Lecture 4 Linear Programming Applications

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ChE 597: Computational Optimization
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Illustrative Example A Linear Programming Problem

The following is a linear programming problem:

$$
\begin{aligned}
\text{minimize} \quad & 2x_1 - x_2 + 4x_3 \\
\text{subject to} \quad & x_1 + x_2 + x_4 \leq 2 \\
& 3x_2 - x_3 = 5 \\
& x_3 + x_4 \geq 3 \\
& x_1 \geq 0 \\
& x_3 \leq 0.
\end{aligned}
$$

The problem has

- inequality constraints ($\geq$, $\leq$)
- equality constraint ($=$)
- variable bounds ($x_i \geq 0$ and $x_i \leq 0$)

Generalized Linear Programming Problem

The following is a generalized linear programming problem:

$$
\begin{aligned}
\text{minimize} \quad & c^T x \\
\text{subject to} \quad & a_i^T x \geq b_i, \quad i \in M_1, \\
& a_i^T x \leq b_i, \quad i \in M_2, \\
& a_i^T x = b_i, \quad i \in M_3, \\
& x_j \geq 0, \qquad j \in N_1, \\
& x_j \leq 0, \qquad j \in N_2.
\end{aligned}
$$

where c, x are vectors, a_i are the coefficient vectors, and b_i are scalars. M_1, M_2, M_3 are index sets for a. N_1, N_2 are index sets for x.

Reformulated Linear Programming Problem

The problem can be reformulated as the succinct form:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \geq b \end{aligned}$$

Where:

- c, x are vectors in $\mathbb{R}^n$
- A is a matrix in $\mathbb{R}^{m \times n}$.
- b is a vector in $\mathbb{R}^m$
- componentwise $\geq, i.e., A_i^T x \geq b_i \quad \forall i = 1, \ldots, m$

$$\begin{aligned} & a_i^T x \geq b_i, && i \in M_1, \\ & -a_i^T x \geq -b_i, \Rightarrow (a_i^T x \leq b_i), && i \in M_2, \\ & a_i^T x \geq b_i, \quad -a_i^T x \geq -b_i \Rightarrow (a_i^T x = b_i), && i \in M_3, \\ & x_j \geq 0, && j \in N_1, \\ & -x_j \geq 0, \Rightarrow (x_j \leq 0) && j \in N_2. \end{aligned}$$

In most applications, A has full column rank; otherwise, the problem can be unbounded (For example, if $\exists v$, $Av = 0$, $c^\top v \neq 0$).

Linear programming

The mixed form with both equality and inequality constraints is:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & Gx \leq h \end{aligned}$$

Where:

- G is a matrix in $\mathbb{R}^{p \times n}$
- h is a vector in $\mathbb{R}^p$

This form combines equality and inequality constraints in the same problem.

Standard Form

The standard form of a linear programming problem is:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Where:

- c, x are vectors in $\mathbb{R}^n$
- A is a matrix in $\mathbb{R}^{m \times n}$
- b is a vector in $\mathbb{R}^m$
- Assuming the rows of A are linearly independent. Typically, we have $m < n$, otherwise, the solution is unique ($m = n$) or the problem is infeasible.

Reduction to standard form

(a) Elimination of free variables: Given an unrestricted variable x_j in a problem in general form, we replace it by $x_j^+ - x_j^-$, where x_j^+ and x_j^- are new variables on which we impose the sign constraints $x_j^+ \geq 0$ and $x_j^- \geq 0$. The underlying idea is that any real number can be written as the difference of two nonnegative numbers.

Reduction to standard form

(b) Elimination of inequality constraints: Given an inequality constraint of the form

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i,$$

we introduce a new variable s_i and the standard form constraints

$$\sum_{j=1}^{n} a_{ij} x_j + s_i = b_i, \quad s_i \geq 0.$$

Such a variable s_i is called a slack variable. Similarly, an inequality constraint

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i,$$

can be put in standard form by introducing a surplus variable s_i and the constraints

$$\sum_{j=1}^{n} a_{ij} x_j - s_i = b_i, \quad s_i \geq 0.$$

Applications of linear program

- Production planning
- Capacity expansion planning
- Model predictive control
- Multicommodity flow
- 1 and ∞ norms

A Production Problem

A firm produces n different goods using m different raw materials. Let b_i, for $i = 1, \ldots, m$, be the available amount of the ith raw material. The jth good, for $j = 1, \ldots, n$, requires a_{ij} units of the ith raw material and results in a revenue of c_j per unit produced. The firm faces the problem of deciding how much of each good to produce in order to maximize its total revenue.

