# Lecture 6 Simplex Algorithm

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ChE 597: Computational Optimization
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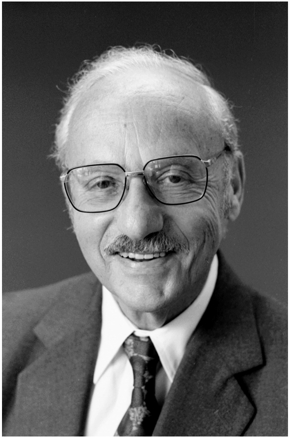
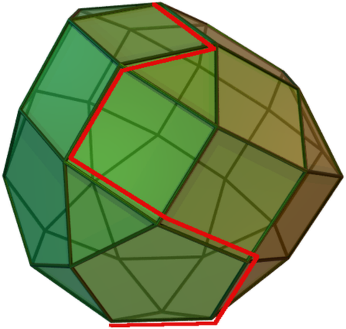
# History





Developed by George Dantzig in 1947.
Intuition: iterate over the extreme points of a polyhedron until an optimality condition is satisfied.

# Important results from Polyhedron theory

When $P$ is nonempty and the optimal objective value is not $-\infty$, the optimal solution can always be obtained at an extreme point.
extreme point $\Leftrightarrow$ basic feasible solution
**Basic feasible solution** Consider a polyhedron $P$ defined by linear equality and inequality constraints, and let $x^*$ be an element of $\Re^n$.

1. The vector $x^*$ is a **basic solution** if:
   - All equality constraints are active;
   - Out of the constraints that are active at $x^*$, there are $n$ of them that are linearly independent.
2. If $x^*$ is a basic solution that satisfies all of the constraints, we say that it is a **basic feasible solution**.

# Basic solution for standard form polyhedron

**Theorem** Consider the constraints $Ax = b$ and $x \geq 0$ and assume that the $m \times n$ matrix $A$ has linearly independent rows. A vector $x \in \Re^n$ is a basic solution if and only if we have $Ax = b$, and there exist indices $B(1), \ldots, B(m)$ such that:
(a) The columns $A_{B(1)}, \ldots, A_{B(m)}$ are linearly independent;
(b) If $i \neq B(1), \ldots, B(m)$, then $x_i = 0$.

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(a) The columns $A_{B(1)}, \ldots, A_{B(m)}$ are linearly independent;
(b) If $i \neq B(1), \ldots, B(m)$, then $x_i = 0$.
Proof sketch: **Sufficiency** If (a) and (b) are satisfied, there are $n$ linearly independent active constraints that uniquely defines a solution and thus is a basic solution.
**Necessity** For a basic solution, there must exist at least $n - m$ active constraints $x_i = 0$, together with $Ax = b$ forming $n$ linearly independent constraints. Consider the active constraints forming the matrix (under permutation) $\begin{bmatrix} B & N \\ 0 & I \end{bmatrix}$ being full rank, where $A = [B\ N]$, $I$ correspond to the indices of the $x_i = 0$. $B$ must be full rank for the matrix to be full rank. d

# Basic variables

**Definition** If $x$ is a basic solution, the variables $x_{B(1)}, \ldots, x_{B(m)}$ are called **basic variables**; the remaining variables are called **nonbasic**. The columns $A_{B(1)}, \ldots, A_{B(m)}$ are called the **basic column** and, since they are linearly independent, they form a basis of $\mathbb{R}^m$

# Basis matrix

**basis matrix** $B \in \mathbb{R}^{m \times m}$ the matrix formed by arranging the $m$ basic columns next to each other.
**basic variables** a vector $x_B$ with the values of the basic variables
**nonbasic variables** $x_N$

$$B = \begin{bmatrix} | & | & & | \\ A_{B(1)} & A_{B(2)} & \cdots & A_{B(m)} \\ | & | & & | \end{bmatrix}, \qquad x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix}.$$

$x = [x_B; x_N]$, $A = [B\ N]$
The basic variables are determined by solving the equation $Bx_B = b$ whose unique solution is given by

$$x_B = B^{-1}b.$$

If $x_B \geq 0$, then it is a basic feasible solution

Recall that for LP with finite optimum, there exists an optimal basic feasible solution. Next, we will derive the conditions under which a basic feasible solution is optimal.

**Feasible direction** Let $x$ be an element of a polyhedron $P$. A vector $d \in \Re^n$ is said to be a feasible direction at $x$, if there exists a positive scalar $\theta$ for which $x + \theta d \in P$.

