Lecture 9 Lagrangian Dual and Optimality Conditions

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General Nonlinear Programming Problem

Consider general minimization problem

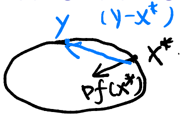
$$\begin{array}{ll} \min_x & f(x) \\ \text{subject to} & h_i(x) \leq 0, i = 1, \ldots, m \\ & \ell_j(x) = 0, j = 1, \ldots, r \end{array}$$

- $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$: optimization variables
- $f : \mathbb{R}^n \to \mathbb{R}$: objective (or cost) function
- $h_i : \mathbb{R}^n \to \mathbb{R}$: inequality constraints
- $\ell_j : \mathbb{R}^n \to \mathbb{R}$: equality constraints
- feasible region:

$$X = \{x \mid h_i(x) \leq 0, i = 1, \cdots, m, \ \ell_j(x) = 0, i = 1, \cdots, r\}$$

Need not be convex, but of course we will pay special attention to convex case (h_i being convex, ℓ_j being linear).

First-Order Optimality Condition



Theorem Suppose that f in a convex optimization problem is differentiable. Let X denote the feasible region. Then x^* is optimal if and only if $x^* \in X$ and

$$\nabla f(x^*)^T(y - x^*) \geq 0 \quad \forall y \in X$$

Interpretation: This condition implies that no feasible direction at x^* leads to a decrease in the value of $f(x)$, making x^* a minimizer of $f(x)$ over the set X.
Caveat: First-order condition is not easy to check.

Proof of first-order optimality condition

Suppose $x^* \in X$. We need to prove

$$f(x^*) \le f(y) \quad \forall y \in X \iff \nabla f(x^*)^T(y - x^*) \ge 0 \quad \forall y \in X$$

$\Leftarrow$ suppose $\nabla f(x^*)^T(y - x^*) \ge 0$ for all $y \in X$. Because f is convex, for all $y \in X$,

$$\underbrace{f(y) \ge f(x^*) + \nabla f(x^*)^T(y - x^*)}_{\text{first-order characterization of a convex function.}} \ge f(x^*)$$

$\Rightarrow$ suppose x^* is optimal, but there exists a $y \in X$ with $\nabla f(x^*)^T(y - x^*) < 0$. Consider the point $z(t) = ty + (1-t)x^*$ with $t \in [0,1]$. Clearly $z(t) \in X$. Because

$$\lim_{t \to 0} \frac{f(z(t)) - f(x^*)}{t} = \nabla f(x^*)^T(y - x^*) < 0$$

For sufficiently small t, $f(z) < f(x^*)$, which contradicts our assumption that x^* is optimal.

Lagrangian

Consider general minimization problem

$$\begin{array}{ll} \min_x & f(x) \\ \text{subject to} & h_i(x) \leq 0, i = 1, \ldots, m \quad \times \quad u_i \geq 0. \\ & \ell_j(x) = 0, j = 1, \ldots, r \quad \times \quad v_j \end{array}$$

The results for Lagrangian will **not be restricted to convex case**. We define the Lagrangian as

$$L(x, u, v) = f(x) + \sum_{i=1}^{m} u_i h_i(x) + \sum_{j=1}^{r} v_j \ell_j(x)$$

New variables $u \in \mathbb{R}^m, v \in \mathbb{R}^r$, with $u \geq 0$ (else $L(x, u, v) = -\infty$)

Langrangian dual function

Important property: for any $u \geq 0$ and v,

$$f(x) \geq L(x, u, v) \text{ at each feasible } x$$

Why? For feasible x,

$u_i \geq 0$ $\quad 0$

$$L(x, u, v) = f(x) + \sum_{i=1}^{m} u_i \underbrace{h_i(x)}_{\leq 0} + \cancel{\sum_{j=1}^{r} v_j \underbrace{\ell_j(x)}_{=0}} \leq f(x)$$