Linear Programming Formulation

In this example, the choice of the decision variables is simple. Let x_j, for $j = 1, \ldots, n$, be the amount of the jth good. Then, the problem facing the firm can be formulated as follows:

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j \le b_i, & i = 1, \ldots, m, \\ & x_j \ge 0, & j = 1, \ldots, n. \end{array}$$

Note that it can also be seen as a continuous version of the knapsack problem.

A Farm Planning Problem

Suppose that a farmer has a piece of farm land, say L km^2, to be planted with either wheat or barley or some combination of the two. The farmer has a limited amount of fertilizer, F kilograms, and pesticide, P kilograms.

Notation	**Description**
L	Total land area available (km^2)
F	Total fertilizer available (kg)
P	Total pesticide available (kg)
F_1	Fertilizer required per km^2 of wheat
P_1	Pesticide required per km^2 of wheat
F_2	Fertilizer required per km^2 of barley
P_2	Pesticide required per km^2 of barley
S_1	Selling price of wheat per km^2
S_2	Selling price of barley per km^2

Linear Programming Formulation

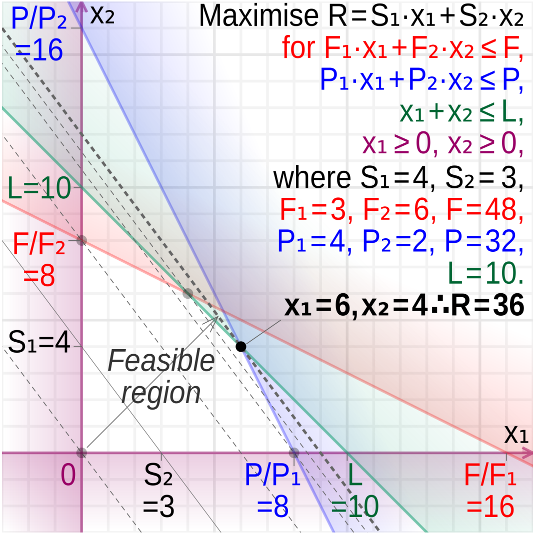
If we denote the area of land planted with wheat and barley by x_1 and x_2 respectively.
Maximize the profit:

$$S_1 \cdot x_1 + S_2 \cdot x_2$$

Subject to:

$$\begin{aligned} x_1 + x_2 &\leq L & \text{(limit on total area)} \\ F_1 \cdot x_1 + F_2 \cdot x_2 &\leq F & \text{(limit on fertilizer)} \\ P_1 \cdot x_1 + P_2 \cdot x_2 &\leq P & \text{(limit on pesticide)} \\ x_1, x_2 &\geq 0 & \text{(non-negativity constraint)} \end{aligned}$$

Geometric intuition



The optimal solution is obtained at a **vertex**.

Multiperiod Planning of Electric Power Capacity

A state wants to plan its electricity capacity for the next T years. The state has a forecast of d_t megawatts, presumed accurate, of the demand for electricity during year $t = 1, \ldots, T$. The existing capacity, which is in oil-fired plants, that will not be retired and will be available during year t, is e_t. There are two alternatives for expanding electric capacity: coalfired or nuclear power plants. There is a capital cost of c_t per megawatt of coal-fired capacity that becomes operational at the beginning of year t. The corresponding capital cost for nuclear power plants is n_t. For various political and safety reasons, it has been decided that no more than 20% of the total capacity should ever be nuclear. Coal plants last for 20 years, while nuclear plants last for 15 years. A least cost capacity expansion plan is desired.

