# Lecture 8 Conic Programming

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# Why conic programming?

- Conic programming including Linear, Conic Quadratic and Semidefinite Programming can be seen as a special class of convex optimization problems that are "well-structured".
- Wide range of applications in fields like control theory, finance, signal processing, engineering, and machine learning

# Cone, convex cone, conic combination

**Cone**: $C \subseteq \mathbb{R}^n$ such that

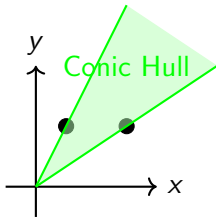$$x \in C \implies tx \in C \text{ for all } t \geq 0$$
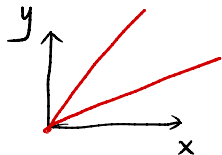
**Convex cone**: cone that is also convex, i.e.,

$$x_1, x_2 \in C \implies t_1 x_1 + t_2 x_2 \in C \text{ for all } t_1, t_2 \geq 0$$

**Conic combination** of $x_1, \ldots, x_k \in \mathbb{R}^n$ :

$$\lambda_1 x_1 + \cdots + \lambda_k x_k$$
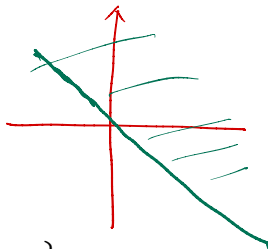
with $\lambda_i \geq 0, i = 1, \ldots, k$.

**Conic hull** collects all conic combinations

# Proper convex cones

We consider proper convex cones $\mathcal{K}$ in $\mathbb{R}^n$ :

- Closed
- Pointed: $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$.
- Non-empty interior.

Dual-cone:
$$\mathcal{K}^* = \left\{ v \in \mathbb{R}^n \mid u^T v \geq 0, \forall u \in \mathcal{K} \right\}$$

If $\mathcal{K}$ is a proper cone, then $\mathcal{K}^*$ is also proper.
We use the notation:

$$K = \{ x \mid x \geqslant 0 \}, \quad x \geqslant_k y$$

$$x \succeq_\mathcal{K} y \iff (x - y) \in \mathcal{K} \qquad x - y \in K$$

$$x \succ_\mathcal{K} y \iff (x - y) \in \operatorname{int} \mathcal{K} \qquad x - y \geqslant 0$$

$$\operatorname{int}(k) = \{ x \mid x > 0 \}$$

# What is conic programming?

$$\min \quad c^T x$$
$$\text{s.t} \quad Ax \geq b \quad (\lambda \geq 0)$$

$$D(\lambda) = \min_x c^T x + \lambda^T (b - Ax)$$
$$= \min_x (c^T - \lambda^T A) x + \lambda^T b$$

Let $\mathcal{K}$ be a proper convex cone in $\mathbb{R}^m$ (pointed, closed, and with a nonempty interior). Given an objective $c \in \mathbb{R}^n$, an $m \times n$ constraint matrix $A$, and a right-hand side $b \in \mathbb{R}^m$, consider the optimization problem

$$\max_\lambda \lambda^T b$$
$$\lambda \geq 0$$
$$c^T - \lambda^T A = 0^T$$

$$\min_x \left\{ c^T x \mid Ax \succeq_{\mathcal{K}} b \right\}$$

where $Ax \succeq_{\mathcal{K}} b \Leftrightarrow Ax - b \in \mathcal{K}$

- Linear program is a special case of conic program (why?).
  For LP, $\mathcal{K} = \{ y \in \mathbb{R}^m \mid y \geq 0 \}$.
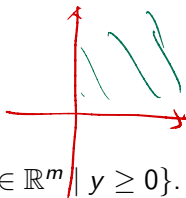
$$D(\lambda) = \min_x c^T x + \lambda^T (b - Ax)$$

$$\lambda \in \{ v \mid v^T u \geq 0, \ \forall u \in \mathcal{K} \}$$

$$\lambda \in \mathcal{K}^* \quad \text{for} \quad D(\lambda) \text{ to be a lower bound}$$

# Dual of conic program

The dual of the conic program is

$$\max\left\{ b^T\lambda \mid A^T\lambda = c, \lambda \succeq_{\mathcal{K}^*} 0 \right\}$$

where $\mathcal{K}^*$ represents the dual cone of $\mathcal{K}$.

