Lecture 7 Linear Programming Duality

Can Li

ChE 597: Computational Optimization
Purdue University

Motivation

$$\min x^2 + y^2$$
$$\text{subject to } x + y = 1$$

we introduce a Lagrange multiplier p and form the Lagrangean $L(x, y, p)$ defined by

$$L(x, y, p) = x^2 + y^2 + p(1 - x - y).$$

While keeping p fixed, we minimize the Lagrangean over all x and y, subject to no constraints, which can be done by setting $\partial L/\partial x$ and $\partial L/\partial y$ to zero. The optimal solution to this unconstrained problem is

$$x = y = \frac{p}{2},$$

and depends on p. The constraint $x + y = 1$ gives us the additional relation $p = 1$, and the optimal solution to the original problem is $x = y = 1/2$.

Lagrangian dual for LP

Consider standard form LP

$$x^* \quad \begin{aligned} &\min c^T x \\ \text{s.t. } &\boxed{Ax = b} \\ &x \geq 0 \end{aligned} \quad p \in \mathbb{R}^m$$

Penalize the violation of linear equalities by $p \in \mathbb{R}^m$

$$g(p) = \min_{x \geq 0} \left[c^T x + p^T (b - Ax) \right] \leq c^T x^* + p^T (b - Ax^*) = c^T x^*$$

$\hat{x}$: the minimum of

$g(p)$ always provide a lower bound of the original LP. Lagrangian dual is defined as the $g(p)$ that provides the "tightest" lower bound.

$$\max_p g(p)$$

$$g(p) = \min_{x \geq 0} [\, c^T x + p^T (b - Ax) \,]$$

$$= p^T b + \min_{x \geq 0} (c^T - p^T A) x.$$

$c^T - p^T A \geq 0^T$

$$\min_{x \geq 0} (c^T - p^T A) x = \begin{cases} 0 & \text{if } c - A^T p \geq 0 \\ -\infty & \text{otherwise.} \end{cases}$$

Goal: $\max_{p} g(p)$

$$\max_{p} \; p^T b$$

$$A^T p \leq c \quad (p^T A \leq c^T)$$

Lagrangian dual of LP

$$
\begin{aligned}
g(p) &= \min_{x \geq 0} \left[c^T x + p^T (b - Ax) \right] \\
&= p^T b + \min_{x \geq 0} \left(c^T - p^T A \right) x
\end{aligned}
$$

Note that

$$
\min_{x \geq 0} \left(c^T - p^T A \right) x = \begin{cases} 0, & \text{if } c^T - p^T A \geq 0^T \\ -\infty, & \text{otherwise} \end{cases}
$$

In maximizing $g(p)$, we only need to consider those values of p for which $g(p)$ is not equal to $-\infty$. Therefore, the dual problem is,

$$
\begin{aligned}
&\max\ p^T b \\
&\text{s.t. } p^T A \leq c^T.
\end{aligned}
$$

Different form of LP

Suppose the primal constraint is in the form of $Ax \geq b$ instead of the standard form. To provide a lower bound of the primal problem, we will need to have $p \geq 0$

$$
\begin{aligned}
g(p) &= \min_x \left[c^T x + p^T \underbrace{(b - Ax)}_{\leq 0} \right] \\
&= p^T b + \min_x \left(c^T - p^T A \right) x
\end{aligned}
$$

Note that

$$
\min_x \left(c^T - p^T A \right) x = \begin{cases} 0, & \text{if } c^T - p^T A = 0^T \\ -\infty, & \text{otherwise} \end{cases}
$$

In maximizing $g(p)$, we only need to consider those values of p for which $g(p)$ is not equal to $-\infty$. Therefore, the dual problem is,

$$
\begin{aligned}
\max\ & p^T b \\
\text{s.t.}\ & p^T A = c^T \\
& p \geq 0
\end{aligned}
$$

The LP dual

$$
\begin{array}{llll}
\min & c^T x & & \\
\text{s.t.} & a_i^T x \geq b_i, & i \in M_1, \\
& a_i^T x \leq b_i, & i \in M_2, \\
& a_i^T x = b_i, & i \in M_3, \\
& x_j \geq 0, & j \in N_1, \\
& x_j \leq 0, & j \in N_2, \\
& x_j \text{ free}, & j \in N_3,
\end{array}
\qquad
\begin{array}{llll}
\max & p^T b & & \\
\text{s.t.} & p_i \geq 0, & i \in M_1, \\
& p_i \leq 0, & i \in M_2, \\
& p_i \text{ free}, & i \in M_3, \\
& p^T A_j \leq c_j, & j \in N_1, \\
& p^T A_j \geq c_j, & j \in N_2, \\
& p^T A_j = c_j, & j \in N_3.
\end{array}
$$

