# ChE 597 Computational Optimization

## Homework 3: Solutions

1. In this question we will model an LP Transsshipment model for a heat exchanger network to determine the minimum utility consumption for the two hot and two cold streams given below:

   Data for the problem:

   | | Fep (MW/C) | Tin (C) | Tout (C) |
   |---|---|---|---|
   | H1 | 1 | 400 | 120 |
   | H2 | 2 | 340 | 120 |
   | C1 | 1.5 | 160 | 400 |
   | C2 | 1.3 | 100 | 250 |

   Steam : 500°C
   Cooling water: 20 – 30°C
   Minimum recovery approach temperature (HRAT): 20°C

   The data for this problem are displayed below in Figure 1, where heat contents of the hot and cold processing streams are shown at each of the temperature intervals, which are based on the inlet and highest and lowest temperatures.

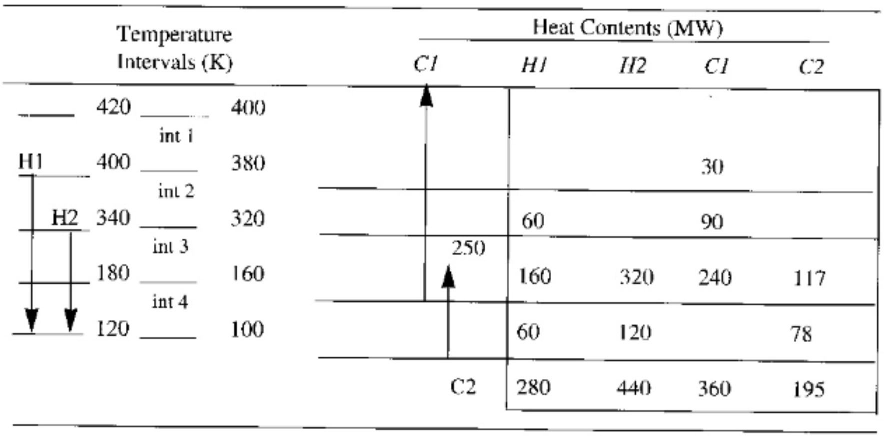


Figure 1: Data interpretation over temperature intervals

The flows of the heat contents can be represented in the heat cascade diagram of Figure 2. Here the heat contents of the hot streams are introduced in the corresponding intervals, while the heat contents of the cold streams are extracted also from their corresponding intervals. If the cold stream cannot absorb all the heat from the heat streams at a given interval, the rest of the heat will be transported to the next heat interval as "heat residuals". The variables $R_1, R_2, R_3$, represent heat residuals, while variables $Q_s Q_w$ represent the heating and cooling loads respectively.

The values in Figure 2 are calculated as the $Fep \times \Delta T$. For example, the heat transported from stream $H_1$ to interval 2 is calculated as $1MW/C \times (400C - 340C) = 60MW$.

