Lecture 6 Simplex Algorithm

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ChE 597: Computational Optimization
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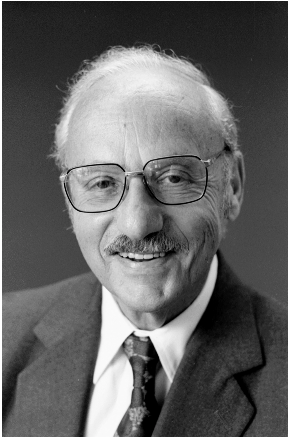
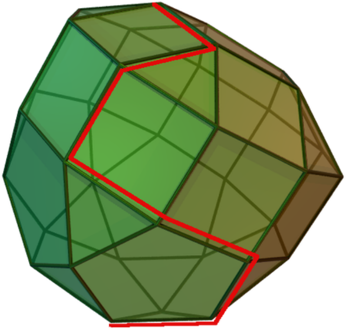
History





Developed by George Dantzig in 1947.
Intuition: iterate over the extreme points of a polyhedron until an optimality condition is satisfied.

Important results from Polyhedron theory

When P is nonempty and the optimal objective value is not $-\infty$, the optimal solution can always be obtained at an extreme point.
extreme point $\Leftrightarrow$ basic feasible solution
Basic feasible solution Consider a polyhedron P defined by linear equality and inequality constraints, and let x^* be an element of $\Re^n$.

1. The vector x^* is a **basic solution** if:
   - All equality constraints are active;
   - Out of the constraints that are active at x^*, there are n of them that are linearly independent.
2. If x^* is a basic solution that satisfies all of the constraints, we say that it is a **basic feasible solution**.

Basic solution for standard form polyhedron

Theorem Consider the constraints $Ax = b$ and $x \geq 0$ and assume that the $m \times n$ matrix A has linearly independent rows. A vector $x \in \Re^n$ is a basic solution if and only if we have $Ax = b$, and there exist indices $B(1), \ldots, B(m)$ such that:
(a) The columns $A_{B(1)}, \ldots, A_{B(m)}$ are linearly independent;
(b) If $i \neq B(1), \ldots, B(m)$, then $x_i = 0$.

BFS $\quad Ax = b \quad A \in \mathbb{R}^{m \times n} \quad m < n$

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} \cdots & \cdots & \cdots \end{bmatrix} \\ & \underbrace{\uparrow \quad \uparrow \quad \uparrow}_{m} \end{array}$$

$x \geq 0 \quad$ pick $(n-m)$ $x_i = 0$

Basic solution for standard form polyhedron

$A = [B \vdots N]$, $B: m\times m$, $N: m\times$

① $Ax = b$

② $x_i = 0 \quad i \in N$

$\Rightarrow \begin{bmatrix} B & N \\ & I \end{bmatrix}$ (n-m

Theorem Consider the constraints $Ax = b$ and $x \geq 0$ and assume that the $m \times n$ matrix A has linearly independent rows. A vector $x \in \Re^n$ is a basic solution if and only if we have $Ax = b$, and there exist indices $B(1), \ldots, B(m)$ such that:

(a) The columns $A_{B(1)}, \ldots, A_{B(m)}$ are linearly independent;

(b) If $i \neq B(1), \ldots, B(m)$, then $x_i = 0$.

Proof sketch: **Sufficiency** If (a) and (b) are satisfied, there are n linearly independent active constraints that uniquely defines a solution and thus is a basic solution.

Necessity For a basic solution, there must exist at least $n - m$ active constraints $x_i = 0$, together with $Ax = b$ forming n linearly independent constraints. Consider the active constraints forming the matrix (under permutation) $\begin{bmatrix} B & N \\ 0 & I \end{bmatrix}$ being full rank, where $A = [B\ N]$, I correspond to the indices of the $x_i = 0$. B must be full rank for the matrix to be full rank.

