ChE 597 Computational Optimization

Practice Exam 2

This is a closed book examination. Only a calculator and a single piece of note-book paper with hand-written notes are allowed. **Please print your name on the upper right corner of each page and on the last page where indicated. Your signature is required on the bottom of this page where indicated.** Read all questions carefully; answer them concisely and **neatly**. Show all work on these pages. Point values in the right-hand margins are based on 100points total for the examination.

Read everything in the following paragraphs, print your name, and sign your name prior to starting the exam. Failure to comply or complete the steps outlined below could result in a score of 0% on the exam.

During this exam, I (the undersigned) agree that I will:

1. Write my name in the upper right hand corner of every page that is to be graded for the exam. If your name is not placed on a page, that page will not be graded.
2. Circle or box the final answer. Any answer that is not circled or boxed will not be graded.
3. Show all work and state all assumptions that are made.
4. Not cheat in any manner whatsoever. I understand that if I am caught cheating by verbal, written, or any other method that I will receive an F for the course. Furthermore, my actions will be reported to the Dean of Students.
5. Submit the completed exam to one of the instructors or teaching assistants at the end of the exam.
6. Not carry the exam out of the examination room.

I understand all of these conditions and will abide by them.

Print Name:

Signature: (Blank page)

1. Derive a valid convex relaxation of the following nonconvex optimization problem using factorization and convex envelopes of univariate functions.

$f(x)$ is convex.

$f(x) \leq 0 \Rightarrow$ convex

$f(x) = 0$ not convex.

$$\min x_1 + x_2$$

$$\text{s.t.} \quad \frac{\exp(\sqrt{x_2} + \log(x_1))}{x_1} \leq x_1^2$$

$$1 \leq x_1 \leq 2, 0 \leq x_2 \leq 1$$

[15 pts]

Factorization: $\exp(\sqrt{x_2} + \log(x_1)) \leq x_1^3$

$y_1 = x_1^3$, $y_2 = \sqrt{x_2}$, $y_3 = \log(x_1)$

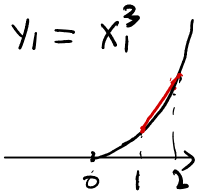
$y_4 = y_2 + y_3$, $y_5 = \exp(y_4)$

$y_5 \leq y_1$

$y_1 \geq x_1^3$

$y_1 \leq x_1^3$

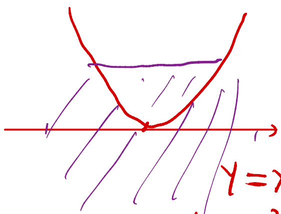
$y_1 = x_1^3$



$y_1 \geq x_1^3$ (convex underestimator)

$x_1 = 1, y_1 = 1$. $x_1 = 2, y_1 = 8$

$y_1 \leq \frac{8-1}{2-1} x_1 - 6 = 7x_1 - 6$

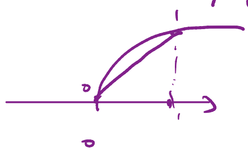


$Y = x^2$.

$Y \geq x^2$

$Y \leq x^2$

$y_2 = \sqrt{x_2}$



$y_2 \geq \sqrt{x_2}$ (nonconvex)

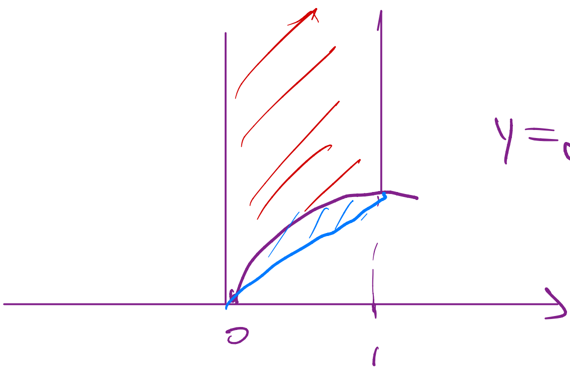
$y_2 \leq \sqrt{x_2}$ (convex)

$y_2 \geq x_2$

convex and concave envelope.