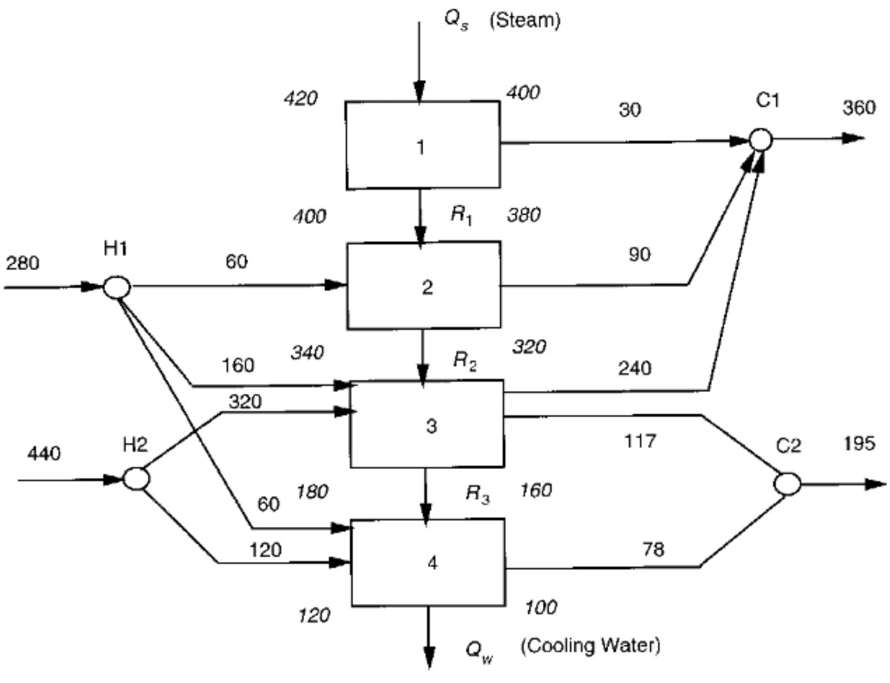


Figure 2: Heat Cascade Diagram

The usefulness of the heat cascade diagram in Figure 2 is that it can be regarded as a transshipment problem that we can formulate as a linear programming problem (Papoulias and Grossmann, 1983). In terms of the transshipment model, hot streams are treated as source nodes, and cold streams as destination nodes. Heat can then be regarded as a commodity that must be transferred from the sources to the destinations through some intermediate "warehouses" that correspond to the temperature intervals that guarantee feasible heat exchange. When not all of the heat can be allocated to the destinations (cold streams) at a given temperature interval, the excess is cascaded down to lower temperature intervals through the heat residuals.

Answer the following questions:

(a). Formulate the minimum utility ($\min Q_s + Q_w$) consumption problem as an LP transshipment problem, utilizing the cascade diagram provided in Figure 2. When formulating the constraints, you need to account for heat balances around each temperature level depicted in the cascade diagram in Figure 2.

(b). Solve the LP problem using Pyomo and determine the values of $R_1, R_2, R_3, Q_s, Q_w$ at optimum

**Solution:** (a) Let us consider first the heat balances around each temperature level cascade diagram. These are given by:

$$R_1 + 30 = Q_s$$
$$R_2 + 90 = R_1 + 60$$
$$R_3 + 357 = R_2 + 480$$
$$Q_w + 78 = R_3 + 180$$

It is clear that we have a system of 4 equations in 5 unknowns: $R_1, R_2, R_3, Q_{s^*} Q_{w^*}$ Thus, there is one degree of freedom, which in turn implies that we have an optimization problem.

By considering the objective of minimization of utility loads, our problem can be formulated as the I.P:

$$\begin{aligned}
\min \quad & Z = Q_s + Q_w \\
\text{s.t.} \quad & R_1 - Q_s = -30 \\
& R_2 - R_1 = -30 \\
& R_3 - R_2 = 123 \\
& Q_w - R_3 = 102 \\
& Q_s, Q_w, R_1, R_2, R_3 \geq 0
\end{aligned}$$

If we solve this problem with a standard Pyomo, we obtain for the utilities $Q_s =$ 60MW, $Q_w = 225$MW, and for the residuals $R_1 = 30$MW, $R_2 = 0, R_3 = 123$MW. Since $R_2 = 0$ this means that we have a pinch point at the temperature level $340° -$ 320°C, which lies between intervals 2 and 3.

(b) The Pyomo code for this problem can be found here.
`https://github.com/li-group/ChE-597-Computational-Optimization/blob/main/HW%203/HW3%20Q1b.ipynb`

2. Consider the following linear programming problem:

   Maximize:

$$Z = x_1 + 2x_2$$

   Subject to the constraints:

$$\begin{aligned} &1. \quad x_1 - x_2 \geq -2 \\ &2. \quad x_1 + x_2 \geq 1 \\ &3. \quad x_1 \geq 0 \\ &4. \quad x_2 \geq 0 \end{aligned}$$

   (a) **Extreme Points:** Determine the vertices (extreme points) of the polyhedron.