- Note that for LP, $\mathcal{K}$ is self-dual. $\mathcal{K} = \mathcal{K}^* = \{y \in \mathbb{R}^m \mid y \geq 0\}$. We reproduce LP duality!

- The duality is **symmetric**: the dual problem is conic, and the problem dual to dual is (equivalent to) the primal.

- **Weak duality**: value of the dual objective at every dual feasible solution $\lambda$ is the value of the primal objective at every primal feasible solution $x$, so that the duality gap
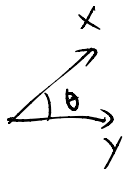
$$c^T x - b^T \lambda \geq 0$$

at every primal-dual feasible pair $(x, \lambda)$.

# Strong duality

- If the primal is bounded below and strictly feasible, i.e., $Ax \succ_{\mathcal{K}}$ for some $x$, then the dual is solvable and the optimal values in the problems are equal to each other.
- If the dual is bounded above and strictly feasible (i.e., exists $\lambda \succ_{\mathcal{K}_*} 0$ such that $A^T \lambda = c$), then the primal is solvable and we have strong duality.
- Assume that at least one of the problems is bounded and strictly feasible. Then a primal-dual feasible pair $(x, \lambda)$ is a pair of optimal solutions to the respective problems if and only if the following condition hold
  - $b^T \lambda = c^T x$    [zero duality gap]
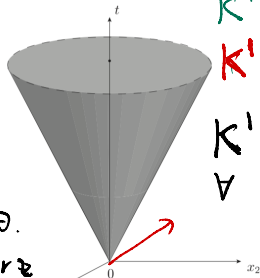  - $\lambda^T [Ax - b] = 0$    [complementary slackness].

# Second-order cone

- The second-order cone (SOC) in $\mathbb{R}^3$ is the set of vectors $(x_1, x_2, t)$ such that $\sqrt{x_1^2 + x_2^2} \leq t$. Horizontal sections of this set at level $\alpha \geq 0$ are disks of radius $\alpha$.



$$K^* = \{ v \mid v^T u \geq 0, \ \forall u \in K \}$$

$$K' = \{ (y, s) \mid \|y\|_2 \leq s \}$$

$$K' \subseteq K^*$$

$$\forall \ (y, s) \in K'.$$

$$(x, t) \in K_n$$

$$x^T y + \underset{\geq 0}{\underline{s \, t}} \geq 0$$

$x$

$\theta$

$y$

$$x^T y = \|x\|_2 \|y\|_2 \cos\theta.$$

Cauchy - Schwarz

$$(x^T y)^2 \leq \|x\|_2^2 \|y\|_2^2 \leq s^2 t^2$$

- In arbitrary dimension: an $(n+1)$-dimensional SOC is the following set:

$$\text{If} \quad \|y\|_2 > s.$$

$$\mathcal{K}_n = \{ (x, t), x \in \mathbb{R}^n, t \in \mathbb{R} : \|x\|_2 \leq t \}. \quad (-y, \|y_2\|)$$

$$\in K_n$$

- SOC is self-dual: $\mathcal{K}_n^* = \mathcal{K}_n$
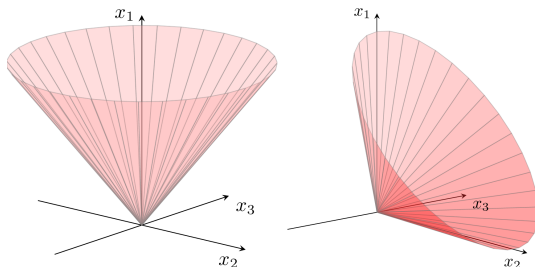
# Rotated second order cone



Figure: Boundary of second order cone $x_1 \geq \sqrt{x_2^2 + x_3^2}$ and rotated second cone $2x_1 x_2 \geq x_3^2, x_1, x_2 \geq 0$.