PRIMAL	minimize	maximize	DUAL
constraints	$\geq b_i$	≥ 0	variables
	$\leq b_i$	≤ 0	
	$= b_i$	free	
variables	≥ 0	$\leq c_j$	constraints
	≤ 0	$\geq c_j$	
	free	$= c_j$	

LP dual example

$$
\begin{aligned}
\min \quad & x_1 + 2x_2 + 3x_3 \\
\text{s.t.} \quad & -x_1 + 3x_2 = 5 \\
& 2x_1 - x_2 + 3x_3 \geq 6 \\
& x_3 \leq 4 \\
& x_1 \geq 0 \\
& x_2 \leq 0 \\
& x_3 \text{ free,}
\end{aligned}
\qquad
\begin{aligned}
\max \quad & 5p_1 + 6p_2 + 4p_3 \\
\text{s.t.} \quad & p_1 \text{ free} \\
& p_2 \geq 0 \\
& p_3 \leq 0 \\
& -p_1 + 2p_2 \leq 1 \\
& 3p_1 - p_2 \geq 2 \\
& 3p_2 + p_3 = 3
\end{aligned}
$$

Weak duality

Theorem If x is a feasible solution to the primal problem and p is a feasible solution to the dual problem, then

$$p^T b \leq c^T x$$

Proof sketch. By construction the Lagrangian dual, for any feasible x to the primal, $g(p) \leq c^T x$ for any p satisfy the sign constraints. When the p satisfy the dual constraints $g(p) = p^T b$. Therefore, $p^T b \leq c^T x$.

$$u_i = p_i(a_i^T x - b_i), \quad v_j = (c_j - p^T A_j) x_j$$

$$u_i \geq 0, \qquad v_j \geq 0.$$

$$\sum_i u_i = p^T A x - p^T b.$$

$$\sum_j v_j = c^T x - p^T A x.$$

$$0 \leq \sum_i u_i + \sum_j v_j = c^T x - p^T b.$$

Weak duality

Proof. For any vectors x and p, we define

$$u_i = p_i \left(a_i^T x - b_i \right),$$
$$v_j = \left(c_j - p^T A_j \right) x_j.$$

Suppose that x and p are primal and dual feasible, respectively. The definition of the dual problem requires the sign of p_i to be the same as the sign of $a_i^T x - b_i$, and the sign of $c_j - p^T A_j$ to be the same as the sign of x_j. Thus, primal and dual feasibility imply that

$$u_i \geq 0, \quad \forall i, \quad v_j \geq 0, \quad \forall j.$$

Notice that

$$\sum_i u_i = p^T A x - p^T b$$

and

$$\sum_j v_j = c^T x - p^T A x.$$

We add these two equalities and use the nonnegativity of u_i, v_j, to obtain

$$0 \leq \sum_i u_i + \sum_j v_j = c^T x - p^T b$$

Weak duality corollary

primal: min $\geq$ dual: max.

$-\infty \Rightarrow -\infty$

(a) If the optimal cost in the primal is $-\infty$, then the dual problem must be infeasible.
(b) If the optimal cost in the dual is $+\infty$, then the primal problem must be infeasible.
Proof. Suppose that the optimal cost in the primal problem is $-\infty$ and that the dual problem has a feasible solution p. By weak duality, p satisfies $p^T b \leq c^T x$ for every primal feasible x. Taking the minimum over all primal feasible x, we conclude that $p^T b \leq -\infty$. This is impossible and shows that the dual cannot have a feasible solution, thus establishing part (a). Part (b) follows by a symmetrical argument.

Weak duality corollary

Let x and p be feasible solutions to the primal and the dual, respectively, and suppose that $p^T b = c^T x$. Then, x and p are optimal solutions to the primal and the dual, respectively.
Proof. Let x and p be as in the statement of the corollary. For every primal feasible solution y, the weak duality theorem yields $c^T x = p^T b \leq c^T y$, which proves that x is optimal. The proof of optimality of p is similar.