   (b) **Extreme Rays:** Determine the extreme rays of the polyhedron.

   (c) **Optimal Solution:** Does this problem have a finite optimal solution? If yes, explain which of the extreme points is the optimal solution. If not, explain why the problem does not have a finite optimal solution.

**Solution:** Given linear programming problem is as follows:

Maximize:

$$Z = x_1 + 2x_2$$

Subject to the constraints:

$$\begin{aligned} &1. \quad x_1 - x_2 \geq -2 \\ &2. \quad x_1 + x_2 \geq 1 \\ &3. \quad x_1 \geq 0 \\ &4. \quad x_2 \geq 0 \end{aligned}$$

A plot of the feasible region indicating the extreme points and extreme rays is shown in Figure 3.

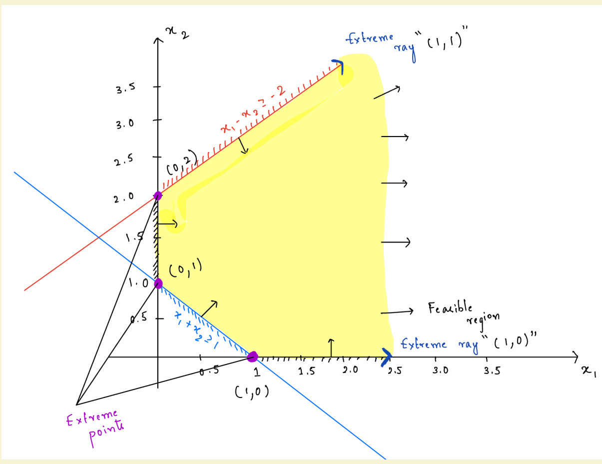


Figure 3: Feasible region with extreme points and extreme rays

(a) **Extreme points**

There are two approaches to finding the extreme points: one involves addressing the problem using the formulation provided in the question, while the other involves addressing the problem in standard form.

i. Approach 1:

In this approach, we solve for extreme points using the formulation presented in the question.
Extreme points of a polyhedron are known to occur at the intersections of half-spaces and/or hyper-planes. Considering our domain is in $\mathbb{R}^2$, we can represent each constraint as a straight line. Therefore, we can determine all intersections by examining the points where these lines intersect, and subsequently, assess whether the point of intersection qualifies as a feasible solution for the problem.

1. Combination 1:

$$\begin{aligned} x_1 - x_2 &= -2 \\ x_1 + x_2 &= 1 \end{aligned}$$

The intersection point of these two lines is (-0.5, 1.5). This point is not a feasible solution to the problem, hence not an extreme point.

2. Combination 2:

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 &= 0 \end{aligned}$$

The intersection point of these two lines is (0, 1). This point is a feasible solution to the problem, hence an extreme point.

3. Combination 3:

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \end{aligned}$$

The intersection point of these two lines is (0, 0). This point is not a feasible solution to the problem, hence not an extreme point.

4. Combination 4:

$$\begin{aligned} x_1 - x_2 &= -2 \\ x_2 &= 0 \end{aligned}$$

The intersection point of these two lines is (-2, 0). This point is not a feasible solution to the problem, hence not an extreme point.

5. Combination 5:

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_2 &= 0 \end{aligned}$$

The intersection point of these two lines is (1, 0). This point is a feasible solution to the problem, hence an extreme point.

6. Combination 6:

$$\begin{aligned} x_1 - x_2 &= -2 \\ x_1 &= 0 \end{aligned}$$

The intersection point of these two lines is (0, 2). This point is a feasible solution to the problem, hence an extreme point.

Hence the extreme points are (0, 1), (1, 0), (0, 2).