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**Basic direction** Let $x_B = B^{-1}b$. be a BFS. We consider the possibility of moving away from $x$, to a new vector $x + \theta d$, by selecting a nonbasic variable $x_j$. Algebraically, $d_j = 1$, and $d_i = 0$ for every nonbasic index $i$ other than $j$. We require $A(x + \theta d) = b$, and since $x$ is feasible, we also have $Ax = b$. Thus, we need $Ad = 0$. Recall now that $d_j = 1$, and that $d_i = 0$ for all other nonbasic indices $i$. Then,

$$0 = Ad = \sum_{i=1}^{n} A_i d_i = \sum_{i=1}^{m} A_{B(i)} d_{B(i)} + A_j = Bd_B + A_j.$$

$$d_B = -B^{-1}A_j, d_j = 1, d_i = 0 \; \forall i \in N, i \neq j \text{ (jth basic direction)}$$

# When is a basic direction a feasible direction?

Since $d_j = 1, d_i = 0 \; \forall i \in N$, the nonbasic variables will remain feasible. We only need to make sure that the basic variables are still nonnegative.

(a) $x$ is a nondegenerate basic feasible solution. Then, $x_B > 0$, from which it follows that $x_B + \theta d_B \geq 0$, and feasibility is maintained, when $\theta$ is sufficiently small. $d$ is a feasible direction.

(b) $x$ is degenerate. Then, $d$ is not always a feasible direction. Indeed, it is possible that a basic variable $x_{B(i)}$ is zero, while the corresponding component $d_{B(i)}$ of $d_B = -B^{-1}A_j$ is negative. In that case, if we follow the $j$ th basic direction, the nonnegativity constraint for $x_{B(i)}$ is immediately violated, and we are led to infeasible solutions;

# Reduced cost

Q: What will be the effect on the cost (objective) function if we move along the $j$th basic direction?

$$d_B = -B^{-1}A_j, d_j = 1, d_i = 0 \; \forall i \in N, i \neq j \text{ (jth basic direction)}$$

$$\bar{c}_j = c^T d^j = c_j - c_B^T B^{-1} A_j$$

where $d^j$ is the $j$th basic direction, $c_B$ the vector of costs of the basic variables. $\bar{c}_j$ is the **reduced cost** of the variable $x_j$

# Optimality condition

**Theorem** Consider a basic feasible solution $x$ associated with a basis matrix $B$, and let $\overline{c}$ be the corresponding vector of reduced costs.
(a) If $\overline{c} \geq 0$, then $x$ is optimal.
(b) If $x$ is optimal and nondegenerate, then $\overline{c} \geq 0$.

# Optimality condition

Proof. (a) We assume that $\overline{c} \geq 0$, we let $y$ be an arbitrary feasible solution, and we define $d = y - x$. Feasibility implies that $Ax = Ay = b$ and, therefore, $Ad = 0$. The latter equality can be rewritten in the form

$$Bd_B + \sum_{i \in N} A_i d_i = 0$$

where $N$ is the set of indices corresponding to the nonbasic variables under the given basis. Since $B$ is invertible, we obtain

$$d_B = -\sum_{i \in N} B^{-1} A_i d_i$$

and

$$c^T d = c_B^T d_B + \sum_{i \in N} c_i d_i = \sum_{i \in N} \left( c_i - c_B^T B^{-1} A_i \right) d_i = \sum_{i \in N} \bar{c}_i d_i$$

For any nonbasic index $i \in N$, we must have $x_i = 0$ and, since $y$ is feasible, $y_i \geq 0$. Thus, $d_i \geq 0$ and $\bar{c}_i d_i \geq 0$, for all $i \in N$. We conclude that $c^T(y - x) = c^T d \geq 0$, and since $y$ was an arbitrary feasible solution, $x$ is optimal.

# Optimality condition

(b) Suppose that $x$ is a nondegenerate basic feasible solution and that $\bar{c}_j < 0$ for some $j$. Since the reduced cost of a basic variable is always zero, $x_j$ must be a nonbasic variable and $\bar{c}_j$ is the rate of cost change along the $j$ th basic direction. Since $x$ is nondegenerate, the $j$ th basic direction is a feasible direction of cost decrease, as discussed earlier. By moving in that direction, we obtain feasible solutions whose cost is less than that of $x$, and $x$ is not optimal.