≤ 0

Lower bound Let X denote primal feasible set, $f^\star$ denote primal optimal value. Minimizing $L(x, u, v)$ over all x gives a lower bound:

$$f^\star = f(x^\star) \geq L(x^\star, u, v) \geq \min_{x \in X} L(x, u, v) \geq \boxed{\min_x L(x, u, v) := g(u, v)}$$

x^* is feasible

We call $g(u, v)$ the **Lagrange dual function**, and it gives a lower bound on $f^\star$ for any $u \geq 0$ and v, called dual feasible u, v

Example: quadratic program (QP)

Consider quadratic program:

$$\underbrace{\begin{array}{ll} \min_x & \frac{1}{2}x^TQx + c^Tx \\ \text{subject to} & \underbrace{Ax = b}_{v}, \underbrace{x \geq 0}_{u} \end{array}}$$

where $Q \succ 0$. Lagrangian:

$$L(x,u,v) = \frac{1}{2}x^TQx + c^Tx - u^Tx + v^T(Ax - b)$$

$$\frac{\partial L}{\partial x} = Qx + (c - u - A^Tv) = 0.$$

$$x^* = Q^{-1}(-c + u + A^Tv)$$

Lagrange dual function:

$$g(u,v) = \min_x L(x,u,v) = -\frac{1}{2}\left(c - u + A^Tv\right)^T Q^{-1}\left(c - u + A^Tv\right) - b^Tv$$

For any $u \geq 0$ and any v, this lower bounds primal optimal value $f^\star$

What if Q is singular?

$$\begin{array}{ll} \min_{x} & \frac{1}{2}x^T Qx + c^T x \\ \text{subject to} & Ax = b, \quad x \geq 0 \end{array}$$

where $Q \succeq 0$

Lagrangian:

$$L(x, u, v) = \frac{1}{2}x^T Qx + c^T x - u^T x + v^T (Ax - b)$$

Lagrange Dual Function:

$$g(u, v) = \begin{cases} -\frac{1}{2}(c - u + A^T v)^T Q^\dagger (c - u + A^T v) - b^T v, & \text{if } c - u + A^T v \in \mathcal{R}(Q) \\ -\infty, & \text{otherwise} \end{cases}$$

where $Q^\dagger$ denotes pseudo inverse of Q. For any $u \geq 0$, v, and $c - u + A^T v \in \mathcal{R}(Q)$, $g(u, v)$ is a nontrivial lower bound on f^*.

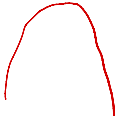
Linear Algebra Review: Vector Spaces of Matrix $A \in \mathbb{R}^{m \times n}$

- **Column Space** ($\mathcal{C}(A)$)
  - Set of all linear combinations of the column vectors of A.
  - Subspace of $\mathbb{R}^m$. $\{Ax \mid \forall x \in \mathbb{R}^n\}$
  - Dimension equals the rank of A.
- **Row Space** ($\mathcal{R}(A)$)
  - Set of all linear combinations of the row vectors of A.
  - Subspace of $\mathbb{R}^n$. $\{A^T y \mid \forall y \in \mathbb{R}^m\}$
  - Dimension equals the rank of A.
  - For $A \in \mathcal{S}^n$ (symmetric matrix), $\mathcal{C}(A) = \mathcal{R}(A)$.
- **Null Space** ($\mathcal{N}(A)$)
  - Set of all vectors x such that $Ax = 0$.
  - Subspace of $\mathbb{R}^n$.
  - Dimension equals the $n-$rank(A).
- **Orthogonal Properties**
  - The row space and null space of A are orthogonal complements in $\mathbb{R}^n$.
  - The column space and the left null space (null space of A^T) are orthogonal complements in $\mathbb{R}^m$.

When is $\min_x \frac{1}{2}x^T Qx + b^T x + c$ unbounded?

- If Q is not PSD, then there exists eigenvector v with negative eigen value $\lambda < 0$. Let $x = \alpha v$

$$\frac{1}{2}x^T Qx + b^T x + c = \frac{1}{2}\lambda\alpha^2 + b^T v\alpha + c$$



Let $\alpha = +\infty$, the problem will be unbounded.