# Rotated second order cone

- The rotated second-order cone in $\mathbb{R}^{n+2}$ is the set

$$\mathcal{K}_n^r = \left\{ (x, y, z), x \in \mathbb{R}^n, y \in \mathbb{R}, z \in \mathbb{R} : x^\top x \leq 2yz, y \geq 0, z \geq 0 \right\}.$$

- The rotated second-order cone in $\mathbb{R}^{n+2}$ can be expressed as a linear transformation (actually, a rotation) of the (plain) second-order cone in $\mathbb{R}^{n+2}$, since

$$\|x\|_2^2 \leq 2yz, y \geq 0, z \geq 0 \iff \left\| \begin{bmatrix} x \\ \frac{1}{\sqrt{2}}(y-z) \end{bmatrix} \right\|_2 \leq \frac{1}{\sqrt{2}}(y+z).$$

That is, $(x, y, z) \in \mathcal{K}_n^r$ if and only if $(w, t) \in \mathcal{K}_n$, where

$$w = (x, (y-z)/\sqrt{2}), \quad t = (y+z)/\sqrt{2}.$$

# Standard SOC constraint

- The standard format of a second-order cone constraint on a variable $x \in \mathbb{R}^n$ expresses the condition that $(y, t) \in \mathcal{K}_m$, with $y \in \mathbb{R}^m, t \in \mathbb{R}$, where $y, t$ are some affine functions of $x$.

- These affine functions can be expressed as $y = Ax + b, t = c^\top x + d$, hence the condition $(y, t) \in \mathcal{K}_m$ becomes

$$\|Ax + b\|_2 \le c^\top x + d,$$

  where $A \in \mathbb{R}^{m,n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$, and $d \in \mathbb{R}$.

- For example, the quadratic constraint

$$x^\top Q x + c^\top x \le t, \quad Q \succeq 0$$

  can be expressed in conic form as

$$\left\| \begin{bmatrix} \sqrt{2} Q^{1/2} x \\ t - c^\top x - 1/2 \end{bmatrix} \right\|_2 \le t - c^\top x + 1/2.$$

# Second-order cone programs

- A second-order cone program is a convex optimization problem having linear objective and SOC constraints. When the SOC constraints have the standard form, we have a SOCP in standard inequality form:

$$\min_{x \in \mathbb{R}^n} \quad c^\top x$$
$$\text{s.t.:} \quad \|A_i x + b_i\|_2 \leq c_i^\top x + d_i, \quad i = 1, \ldots, m,$$

  where $A_i \in \mathbb{R}^{m_i, n}$ are given matrices, $b_i \in \mathbb{R}^{m_i}$, $c_i \in \mathbb{R}^n$ are vectors, and $d_i$ are given scalars.
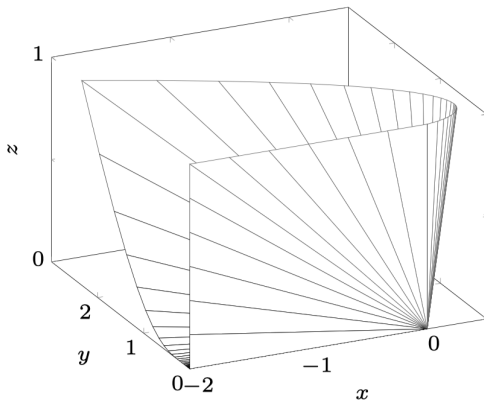
- SOCPs are representative of a quite large class of convex optimization problems. Indeed, LPs, convex QPs, and convex QCQPs can all be represented as SOCPs.