# Optimal basis

A basis matrix $B$ is said to be optimal if:
(a) $B^{-1}b \geq 0$, and
(b) $\overline{c}^T = c^T - c_B^T B^{-1} A \geq 0$

# Simplex Algorithm

We assume all the basic feasible solutions are nondegenerate. Under this assumption, the reduced costs being nonnegative are both necessary and sufficient for a BFS to be optimal.
If a reduced cost $\bar{c}_j$ of a nonbasic variable $x_j$ is negative, we can move toward the $j$th feasible direction $d^j$ to further decrease the cost. It is desirable to move as far as possible:

$$\theta^* = \max\{\theta \geq 0 \mid x + \theta d \in P\}$$

The resulting cost change is $\theta^* c^T d$, which is the same as $\theta^* \bar{c}_j$. Given that $Ad = 0$, we have $A(x + \theta d) = Ax = b$ for all $\theta$, and the equality constraints will never be violated. Thus, $x + \theta d$ can become infeasible only if one of its components becomes negative.

# Simplex Algorithm

Two cases can occur when we move along direction $d$:
(a) If $d \geq 0$, then $x + \theta d \geq 0$ for all $\theta \geq 0$, the vector $x + \theta d$ never becomes infeasible, and we let $\theta^* = \infty$.
(b) If $d_i < 0$ for some $i$, the constraint $x_i + \theta d_i \geq 0$ becomes $\theta \leq -x_i/d_i$. This constraint on $\theta$ must be satisfied for every $i$ with $d_i < 0$. Thus, the largest possible value of $\theta$ is

$$\theta^* = \min_{\{i|d_i<0\}} \left( -\frac{x_i}{d_i} \right).$$

Recall that if $x_i$ is a nonbasic variable, then either $x_i$ is the entering variable and $d_i = 1$, or else $d_i = 0$. In either case, $d_i$ is nonnegative. Thus, we only need to consider the basic variables and we have the equivalent formula

$$\theta^* = \min_{\left\{i=1,\ldots,m|d_{B(i)}<0\right\}} \left( -\frac{x_{B(i)}}{d_{B(i)}} \right).$$

Note that $\theta^* > 0$, because $x_{B(i)} > 0$ for all $i$, as a consequence of nondegeneracy.

# Simplex Algorithm

Once $\theta^*$ is chosen, and assuming it is finite, we move to the new feasible solution $y = x + \theta^* d$. Since $x_j = 0$ and $d_j = 1$, we have $y_j = \theta^* > 0$. Let $\ell$ be a minimizing index of

$$-\frac{x_{B(\ell)}}{d_{B(\ell)}} = \min_{\{i=1,\dots,m \mid d_{B(i)}<0\}} \left(-\frac{x_{B(i)}}{d_{B(i)}}\right) = \theta^*;$$

$$x_{B(\ell)} + \theta^* d_{B(\ell)} = 0.$$

We observe that the basic variable $x_{B(\ell)}$ has become zero, whereas the nonbasic variable $x_j$ has now become positive, which suggests that $x_j$ should replace $x_{B(\ell)}$ in the basis. The new basis matrix is

$$\bar{B}(i) = \begin{cases} B(i), & i \neq \ell \\ j, & i = \ell \end{cases}$$

In other words, $\ell$ leaves the basis, $j$ enters the basis.

# One simplex iteration (pivot) summary

1. In a typical iteration, we start with a basis consisting of the basic columns $A_{B(1)}, \ldots, A_{B(m)}$, and an associated basic feasible solution $x$.
2. Compute the reduced costs $\bar{c}_j = c_j - c_B^T B^{-1} A_j$ for all nonbasic indices $j$. If they are all nonnegative, the current basic feasible solution is optimal, and the algorithm terminates; else, choose some $j$ for which $\bar{c}_j < 0$.
3. Compute $u = -d_B = B^{-1} A_j$. If no component of $u$ is positive, we have $\theta^* = \infty$, the optimal cost is $-\infty$, and the algorithm terminates.
4. If some component of $u$ is positive, let

$$\theta^* = \min_{\{i=1,\ldots,m | u_i > 0\}} \frac{x_{B(i)}}{u_i}$$

5. Let $\ell$ be such that $\theta^* = x_{B(\ell)}/u_\ell$. Form a new basis by replacing $A_{B(\ell)}$ with $A_j$. If $y$ is the new basic feasible solution, the values of the new basic variables are $y_j = \theta^*$ and $y_{B(i)} = x_{B(i)} - \theta^* u_i$, $i \neq \ell$.

