\, |y| \leq x_1^{\alpha_1} \cdots x_m^{\alpha_m}\right\}$$

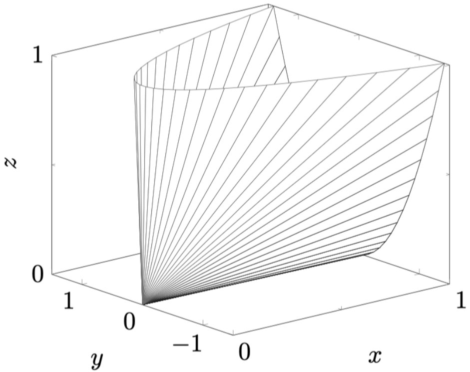
Examples for $m = 2$



Positive semidefinite cone

$$\mathcal{S}^p = \{\mathrm{vec}(X) \mid X \in \mathcal{S}_+^p\} = \left\{x \in \mathbb{R}^{p(p+1)/2} \mid \mathrm{mat}(x) \succeq 0\right\}$$

$$\mathcal{S}^2 = \left\{(x, y, z) \,\middle|\, \begin{bmatrix} x & y/\sqrt{2} \\ y/\sqrt{2} & z \end{bmatrix} \succeq 0\right\}$$



Basic properties of a PSD matrix

- We denote $n \times n$ symmetric matrices by $\mathcal{S}^n$.
- Standard inner product for matrices:

$$\langle V, W \rangle := \operatorname{tr}\left(V^T W\right) = \sum_{ij} V_{ij} W_{ij} = \operatorname{vec}(V)^T \operatorname{vec}(W)$$

- X is positive semidefinite if and only if
  1. $z^T X z \geq 0, \forall z \in \mathbb{R}^n$.
  2. All the eigenvalues of X are nonnegative.
  3. There exist $V \in \mathbb{R}^{r \times n}$ $r \leq n$ such that $X = V^T V$
- The positive (semi)definite matrices form a cone $(\mathcal{S}_+)\, \mathcal{S}_{++}$.

Eigenvalue decomposition

The eigenvalue decomposition of a positive semi-definite (PSD) matrix A is given by:

$$A = Q\Lambda Q^T$$

where,

- A is the PSD matrix.
- Q is an orthogonal matrix whose columns are the eigenvectors of A.
- Λ is a diagonal matrix with the eigenvalues of A on its diagonal.
- Q^T is the transpose of Q, ensuring the reconstruction of A through the product $Q\Lambda Q^T$.

Pseudo-Inverse of a PSD Matrix

The pseudo-inverse of a PSD matrix A, denoted $A^\dagger$, holds special interest due to A's properties:

$$A^\dagger = Q\Lambda^\dagger Q^T$$

where,

- $A = Q\Lambda Q^T$ is the eigenvalue decomposition of A, with A being PSD.
- $\Lambda^\dagger$ contains the reciprocals of the non-zero eigenvalues of A on its diagonal and zeros elsewhere.
- Vector product form: Suppose we have an eigenvalue decomposition of $Q = \sum_{i=1}^{k} \lambda_i q_i q_i^T$ where the λ_i, $i = 1, \ldots, k$ are the nonzero eigenvalues. q_i, $i = 1, \ldots, k$ are the corresponding orthonormal eigenvectors.

$$Q^\dagger := \sum_{i=1}^{k} \lambda_i^{-1} q_i q_i^T$$

Schur's Lemma

Given:

- $Q \in \mathcal{S}_p$, $P \in \mathcal{S}_n$
- $S = [s_1, \ldots, s_p] \in \mathbb{R}^{n \times p}$

Lemma Statement: The following are equivalent:

(i) The matrix $\begin{bmatrix} P & S^\top \\ S & Q \end{bmatrix}$ is positive [resp. semi-]definite.

(ii) The matrices Q and $P - S^\top Q^\dagger S$ are both positive [resp. semi-]definite, and $s_i \in \mathcal{R}(Q)$ for $i = 1, \ldots, p$; these range conditions are automatically satisfied when $Q \succ 0$ (then $\mathcal{R}(Q) = \mathbb{R}^p$).

Implications:

- Establishes a condition under which a block matrix is positive (semi-)definite.
- Highlights the relationship between the definiteness of a block matrix and its submatrices.

SOC as SDP

The SOC constraint can be converted to the PSD contraint,
Consider the SOC constraint:

$$\|x\|_2 \leq t$$

This can be represented as a PSD constraint using the Schur's lemma:

$$\begin{bmatrix} t & x^T \\ x & tI \end{bmatrix} \succeq 0$$

This indicates SDP is more “expressive” than SOCP.