Parameters and Decision Variables

Parameter	Description
T	Number of years for planning
d_t	Forecasted demand in year t
e_t	Existing capacity in year t
c_t	Capital cost per megawatt of coal capacity
n_t	Capital cost for nuclear power plants

Variable	Description
x_t	Coal capacity brought online in year t
y_t	Nuclear capacity brought online in year t
w_t	Total coal capacity available in year t
z_t	Total nuclear capacity available in year t

Problem Formulation

The cost of a capacity expansion plan is:

$$\sum_{t=1}^{T} (c_t x_t + n_t y_t)$$

Conditions for coal and nuclear power plants capacity:

$$w_t = \sum_{s=\max\{1,t-19\}}^{t} x_s, \quad t = 1, \ldots, T.$$

$$z_t = \sum_{s=\max\{1,t-14\}}^{t} y_s, \quad t = 1, \ldots, T.$$

Capacity must meet the forecasted demand:

$$w_t + z_t + e_t \geq d_t, \quad t = 1, \ldots, T.$$

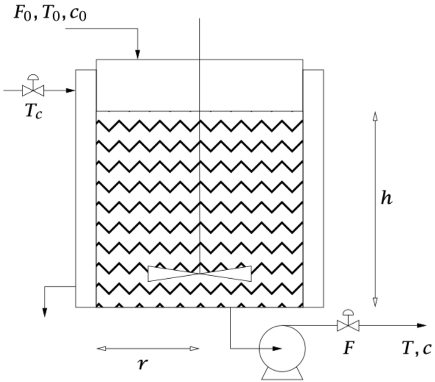
Limitation for nuclear capacity:

$$\frac{z_t}{w_t + z_t + e_t} \leq 0.2$$

$$0.8 z_t - 0.2 w_t \leq 0.2 e_t.$$

Continuous Stirred-Tank Reactor (CSTR)

- It ensures uniform composition due to continuous stirring, which is critical for consistent product quality.
- The CSTR operates with a steady flow of reactants in, flow of product out, and temperature of coolant provided, allowing for steady-state operating conditions.



CSTR Variables and Control Parameters

Variable	Description
F_0, T_0, c_0	Inflow rate, temperature, and concentration
T_c	Coolant temperature
F	Effluent outflow rate
T, c	Temperature and concentration of the effluent
h	Height of the liquid in the reactor

The systems follow the mass and energy balances.

$$\frac{dc}{dt} = \frac{F_0\left(c_0 - c\right)}{\pi r^2 h} - k_0 \exp\left(-\frac{E}{RT}\right) c$$

$$\frac{dT}{dt} = \frac{F_0\left(T_0 - T\right)}{\pi r^2 h} + \frac{-\Delta H}{\rho C_p} k_0 \exp\left(-\frac{E}{RT}\right) c + \frac{2U}{r \rho C_p}\left(T_c - T\right)$$

$$\frac{dh}{dt} = \frac{F_0 - F}{\pi r^2}$$

Model Predictive Control

- Goal: to take the state of the system to a specified constant setpoint or time-varying setpoint trajectory, by manipulating the inputs.
- Assume the coolant temperature T_c and the effluent outflow rate F can be manipulated to make the CSTR states (e.g., c, h) move to their setpoints.
- The ODEs are difficult to optimize directly. $\Rightarrow$ Use a linear model to approximate the dynamics of the ODE.

Model Predictive Control

$$\min_{x(\cdot),u(\cdot),y(\cdot)} \quad ||Qy(\cdot)|| + ||Ru(\cdot)||$$

$$x(t+1) = Ax(t) + Bu(t) \quad t = 0, \ldots, T-1$$
$$y(t) = Cx(t) \quad t = 0, \ldots, T$$

- $x(t)$: the states of the system at time t. The initial state should be provided from measurement (or estimator)
- $y(t)$: system outputs. variables that can be measured. If all the states can be measured, $C = I, y(t) = x(t)$
- $u(t)$: input variables that can be manipulated to determine the future states.
- Assume at steady state, $x(\cdot), u(\cdot), y(\cdot) = 0$.
- The objective penalizes the deviation of the outputs from the steady states and the change in control actions (i.e., input).
- Both the states and inputs may be subject to linear constraints of the form $Du(t) \leq d$ and $ECx(t) \leq e$ (e.g., saturation and safety constraints)