Strong duality

Theorem If a linear programming problem has an optimal solution (neither infeasible nor unbounded), then its dual also has an optimal solution, and the respective optimal costs are equal.

Strong duality

Proof. Consider the standard form problem

$$\begin{aligned} \min \quad & c^T x \\ \text{subject to } & Ax = b \\ & x \geq 0. \end{aligned}$$

Let us assume temporarily that the rows of A are linearly independent and that there exists an optimal solution. It can be shown that there exists an optimal solution x and an optimal basis B. Let $x_B = B^{-1}b$ be the corresponding vector of basic variables with nonnegative reduced cost (proof omitted). This can be easily shown in the case of nondegeneracy.

$$c^T - c_B^T B^{-1} A \geq 0^T$$

Let us define a vector p by letting $p^T = c_B^T B^{-1}$. We then have $p^T A \leq c^T$, which shows that p is a feasible solution to the dual problem

$$\begin{aligned} \max \quad & p^T b \\ \text{subject to } & p^T A \leq c^T. \end{aligned}$$

In addition,

$$p^T b = c_B^T B^{-1} b = c_B^T x_B = c^T x$$

It follows that p is an optimal solution to the dual, and the optimal dual cost is equal to the optimal primal cost.

Primal dual possibilities

Three possibilitiies can occur for any linear program.
(a) There is an optimal solution.
(b) The problem is "unbounded"; that is, the optimal cost is $-\infty$ (for minimization problems), or $+\infty$ (for maximization problems).
(c) The problem is infeasible.

	Finite optimum	Unbounded	Infeasible
Finite optimum	Possible	Impossible	Impossible
Unbounded	Impossible	Impossible	Possible
Infeasible	Impossible	Possible	Possible

Complementary slackness

Theorem Let x and p be feasible solutions to the primal and the dual problem, respectively. The vectors x and p are optimal solutions for the two respective problems if and only if:

$$p_i \left(a_i^T x - b_i \right) = 0, \forall i$$
$$\left(c_j - p^T A_j \right) x_j = 0, \forall j$$

Proof. In the proof of weak duality, we defined $u_i = p_i \left(a_i^T x - b_i \right)$ and $v_j = \left(c_j - p^T A_j \right) x_j$, and noted that for x primal feasible and p dual feasible, we have $u_i \geq 0$ and $v_j \geq 0$ for all i and j. In addition, we showed that

$$c^T x - p^T b = \sum_i u_i + \sum_j v_j.$$

By the strong duality theorem, if x and p are optimal, then $c^T x = p^T b$, which implies that $u_i = v_j = 0$ for all i, j. Conversely, if $u_i = v_j = 0$ for all i, j, then $c^T x = p^T b$, which implies that x and p are optimal.

Optimality conditions for LP

Remark The theorem can be used as an optimality condition for linear programs. If the primal and dual solutions satisfy

1. primal feasibility
2. dual feasibility
3. complementary slackness

then the primal and dual solution are both optimal. This is known as the KKT conditions in general convex optimization.

Remark A slightly different optimality condition is

1. primal feasibility
2. dual feasibility
3. primal and dual objectives are equal ($c^T x = p^T b$)

Marginal cost (shadow price)

$$\begin{aligned} \min \quad & c^T x \\ \text{subject to } & Ax = (b + d) \\ & x \geq 0 \end{aligned} \quad \Longleftrightarrow \quad \max \; p^T (b+d)$$

Motivation: Estimate how much the objective changes when the right hand side b is perturbed slightly by d.

- For nondegenerate problem, we have $B^{-1}b > 0$. We also have $B^{-1}(b + d) > 0$, as long as d is small.
- The reduced costs $c^T - c_B^T B^{-1} A$ does not depend on b. $p = c_B^T B^{-1}$ remains dual feasible.
- B remains to be the optimal basis when d is small
- The new objective will be $c_B^T B^{-1}(b + d) = p^T (b + d)$

The dual variable p_i can be interpreted as the marginal cost, also known as the shadow price per unit increase of the ith requirement b_i

Interpretation using the diet problem and economic dispatch

- Recall that the constraints of the diet problems are to ensure all dietary requirements for calories, protein, and vitamins are met. The RHS parameters b are the requirements for calories, protein, and vitamins. The marginal costs are the cost we need to pay for getting more nutrients. $p^T b = c^T x$ implies for cost for nutrients equal to the cost we paid for food.
- The Economic Dispatch Problem focuses on minimizing the total cost of generating power across all generating units while meeting the demand and adhering to operational constraints of the generators. The demand are the RHS parameters for energy balance constraints. The electricity price are the dual variables for these constraints. The interpretation is when the demand increase by 1 unit, how much additional price when need to pay.