(b) **Extreme Rays**
Consider the original formulation

Maximize:

$$Z = x_1 + 2x_2$$

Subject to the constraints:

1. $x_1 - x_2 \geq -2$
2. $x_1 + x_2 \geq 1$
3. $x_1 \geq 0$
4. $x_2 \geq 0$

The recession cone for the above inequalities is defined as follows: $C = \{(d_1, d_2) | \, Constraints\}$, with the constraints being the following:

1. $d_1 - d_2 \geq 0$
2. $d_1 + d_2 \geq 0$
3. $d_1 \geq 0$
4. $d_2 \geq 0$

Now, it is established that the extreme rays of the recession cone coincide with the extreme rays of our original polyhedron.
Additionally, if $(d_1, d_2)$ represents an extreme ray, only n-1 active constraints should be present. As our problem is in $\mathbb{R}^2$ (i.e., n = 2), we will have one active constraint.

Taking $d_1 - d_2 \geq 0$ as the active constraint, we find that $d_1 = d_2$, resulting in the normalized point (1,1). This point satisfies all other constraints in the recession cone, making it an extreme ray.

Considering $d_1 + d_2 \geq 0$ as the active constraint, we have $d_1 = -d_2$, leading to the normalized point (-1,1). However, this point does not satisfy the other constraints in the recession cone, rendering it not an extreme ray.

Selecting $d_1 \geq 0$ as the active constraint, we get $d_1 = 0$, yielding the normalized point (0,1). Nonetheless, this point fails to satisfy the other constraints in the recession cone, making it not an extreme ray.

Opting for $d_2 \geq 0$ as the active constraint, we find that $d_2 = 0$, resulting in the normalized point (1,0). This point satisfies the other constraints in the recession cone, classifying it as an extreme ray.

Therefore the extreme rays are (1,1) and (1,0).

(c) **Optimal Solution**

We have the following corollary applicable to a minimization problem.

**Corollary**: For the minimization problem of minimizing $c^T x$ over a pointed polyhedral cone $C = \{x \in \mathbb{R}^n | a_i^T x \geq 0,\ i = (1,m)\}$, the optimal cost is equal to $-\infty$ if and only if there exists some extreme ray $d$ of $C$ such that $c^T d < 0$.

Rewriting this corollary for a maximization problem, we obtain:

**Corollary**: For the maximization problem of maximizing $c^T x$ over a pointed polyhedral cone $C = \{x \in \mathbb{R}^n | a_i^T x \geq 0,\ i = (1,m)\}$, the optimal cost is equal to $\infty$ if and only if there exists some extreme ray $d$ of $C$ such that $c^T d > 0$.

Given that the extreme rays for the polyhedron are $r_1 = [1,1]$ and $r_2 = [1,0]$, and the objective vector is $c = [1,2]$, we observe that both $c^T r_1 > 0$ and $c^T r_2 > 0$. This implies that the objective function grows without bounds, indicating an unbounded problem. Consequently, the optimal solution is $\infty$.

In conclusion, the problem lacks a finite optimal solution.

3. Given is the following linear programming problem:

$$
\begin{aligned}
\min \quad & Z = -2x_1 - 3x_2 \\
\text{s.t.} \quad & x_1 + x_2 \leq 4 \\
& x_1 + 2x_2 \leq 6 \\
& x_1 \geq 0, x_2 \geq 0
\end{aligned}
$$

(a) Plot the contours of the objective and the feasible region. Determine the optimum by inspection.

(hint: do not add slack variables. Just plot it on a 2D plane)

(b) Solve the above problem using the simplex algorithm.