# Exponential cone

the epigraph of the perspective of $\exp x$ is a non-proper cone

$$\mathcal{K} = \left\{ (x, y, z) \in \mathbb{R}^3 \mid y e^{x/y} \leq z, y > 0 \right\}$$

the exponential cone is $\mathcal{K}_{\exp} = \operatorname{cl} \mathcal{K} = \mathcal{K} \cup \{(x, 0, z) \mid x \leq 0, z \geq 0\}$

# Exponential cone representable functions

- The epigraph $t \geq e^x$ is a section of $\mathcal{K}_{\exp}$ :

$$t \geq e^x \iff (x, 1, t) \in \mathcal{K}_{\exp} .$$

- Similarly, we can express the hypograph $t \leq \log x, x \geq 0$ :

$$t \leq \log x \iff (t, 1, x) \in \mathcal{K}_{\exp} .$$

- The entropy function $H(x) = -x \log x$ can be maximized using the following representation.:
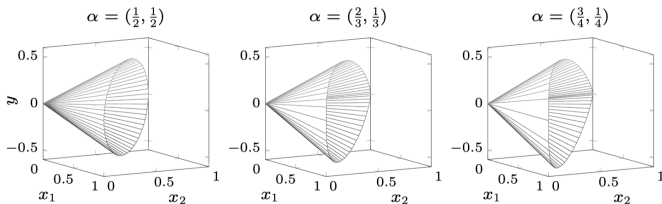
$$t \leq -x \log x \iff t \leq x \log(1/x) \iff (t, x, 1) \in \mathcal{K}_{\exp} .$$

# Power cone

Definition: for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) > 0$ and $\sum_{i=1}^{m} \alpha_i = 1$

$$K_\alpha = \left\{ (x, y) \in \mathbb{R}_+^m \times \mathbb{R} \mid |y| \leq x_1^{\alpha_1} \cdots x_m^{\alpha_m} \right\}$$

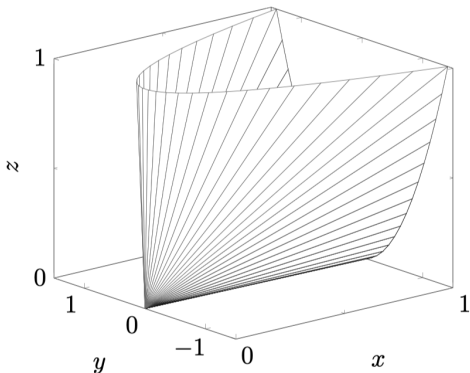Examples for $m = 2$

# Positive semidefinite cone

$$\mathcal{S}^p = \left\{ \text{vec}(X) \mid X \in \mathcal{S}^p_+ \right\} = \left\{ x \in \mathbb{R}^{p(p+1)/2} \mid \text{mat}(x) \succeq 0 \right\}$$

$$\mathcal{S}^2 = \left\{ (x, y, z) \,\middle|\, \begin{bmatrix} x & y/\sqrt{2} \\ y/\sqrt{2} & z \end{bmatrix} \succeq 0 \right\}$$

$X \geq 0, \quad z \geq 0$

$xz - \dfrac{y^2}{2} \geq 0$

# Basic properties of a PSD matrix

- We denote $n \times n$ symmetric matrices by $\mathcal{S}^n$.
- Standard inner product for matrices:

$$\langle V, W \rangle := \operatorname{tr}\left(V^T W\right) = \sum_{ij} V_{ij} W_{ij} = \operatorname{vec}(V)^T \operatorname{vec}(W)$$

- $X$ is positive semidefinite if and only if
  1. $z^T X z \geq 0, \forall z \in \mathbb{R}^n$.
  2. All the eigenvalues of $X$ are nonnegative.
  3. There exist $V \in \mathbb{R}^{r \times n}$ $r \leq n$ such that $X = V^T V$
- The positive (semi)definite matrices form a cone $(\mathcal{S}_+)\,\mathcal{S}_{++}$.

# Eigenvalue decomposition

The eigenvalue decomposition of a positive semi-definite (PSD)
matrix $A$ is given by:

$$A = Q \Lambda Q^T$$

where,

- $A$ is the PSD matrix.

- $Q$ is an orthogonal matrix whose columns are the eigenvectors
  of $A$.

- $\Lambda$ is a diagonal matrix with the eigenvalues of $A$ on its
  diagonal.