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  Let $\alpha = +\infty$, the problem will be unbounded.

- If b is not in the row (column) space of Q, there exists $\cancel{x}$ y such that $Qy = 0$ and $b^T y < 0$. Let $x = \alpha y$, $\alpha = +\infty$ gives unboundedness.

$$\frac{1}{2}\underbrace{(\alpha y)^T Q(\alpha y)}_{0} + \alpha b^T y + c.$$

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- If b is not in the row (column) space of Q, there exists x such that $Qy = 0$ and $b^T y < 0$. Let $x = \alpha y$, $\alpha = +\infty$ gives unboundedness.
- If $Q \succeq 0$ and $b \in \mathcal{R}(Q)$, the optimum is obtained at the solution of

$$\nabla_x\Big(\frac{1}{2}x^T Qx + b^T x + c\Big) = Qx + b = 0$$

The solution of $Qx = b$, $Q \in \mathcal{S}^n$

- Q is full rank. $x = Q^{-1}b$
- Q is not full rank and $b \in \mathcal{R}(Q)$. $x = Q^{\dagger}b + z$. where $Q^{\dagger}$ is the pseudo inverse of Q, z is any vector in the null space of Q, $z \in \mathcal{N}(Q)$.
- Recall the definition of pseudo inverse. Suppose we have an eigenvalue decomposition of $Q = \sum_{i=1}^{k} \lambda_i q_i q_i^T$ where the λ_i, $i = 1, \ldots, k$ are the nonzero eigenvalues. q_i, $i = 1, \ldots, k$ are the corresponding orthonormal eigenvectors.

$$Q^{\dagger} := \sum_{i=1}^{k} \lambda_i^{-1} q_i q_i^T$$

The solution of $Qx = b$, $Q \in \mathcal{S}^n$

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$$Q^{\dagger} := \sum_{i=1}^{k} \lambda_i^{-1} q_i q_i^T$$

Proof

$$QQ^{\dagger} = \sum_{i,j=1}^{k} \lambda_i \lambda_j^{-1} q_i \left(q_i^{\top} q_j\right) q_j^{\top} = \sum_{i=1}^{k} q_i q_i^{\top}$$

Because the q_i 's form an orthonormal basis of $\mathcal{R}(Q)$, we can write $b = \sum \beta_i q_i$, with $\beta_i = q_i^{\top} b$. This gives

$$QQ^{\dagger} b = \sum_{i,j=1}^{k} q_i q_i^{\top} q_j q_j^{\top} b = \sum_{i=1}^{k} q_i q_i^{\top} b = b$$

Lagrangian dual of QCQP

We consider the QCQP

$$\begin{array}{ll} \min_x & \frac{1}{2}x^T Q_0 x + q_0^T x + r_0 \\ \text{s.t.} & \frac{1}{2}x^T Q_i x + q_i^T x + r_i \le 0, \quad i = 1, \dots, m, \quad \times\ \lambda_i \end{array}$$

with $Q_0 \in \mathcal{S}^n$, and $Q_i \in \mathcal{S}^n, i = 1, \dots, m$ (does not have to be PSD).
The Lagrangian is

$$L(x, \lambda) = \frac{1}{2}x^T Q(\lambda)x + q(\lambda)^T x + r(\lambda),$$

where

$$Q(\lambda) = Q_0 + \sum_{i=1}^m \lambda_i Q_i, \quad q(\lambda) = q_0 + \sum_{i=1}^m \lambda_i q_i, \quad r(\lambda) = r_0 + \sum_{i=1}^m \lambda_i r_i.$$

where $\lambda \ge 0$

Lagrangian dual of QCQP

$$g(\lambda) = \min_x L(x, \lambda) = \min_x \frac{1}{2} x^T Q(\lambda) x + q(\lambda)^T x + r(\lambda)$$

- If $Q(\lambda)$ is not PSD, $g(\lambda)$ is $-\infty$.
- If $Q(\lambda)$ is PSD but $q(\lambda) \notin \mathcal{R}\big(Q(\lambda)\big)$, $g(\lambda)$ is $-\infty$.

