ChE 597 Homework 6 Solutions

Problem 1

Convert the logical expression into a system of inequalities using binary (0-1) variables.

(a)

Given:

$$(p_1 \vee \neg p_2) \Rightarrow (p_3 \vee p_4)$$

Use implication equivalence:

$$\neg(p_1 \vee \neg p_2) \vee (p_3 \vee p_4)$$

Apply DeMorgan's law:

$$(\neg p_1 \wedge p_2) \vee (p_3 \vee p_4)$$

Distribute the OR across the AND:

$$(\neg p_1 \vee p_3 \vee p_4) \wedge (p_2 \vee p_3 \vee p_4)$$

Let $y_i = 1$ if p_i is true, 0 otherwise.
The inequalities become:

$$\begin{aligned} 1 - y_1 + y_3 + y_4 &\geq 1 \\ y_2 + y_3 + y_4 &\geq 1 \end{aligned}$$

Simplifying:

$$\begin{aligned} -y_1 + y_3 + y_4 &\geq 0 \\ y_2 + y_3 + y_4 &\geq 1 \end{aligned}$$

(b)

Given:

$$((p_1 \wedge p_2) \Rightarrow p_3) \Rightarrow (p_5 \vee p_6)$$

Step-by-step:

$$\begin{gathered} ((p_1 \wedge p_2) \Rightarrow p_3) \Rightarrow (p_5 \vee p_6) \\ \neg[(\neg(p_1 \wedge p_2) \vee p_3)] \vee (p_5 \vee p_6) \\ (p_1 \wedge p_2) \wedge \neg p_3)] \vee (p_5 \vee p_6) \end{gathered}$$

Which becomes:

$$(p_1 \vee p_5 \vee p_6) \wedge (p_2 \vee p_5 \vee p_6) \wedge (\neg p_3 \vee p_5 \vee p_6)$$

Translate to inequalities:

$$\begin{aligned} y_1 + y_5 + y_6 &\geq 1 \\ y_2 + y_5 + y_6 &\geq 1 \\ 1 - y_3 + y_5 + y_6 &\geq 1 \end{aligned}$$

Or after simplification:

$$\begin{aligned} y_1 + y_5 + y_6 &\geq 1 \\ y_2 + y_5 + y_6 &\geq 1 \\ -y_3 + y_5 + y_6 &\geq 0 \end{aligned}$$

Problem 2

(a)

If A and B are true, then C or D must be true.

$$(A \wedge B) \Rightarrow (C \vee D)$$

Remove implication:

$$\neg(A \wedge B) \vee (C \vee D)$$

Equivalently:

$$\neg A \vee \neg B \vee C \vee D$$

Using binary variables:

$$-y_A - y_B + y_C + y_D \geq -1$$

(b)

All combinations are feasible **except** $y_j = 0$ for $j \in N$ and $y_j = 1$ for $j \in B$. To eliminate that combination:

$$\sum_{j \in N} y_j + \sum_{j \in B} (1 - y_j) \geq 1$$

(c)

If power is generated at any time (1, 2, or 3), then install a gas turbine. Let y_1, y_2, y_3 be generation at times 1–3. Let t be turbine install.

Logical form:

$$(y_1 \vee y_2 \vee y_3) \Rightarrow t$$

Remove implication:

$$\neg(y_1 \vee y_2 \vee y_3) \vee t$$

Equivalent to:

$$(\neg y_1 \wedge \neg y_2 \wedge \neg y_3) \vee t$$

Distribute or:

$$(\neg y_1 \vee t) \wedge (\neg y_2 \vee t) \wedge (\neg y_3 \vee t)$$

Inequalities:

$$\begin{aligned}
1 - y_1 + t &\geq 0te \\
1 - y_2 + t &\geq 1 \\
1 - y_3 + t &\geq 1
\end{aligned}$$

Or:

$$\begin{aligned}
-y_1 + t &\geq 0 \\
-y_2 + t &\geq 0 \\
-y_3 + t &\geq 0
\end{aligned}$$

Problem 3

(a)

Either $0 \leq x \leq 10$ **or** $20 \leq x \leq 30$

Big-M Formulation: Introduce binary variables y_1, y_2 for each disjunct:

$$\begin{aligned}
-x &\leq M(1 - y_1) \\
x &\leq 10 + M(1 - y_1) \\
-x + 20 &\leq M(1 - y_2) \\
-x + 30 &\geq -M(1 - y_2) \\
y_1 + y_2 &= 1
\end{aligned}$$

Convex-Hull: Let $x = z_1 + z_2$

$$\begin{aligned}
0 \leq z_1 &\leq 10y_1 \\
20y_2 \leq z_2 &\leq 30y_2 \\
y_1 + y_2 &= 1
\end{aligned}$$

(b)

Temperature constraint holds only if unit selected ($y = 1$):

Big-M:

$$T_{in} - T_{out} \geq \Delta T_{min} - M(1 - y)$$

Convex-Hull:

$$\begin{aligned}
T_{in} - T_{out} &= z \\
z &\geq \Delta T_{min} y \\
0 \leq z &\leq M \text{ where M is upper bound for } \Delta T
\end{aligned}$$

Problem 4

Given:

$$K = \{x \in \{0,1\}^3 : 2x_1 + 3x_2 + 4x_3 \leq 5\}$$

(a) Reformulation using minimal cover inequalities.