Farkas' lemma

$Ax \leq b$

Motivation: How to certify that a systems of linear inequalities are infeasible.

Theorem Let A be a matrix of dimensions $m \times n$ and let b be a vector in $\Re^m$. Then, exactly one of the following two alternatives holds:

this system is feasible

(a) There exists some $x \geq 0$ such that $Ax = b$.

(b) There exists some vector p such that $p^T A \geq 0^T$ and $p^T b < 0$.

p as a certificate of infeasibility

Farkas' lemma

Proof. One direction is easy. If there exists some $x \geq 0$ satisfying $Ax = b$, and if $p^T A \geq 0^T$, then $p^T b = p^T Ax \geq 0$, which shows that the second alternative cannot hold.
Let us now assume that there exists no vector $x \geq 0$ satisfying $Ax =$ b. Consider the pair of problems

$$\begin{aligned} \max \quad & 0^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0, \end{aligned} \qquad\qquad \begin{aligned} \min \quad & p^T b \\ \text{s.t.} \quad & p^T A \geq 0^T, \end{aligned}$$

and note that the first is the dual of the second. The maximization problem is infeasible, which implies that the minimization problem is either unbounded (the optimal cost is $-\infty$) or infeasible. Since $p = 0$ is a feasible solution to the minimization problem, it follows that the minimization problem is unbounded. Therefore, there exists some p which is feasible, that is, $p^T A \geq 0^T$, and whose cost is negative, that is, $p^T b < 0$.

Variants of Farkas' lemma

- Either the system $Ax = b$ has a solution with $x \geq 0$, or the system $A^\top p \geq 0$ has a solution with $b^\top p < 0$.
- Either the system $Ax \leq b$ has a solution with $x \geq 0$, or the system $A^\top p \geq 0$ has a solution with $b^\top p < 0$ and $p \geq 0$.
- Either the system $Ax \leq b$ has a solution with $x \in \mathbb{R}^n$, or the system $A^\top p = 0$ has a solution with $b^\top p < 0$ and $p \geq 0$.
- Either the system $Ax = b$ has a solution with $x \in \mathbb{R}^n$, or the system $A^\top p = 0$ has a solution with $b^\top p \neq 0$.

Theorem of the alternative

Theorem Suppose that the system of linear inequalities $Ax \leq b$ has at least one solution, and let d be some scalar. Then, the following are equivalent:

(a) Every feasible solution to the system $Ax \leq b$ satisfies $c^T x \leq d$.

(b) There exists some $p \geq 0$ such that $p^T A = c^T$ and $p^T b \leq d$.

if we have $Ax \leq b$

implies $c^T x \leq d$

Proof of theorem of the alternative

Proof. Consider the following pair of problems

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \end{aligned} \qquad\qquad \begin{aligned} \min \quad & p^T b \\ \text{s.t.} \quad & p^T A = c^T \\ & p \ge 0 \end{aligned}$$

and note that the first is the dual of the second. If the system $Ax \le b$ has a feasible solution and if every feasible solution satisfies $c^T x \le d$, then the first problem has an optimal solution and the optimal cost is bounded above by d. By the strong duality theorem, the second problem also has an optimal solution p whose cost is bounded above by d. This optimal solution satisfies $p^T A = c^T, p \ge 0$, and $p^T b \le d$.

Conversely, if some p satisfies $p^T A = c^T, p \ge 0$, and $p^T b \le d$, then the weak duality theorem asserts that every feasible solution to the first problem must also satisfy $c^T x \le d$

Dual simplex algorithm

- Idea: start with the dual feasible solution that satisfy $c^T - c_B^T B^{-1} A \geq 0$, move to a primal feasible solution.
- Dual simplex algorithm differs from applying primal simplex algorithm to the dual. Note that in dual simplex, only the primal problem is in the standard form. The dual problem is not in the standard form.
- Dual simplex algorithm is especially useful when we are repeatedly solving problems with the same A and c but different right hand side b. The change of b does not change dual feasibility.

Reference

1. Chapter 4. Bertsimas, D., & Tsitsiklis, J. N. (1997). Introduction to linear optimization. Belmont, MA: Athena scientific.