Basic variables

Definition If x is a basic solution, the variables $x_{B(1)}, \ldots, x_{B(m)}$ are called **basic variables**; the remaining variables are called **nonbasic**. The columns $A_{B(1)}, \ldots, A_{B(m)}$ are called the **basic column** and, since they are linearly independent, they form a basis of $\mathbb{R}^m$

Simpler way to put:

① pick $(n-m)$ variables x_i, $i \in N$. set them to zero.

② enforce the constraints $Ax = b$ (m)

Basis matrix

basis matrix $B \in \mathbb{R}^{m\times m}$ the matrix formed by arranging he m basic columns next to each other.
basic variables a vector x_B with the values of the basic variables
nonbasic variables x_N $(n-m)$

$$B = \begin{bmatrix} | & | & & | \\ A_{B(1)} & A_{B(2)} & \cdots & A_{B(m)} \\ | & | & & | \end{bmatrix}, \qquad x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix}.$$

$N \in \mathbb{R}^{m\times(n-m)}$

$x = [x_B; x_N]$, $A = [B\ N]$ $\quad Ax = b \quad [B\ N]\begin{bmatrix} x_B \\ x_N \end{bmatrix} = b$

The basic variables are determined by solving the equation $Bx_B = b$ whose unique solution is given by

$$\Downarrow$$

$$Bx_B + N x_N = b$$

$$\Downarrow$$

$$Bx_B = b$$

$$x_B = B^{-1}b.$$

If $x_B \geq 0$, then it is a basic feasible solution

Recall that for LP with finite optimum, there exists an optimal basic feasible solution. Next, we will derive the conditions under which a basic feasible solution is optimal.

Feasible direction Let x be an element of a polyhedron P. A vector $d \in \Re^n$ is said to be a feasible direction at x, if there exists a positive scalar θ for which $x + \theta d \in P$.

$x_B = B^{-1} b \qquad x_N = 0.$

pick an index $j \in N$, $\quad d_j = 1,$

$d_i = 0 \quad \forall i \in N, \ i \neq j$

What about the basic variables?

$Ax = b, \qquad x + d \qquad d_j = 1. \quad d_i = 0, \ i \in N, \ i \neq j.$

$A(x+d) = b$

$\Downarrow$

$Ad = 0$

$[B \ N] \begin{bmatrix} d_B \\ d_N \end{bmatrix} = 0 \Leftrightarrow B d_B + A_j = 0$

$d_B = -B^{-1} A_j$

Recall that for LP with finite optimum, there exists an optimal basic feasible solution. Next, we will derive the conditions under which a basic feasible solution is optimal.

Feasible direction Let x be an element of a polyhedron P. A vector $d \in \Re^n$ is said to be a feasible direction at x, if there exists a positive scalar θ for which $x + \theta d \in P$.

Basic direction Let $x_B = B^{-1}b$. be a BFS. We consider the possibility of moving away from x, to a new vector $x + \theta d$, by selecting a nonbasic variable x_j. Algebraically, $d_j = 1$, and $d_i = 0$ for every nonbasic index i other than j. We require $A(x + \theta d) = b$, and since x is feasible, we also have $Ax = b$. Thus, we need $Ad = 0$. Recall now that $d_j = 1$, and that $d_i = 0$ for all other nonbasic indices i. Then,

$$0 = Ad = \sum_{i=1}^{n} A_i d_i = \sum_{i=1}^{m} A_{B(i)} d_{B(i)} + A_j = B d_B + A_j.$$

$$d_B = -B^{-1}A_j, d_j = 1, d_i = 0 \; \forall i \in N, i \neq j \text{ (jth basic direction)}$$

$x + \theta d \geq 0$

When is a basic direction a feasible direction?

Degeneracy: At BFS, more than n constraints are active.

Since $d_j = 1, d_i = 0 \ \forall i \in N$, the nonbasic variables will remain feasible. We only need to make sure that the basic variables are still nonnegative.

(a) x is a nondegenerate basic feasible solution. Then, $x_B > 0$, from which it follows that $x_B + \theta d_B \geq 0$, and feasibility is maintained, when θ is sufficiently small. d is a feasible direction.