$$g(\lambda) = \begin{cases} -\frac{1}{2} q(\lambda)^T Q^\dagger(\lambda) q(\lambda) + r(\lambda), & \text{if } q(\lambda) \in \mathcal{R}\big(Q(\lambda)\big) \text{ and } Q(\lambda) \succeq 0 \\ -\infty, & \text{otherwise} \end{cases}$$

Lagrangian dual of QCQP is an SDP

$$g(\lambda) = \begin{cases} -\frac{1}{2}q(\lambda)^T Q^\dagger(\lambda) q(\lambda) + r(\lambda), & \text{if } q(\lambda) \in \mathcal{R}\big(Q(\lambda)\big) \text{ and } Q(\lambda) \succeq 0 \\ -\infty, & \text{otherwise} \end{cases}$$

$\max_\lambda g(\lambda)$

The dual problem $\max_\lambda g(\lambda)$ is equivalent to

$$\max_{\lambda, d} d,$$

$$\begin{bmatrix} r(\lambda) - d & \frac{1}{\sqrt{2}} q(\lambda)^\top \\ \frac{1}{\sqrt{2}} q(\lambda) & Q(\lambda) \end{bmatrix} \succeq 0.$$

$\Leftrightarrow \quad Q(\lambda) \succeq 0, \quad r(\lambda) - d - \frac{1}{2} q(\lambda) Q^\dagger q(\lambda) \geq 0.$

$q(\lambda) \in R(Q(\lambda))$

$d \leq r(\lambda) - \frac{1}{2} q(\lambda) Q^\dagger q(\lambda)$

Lagrangian dual of QCQP is an SDP

$$g(\lambda) = \begin{cases} -\frac{1}{2}q(\lambda)^T Q^\dagger(\lambda) q(\lambda) + r(\lambda), & \text{if } q(\lambda) \in \mathcal{R}\big(Q(\lambda)\big) \text{ and } Q(\lambda) \succeq 0 \\ -\infty, & \text{otherwise} \end{cases}$$

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$$\max_{\lambda, d} d,$$
$$\begin{bmatrix} r(\lambda) - d & \frac{1}{\sqrt{2}}q(\lambda)^\top \\ \frac{1}{\sqrt{2}}q(\lambda) & Q(\lambda) \end{bmatrix} \succeq 0.$$

Proof Using the Schur lemma, the PSD constraint is equivalent to $Q(u) \succeq 0$ and $r(\lambda) - d - \frac{1}{2}q(\lambda)^T Q^\dagger(\lambda) q(\lambda) \geq 0$ for all $q(\lambda) \in \mathcal{R}(Q)$. Rearrange the inequality we have $d \leq -\frac{1}{2}q(\lambda)^T Q^\dagger(\lambda) q(\lambda) + r(\lambda)$

Langrangian dual problem

Given primal problem

$$\begin{array}{ll} \min_x & f(x) \\ \text{subject to} & h_i(x) \leq 0, i = 1, \dots, m \\ & \ell_j(x) = 0, j = 1, \dots, r \end{array}$$

Our dual function $g(u, v)$ satisfies $f^\star \geq g(u, v)$ for all $u \geq 0$ and v. Hence best lower bound: maximize $g(u, v)$ over dual feasible u, v, yielding Lagrange dual problem:

$$\begin{array}{ll} \max_{u,v} & g(u, v) \\ \text{subject to} & u \geq 0 \end{array}$$

Key property, called weak duality: if dual optimal value is $g^\star$, then

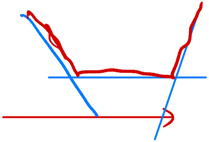
$$f^\star \geq g^\star$$

Note that this always holds (even if primal problem is nonconvex)