Dual cone of the PSD cone

Dual cone:

$$\left(\mathcal{S}_+^n\right)^* = \left\{Z \in \mathbb{R}^{n\times n} \mid \langle X, Z\rangle \geq 0, \forall X \in \mathcal{S}_+^n\right\}.$$

The semidefinite is self-dual: $\left(\mathcal{S}_+^n\right)^* = \mathcal{S}_+^n$.
Proof: Assume $Z \succeq 0$ so that $Z = U^T U$ and $X = V^T V$.

$$\langle X, Z\rangle = \left\langle V^T V, U^T U\right\rangle = \operatorname{tr}\left(V^T V U^T U\right) = \operatorname{tr}\left(U V^T V U^T\right) = \left\|U V^T\right\|_F^2 \geq 0.$$

The equalities come from the following properties of trace

$$\langle A, B\rangle = \operatorname{tr}(A^T B)$$

$$\operatorname{tr}(ABCD) = \operatorname{tr}(BCDA) = \operatorname{tr}(CDAB) = \operatorname{tr}(DABC) \qquad \text{(cyclic property)}$$

$$tr(A^T A) = \langle A, A\rangle = \sum_{i,j} a_{i,j}^2 = \|A\|_F^2$$

Conversely assume $Z \not\succeq 0$. Then $\exists w \in \mathbb{R}^n$ such that

$$w^T Z w = \left\langle w w^T, Z\right\rangle = \langle X, Z\rangle < 0.$$

Applications of SDP-Eigenvalue optimization

$$F(x) = F_0 + x_1 F_1 + \cdots + x_m F_m, \quad F_i \in \mathcal{S}_m.$$

- Minimize largest eigenvalue $\lambda_1(F(x))$:

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & \gamma I \succeq F(x), \end{array}$$

- Maximize smallest eigenvalue $\lambda_n(F(x))$:

$$\begin{array}{ll} \text{maximize} & \gamma \\ \text{subject to} & F(x) \succeq \gamma I, \end{array}$$

- Minimize eigenvalue spread $\lambda_1(F(x)) - \lambda_n(F(x))$:

$$\begin{array}{ll} \text{minimize} & \gamma - \lambda \\ \text{subject to} & \gamma I \succeq F(x) \succeq \lambda I \end{array}$$

Relaxation of QUBO

Consider an quadratic unconstrained binary optimization (QUBO) problem

$$\begin{array}{ll} \text{minimize} & x^T Q x + c^T x \\ \text{subject to} & x_i \in \{0, 1\}, \quad i = 1, \ldots, n \end{array}$$

where $Q \in \mathcal{S}^n$ can be indefinite. QUBO is NP-hard in general.

- Rewrite binary constraints $x_i \in \{0, 1\}$ and introduce lifted space variables $X \in \mathcal{S}^n$:

$$X_{ij} = x_i x_j, \quad x_i^2 = x_i \Longleftrightarrow X = xx^T, \quad \mathrm{diag}(X) = x.$$

- Semidefinite relaxation:

$$X \succeq xx^T, \quad \mathrm{diag}(X) = x.$$

SDP relaxation of QUBO

Lifted non-convex problem:

$$\begin{array}{ll} \text{minimize} & \langle Q, X \rangle + c^T x \\ \text{subject to} & \text{diag}(X) = x \\ & X = xx^T. \end{array}$$

Note that

$$\left(\begin{array}{cc} X & x \\ x^T & 1 \end{array} \right) \succeq 0 \Leftrightarrow X \succeq xx^T \text{ (Schur's lemma)}$$

Semidefinite relaxation:

$$\begin{array}{ll} \text{minimize} & \langle Q, X \rangle + c^T x \\ \text{subject to} & \text{diag}(X) = x \\ & \left(\begin{array}{cc} X & x \\ x^T & 1 \end{array} \right) \succeq 0 \end{array}$$

Relaxation is exact if $X = xx^T$.

A simple cutting plane algorithm for solving SDP

Semidefinite programming (SDP) problems can be formulated as:

$$\min_{X} \langle C, X \rangle$$

$$\text{s.t. } \langle A^i, X \rangle \geq b_i, \quad i = 1, \ldots, m,$$

$$X \succeq 0,$$

where X is the semidefinite matrix variable, $C \in \mathcal{S}^n$ and $A^i \in \mathcal{S}^n$ are given matrices.

1. Start with only the m linear constraints on X.
2. Compute the eigenvalues and eigenvectors of the current solution $\hat{X}$. If all eigenvalues are non-negative, $\hat{X}$ is feasible and thus optimal.
3. For each negative eigenvalue λ_i, $i = 1, \ldots, r$, use its corresponding eigenvector v^i, $i = 1, \ldots, r$ to add a linear cut $(v^i)^T X v^i \geq 0$ that excludes the current infeasible solution $\hat{X}$.
4. Add the linear constraints and repeat from step 2 until a feasible X is found or the algorithm converges to a solution within a desired tolerance.

Software for conic programming

- Gurobi, Cplex ((MI)SOCP)
- Mosek: mostly commonly used conic solver (SOCP, EXP, POW, SDP).
- COPT (SOCP, SDP)
- other solvers: see
  `https://jump.dev/JuMP.jl/stable/installation/`
  `https://plato.asu.edu/bench.html`
- most of the algorithms for conic programs are variants of the interior point algorithm.
- Although SDP is convex, solving it is usually "slow". $n = 100 - 1,000$ are the problem size we are usually talking about.

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