Lecture 15 Cutting Planes

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ChE 597: Computational Optimization
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Why cutting planes?

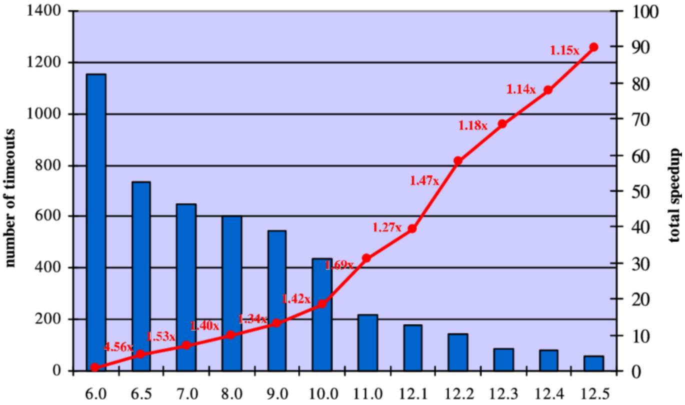
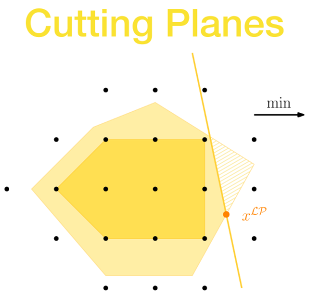


Figure: Comparison of CPLEX versions 6.0 to 12.5 on 1753 problem instances

Version 6.5 is when CPLEX started to add cutting planes (4.56× speedup).

Why cutting planes?



- Cutting planes (valid inequalities) improve the LP relaxation, significantly reducing the size of the branch and bound tree.
- Cutting planes are derived from the fact that some of the variables are integer. (They cannot be derived from the LP relaxation itself.)

Theoretical motivation

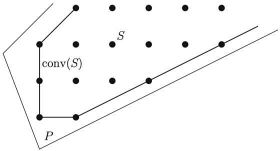
Theorem. Let $S \subset \mathbb{R}^n$ and $c \in \mathbb{R}^n$. Then
$\sup\{c^T x : x \in S\} = \sup\{c^T x : x \in \text{conv}(S)\}$.
Furthermore, the supremum of $c^T x$ is attained over S if and only if it is attained over $\text{conv}(S)$.
Interpretation: Any nonconvex optimization problem with linear objective function such as MILP is equivalent to optimizing over the convex hull of its feasible region. If we can find $\text{conv}(S)$, it will become a convex optimization problem.

Meyer's Theorem (The Fundamental Theorem of Integer Programming)

Theorem Given rational matrices A, G and a rational vector b, let $P := \{(x, y) : Ax + Gy \leq b\}$ and let $S := \{(x, y) \in P : x \text{ integral}\}$.

1. There exist rational matrices A', G' and a rational vector b' such that $\text{conv}(S) = \{(x, y) : A'x + G'y \leq b'\}$.
2. If S is nonempty, the recession cones of $\text{conv}(S)$ and P coincide.

Interpretation: Meyer's theorem shows that there exist a polyhedral representation of $\text{conv}(S)$. The next question is how to find the $\text{conv}(S)$ or a relaxation of $\text{conv}(S)$. Answer: cutting planes (valid inequalities).



Illustrative example: tightened big-M

$$X = \{(x, y) : y \leq 9999x, 0 \leq y \leq 5, x \in \{0, 1\}\}.$$

It is easily checked that the following inequality is valid.

$$y \leq 5x$$

Remark Note that this valid inequality cannot be derived based on the LP relaxation itself. We need to take the fact that x is binary into account. On the other hand, a valid inequality derived from LP relaxation will not improve the bound.

Application to facility location

Such constraints arise often. For instance, in the capacitated facility location problem the feasible region is:

$$\sum_{i \in M} y_{ij} \leq b_j x_j \quad \text{for } j \in N$$

$$\sum_{j \in N} y_{ij} = a_i \quad \text{for } i \in M$$

$$y_{ij} \geq 0 \quad \text{for } i \in M, j \in N, \quad x_j \in \{0,1\} \quad \text{for } j \in N.$$

All feasible solutions satisfy $y_{ij} \leq b_j x_j$ and $y_{ij} \leq a_i$ with $x_j \in \{0,1\}$. This is precisely the situation above leading to the family of valid inequalities $y_{ij} \leq \min\{a_i, b_j\} x_j$.

Clique inequalities in stable sets

Consider a stable sets (a set of vertices in a graph, no two of which are adjacent). Consider the set X of incidence vectors of stable sets in a graph $G = (V, E)$:

$$x_i + x_j \leq 1 \quad \text{for } (i,j) \in E$$
$$x \in \{0,1\}^{|V|}.$$

Take a clique $U \subseteq V$. As there is an edge between every pair of nodes in U, any stable set contains at most one node of U. Therefore,

$$\sum_{j \in U} x_j \leq 1$$

is a valid inequality for X.