$y_2 \geq x_2$ (convex envelop

$y_2 \leq \sqrt{x_2}$ (concave envelop)



$y = \sqrt{x}$

2. (a) Given are the two following optimization problems:

P1:

$$\begin{aligned} \min Z^1 \quad &= f(x) \\ \text{s.t.} \quad & g(x) \leq 0 \\ & h(x) \leq 0 \\ & x \in R^n \end{aligned}$$

$A = \{x \mid g(x) \leq 0\}$
$B = \{x \mid h(x) \leq 0\}$

Feasible region
$x \in A \cap B$

P2:

$$\begin{aligned} \min Z^2 = f(x) \\ \text{s.t. } g(x) \leq 0 \vee h(x) \leq 0 \\ x \in R^n \end{aligned}$$

where $f(x), g(x)$ and $h(x)$ can be any continuous functions (may not be convex). Prove whether the optimal objective function values of the above problem obey either of the two following inequalities:

$$(Z^1)^* \geq (Z^2)^* \text{ or } (Z^1)^* \leq (Z^2)^*$$

Feasible region
$x \in A \cup B$

$A \cap B \subseteq A \cup B$

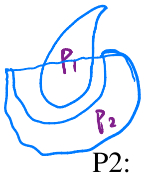
$\Rightarrow (Z^1)^* \geq (Z^2)^*$

$A = \{x \mid g(x) \le 0\}$

$B = \{x \mid h(x) \le 0\}$

(b) Consider the following two problems,

P1:



$\mathrm{conv}(P_1 \cap P_2) = P$

$\mathrm{conv}(P_1) \cap \mathrm{conv}(P_2)$

$$\begin{aligned} \min Z^1 &= f(x) \\ \text{s.t.} \quad & x \in \mathrm{conv}\big(\{x \mid g(x) \le 0\}\big) \\ & x \in \mathrm{conv}\big(\{x \mid h(x) \le 0\}\big) \\ & x \in R^n \end{aligned}$$

$x \in \mathrm{conv}(A) \cap \mathrm{conv}(B)$

P2:

$$\begin{aligned} \min Z^2 &= f(x) \\ \text{s.t.} \quad & x \in \mathrm{conv}\big(\{x \mid g(x) \le 0, h(x) \le 0\}\big) \\ & x \in R^n \end{aligned}$$

$x \in \mathrm{conv}(A \cap B)$

where $\mathrm{f(x)}, \mathrm{g(x)}$ and $\mathrm{h(x)}$ can be any continuous functions (may not be convex). Prove whether the optimal objective function values of the above problem obey either of the two following inequalities:

$$(Z^1)^* \ge (Z^2)^* \text{ or } (Z^1)^* \le (Z^2)^*$$

[20 pts]

Goal: Take any arbitrary point $\hat{x} \in \mathrm{conv}(A \cap B)$

$\Rightarrow \hat{x} \in \mathrm{conv}(A) \cap \mathrm{conv}(B)$

$\exists\ x_1 \dots x_n \in A \cap B$, $\lambda_i \ge 0$, $\sum_{i=1}^{n} \lambda_i = 1$

s.t. $\hat{x} = \sum_{i=1}^{n} \lambda_i x_i$

$x_1 \dots x_n \in A$. $\quad x_1 \dots x_n \in B$

pick same λ_i $\quad \hat{x} = \sum_{i=1}^{n} \lambda_i x_i \quad \hat{x} = \sum_{i=1}^{n} \lambda_i x_i$

$\Downarrow \quad \Downarrow$

$\hat{x} \in \mathrm{conv}(A) \quad \hat{x} \in \mathrm{conv}(B)$

$\Rightarrow \hat{x} \in \mathrm{conv}(A) \cap \mathrm{conv}(B)$

$(Z^1)^* \le (Z^2)^*$

3. Given is the linear bi-level optimization problem:

$$
\begin{aligned}
&\min Z = a^T u + b^T v \\
&\text{s.t.} \quad Au + Bv \leq d \\
&u \geq 0, \cancel{v \geq 0} \\
&v \in \arg\min_w c^T w \\
&\quad \text{s.t.} \quad \alpha_i^T w + \beta_i^T u \leq \gamma_i \quad i = 1 \ldots n \\
&\quad w \geq 0
\end{aligned}
$$

optimality condition
(KKT condition
strong duality)

in which the variable v is constrained to satisfy an inner minimization problem. Show that the above problem can be reformulated as an MILP.

