ChE 597 Computational Optimization

Homework 5

Feb 21st 11:59 pm

1. Consider the following linear program where the linear equalities and inequalities are separated.

$$\begin{aligned} \min_x \ & c^T x \\ \text{s.t. } & Ax = b \\ & Gx \leq h \end{aligned}$$

   (a) Write down the KKT conditions for this problem.

   (b) Compare the KKT conditions with the results we got from linear programming optimality conditions in the LP duality lecture (Lecture 7). What can you observe? Are they the same?

2. Consider the following problem with a quadratic objective (without linear and constant term) subject to one quadratic and m linear inequalities constraints.

$$\begin{aligned} \min_x \, & x^T Q^0 x \\ \text{s.t. } & x^T Q^1 x + q^T x + r \leq 0 \\ & Ax \leq b \end{aligned}$$

where Q^0 and Q^1 can be any symmetric matrices (they are not necessarily PSD).

(a) Derive the Lagrangian dual function.

(b) Derive the dual maximization problem as SDP using Schur's lemma.

3. Consider the nonlinear programming problem

$$
\begin{aligned}
&\min f(x) \\
&\text{s.t. } g_j(x) \leq 0 \quad j = 1\ldots,r \\
&x \in R^n
\end{aligned}
$$

where the functions f and g are monotone in each variable x_i (i.e. $\partial \mathrm{f}/\partial \mathrm{x_i}$ and $\partial \mathrm{g_j}/\partial \mathrm{x_i}$ are one-signed if the derivatives are non-zero).

Show that the following holds true for the optimal solutions of this problem if the problem satisfies some constraint qualification (these are the so called principles of monotonicity analysis):

(a) If a variable x_i is present in the objective and in some of the constraints, there is at least one active constraint involving x_i and whose derivative has opposite sign from the objective.

(b) If a variable x_i is not present in the objective, it is either involved in at least two active constraints with opposite sign in the derivatives, or else it is not involved in any active constraints.

4. Consider the design of a storage vessel that has the form of a cylinder. The required volume is 25 m^3. The cost of the side of the cylinder is \$150/m^2, while the top and bottom cost \$190/m^2 and 260/m^2 respectively. Formulate an NLP problem to determine the optimal dimensions of this vessel, and solve with Pyomo using the interior point solver IPOPT (check our pyomo installation guide if you have not installed IPOPT).

5. Consider the following convex optimization problem

$$
\begin{aligned}
\min_{x_1,x_2} \quad & \frac{3}{x_1+x_2} + e^{x_1} + (x_1-x_2)^2 \\
\text{s.t.} \quad & x_1 \geq 0 \\
& x_2 \geq 0 \\
& x_1^2 + x_2^2 \leq 2 \\
& x_1 - x_2 \leq 1
\end{aligned}
$$

You will implement different versions of the interior point method to solve the problem. Use (0.5, 0.5) as the starting point for all the algorithms.

(a) Plot the feasible region using python.

(b) Implement the barrier algorithm to solve the problem. Use $t^0 = 1$, $\mu = 4$, $\varepsilon = 10^{-4}$. For the Newton's method- use $\alpha = 0.2$, $\beta = 0.5$ and for convergence $\varepsilon_{newton} = 10^{-5}$. Also, plot path of the variables on the plot conceived in (a).

(c) Implement v1 of the primal dual interior point method. Use $t^0 = 1$, $\mu = 1.2$, $\varepsilon = 10^{-5}$. For backtracking, use $\alpha = 0.5$ and $\beta = 0.3$.

(d) Implement v2 of the primal dual interior point method. Use $u^0 = 1^T$, $\mu = 1.2$, $\varepsilon = 10^{-5}$. For backtracking, use $\alpha = 0.5$ and $\beta = 0.3$. Note: here $1^T = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^T$