Dual problem is convex

The dual problem is a convex optimization problem (as written, it is a concave maximization problem)
Again, this is always true (**even when primal problem is not convex**) By definition:

$$g(u,v) = \min_x \left\{ f(x) + \sum_{i=1}^m u_i h_i(x) + \sum_{j=1}^r v_j \ell_j(x) \right\}$$



$$= -\underbrace{\max_x \left\{ -f(x) - \sum_{i=1}^m u_i h_i(x) - \sum_{j=1}^r v_j \ell_j(x) \right\}}_{\text{pointwise maximum of convex functions in } (u,v)}$$

That is, g is concave in (u,v), and $u \geq 0$ is a convex constraint, so dual problem is a concave maximization problem

Strong duality

Recall that we always have $f^\star \geq g^\star$ (weak duality). On the other hand, in some problems we have observed that actually

$$f^\star = g^\star$$

which is called strong duality.

Slater's condition: if the primal is a convex problem (i.e., f and $h_1, \ldots, h_m$ are convex, $\ell_1, \ldots, \ell_r$ are affine), and there exists at least one strictly feasible $x \in \mathbb{R}^n$, meaning

$$h_1(x) < 0, \ldots, h_m(x) < 0 \text{ and } \ell_1(x) = 0, \ldots, \ell_r(x) = 0$$

then strong duality holds.

Duality gap

Given primal feasible x and dual feasible u, v, the quantity

$$f(x) - g(u, v)$$

is called the duality gap between x and u, v. Note that

$$f(x) - f^\star \le f(x) - g(u, v)$$

so if the duality gap is zero, then x is primal optimal (and similarly, u, v are dual optimal)

Also from an algorithmic viewpoint, provides a stopping criterion: if $f(x) - g(u, v) \le \epsilon$, then we are guaranteed that $f(x) - f^\star \le \epsilon$

Very useful, especially in conjunction with iterative methods.

Karush-Kuhn-Tucker conditions

Given general problem

$$\begin{array}{ll} \min_x & f(x) \\ \text{subject to} & h_i(x) \leq 0, i = 1, \ldots, m \\ & \ell_j(x) = 0, j = 1, \ldots, r \end{array}$$

The Karush-Kuhn-Tucker conditions or KKT conditions are:

- $0 \in \partial_x \Big(f(x) + \sum_{i=1}^m u_i h_i(x) + \sum_{j=1}^r v_j \ell_j(x) \Big)$ (stationarity)
- $u_i \cdot h_i(x) = 0$ for all i (complementary slackness)
- $h_i(x) \leq 0, \ell_j(x) = 0$ for all i, j (primal feasibility)
- $u_i \geq 0$ for all i (dual feasibility)

Necessity of KKT condition

Theorem If $x^\star$ and $u^\star, v^\star$ are primal and dual solutions, with zero duality gap, then $x^\star, u^\star, v^\star$ satisfy the KKT conditions.

$f(x^*) = g(u^*, v^*)$ (zero duality gap)

$= \min_x f(x) + \sum_{i=1}^m u_i^* h_i(x) + \sum_{j=1}^r v_j^* l_j(x)$

$(=) \le f(x^*) + \sum_{i=1}^m \underbrace{u_i^*}_{\ge 0} \underbrace{h_i(x^*)}_{\le 0} + \sum_{j=1}^r v_j^* \underbrace{l_j(x^*)}_{=0}$

$(=) \le f(x^*)$

"=" : x^* minimizes $L(x, u^*, v^*)$

$\Leftrightarrow \nabla_x L = 0$

"=" $u_i^* h_i(x^*) = 0 \quad \forall i = 1 \cdots m$

Necessity of KKT condition

Theorem If $x^\star$ and $u^\star, v^\star$ are primal and dual solutions, with zero duality gap, then $x^\star, u^\star, v^\star$ satisfy the KKT conditions.
Proof Let $x^\star$ and $u^\star, v^\star$ be primal and dual feasible solutions with zero duality gap (strong duality holds, e.g., under Slater's condition). Then

$$
\begin{aligned}
f\left(x^\star\right) &= g\left(u^\star, v^\star\right) \\
&= \min_x f(x) + \sum_{i=1}^m u_i^\star h_i(x) + \sum_{j=1}^r v_j^\star \ell_j(x) \\
&\leq f\left(x^\star\right) + \sum_{i=1}^m u_i^\star h_i\left(x^\star\right) + \sum_{j=1}^r v_j^\star \ell_j\left(x^\star\right) \\
&\leq f\left(x^\star\right)
\end{aligned}
$$

In other words, all these inequalities are actually equalities.