[15 pts]

4. Consider the following problem,

Ay

$Ay^k + A(y - y^k)$

$$\begin{aligned} \min_{x,y} \quad & c^T x + d^T y \\ \text{s.t.} \quad & Ay + g(x) \leq 0 \\ & Bx \leq b \\ & y \in \{0,1\}^{n_y} \\ & x \geq 0 \end{aligned}$$

$A \in \mathbb{R}^{m \times n}$

$g \quad \mathbb{R}^n \to \mathbb{R}^m$

Derive a generalized Benders cut for $y = y^k$.

[15 pts]

$$\min \; d^T y + V(y) \qquad V(y) := \min_x \; c^T x$$

$$y \in \{0,1\}^{n_y} \qquad Ay + g(x) \leq 0, \quad Bx \leq b, \quad x \geq 0.$$

$$\min \; d^T y + \alpha$$

$$\alpha \geq (\beta^k)^T y + r^k \quad \forall k = 1..K.$$

$$y \in \{0,1\}^{n_y}$$

$$V(y^k) = \min_x \; c^T x.$$

$$Ay^k + g(x) \leq 0 \quad (\lambda_i^k \geq 0)$$

$$Bx \leq b \quad (\mu^k \geq 0)$$

$$x \geq 0 \quad (v^k \geq)$$

x^k

Stationarity condition:

$$c + \sum_{i=1}^m \lambda_i^k \nabla g_i(x^k) + B^T \mu^k - v^k = 0. \quad (1)$$

OA cuts at x^k.

$$\alpha \geq c^T x, \quad (\times 1)$$

$$(Ay^k)_i + g_i(x^k) + \nabla g_i^T(x^k)(x - x^k) \leq 0, \quad (\lambda_i^k)$$

$$Bx \leq b \quad (\mu^k)$$

$$x \geq 0 \quad (v^k)$$

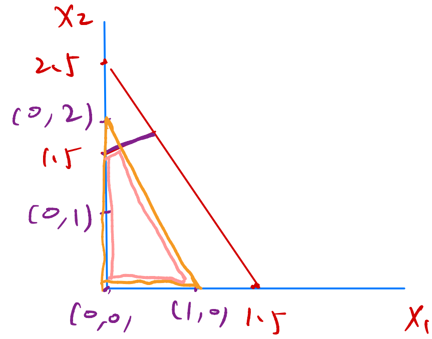
$$(1)^T (x - x^k)$$

$$c^T(x - x^k) + \sum_{i=1}^m \lambda_i^k \nabla g_i^T(x^k)(x - x_k) + (\mu^k)^T B(x - x_k) - (v^k)^T(x - x_k) = 0.$$

5. Consider the following MILP,

primal characterization of Lagrangian relaxation

$$
\begin{aligned}
\max\ \cancel{\min}\quad & x_1 + x_2 \\
\text{s.t. }\ & 5x_1 + 3x_2 \leq 7.5 \\
& x_2 \leq \frac{3}{2} + x_1 \\
& x_1 \in \mathbb{Z}_+, \quad x_2 \in \mathbb{Z}_+
\end{aligned}
$$



where $\mathbb{Z}_+$ means nonnegative integer.

(a) What is the optimal solution of the LP relaxation of this problem and its optimal objective value.

(b) Consider the Lagrangian relaxation where we dualize $x_2 \leq \frac{3}{2} + x_1$. What is the value of the Lagrangian relaxation? How does it compare with the LP relaxation?

Hint: you can draw a 2-D picture to help you visualize this problem.

6. *Short Questions* Bilevel optimization

(a) It is desired to formulate as follows a robust optimization problem with a minimax criterion for the objective function where θ is the vector of uncertain parameters defined by the uncertainty set he set $\mathrm{T} = \{\theta \mid \theta^L \leq \theta \leq \theta^U\}$:

$$\begin{aligned} &\min_{x} \max_{\theta \in T} Z = f(x,\theta) \\ &\text{s.t.} \quad g(x,\theta) \leq 0 \quad \forall \theta \in T \\ &\quad x \in R^n \end{aligned}$$

Assuming that you have available a solver for solving semi-infinite programming problems, is it possible to apply it to the above problem? If so, how would you reformulate the above problem?