- $Q^T$ is the transpose of $Q$, ensuring the reconstruction of $A$
  through the product $Q \Lambda Q^T$.

# Pseudo-Inverse of a PSD Matrix

The pseudo-inverse of a PSD matrix $A$, denoted $A^\dagger$, holds special interest due to $A$'s properties:

$$A^\dagger = Q \Lambda^\dagger Q^T$$

where,

- $A = Q \Lambda Q^T$ is the eigenvalue decomposition of $A$, with $A$ being PSD.
- $\Lambda^\dagger$ contains the reciprocals of the non-zero eigenvalues of $A$ on its diagonal and zeros elsewhere.
- Vector product form: Suppose we have an eigenvalue decomposition of $Q = \sum_{i=1}^{k} \lambda_i q_i q_i^T$ where the $\lambda_i$, $i = 1, \ldots, k$ are the nonzero eigenvalues. $q_i$, $i = 1, \ldots, k$ are the corresponding orthonormal eigenvectors.

$$Q^\dagger := \sum_{i=1}^{k} \lambda_i^{-1} q_i q_i^T$$

# Schur's Lemma

**Given:**

- $Q \in \mathcal{S}_p$, $P \in \mathcal{S}_n$
- $S = [s_1, \ldots, s_n] \in \mathbb{R}^{p \times n}$

**Lemma Statement:** The following are equivalent:

(i) The matrix $\begin{bmatrix} P & S^\top \\ S & Q \end{bmatrix}$ is positive [resp. semi-]definite.

(ii) The matrices $Q$ and $P - S^\top Q^\dagger S$ are both positive [resp. semi-]definite, and $s_i \in \text{Range}(Q)$ for $i = 1, \ldots, n$; these range conditions are automatically satisfied when $Q \succ 0$ (then $\text{Range}(Q) = \mathbb{R}^p$).

**Implications:**

- Establishes a condition under which a block matrix is positive (semi-)definite.
- Highlights the relationship between the definiteness of a block matrix and its submatrices.

# SOC as SDP

The SOC constraint can be converted to the PSD contraint,
Consider the SOC constraint:

$$\|x\|_2 \leq t$$

This can be represented as a PSD constraint using the Schur's lemma:

$$\left[ \begin{array}{cc} t & x^T \\ x & tI \end{array} \right] \succeq 0$$

This indicates SDP is more "expressive" than SOCP.

# Dual cone of the PSD cone

Dual cone:

$$\left(\mathcal{S}_+^n\right)^* = \left\{Z \in \mathbb{R}^{n \times n} \mid \langle X, Z \rangle \geq 0, \forall X \in \mathcal{S}_+^n\right\}.$$

The semidefinite is self-dual: $\left(\mathcal{S}_+^n\right)^* = \mathcal{S}_+^n$.

**Proof**: Assume $Z \succeq 0$ so that $Z = U^T U$ and $X = V^T V$.

$$\langle X, Z \rangle = \left\langle V^T V, U^T U \right\rangle = \text{tr}\left(V^T V U^T U\right) = \text{tr}\left(U V^T V U^T\right) = \left\|U V^T\right\|_F^2 \geq 0.$$

The equalities come from the following properties of trace

$$\langle A, B \rangle = \text{tr}(A^T B)$$

$$\text{tr}(ABCD) = \text{tr}(BCDA) = \text{tr}(CDAB) = \text{tr}(DABC) \qquad \text{(cyclic property)}$$

$$tr(A^T A) = \langle A, A \rangle = \sum_{i,j} a_{i,j}^2 = \|A\|_F^2$$

Conversely assume $Z \not\succeq 0$. Then $\exists w \in \mathbb{R}^n$ such that

$$w^T Z w = \left\langle w w^T, Z \right\rangle = \langle X, Z \rangle < 0.$$

# Applications of SDP-Eigenvalue optimization

$$F(x) = \underbrace{F_0 + x_1 F_1 + \cdots + x_m F_m}_{\text{linear matrix inequality}}, \quad F_i \in \mathcal{S}_m.$$