(hint: add slack variables first. You can either use the vanilla simplex algorithm or the full tableau method. Solving it by hand is enough. However, if you would like to try solving it by implementing a simplex algorithm-based solver in Python, it is encouraged as well. )

**Solution:** (a) Plot the contours of the objective and the feasible region. Determine the optimum by inspection.

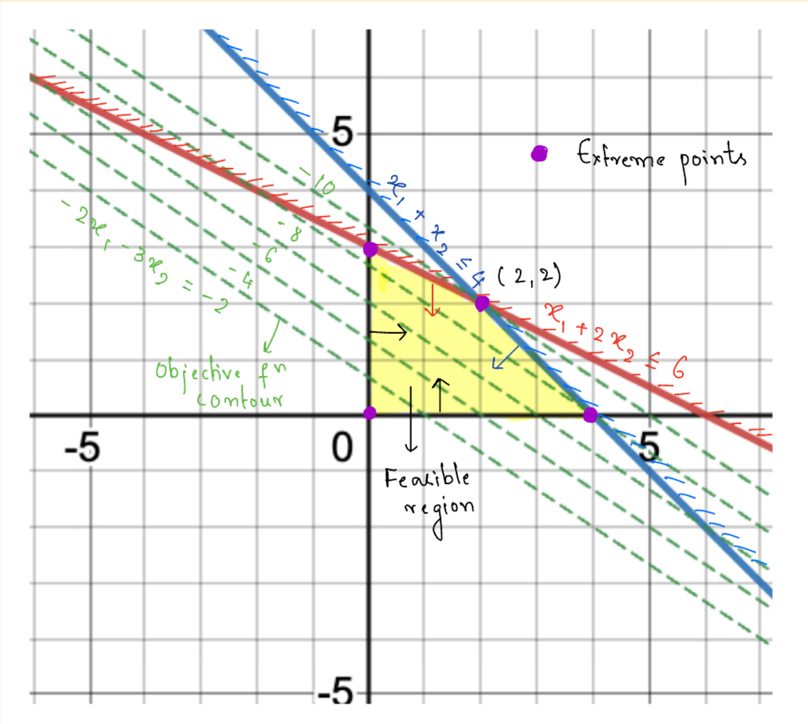


Figure 4: Feasible region with contours of objective function

Upon inspecting Figure 4, it is evident that the optimal solution is (2,2). This determination is based on the observation that the contour of the objective, with a value of -10, intersects this point and represents the minimum value of the objective function across the entire feasible region.

(b) Solve the above problem using the simplex algorithm.

Given the following linear programming problem:

$$\begin{aligned} \min \quad & Z = -2x_1 - 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 4 \\ & x_1 + 2x_2 \leq 6 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

After introducing slack variables, we obtain the following standard form problem:

$$\begin{aligned} \min \quad & Z = -2x_1 - 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 4 \\ & x_1 + 2x_2 + x_4 = 6 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0 \end{aligned}$$

I am using the **Tableau Simplex method** to solve the problem.

Note that $\mathbf{x} = (0,0,4,6)$ is a basic feasible solution and can be used to start the algorithm. Let accordingly, $B(1) = 3, B(2) = 4$ be the indices for the basic variables. The corresponding basis matrix is the identity matrix I. To obtain the zeroth row of the initial tableau, we note that $\mathbf{c}_B = \mathbf{0}$ and, therefore, $\mathbf{c}_B'\mathbf{x}_B = 0$ and $\bar{\mathbf{c}} = \mathbf{c}$ Hence, we have the following initial tableau:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| 0 | -2 | -3 | 0 | 0 |
| $x_3$ = 4 | 1* | 1 | 1 | 0 |
| $x_4$ = 6 | 1 | 2 | 0 | 1 |

We note a few conventions in the format of the above tableau: the label $x_i$ on top of the ith column indicates the variable associated with that column. The labels " $x_i =$ " to the left of the tableau tell us which are the basic variables and in what order. For example, the first basic variable $x_{B(1)}$ is $x_3$, and its value is 4.