Proof continued

- The point $x^\star$ minimizes $L(x, u^\star, v^\star)$ over $x \in \mathbb{R}^n$. Hence the subdifferential of $L(x, u^\star, v^\star)$ must contain 0 at $x = x^\star$-this is exactly the stationarity condition
- We must have $\sum_{i=1}^m u_i^\star h_i(x^\star) = 0$, and since each term here is ≤ 0, this implies $u_i^\star h_i(x^\star) = 0$ for every i-this is exactly complementary slackness.
- Note that this statement assumes nothing a priori about convexity of our problem, i.e., of f, h_i, ℓ_j.

Constraint qualifications

Besides the Slater's condition, there are other conditions under which the KKT condition has to be satisfied in order for the primal and dual solutions to be optimal. These conditions are called **constraint qualifications**. See Wikipedia for details `https://en.wikipedia.org/wiki/KarushKuhnTucker_conditions`

- Linearity constraint qualification (LCQ)
- Linear independence constraint qualification (LICQ)
- Mangasarian-Fromovitz constraint qualification (MFCQ)
- Constant rank constraint qualification (CRCQ)
- Constant positive linear dependence constraint qualification (CPLD)
- Quasi-normality constraint qualification (QNCQ)
- Slater's condition (SC)

Sufficiency of KKT condition

Theorem When the problem is convex, if $x^\star$ and $u^\star, v^\star$ satisfy the KKT conditions, then $x^\star$ and $u^\star, v^\star$ are primal and dual solutions
Proof If there exists $x^\star, u^\star, v^\star$ that satisfy the KKT conditions, then

$$\begin{aligned} g\left(u^\star, v^\star\right) &= f\left(x^\star\right) + \sum_{i=1}^{m} u_i^\star h_i\left(x^\star\right) + \sum_{j=1}^{r} v_j^\star \ell_j\left(x^\star\right) \\ &= f\left(x^\star\right) \end{aligned}$$

where the first equality holds from stationarity, and the second holds from complementary slackness
Therefore the duality gap is zero (and $x^\star$ and $u^\star, v^\star$ are primal and dual feasible) so $x^\star$ and $u^\star, v^\star$ are primal and dual optimal.

Convex problem with Slater's condition

Theorem For a problem with strong duality (e.g., assume Slater's condition: convex problem and there exists x strictly satisfying nonaffine inequality constraints),

$$x^\star \text{ and } u^\star, v^\star \text{ are primal and dual optimal solutions}$$
$$\iff x^\star \text{ and } u^\star, v^\star \text{ satisfy the KKT conditions}$$

Example: quadratic with equality constraints

Consider for $Q \succeq 0$,

$$\begin{array}{ll} \min_x & \frac{1}{2}x^TQx + c^Tx \\ \text{subject to} & Ax = b \quad (\times u) \end{array}$$

primal feasible : $Ax = b$

Stationarity. $L(x, u) = \frac{1}{2}x^TQx + c^Tx + u^T(b - Ax)$

$\nabla_x L = Qx + c - A^Tu = 0$

Example: quadratic with equality constraints

Consider for $Q \succeq 0$,

$$\begin{array}{ll} \min_x & \frac{1}{2}x^T Qx + c^T x \\ \text{subject to} & Ax = b \end{array}$$

Convex problem, no inequality constraints, so by KKT conditions: x is a solution if and only if

$$\begin{bmatrix} Q & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix}$$

for some u. Linear system combines stationarity, primal feasibility (complementary slackness and dual feasibility are vacuous)

References

- Convex optimization notes by Ryan Tibshirani
  `https://www.stat.cmu.edu/~ryantibs/convexopt/`
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