Integer rounding

Consider the integer region $X = P \cap \mathbb{Z}^4$, where

$$P = \left\{x \in \mathbb{R}^4_+ : 13x_1 + 20x_2 + 11x_3 + 6x_4 \geq 72\right\}.$$

Dividing by 11 gives the valid inequality for P :

$$\frac{13}{11}x_1 + \frac{20}{11}x_2 + x_3 + \frac{6}{11}x_4 \geq 6\frac{6}{11}.$$

As $x \geq 0$, rounding up the coefficients on the left to the nearest integer gives $2x_1 + 2x_2 + x_3 + x_4 \geq \frac{13}{11}x_1 + \frac{20}{11}x_2 + x_3 + \frac{6}{11}x_4 \geq 6\frac{6}{11}$, and so we get a weaker valid inequality for P :

$$2x_1 + 2x_2 + x_3 + x_4 \geq 6\frac{6}{11}.$$

As x is an integer, and all the coefficients are integers, the lhs must be an integer. An integer that is greater than or equal to $6\frac{6}{11}$ must be at least 7 , and so we can round the rhs up to the nearest integer giving the valid inequality for X :

$$2x_1 + 2x_2 + x_3 + x_4 \geq 7.$$

Valid inequality for linear program

Theorem of alternative $\pi x \leq \pi_0$ is valid for $P = \{x : Ax \leq b, x \geq 0\} \neq \emptyset$ if and only if there exist $u \geq 0, v \geq 0$ such that $uA - v = \pi$ and $ub \leq \pi_0$, or alternatively there exists $u \geq 0$ such that $uA \geq \pi$ and $ub \leq \pi_0$.

Chvátal-Gomory procedure to construct a valid inequality

Consider the set $X = P \cap \mathbb{Z}^n$, where $P = \left\{x \in \mathbb{R}^n_+ : Ax \leq b\right\}, A$ is an $m \times n$ matrix with columns $\{a_1, a_2, \ldots, a_n\}$ and $u \in \mathbb{R}^m_+$:

(i) the inequality

$$\sum_{j=1}^{n} u a_j x_j \leq ub$$

is valid for P as $u \geq 0$ and $\sum_{j=1}^{n} a_j x_j \leq b$,

(ii) the inequality

$$\sum_{j=1}^{n} \lfloor u a_j \rfloor x_j \leq ub$$

is valid for P as $x \geq 0$,

(iii) the inequality

$$\sum_{j=1}^{n} \lfloor u a_j \rfloor x_j \leq \lfloor ub \rfloor$$

is valid for X as x is integer, and thus $\sum_{j=1}^{n} \lfloor u a_j \rfloor x_j$ is integer.

Theorem Every valid inequality for X can be obtained by applying the Chvátal-Gomory procedure a finite number of times.
Remark It is impractical to start with all the cuts one can add. In practice, one usually start with the solution to the LP relaxation and gradually cut off the fractional solution. One such scheme is the Gomory's Fractional Cutting Plane Algorithm.

Start from the optimal basis

Given a rational $m \times n$ matrix A and a rational vector $b \in \mathbb{R}^n$, let $P := \{x \in \mathbb{R}^n_+ : Ax = b\}$ and let $S := P \cap \mathbb{Z}^n$.
Let $c \in \mathbb{R}^n$, and consider the pure integer programming problem $\max\{c^T x : x \in S\}$. Let B be an optimal basis of the linear programming relaxation $\max\{c^T x : x \in P\}$, and let $N := \{1, \ldots, n\} \setminus B$. The tableau relative to B is of the form

$$x_i + \sum_{j \in N} \bar{a}_{ij} x_j = \bar{b}_i, \quad i \in B.$$

The Chvátal Inequality

The corresponding optimal solution to the linear programming relaxation is $x_i^* = \bar{b}_i, i \in B, x_j^* = 0, j \in N$. If x^* is integral, it is an optimal solution to the integer programming problem. Otherwise, there exists some $h \in B$ such that $\bar{b}_h \notin \mathbb{Z}$.
The Chvátal inequality relative to the hth row of the tableau is

$$x_h + \sum_{j \in N} \lfloor \bar{a}_{hj} \rfloor x_j \leq \lfloor \bar{b}_h \rfloor .$$

Introducing a nonnegative slack variables x_{n+1}, the above inequality becomes

$$x_h + \sum_{j \in N} \lfloor \bar{a}_{hj} \rfloor x_j + x_{n+1} = \lfloor \bar{b}_h \rfloor .$$

Gomory Fractional Cut

By subtracting the hth tableau row from the Chvátal inequality, we obtain the Gomory fractional cut:

$$\sum_{j \in N} -f_j x_j + x_{n+1} = -f_0,$$

where $f_j := \bar{a}_{hj} - \lfloor \bar{a}_{hj} \rfloor$ and $f_0 := \bar{b}_h - \lfloor \bar{b}_h \rfloor$. Juxtaposing the latter equation at the bottom of the tableau, we obtain the tableau with respect to the basis $B' := B \cup \{n+1\}$. This tableau is not primal feasible but is dual feasible, allowing us to apply the dual simplex method starting from the basis B' for the new problem.