The reduced cost of $x_1$ is negative and we let that variable enter the basis. The pivot column is $\mathbf{u} = (1,1)$. We form the ratios $x_{B(i)}/u_i, i = 1,2$; the smallest ratio corresponds to $i = 1$ . Hence, we have $\ell = 1$. This determines the pivot element, which we indicate by an asterisk. The first basic variable $x_{B(1)}$, which is $x_3$, exits the basis. The new basis is given by $\bar{B}(1) = 1, \bar{B}(2) = 4$. We multiply the pivot row by 2 and add it to the zeroth row. We subtract the pivot row from the second row. This leads us to the new tableau:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| 8 | 0 | -1 | 2 | 0 |
| $x_1$ = 4 | 1 | 1 | 1 | 0 |
| $x_4$ = 2 | 0 | 1* | -1 | 1 |

The corresponding basic feasible solution is $\mathbf{x} = (4,0,0,2)$. In terms of the original variables $x_1, x_2$, we have moved to point $D = (4,0)$ .

Now, the reduced cost of $x_2$ is negative and we let that variable enter the basis. The pivot column is $\mathbf{u} = (1,1)$. We form the ratios $x_{B(i)}/u_i, i = 1,2$; the smallest ratio corresponds to $i = 2$. Hence, we have $\ell = 2$. This determines the pivot element, which we indicate by an asterisk. The second basic variable $x_{B(2)}$, which is $x_4$, exits the basis. The new basis is given by $\bar{B}(1) = 1, \bar{B}(2) = 2$. We multiply the pivot row by 1 and add it to the zeroth row. We subtract the pivot row from the first row. This leads us to the new tableau:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| 10 | 0 | 0 | 1 | 1 |
| $x_1$ = 2 | 1 | 0 | 2 | -1 |
| $x_2$ = 2 | 0 | 1 | -1 | 1 |

Observing that the reduced cost of each variable is non-negative, we conclude that the optimal solution, specifically (2, 2, 0, 0), has been attained, leading to the termination of the simplex algorithm at this stage. Expressing the result in the original variables x1 and x2, we have transitioned to the coordinates (2, 2). The optimal objective value at this point is -10.

4. Let $x$ be an element of the standard form polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$. Prove that a vector $d \in \mathbb{R}^n$ is a feasible direction at $x$ if and only if $Ad = 0$ and $d_i \geq 0$ for every $i$ such that $x_i = 0$.

**Solution:** Let $\mathbf{x} \in P$ be arbitrary feasible solution, i.e., $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

Suppose that $\mathbf{d} \in \Re^n$ is a feasible direction at $\mathbf{x}$, i.e., there exists a $\theta > 0$ such that $x + \theta\mathbf{d} \in P$. Then, we must have $\mathbf{A}(\mathbf{x} + \theta\mathbf{d}) = \mathbf{b}$. Since $\mathbf{Ax} = \mathbf{b}$, we obtain $\mathbf{Ad} = (\mathbf{b} - \mathbf{Ax})/\theta = \mathbf{0}$. This proves the first part of our problem.

Now, $\mathbf{x} + \theta\mathbf{d}$ is nonnegative, i.e., $\mathbf{x} + \theta\mathbf{d} \geq \mathbf{0}$. At any index $i$ such that $x_i = 0$, we have $x_i + \theta d_i = \theta d_i \geq 0$. Dividing by the positive $\theta > 0$, we get $d_i \geq 0$, as desired. This proves the second part of the problem.

5. Let $x$ be a basic feasible solution associated with some basis matrix $B$. Prove the following:

   (a) If the reduced cost of every nonbasic variable is positive, then $x$ is the unique optimal solution.

   (b) If $x$ is the unique optimal solution and is nondegenerate, then the reduced cost of every nonbasic variable is positive.