# Compute matrix inverse faster

**Motivation**: Taking matrix inverse ($B^{-1}$) at each iteration is expensive ($O(m^3)$). We would like a more efficient implementation.
**Observation**: The new basis $\overline{B}$ and the old basis $B$ only differ at the $\ell$th column

$$B = \left[A_{B(1)} \cdots A_{B(m)}\right]$$

$$\overline{B} = \left[\ A_{B(1)} \cdots A_{B(\ell-1)} \quad A_j \quad A_{B(\ell+1)} \quad \cdots A_{B(m)}\ \right]$$

**Idea:** Exploit the information of $B^{-1}$ to compute $\overline{B}^{-1}$

# Row operations

$$B^{-1}\overline{B} = \begin{bmatrix} | & & | & | & | & & | \\ e_1 & \cdots & e_{\ell-1} & u & e_{\ell+1} & \cdots & e_m \\ | & & | & | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & u_1 & & \\ & \ddots & \vdots & & \\ & & u_\ell & & \\ & & \vdots & \ddots & \\ & & u_m & & 1 \end{bmatrix}$$

where $u = B^{-1}A_j$.

Let us apply a sequence of elementary row operations that will change the above matrix to the identity matrix.

(a) For each $i \neq \ell$, we add the $\ell$ th row times $-u_i/u_\ell$ to the $i$ th row. (Recall that $u_\ell > 0$.) This replaces $u_i$ by zero.

(b) We divide the $\ell$ th row by $u_\ell$. This replaces $u_\ell$ by one.

Let these row operations be matrix $Q$. $QB^{-1}\overline{B} = I$, which yields $QB^{-1} = \overline{B}^{-1}$. This shows that if we apply the same sequence of row operations to the matrix $B^{-1}$, we obtain $\overline{B}^{-1}$.

# Full tableau implementation

$$B^{-1}[b \mid A]$$

with columns $B^{-1}b$ and $B^{-1}A_1, \ldots, B^{-1}A_n$.

- This matrix is called the **simplex tableau**.
- $B^{-1}b$ is called the zeroth column
- The column $u = B^{-1}A_j$ corresponding to the variable that enters the basis is called the **pivot column**.
- If the $\ell$ th basic variable exits the basis, the $\ell$ th row of the tableau is called the **pivot row**.

At the end of each iteration, we need to update the tableau $B^{-1}[b \mid A]$ and compute $\overline{B}^{-1}[b \mid A]$. This can be accomplished by left-multiplying the simplex tableau with a matrix $Q$ satisfying $QB^{-1} = \overline{B}^{-1}$

# Full tableau implementation

It is customary to add the objective and the reduced cost as a zeroth row.

| | |
|---|---|
| $-c_B^T B^{-1} b$ | $c^T - c_B^T B^{-1} A$ |
| $B^{-1} b$ | $B^{-1} A$ |

We've already shown how to update the rows of $B^{-1}$ to pivot to $\overline{B}^{-1}$.

It can be shown that the rule for updating the zeroth row turns out to be identical to the rule used for the other rows of the tableau: add a multiple of the pivot row to the zeroth row to set the reduced cost of the entering variable to zero.

# An iteration of the full tableau implementation

1. A typical iteration starts with the tableau associated with a basis matrix $B$ and the corresponding basic feasible solution $x$.
2. Examine the reduced costs in the zeroth row of the tableau. If they are all nonnegative, the current basic feasible solution is optimal, and the algorithm terminates; else, choose some $j$ for which $\bar{c}_j < 0$.
3. Consider the vector $u = B^{-1}A_j$, which is the $j$ th column (the pivot column) of the tableau. If no component of $u$ is positive, the optimal cost is $-\infty$, and the algorithm terminates.
4. For each $i$ for which $u_i$ is positive, compute the ratio $x_{B(i)}/u_i$. Let $\ell$ be the index of a row that corresponds to the smallest ratio. The column $A_{B(\ell)}$ exits the basis and the column $A_j$ enters the basis.
5. Add to each row of the tableau a constant multiple of the $\ell$ th row (the pivot row) so that $u_\ell$ (the pivot element) becomes one and all other entries of the pivot column become zero.