$$\min_{x} \ \alpha$$

$$\alpha \geq f(x,\theta) \quad \forall \theta \in T$$

$$g(x,\theta) \leq 0 \quad \forall \theta \in T$$

$$x \in R^n$$

(b) The logic condition

$$y_1 \Rightarrow y_2, \quad \neg y_1 \vee \neg y_2$$

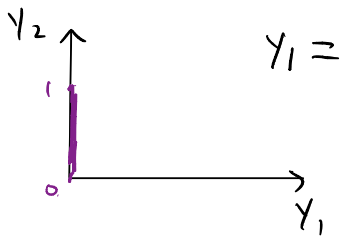
can be represented with the constraints $y_1 = 0$, $y_2 = \delta$, $\delta \in \{0,1\}$, $0 \le y_2 \le 1$. Is this tighter model than using the following linear inequalities that are derived from CNF form, $y_1 \le y_2$, $y_1 + y_2 \le 1$, $y_1 \in \{0,1\}$, $y_2 \in \{0,1\}$?

① LP relaxation.

$$y_1 = 0, \quad y_2 = \delta, \quad 0 \le \delta \le 1, \quad 0 \le y_2 \le 1$$

$$\Downarrow$$

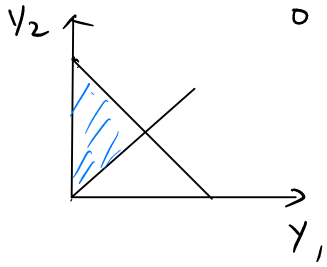
$$y_1 = 0, \quad 0 \le y_2 \le 1$$



Show that all the constraints in the second formulation is implied by the constraints in the first one.

② $y_1 \le y_2, \quad y_1 + y_2 \le 1$

$0 \le y_1 \le 1, \quad 0 \le y_2 \le 1$



① $y_1 \le y_2$

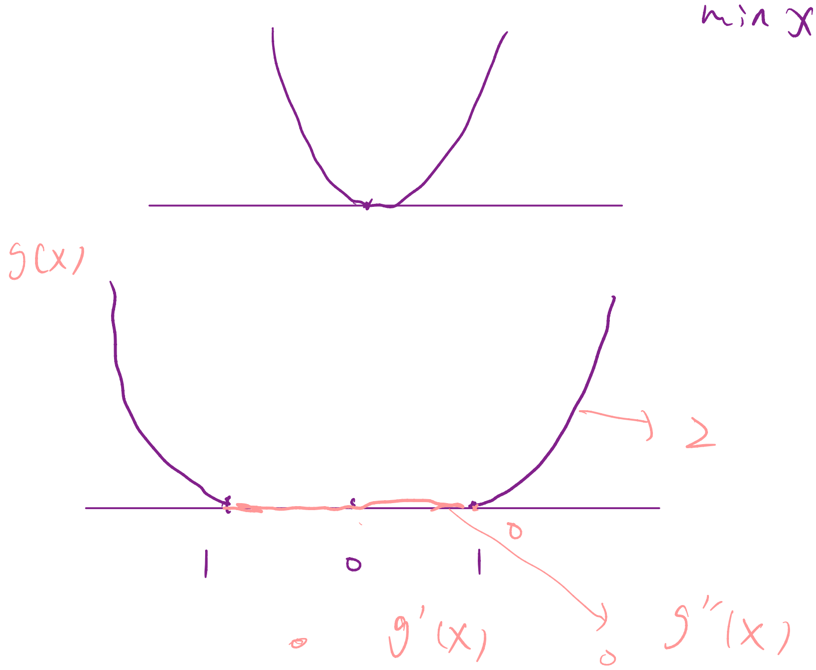
$y_1 = 0 \quad (\times 1)$

$0 \le y_2 \quad (\times 1)$

$y_1 \le y_2$

(c) Is it possible for a nonlinear programming problem that has a convex objective function and convex feasible region to have multiple optima? If so, give a small example. If not, write a formal proof that the solution is always unique.

min x^2



(d) Replace the following nonlinear inequality by linear inequalities that rigorously bounds the feasible space defined by that inequality:

$$x_1 x_2 \leq 8 \quad \text{where } 0 \leq x_1 \leq 4 \quad 0 \leq x_2 \leq 6$$

[20 pts]

Macormick envelopes

$w \leq 8$

$w \geq$ - - -

$w \geq$ - - -

$w \leq$ - - -

$w \leq$ - - -