A *cover* is a subset of items whose total weight strictly exceeds the capacity (here, the capacity is 5). The weights of the items are $w_1 = 2$, $w_2 = 3$, $w_3 = 4$. A subset $\{ i_1, i_2, \ldots\}$ is a cover if

$$w_{i_1} + w_{i_2} + \ldots \; > \; 5.$$

A cover is *minimal* if by removing any item from it, it ceases to be a cover.

Step 1: Find the minimal covers.

- Singletons:
$$\{1\} \quad (2 \leq 5), \quad \{2\} \quad (3 \leq 5), \quad \{3\} \quad (4 \leq 5).$$
None of these exceed 5, so no singleton is a cover.
- Pairs:
$$\begin{aligned} \{1,2\}: &\quad 2+3=5 \;\; (\text{not} > 5), \text{ not a cover;} \\ \{1,3\}: &\quad 2+4=6 \; > \; 5, \quad \text{so this is a cover;} \\ \{2,3\}: &\quad 3+4=7 \; > \; 5, \quad \text{so this is a cover.} \end{aligned}$$
- Triple: $\{1,2,3\}$ sums to $9 > 5$, but is *not* minimal because, for instance, removing item 1 from $\{1,2,3\}$ still leaves $\{2,3\}$ which is already a cover.

Hence, the minimal covers are precisely

$$\{1,3\} \quad \text{and} \quad \{2,3\}.$$

Step 2: Derive the minimal cover inequalities. Each minimal cover $\{i,j\}$ with $w_i + w_j > 5$ yields an inequality

$$x_i + x_j \; \leq \; 1.$$

From the two minimal covers, we obtain:

$$x_1 + x_3 \; \leq \; 1, \quad x_2 + x_3 \; \leq \; 1.$$

Thus, we can refine the original knapsack set by adding these constraints:

$$K_C \; := \; \big\{ x \in \{0,1\}^3 \; : \; 2x_1 + 3x_2 + 4x_3 \; \leq \; 5, \; x_1 + x_3 \; \leq \; 1, \; x_2 + x_3 \; \leq \; 1 \big\}.$$

(b) Showing $K = K_C$.

We claim that for *binary* $x \in \{0,1\}^3$, the constraints

$$x_1 + x_3 \; \leq \; 1 \quad \text{and} \quad x_2 + x_3 \; \leq \; 1$$

are already implied by

$$2x_1 + 3x_2 + 4x_3 \; \leq \; 5.$$

Hence, these extra constraints do not cut off any integer-feasible points.

Reasoning: If $x_3 = 0$, then $x_1 + x_3 = x_1 \le 1$ and $x_2 + x_3 = x_2 \le 1$ trivially (since $x_i \in \{0, 1\}$). If instead $x_3 = 1$, then:

$$2x_1 + 3x_2 + 4(1) \;\le\; 5 \quad\Longrightarrow\quad 2x_1 + 3x_2 \;\le\; 1.$$

In particular, x_1 or x_2 cannot be 1 (otherwise, $2 + 3 = 5$ or $3 + 4 = 7$, both > 1). Thus $x_3 = 1$ forces $x_1 = x_2 = 0$, implying $x_1 + x_3 = 1 \le 1$ and $x_2 + x_3 = 1 \le 1$.

Therefore, among binary vectors, the constraints $x_1 + x_3 \le 1$ and $x_2 + x_3 \le 1$ are automatically enforced by $2x_1 + 3x_2 + 4x_3 \le 5$. This shows

$$K \;=\; \{x \in \{0,1\}^3 : 2x_1{+}3x_2{+}4x_3 \le 5\} \;=\; \big\{x \in \{0,1\}^3 : 2x_1{+}3x_2{+}4x_3 \le 5,\; x_1{+}x_3 \le 1,\; x_2{+}x_3 \le 1\big\} \;=\; K_C.$$

(c)

LP Relaxations:

- P: $2x_1 + 3x_2 + 4x_3 \le 5$
- P^C:

$$x_2 + x_3 \le 1$$
$$x_1 + x_3 \le 1$$

We can add the first inequality in P^C multiplied by 3 to the second inequality multiplied by 2. We get:

$$2x_1 + 3x_2 + 5x_3 \le 5$$

So P^C is implied by P^C and:

$$P^C \subseteq P$$

We will find a point in P that violates one of the new cover inequalities, proving strict containment.

Example: Let

$$x_1 = 0.5, \quad x_2 = \tfrac{2}{3}, \quad x_3 = 0.5.$$

Check the knapsack constraint:

$$2(0.5) + 3\big(\tfrac{2}{3}\big) + 4(0.5) = 1 + 2 + 2 = 5,$$

so $(0.5, \frac{2}{3}, 0.5)$ indeed lies in P. However,

$$x_2 + x_3 \;=\; \tfrac{2}{3} \;+\; 0.5 \;=\; 1.1666\ldots \;>\; 1,$$

thus violating $x_2 + x_3 \le 1$. Hence the point is *not* in P_C. This shows $P_C \subsetneq P$ strictly.