ChE 597 Computational Optimization

Homework 9

March 22nd 11:59 pm

1. For the pooling problem you implemented in Homework 8, replace all the bilinear terms with the McCormick envelopes. Solve the McCormick relaxation using Gurobi. Compare the McCormick relaxations of the P formulation, Q formulation, and the PQ formulation.

   Solution: The code is in . .. The objective with the McCormick relaxation for the P formulation is -599.48, for the Q formulation is -2450 and for the PQ formulation is -500.

2. Consider the following nonconvex quadratic constraint

$$-x_1^2 + x_2^2 + 4x_1x_2 \leq 7$$

Let

$$f(x) := -x_1^2 + x_2^2 + 4x_1x_2$$

(a) Write this nonconvex constraint as a difference of convex function, $f(x) = p(x) - q(x)$ where both p and q are convex. Hint: using uniform perturbation of the Hessian.

(b) Relax the concave function $-q(x)$ by McCormick envelopes. You can assume $0 \leq x_1 \leq 3$, $0 \leq x_2 \leq 5$

(c) Take the relaxation you derived, solve the problem with Gurobi using the objective $\min x_1 - 2x_2$

Solution: We can write $f(x), x \in \mathbb{R}^2$ in the following way:

(a) $f(x) := -x_1^2 + x_2^2 + 4x_1x_2 = x^T Qx + q^T x$

$$Q = \frac{\nabla^2 f}{2} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$q^T x = f(x) - x^T Qx = -x_1^2 + x_2^2 + 4x_1x_2 = 0 \implies q = 0$$

The eigenvalues of Q are $\pm\sqrt{5}$. We use the uniform perturbation of the Hessian.

$$f(x) = x^T Qx + q^T x + \mu \sum_i x_i^2 - \mu \sum_i x_i^2$$

We let $\mu = -\lambda_{\min}(Q) = \sqrt{5}$.. Therefore $p(x) = -x_1^2 + x_2^2 + 4x_1x_2 + \sqrt{5}(x_1^2 + x_2^2), q(x) = \sqrt{5}(x_1^2 + x_2^2)$.

(b) Let $-q(x) = z + y, z = -\mu x_1 x_1, y = -\mu x_2 x_2$. Let us consider the function $g = -\mu x_i x_i$.

$$x_i^L \leq x_i \leq x_i^U$$

$$-\mu x_i^U \leq -\mu x_i \leq -\mu x_i^L$$

We relax g using McCormick envelopes. We get the following inequalities:

$$g \geq -\mu x_i x_i^L - \mu x_i x_i^U + \mu x_i^U x_i^L$$

$$g \leq -2\mu x_i x_i^L + \mu (x_i^L)^2$$

$$g \leq -2\mu x_i x_i^U + \mu (x_i^U)^2$$

Using the above we can write the following:

$$
\begin{aligned}
-q(x) &= z + y \\
z &\geq -3\sqrt{5}x_1 \\
z &\leq 0 \\
z &\leq -6\sqrt{5}x_1 + 9\sqrt{5} \\
y &\geq -5\sqrt{5}x_2 \\
y &\leq 0 \\
y &\leq -10\sqrt{5}x_2 + 25\sqrt{5}
\end{aligned}
$$

The optimum objective we obtain is -7.99

3. Assume the following constraint is in a convex relaxation of an optimization problem

$$x_1 = 2x_2 - x_3$$

where $0 \le x_2 \le 2$, $-1 \le x_3 \le 1$. Given $0 \le x_1 \le 1$, try using FBBT to tighten the bounds of x_2 and x_3.