(b) x is degenerate. Then, d is not always a feasible direction. Indeed, it is possible that a basic variable $x_{B(i)}$ is zero, while the corresponding component $d_{B(i)}$ of $d_B = -B^{-1}A_j$ is negative. In that case, if we follow the j th basic direction, the nonnegativity constraint for $x_{B(i)}$ is immediately violated, and we are led to infeasible solutions;

Set N $(n-m)$ $x_i = 0$ $\forall i \in N$

$Ax = b$ (m)

$\exists x_i$ s.t. $i \in B$

$x_i = 0$

nondeger : $x_i > 0.$ $\forall i \in B$

Reduced cost

Q: What will be the effect on the cost (objective) function if we move along the jth basic direction?

$$d_B = -B^{-1}A_j, d_j = 1, d_i = 0 \; \forall i \in N, i \neq j \text{ (jth basic direction)}$$

$$\bar{c}_j = c^T d^j = c_j - c_B^T B^{-1} A_j$$

where d^j is the jth basic direction, c_B the vector of costs of the basic variables. $\bar{c}_j$ is the **reduced cost** of the variable x_j

Optimality condition

Theorem Consider a basic feasible solution x associated with a basis matrix B, and let $\overline{c}$ be the corresponding vector of reduced costs.
(a) If $\overline{c} \geq 0$, then x is optimal.
(b) If x is optimal and nondegenerate, then $\overline{c} \geq 0$.

Optimality condition

Proof. (a) We assume that $\overline{c} \geq 0$, we let y be an arbitrary feasible solution, and we define $d = y - x$. Feasibility implies that $Ax = Ay = b$ and, therefore, $Ad = 0$. The latter equality can be rewritten in the form

$$Bd_B + \sum_{i \in N} A_i d_i = 0$$

where N is the set of indices corresponding to the nonbasic variables under the given basis. Since B is invertible, we obtain

$$d_B = -\sum_{i \in N} B^{-1} A_i d_i$$

and

$$c^T d = c_B^T d_B + \sum_{i \in N} c_i d_i = \sum_{i \in N} \left(c_i - c_B^T B^{-1} A_i \right) d_i = \sum_{i \in N} \bar{c}_i d_i$$

For any nonbasic index $i \in N$, we must have $x_i = 0$ and, since y is feasible, $y_i \geq 0$. Thus, $d_i \geq 0$ and $\bar{c}_i d_i \geq 0$, for all $i \in N$. We conclude that $c^T(y - x) = c^T d \geq 0$, and since y was an arbitrary feasible solution, x is optimal.

Optimality condition

(b) Suppose that x is a nondegenerate basic feasible solution and that $\bar{c}_j < 0$ for some j. Since the reduced cost of a basic variable is always zero, x_j must be a nonbasic variable and $\bar{c}_j$ is the rate of cost change along the j th basic direction. Since x is nondegenerate, the j th basic direction is a feasible direction of cost decrease, as discussed earlier. By moving in that direction, we obtain feasible solutions whose cost is less than that of x, and x is not optimal.

Optimal basis

A basis matrix B is said to be optimal if:
(a) $B^{-1}b \geq 0$, and
(b) $\overline{c}^T = c^T - c_B^T B^{-1} A \geq 0$

Simplex Algorithm

We assume all the basic feasible solutions are nondegenerate. Under this assumption, the reduced costs being nonnegative are both necessary and sufficient for a BFS to be optimal.
If a reduced cost $\bar{c}_j$ of a nonbasic variable x_j is negative, we can move toward the jth feasible direction d^j to further decrease the cost. It is desirable to move as far as possible:

$$\theta^* = \max\{\theta \geq 0 \mid x + \theta d \in P\}$$

The resulting cost change is $\theta^* c^T d$, which is the same as $\theta^* \bar{c}_j$.
Given that $Ad = 0$, we have $A(x+ \theta d) = Ax = b$ for all θ, and the equality constraints will never be violated. Thus, $x + \theta d$ can become infeasible only if one of its components becomes negative.

move along a direction d.