# Example

$$
\begin{array}{rr}
\text{minimize} & -10x_1 - 12x_2 - 12x_3 \\
\text{subject to} & x_1 + 2x_2 + 2x_3 \leq 20 \\
& 2x_1 + x_2 + 2x_3 \leq 20 \\
& 2x_1 + 2x_2 + x_3 \leq 20 \\
& x_1, x_2, x_3 \geq 0.
\end{array}
$$

Add slack variables to transform to the standard form.

$$
\begin{array}{rrl}
\text{minimize} & -10x_1 - 12x_2 - 12x_3 & \\
\text{subject to} & x_1 + 2x_2 + 2x_3 + x_4 & = 20 \\
& 2x_1 + x_2 + 2x_3 + x_5 & = 20 \\
& 2x_1 + 2x_2 + x_3 + x_6 = 20 & \\
& x_1, \dots, x_6 \geq 0. &
\end{array}
$$

Note that $x = (0, 0, 0, 20, 20, 20)$ is a basic feasible solution and can be used to start the algorithm. Let accordingly, $B(1) = 4, B(2) = 5$, and $B(3) = 6$.

# Example

| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|---|
| | 0 | $-10$ | $-12$ | $-12$ | 0 | 0 | 0 |
| $x_4 =$ | 20 | 1 | 2 | 2 | 1 | 0 | 0 |
| $x_5 =$ | 20 | $2^*$ | 1 | 2 | 0 | 1 | 0 |
| $x_6 =$ | 20 | 2 | 2 | 1 | 0 | 0 | 1 |

$\overline{c}_1 < 0$. choose $x_1$ to enter the basis

$\frac{x_{B(1)}}{u_1} = \frac{x_4}{u_1} = \frac{20}{1} = 20$

$\frac{x_{B(2)}}{u_2} = \frac{x_5}{u_2} = \frac{20}{2} = 10$

$\frac{x_{B(3)}}{u_3} = \frac{x_6}{u_3} = \frac{20}{2} = 10$

The second row can be selected as the pivot row.

Row operations:

Add 5 times the second row to the zeroth row.

Add $-\frac{1}{2}$ times the second row to the first row.

Add $-1$ times the second row to the third row.

Divide the second row by 2.

# Example

| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|---|
| | 100 | 0 | $-7$ | $-2$ | 0 | 5 | 0 |
| $x_4 =$ | 10 | 0 | 1.5 | $1^*$ | 1 | $-0.5$ | 0 |
| $x_1 =$ | 10 | 1 | 0.5 | 1 | 0 | 0.5 | 0 |
| $x_6 =$ | 0 | 0 | 1 | $-1$ | 0 | $-1$ | 1 |

choose $x_3$ to enter the basis.

$\frac{x_{B(1)}}{u_1} = \frac{x_4}{u_1} = \frac{10}{1} = 10$

$\frac{x_{B(2)}}{u_2} = \frac{x_1}{u_2} = \frac{10}{1} = 10$

$u_3$ is negative.

Choose the first row to be the pivot row.

Add 2 times the first row to the zeroth row.

Add $-1$ times the first row to the second row.

Add 1 times the first row to the third row.

Divide the first row by 1.

# Example

| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|---|
| | 120 | 0 | $-4$ | 0 | 2 | 4 | 0 |
| $x_3 =$ | 10 | 0 | 1.5 | 1 | 1 | $-0.5$ | 0 |
| $x_1 =$ | 0 | 1 | $-1$ | 0 | $-1$ | 1 | 0 |
| $x_6 =$ | 10 | 0 | $2.5^*$ | 0 | 1 | $-1.5$ | 1 |

$x_2$ has negative reduced cost and enters the basis.
$\frac{x_{B(1)}}{u_1} = \frac{x_3}{u_1} = \frac{10}{1.5}$
$u_2$ is negative.
$\frac{x_{B(3)}}{u_3} = \frac{x_6}{u_3} = \frac{10}{2.5}$
Choose the third row to be the pivot row.
Add $\frac{4}{2.5}$ times the third row to the zeroth row.
Add $-\frac{1.5}{2.5}$ times the third row to the first row.
Add $\frac{1}{2.5}$ times the third row to the second row.
Divide the third row by 2.5.

# Example

| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|---|
| | 136 | 0 | 0 | 0 | 3.6 | 1.6 | 1.6 |
| $x_3 =$ | 4 | 0 | 0 | 1 | 0.4 | 0.4 | $-0.6$ |
| $x_1 =$ | 4 | 1 | 0 | 0 | $-0.6$ | 0.4 | 0.4 |
| $x_2 =$ | 4 | 0 | 1 | 0 | 0.4 | $-0.6$ | 0.4 |

Optimal solution found. Optimal objective is $-136$.
$x^* = (4, 4, 4, 0, 0, 0)$

# Reference

1. Chapter 3. Bertsimas, D., & Tsitsiklis, J. N. (1997). Introduction to linear optimization (Vol. 6, pp. 479-530). Belmont, MA: Athena scientific.