Solution: We are given the constraint

$$x_1 = 2x_2 - x_3,$$

with the following bounds:

$$x_1 \in [0,1], \quad x_2 \in [0,2], \quad x_3 \in [-1,1].$$

We can write the equation as

$$x_1 = a_0 + a_1 x_2 + a_2 x_3,$$

with

$$a_0 = 0, \quad a_1 = 2\,(>0), \quad a_2 = -1\,(<0).$$

Thus, the index sets are

$$J^+ = \{1\} \quad (\text{corresponding to } x_2), \qquad J^- = \{2\} \quad (\text{corresponding to } x_3).$$

The FBBT update formulas for the bounds are given by:

$$\text{For } a_j > 0: \quad l'_j = \frac{1}{a_j}\left(l_k - \left(a_0 + \sum_{i \in J^+ \setminus \{j\}} a_i u_i + \sum_{i \in J^-} a_i l_i\right)\right),$$

$$u'_j = \frac{1}{a_j}\left(u_k - \left(a_0 + \sum_{i \in J^+ \setminus \{j\}} a_i l_i + \sum_{i \in J^-} a_i u_i\right)\right),$$

$$\text{For } a_j < 0: \quad l'_j = \frac{1}{a_j}\left(u_k - \left(a_0 + \sum_{i \in J^+} a_i l_i + \sum_{i \in J^- \setminus \{j\}} a_i u_i\right)\right),$$

$$u'_j = \frac{1}{a_j}\left(l_k - \left(a_0 + \sum_{i \in J^+} a_i u_i + \sum_{i \in J^- \setminus \{j\}} a_i l_i\right)\right).$$

In our case, $k = 1$ (corresponding to x_1), and we update x_2 (with $a_1 = 2 > 0$) and x_3 (with $a_2 = -1 < 0$).

Updating the bound for x_2:

Since x_2 has a positive coefficient ($a_1 = 2$), we have

$$l'_2 = \frac{1}{2}\left(l_1 - \left(\sum_{i \in J^+ \setminus \{1\}} a_i u_i + \sum_{i \in J^-} a_i l_i\right)\right),$$

$$u_2' = \frac{1}{2}\Big(u_1 - \Big(\sum_{i\in J^+\setminus\{1\}} a_i l_i + \sum_{i\in J^-} a_i u_i\Big)\Big).$$

Since $J^+ \setminus \{1\}$ is empty, these simplify to:

$$l_2' = \frac{1}{2}\Big(0 - \big((-1)(-1)\big)\Big) = \frac{1}{2}(0-1) = -0.5,$$

$$u_2' = \frac{1}{2}\Big(1 - \big((-1)(1)\big)\Big) = \frac{1}{2}(1+1) = 1.$$

Since the original bounds for x_2 are $[0,2]$, the new (tighter) bounds are:

$$x_2 \in \Big[\max\{0,-0.5\}, \min\{2,1\}\Big] = [0,1].$$

Updating the bound for x_3:

Since x_3 has a negative coefficient ($a_2 = -1$), the update formulas become:

$$l_3' = \frac{1}{-1}\Big(u_1 - \Big(\sum_{i\in J^+} a_i l_i + \sum_{i\in J^-\setminus\{2\}} a_i u_i\Big)\Big),$$

$$u_3' = \frac{1}{-1}\Big(l_1 - \Big(\sum_{i\in J^+} a_i u_i + \sum_{i\in J^-\setminus\{2\}} a_i l_i\Big)\Big).$$

Here, $J^+ = \{1\}$ and $J^- \setminus \{2\}$ is empty. Thus,

$$l_3' = \frac{1}{-1}\Big(1 - \big(2\cdot 0\big)\Big) = -(1-0) = -1,$$

$$u_3' = \frac{1}{-1}\Big(0 - \big(2\cdot 1\big)\Big) = -(0-2) = 2.$$

Given the original bounds $x_3 \in [-1,1]$, the new bounds are:

$$x_3 \in \Big[\max\{-1,-1\}, \min\{1,2\}\Big] = [-1,1].$$

Summary of Tightened Bounds:

$$\boxed{x_2 \in [0,1] \quad \text{and} \quad x_3 \in [-1,1].}$$

4. Derive a valid convex relaxation of the following nonconvex optimization problem using factorization and convex envelopes of univariate functions.