(a) $d \geq 0$, $x + \theta d \geq 0$ $\forall \theta \geq 0$.

$x + \theta d \geq 0$, $\theta^* = +\infty$

$\Rightarrow$ problem is unbounded.

(b) $d_i < 0$ for some i,

$x_i + \theta d_i \geq 0$.

$\theta \leq \dfrac{-x_i}{d_i}$.

check this for all i, $d_i < 0$.

$$\theta^* = \min_{\{i \mid d_i < 0\}} \left(\frac{-x_i}{d_i} \right).$$

Simplex Algorithm

Two cases can occur when we move along direction d:

(a) If $d \geq 0$, then $x + \theta d \geq 0$ for all $\theta \geq 0$, the vector $x + \theta d$ never becomes infeasible, and we let $\theta^* = \infty$.

(b) If $d_i < 0$ for some i, the constraint $x_i + \theta d_i \geq 0$ becomes $\theta \leq -x_i/d_i$. This constraint on θ must be satisfied for every i with $d_i < 0$. Thus, the largest possible value of θ is

$$\theta^* = \min_{\{i|d_i<0\}} \left(-\frac{x_i}{d_i} \right).$$

Recall that if x_i is a nonbasic variable, then either x_i is the entering variable and $d_i = 1$, or else $d_i = 0$. In either case, d_i is nonnegative. Thus, we only need to consider the basic variables and we have the equivalent formula

$$\theta^* = \min_{\left\{i=1,\ldots,m|d_{B(i)}<0\right\}} \left(-\frac{x_{B(i)}}{d_{B(i)}} \right).$$

Note that $\theta^* > 0$, because $x_{B(i)} > 0$ for all i, as a consequence of nondegeneracy.

Simplex Algorithm

Once θ^* is chosen, and assuming it is finite, we move to the new feasible solution $y = x + \theta^* d$. Since $x_j = 0$ and $d_j = 1$, we have $y_j = \theta^* > 0$. Let ℓ be a minimizing index of

$$-\frac{x_{B(\ell)}}{d_{B(\ell)}} = \min_{\{i=1,\ldots,m \mid d_{B(i)}<0\}} \left(-\frac{x_{B(i)}}{d_{B(i)}}\right) = \theta^*;$$

$$x_{B(\ell)} + \theta^* d_{B(\ell)} = 0.$$

We observe that the basic variable $x_{B(\ell)}$ has become zero, whereas the nonbasic variable x_j has now become positive, which suggests that x_j should replace $x_{B(\ell)}$ in the basis. The new basis matrix is

$$\bar{B}(i) = \begin{cases} B(i), & i \neq \ell \\ j, & i = \ell \end{cases}$$

In other words, ℓ leaves the basis, j enters the basis.

One simplex iteration (pivot) summary

1. In a typical iteration, we start with a basis consisting of the basic columns $A_{B(1)}, \ldots, A_{B(m)}$, and an associated basic feasible solution x.
2. Compute the reduced costs $\bar{c}_j = c_j - c_B^T B^{-1} A_j$ for all nonbasic indices j. If they are all nonnegative, the current basic feasible solution is optimal, and the algorithm terminates; else, choose some j for which $\bar{c}_j < 0$.
3. Compute $u = -d_B = B^{-1} A_j$. If no component of u is positive, we have $\theta^* = \infty$, the optimal cost is $-\infty$, and the algorithm terminates.
4. If some component of u is positive, let

$$\theta^* = \min_{\{i=1,\ldots,m \mid u_i > 0\}} \frac{x_{B(i)}}{u_i}$$

5. Let ℓ be such that $\theta^* = x_{B(\ell)}/u_\ell$. Form a new basis by replacing $A_{B(\ell)}$ with A_j. If y is the new basic feasible solution, the values of the new basic variables are $y_j = \theta^*$ and $y_{B(i)} = x_{B(i)} - \theta^* u_i$, $i \neq \ell$.