**Solution:** (a) If the reduced cost of the nonbasic variable is positive, then $x$ is the unique optimal solution.

Let $\mathbf{y} \in P : \mathbf{y} \neq \mathbf{x}$ be an arbitrary feasible solution, and $\mathbf{d} = \mathbf{y} - \mathbf{x}$. Feasibility implies that $\mathbf{Ax} = \mathbf{Ay} = \mathbf{b}$ and, therefore, $\mathbf{Ad} = \mathbf{0}$. The latter equality can be rewritten in the form

$$\mathbf{B}\mathbf{d}_B + \sum_{j \in N} \mathbf{A}_j d_j = 0$$

where $N$ is the set of indices corresponding to the nonbasic variables under the given basis. Since $\mathbf{B}$ is invertible, we obtain

$$\mathbf{d}_B = -\mathbf{B}^{-1} \sum_{j \in N} \mathbf{A}_j d_j,$$

and

$$\mathbf{c}'\mathbf{d} = \mathbf{c}_B'\mathbf{d}_B + \sum_{j \in N} c_j d_j = \sum_{j \in N} \left(c_j - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{A}_j\right) d_j = \sum_{j \in N} \bar{c}_j d_j$$

For any nonbasic index $j \in N$, we must have $x_j = 0$; thus,

$$\mathbf{c}'\mathbf{d} = \sum_{j \in N} \bar{c}_j y_j.$$

Since $\mathbf{y}$ is feasible, we have $y_j \geq 0$. Suppose for the sake of contradiction that $y_j = 0$ for all nonbasic indices $j \in N$. Then $\mathbf{Ay} = \mathbf{By} = \mathbf{b}$ and thus $\mathbf{y}_B = -\mathbf{B}^{-1}\mathbf{b} = \mathbf{x}_B$. But this would imply that $\mathbf{y}$ and $\mathbf{x}$ agree both on basic and non-basic variables and are thus equal - a contradiction. Thus, there exists at least one $k \notin B$ such that $y_k \neq 0$ and thus

$$\mathbf{c}'\mathbf{d} = \sum_{j \in N} \bar{c}_j y_j \geq \bar{c}_k y_k > 0$$

The above equation holds because we have assumed $c_j$ (reduced-cost) for every non-basic variable $j \in N$ to be positive (given in the problem statement). Therefore, we conclude that $\mathbf{c}'(\mathbf{y} - \mathbf{x}) = \mathbf{c}'\mathbf{d} > 0$, and since $\mathbf{y}$ was an arbitrary feasible solution, $\mathbf{x}$ is the unique optimal solution.

(b) If $x$ is the unique optimal solution and is nondegenerate, then the reduced cost of every nonbasic variable is positive.

Suppose that $\mathbf{x}$ is a nondegenerate basic feasible solution. Suppose for the sake of contradiction that $\bar{c}_k \leq 0$ for some $k$. Since the reduced cost of a basic variable is always zero, $x_k$ must be a nonbasic variable and $\bar{c}_k$ is the rate of cost change along the $k$th basic direction $\mathbf{d}$. We use $\mathbf{d}$ to construct a contradiction. Since $\mathbf{x}$ is nondegenerate, the $k$ th basic direction $\mathbf{d}$ is a feasible direction of cost decrease; thus, there exists a $\theta > 0$ such that $\mathbf{x} + \theta\mathbf{d} \in P$. We have

$$\mathbf{c}'\mathbf{d} = \sum_{j \in N} \bar{c}_j d_j = \bar{c}_k * 1 \leq 0$$

since $\mathbf{d}$ is everywhere zero for nonbasic indices except for $d_k = 1$. But then we have

$$\mathbf{c}'(\mathbf{x} + \theta\mathbf{d}) = \mathbf{c}'\mathbf{x} + \theta\mathbf{c}'\mathbf{d} \leq \mathbf{c}'\mathbf{x}$$

In other words, we have constructed an element $\mathbf{x} + \theta\mathbf{d}$ of $P$ whose costs are at most the costs of $\mathbf{x}$ - a contradiction to the fact that $\mathbf{x}$ is a unique optimal solution. Therefore, the reduced cost of every non-basic solution should be positive for $\mathbf{x}$ to be the unique optimal solution.