- Minimize largest eigenvalue $\lambda_1(F(x))$ :

$$\begin{aligned} \text{minimize} \quad & \gamma \\ \text{subject to} \quad & \gamma I \succeq F(x), \end{aligned}$$

- Maximize smallest eigenvalue $\lambda_n(F(x))$ :

$$\begin{aligned} \text{maximize} \quad & \gamma \\ \text{subject to} \quad & F(x) \succeq \gamma I, \end{aligned}$$

- Minimize eigenvalue spread $\lambda_1(F(x)) - \lambda_n(F(x))$ :

$$\begin{aligned} \text{minimize} \quad & \gamma - \lambda \\ \text{subject to} \quad & \gamma I \succeq F(x) \succeq \lambda I \end{aligned}$$

# Relaxation of QUBO

Consider an quadratic unconstrained binary optimization (QUBO) problem

$$\begin{array}{ll}
\text{minimize} & x^T Q x + c^T x \\
\text{subject to} & x_i \in \{0,1\}, \quad i = 1, \dots, n
\end{array}$$

where $Q \in \mathcal{S}^n$ can be indefinite. QUBO is NP-hard in general.

- Rewrite binary constraints $x_i \in \{0,1\}$ and introduce lifted space variables $X \in \mathcal{S}^n$:

$$X_{ij} = x_i x_j, \quad x_i^2 = x_i \iff X = xx^T, \quad \text{diag}(X) = x.$$

- Semidefinite relaxation: *convex relaxation.*

$$X \succeq xx^T, \quad \text{diag}(X) = x.$$

# SDP relaxation of QUBO

Lifted non-convex problem:

$$\begin{aligned}
\text{minimize} \quad & \langle Q, X \rangle + c^T x \\
\text{subject to} \quad & \text{diag}(X) = x \\
& X = xx^T.
\end{aligned}$$

Note that

*relax*

$$\begin{pmatrix} X & x \\ x^T & 1 \end{pmatrix} \succeq 0 \Leftrightarrow X \succeq xx^T \text{ (Schur's lemma)}$$

Semidefinite relaxation:

$$\begin{aligned}
\text{minimize} \quad & \langle Q, X \rangle + c^T x \\
\text{subject to} \quad & \text{diag}(X) = x \\
& \begin{pmatrix} X & x \\ x^T & 1 \end{pmatrix} \succeq 0
\end{aligned}$$

Relaxation is exact if $X = xx^T$.

# A simple cutting plane algorithm for solving SDP

Semidefinite programming (SDP) problems can be formulated as:

$$\min_X \langle C, X \rangle$$

$$\text{s.t. } \langle A^i, X \rangle \geq b_i, \quad i = 1, \ldots, m,$$

$$X \succeq 0,$$

where $X$ is the semidefinite matrix variable, $C \in \mathcal{S}^n$ and $A^i \in \mathcal{S}^n$ are given matrices.

1. Start with only the $m$ linear constraints on $X$.

2. Compute the eigenvalues and eigenvectors of the current solution $\hat{X}$. If all eigenvalues are non-negative, $\hat{X}$ is feasible and thus optimal.

3. For each negative eigenvalue $\lambda_i$, $i = 1, \ldots, r$, use its corresponding eigenvector $v^i$, $i = 1, \ldots, r$ to add a linear cut $(v^i)^T X v^i \geq 0$ that excludes the current infeasible solution $\hat{X}$.

4. Add the linear constraints and repeat from step 2 until a feasible $X$ is found or the algorithm converges to a solution within a desired tolerance.

# Software for conic programming

- Gurobi, Cplex ((MI)SOCP)
- Mosek: mostly commonly used conic solver (SOCP, EXP, POW, SDP).
- COPT (SOCP, SDP)
- other solvers: see
  https://jump.dev/JuMP.jl/stable/installation/
  https://plato.asu.edu/bench.html
- most of the algorithms for conic programs are variants of the interior point algorithm.
- Although SDP is convex, solving it is usually "slow".
  $n = 100 - 1,000$ are the problem size we are usually talking about.

# References

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