Compute matrix inverse faster

Motivation: Taking matrix inverse (B^{-1}) at each iteration is expensive ($O(m^3)$). We would like a more efficient implementation.
Observation: The new basis $\overline{B}$ and the old basis B only differ at the ℓth column

$$B = \left[A_{B(1)} \cdots A_{B(m)}\right]$$

$$\overline{B} = \left[\; A_{B(1)} \cdots A_{B(\ell-1)} \quad A_j \quad A_{B(\ell+1)} \quad \cdots A_{B(m)} \;\right]$$

Idea: Exploit the information of B^{-1} to compute $\overline{B}^{-1}$

Row operations

$$B^{-1}\overline{B} = \begin{bmatrix} | & & | & | & | & & | \\ e_1 & \cdots & e_{\ell-1} & u & e_{\ell+1} & \cdots & e_m \\ | & & | & | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & u_1 & & \\ & \ddots & \vdots & & \\ & & u_\ell & & \\ & & \vdots & \ddots & \\ & & u_m & & 1 \end{bmatrix}$$

$O(m^2)$

where $u = B^{-1}A_j$.
Let us apply a sequence of elementary row operations that will change the above matrix to the identity matrix.
(a) For each $i \neq \ell$, we add the ℓ th row times $-u_i/u_\ell$ to the i th row. (Recall that $u_\ell > 0$.) This replaces u_i by zero.
(b) We divide the ℓ th row by u_ℓ. This replaces u_ℓ by one.
Let these row operations be matrix Q. $QB^{-1}\overline{B} = I$, which yields $QB^{-1} = \overline{B}^{-1}$. This shows that if we apply the same sequence of row operations to the matrix B^{-1}, we obtain $\overline{B}^{-1}$.

Full tableau implementation

$$B^{-1}[b \mid A]$$

with columns $B^{-1}b$ and $B^{-1}A_1, \ldots, B^{-1}A_n$.

- This matrix is called the **simplex tableau**.
- $B^{-1}b$ is called the zeroth column
- The column $u = B^{-1}A_j$ corresponding to the variable that enters the basis is called the **pivot column**.
- If the ℓ th basic variable exits the basis, the ℓ th row of the tableau is called the **pivot row**.

At the end of each iteration, we need to update the tableau $B^{-1}[b \mid A]$ and compute $\overline{B}^{-1}[b \mid A]$. This can be accomplished by left-multiplying the simplex tableau with a matrix Q satisfying $QB^{-1} = \overline{B}^{-1}$

Full tableau implementation

It is customary to add the objective and the reduced cost as a zeroth row.

$-c_B^T B^{-1} b$	$c^T - c_B^T B^{-1} A$
$B^{-1} b$	$B^{-1} A$

We've already shown how to update the rows of B^{-1} to pivot to $\overline{B}^{-1}$.

It can be shown that the rule for updating the zeroth row turns out to be identical to the rule used for the other rows of the tableau: add a multiple of the pivot row to the zeroth row to set the reduced cost of the entering variable to zero.

An iteration of the full tableau implementation

1. A typical iteration starts with the tableau associated with a basis matrix B and the corresponding basic feasible solution x.
2. Examine the reduced costs in the zeroth row of the tableau. If they are all nonnegative, the current basic feasible solution is optimal, and the algorithm terminates; else, choose some j for which $\bar{c}_j < 0$.
3. Consider the vector $u = B^{-1}A_j$, which is the j th column (the pivot column) of the tableau. If no component of u is positive, the optimal cost is $-\infty$, and the algorithm terminates.
4. For each i for which u_i is positive, compute the ratio $x_{B(i)}/u_i$. Let ℓ be the index of a row that corresponds to the smallest ratio. The column $A_{B(\ell)}$ exits the basis and the column A_j enters the basis.
5. Add to each row of the tableau a constant multiple of the ℓ th row (the pivot row) so that u_ℓ (the pivot element) becomes one and all other entries of the pivot column become zero.