$$\min x_1 + x_2$$

$$\text{s.t.} \quad \exp(x_2\sqrt{x_1x_2} + \log(x_1)) \leq x_1^2$$

$$1 \leq x_1 \leq 2, 0 \leq x_2 \leq 1$$

Solution: We need to convexify the terms $\exp(x_2\sqrt{x_1x_2} + \log(x_1))$ and x_1^2. We introduce the following variables and relations:

$$y_1 = x_1^2$$

$$y_2 = log(x_1)$$

$$y_3 = x_1x_2$$

$$y_4 = \sqrt{y_3}$$

$$y_5 = x_2y_4$$

$$y_6 = y_5 + y_2$$

$$y_7 = exp(y_6)$$

We need $y_7 \leq y_1$. The convex relaxations using the envolopes is as follows:

$$y_1 \geq 2x_1 - 1, y_1 \geq 4x_1 - 4, y_1 \leq 3x_1 - 2$$

$$y_2 \geq log(1) + \frac{(log(2) - log(1))}{2-1}(x_1 - 1) = log(2)(x_1 - 1)$$

$$y_2 \leq log(x_1)$$

$$y_3 \geq x_1 + 2x_2 - 2, y_3 \geq x_2, y_3 \leq x_2 + x_1 - 1, y_3 \leq 2x_2$$

The least value of y_3 is 0 and the highest value of y_3 is 2.

$$y_4 \geq \sqrt{0} + \frac{\sqrt{2} - \sqrt{0}}{2-0}(y_3 - 0) = \frac{1}{\sqrt{2}}y_3$$

$$y_4 \leq \sqrt{y_3}$$

We have $0 \leq x_2 \leq 1, 0 \leq y_4 \leq \sqrt{2}$.

$$y_5 \geq 0, y_5 \geq y_4 + \sqrt{2}x_2 - \sqrt{2}, y_5 \leq \sqrt{2}x_2, y_5 \leq y_4$$

$$y_6 \leq y_5 + y_2, y_6 \geq y_5 + y_2$$

We have $0 \leq y_2 \leq log(2), 0 \leq y_5 \leq \sqrt{2} \implies 0 \leq y_6 \leq log(2) + \sqrt{2}$ The below is true due to the convexity of the exponential function

$$y_7 \geq e^0 + e^0(y_6 - 0) = 1 + y_6$$

$$y_7 \leq 2exp(\sqrt{2})$$

5. Consider the following convex relaxation,

$$
\begin{aligned}
2x_1 + x_2 - x_3 - 2x_4 &= 1 \\
3x_2 + x_4 &= 5 \\
0 \le x_1 &\le 4 \\
-1 \le x_2 &\le 2 \\
0 \le x_3 &\le 3 \\
-1 \le x_4 &\le 1
\end{aligned}
$$

Use OBBT to tighten the bounds of all the variables. You can solve the LPs using Gurobi.

Solution: OBBT involves finding tighter bound for the variables. Let us define the set S in the following way:

$$
S = \left\{ x : \begin{array}{c} x \in \mathbb{R}^4 \\ 2x_1 + x_2 - x_3 - 2x_4 = 1 \\ 3x_2 + x_4 = 5 \\ 0 \le x_1 \le 4 \\ -1 \le x_2 \le 2 \\ 0 \le x_3 \le 3 \\ -1 \le x_4 \le 1 \end{array} \right\}
$$

For each variable $x_i, i = 1, 2, \ldots, 4$, updated lower and upper bounds can be computed by solving the following optimization problems:

$$
l_i' = \min \{x_i : x \in S\}; \quad u_i' = \max \{x_i : x \in S\}.
$$

The tightened bounds are as follows

Variable	Lower bound	Upper bound
x_1	0	2.33
x_2	1.33	2
x_3	0	3
x_4	-1	1