Multicommodity Flow Problem: Problem Statement

- Consider a network represented as a directed graph $G = (V, E)$
- Multiple commodities need to be transported across the network
- Each edge $e \in E$ has a capacity u_e and a cost c_e per unit commodity.
- Each commodity k has a demand d_k from a source s_k to a sink t_k

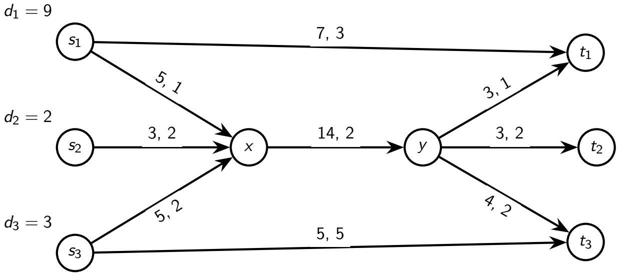


Figure: (u_e, c_e) are shown above the edge

Formulation

Variables

- Let $x_{e,k} \geq 0$ represent the flow of commodity k through edge e

Constraints:

- Capacity constraints: $\sum_k x_{e,k} \leq u_e$ for all $e \in E$
- Flow conservation: For each node v and each commodity k,

$$\sum_{(u,v)\in E} x_{u,v,k} - \sum_{(v,w)\in E} x_{v,w,k} = \begin{cases} d_k, & \text{if } v = s_k \\ -d_k, & \text{if } v = t_k \\ 0, & \text{otherwise} \end{cases}$$

Objective Function:

- minimize the total cost: $\min \sum_{e\in E} \sum_k c_e x_{e,k}$

1 and ∞ norm

Recall what we discussed in Lecture 2.
Norm: $\|x\|$ is convex for any norm; e.g., ℓ_p norms,

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \quad \text{for } p \geq 1,$$

$$\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$$

$$\|x\|_1 = \sum_{i=1,\ldots,n} |x_i|$$

Reformulating $\min ||x||_1$ into LP

To reformulate the problem

$$\min ||x||_1 \quad \text{s.t.} \quad Ax \leq b$$

into a linear program, we introduce a new variable z such that $z_i \geq |x_i|$. The problem becomes

$$\begin{aligned} \min \sum_i z_i \\ \text{s.t.} \quad & Ax \leq b \\ & x_i \leq z_i \quad \forall i \\ & -x_i \leq z_i \quad \forall i \end{aligned}$$

Here z_i represents an upper bound of the absolute value of x_i, making the objective linear. Since we are minimizing, z_i always equal to $|x_i|$ at optimum.

Reformulating $\min ||x||_\infty$ into LP

To reformulate the problem

$$\min ||x||_\infty \quad \text{s.t.} \quad Ax \leq b$$

into a linear program, we introduce a new variable t such that $t \geq |x_i|$ for all i. The problem becomes

$$\begin{aligned} & \min t \\ \text{s.t.} \quad & Ax \leq b \\ & x_i \leq t \quad \forall i \\ & -x_i \leq t \quad \forall i \end{aligned}$$

Reformulating constraints that involve absolute values into LP

$$\begin{aligned} & \min c^T x \\ \text{s.t.} \quad & Gx \leq h \\ & \sum a_i |x_i| \leq b_i \end{aligned}$$

if $a_i \geq 0 \quad \forall i$.
We introduce a new variable $z \in \mathbb{R}^n$ such that $z_i \geq |x_i|$. The problem becomes

$$\begin{aligned} & \min c^T x \\ \text{s.t.} \quad & Gx \leq h \\ & \sum a_i z_i \leq b_i \\ & x_i \leq z_i, \quad -x_i \leq z_i \quad \forall i \end{aligned}$$

This does not hold if there exists $a_i < 0$

References

1. Bertsimas, D., & Tsitsiklis, J. N. (1997). Introduction to linear optimization (Vol. 6, pp. 479-530). Belmont, MA: Athena scientific.