Example

$$\begin{aligned} \text{minimize} \quad & -10x_1 - 12x_2 - 12x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + 2x_3 \leq 20 \\ & 2x_1 + x_2 + 2x_3 \leq 20 \\ & 2x_1 + 2x_2 + x_3 \leq 20 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

Add slack variables to transform to the standard form.

$$\begin{aligned} \text{minimize} \quad & -10x_1 - 12x_2 - 12x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + 2x_3 + x_4 = 20 \\ & 2x_1 + x_2 + 2x_3 + x_5 = 20 \\ & 2x_1 + 2x_2 + x_3 + x_6 = 20 \\ & x_1, \ldots, x_6 \geq 0. \end{aligned}$$

Note that $x = (0, 0, 0, 20, 20, 20)$ is a basic feasible solution and can be used to start the algorithm. Let accordingly, $B(1) = 4, B(2) = 5$, and $B(3) = 6$.

Example

		x_1	x_2	x_3	x_4	x_5	x_6
	0	-10	-12	-12	0	0	0
$x_4 =$	20	1	2	2	1	0	0
$x_5 =$	20	2^*	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

reduced cost.

$B^{-1}A$

$\bar{c}_1 < 0$. choose x_1 to enter the basis

$\frac{x_{B(1)}}{u_1} = \frac{x_4}{u_1} = \frac{20}{1} = 20$

$\frac{x_{B(2)}}{u_2} = \frac{x_5}{u_2} = \frac{20}{2} = 10$

$\frac{x_{B(3)}}{u_3} = \frac{x_6}{u_3} = \frac{20}{2} = 10$

The second row can be selected as the pivot row.

Row operations:

Add 5 times the second row to the zeroth row.

Add $-\frac{1}{2}$ times the second row to the first row.

Add -1 times the second row to the third row.

Divide the second row by 2.

Example

		x_1	x_2	x_3	x_4	x_5	x_6
	100	0	-7	-2	0	5	0
$x_4 =$	10	0	1.5	1^*	1	-0.5	0
$x_1 =$	10	1	0.5	1	0	0.5	0
$x_6 =$	0	0	1	-1	0	-1	1

choose x_3 to enter the basis.

$\frac{x_{B(1)}}{u_1} = \frac{x_4}{u_1} = \frac{10}{1} = 10$

$\frac{x_{B(2)}}{u_2} = \frac{x_1}{u_2} = \frac{10}{1} = 10$

u_3 is negative.

Choose the first row to be the pivot row.

Add 2 times the first row to the zeroth row.

Add -1 times the first row to the second row.

Add 1 times the first row to the third row.

Divide the first row by 1.

Example

		x_1	x_2	x_3	x_4	x_5	x_6
	120	0	-4	0	2	4	0
$x_3 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	0	1	-1	0	-1	1	0
$x_6 =$	10	0	2.5^*	0	1	-1.5	1

x_2 has negative reduced cost and enters the basis.
$\frac{x_{B(1)}}{u_1} = \frac{x_3}{u_1} = \frac{10}{1.5}$
u_2 is negative.
$\frac{x_{B(3)}}{u_3} = \frac{x_6}{u_3} = \frac{10}{2.5}$
Choose the third row to be the pivot row.
Add $\frac{4}{2.5}$ times the third row to the zeroth row.
Add $-\frac{1.5}{2.5}$ times the third row to the first row.
Add $\frac{1}{2.5}$ times the third row to the second row.
Divide the third row by 2.5.

Example

		x_1	x_2	x_3	x_4	x_5	x_6
	136	0	0	0	3.6	1.6	1.6
$x_3 =$	4	0	0	1	0.4	0.4	-0.6
$x_1 =$	4	1	0	0	-0.6	0.4	0.4
$x_2 =$	4	0	1	0	0.4	-0.6	0.4

Optimal solution found. Optimal objective is -136.
$x^* = (4, 4, 4, 0, 0, 0)$

Reference

1. Chapter 3. Bertsimas, D., & Tsitsiklis, J. N. (1997). Introduction to linear optimization (Vol. 6, pp. 479-530). Belmont, MA: Athena scientific.