Example

$$Z = \max 4x_1 - x_2$$

Subject to:

$$7x_1 - 2x_2 \leq 14,$$
$$x_2 \leq 3,$$
$$2x_1 - 2x_2 \leq 3,$$
$$x_1,\ x_2 \geq 0 \text{ and integer.}$$

Adding slack variables x_3, x_4, x_5, the linear program equivalent representation where x_1, x_2, x_5 are basic, x_3, x_3 are nonbasic:

$$Z = \max \frac{59}{7} - \frac{4}{7}x_3 - \frac{1}{7}x_4$$

With constraints:

$$x_1 + \frac{1}{7}x_3 + \frac{2}{7}x_4 = \frac{20}{7},$$
$$x_2 + x_4 = 3,$$
$$-\frac{2}{7}x_3 + \frac{10}{7}x_4 + x_5 = \frac{23}{7}$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0 \text{ and integer.}$$

Solution and Cut Generation

The optimal linear programming solution is
$x = \left(\frac{20}{7}, 3, 0, 0, \frac{23}{7}\right) \notin Z_+^5$, indicating the solution is not entirely integer.
To address this, we generate a cut from the first row, where the basic variable x_1 is fractional:

$$\frac{1}{7}x_3 + \frac{2}{7}x_4 \geq \frac{6}{7}$$

Or equivalently:

$$s = -\frac{6}{7} + \frac{1}{7}x_3 + \frac{2}{7}x_4$$

with $s, x_3, x_4 \geq 0$ and integer, aiming to steer the solution towards integrality.

Basic mixed-integer inequality

Theorem Let $X^{\geq} = \left\{(x, y) \in \mathbb{Z}^1 \times \mathbb{R}^1_+ : x + y \geq b\right\}$, and $f = b - \lfloor b \rfloor > 0$. The inequality

$$y \geq f(\lceil b \rceil - x) \quad \text{or} \quad \frac{y}{f} + x \geq \lceil b \rceil$$

is valid for $X^{\geq}$.

Proof. If $x \geq \lceil b \rceil$, then $y \geq 0 \geq f(\lceil b \rceil - x)$. If $x \leq \lfloor b \rfloor$, then

$$\begin{aligned} y &\geq b - x = f + (\lfloor b \rfloor - x) \\ &\geq f + f(\lfloor b \rfloor - x), \quad \text{as } \lfloor b \rfloor - x \geq 0 \text{ and } f < 1, \\ &= f(\lceil b \rceil - x). \end{aligned}$$

Basic mixed-integer inequality

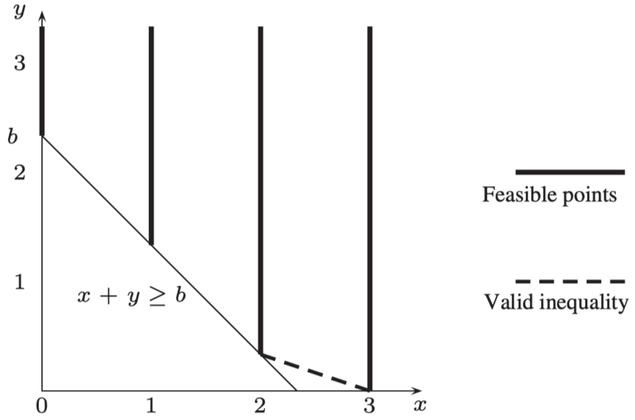


Figure: ref: Wolsey, 2020

Corollary If $X^{\leq} = \left\{(x, y) \in \mathbb{Z}^1 \times \mathbb{R}^1_+ : x \leq b + y\right\}$ and $f = b - \lfloor b \rfloor > 0$, the inequality

$$x \leq \lfloor b \rfloor + \frac{y}{1 - f}$$

is valid for $X^{\leq}$.
Proof. Rewriting $x \leq b + y$ as $y - x \geq -b$ and observing that $-b - \lfloor -b \rfloor = 1 - f$, we obtain from the previous theorem that $\frac{y}{1-f} - x \geq \lceil -b \rceil = -\lfloor b \rfloor$.

Mixed Integer Rounding (MIR) Inequality

To obtain a slight variant of the basic inequality, we consider a set

$$X^{\mathrm{MIR}} = \left\{(x, y) \in \mathbb{Z}_+^2 \times \mathbb{R}_+^1 : a_1 x_1 + a_2 x_2 \leq b + y\right\},$$

where a_1, a_2, and b are scalars with $b \notin \mathbb{Z}^1$.

Theorem Let $f = b - \lfloor b \rfloor$ and $f_i = a_i - \lfloor a_i \rfloor$ for $i = 1, 2$. Suppose $f_1 \leq f < f_2$, then

$$\lfloor a_1 \rfloor x_1 + \left(\lfloor a_2 \rfloor + \frac{f_2 - f}{1 - f}\right) x_2 \leq \lfloor b \rfloor + \frac{y}{1 - f}$$

is valid for X^{MIR}.

Proof. $(x, y) \in X^{\mathrm{MIR}}$ satisfies
$\lfloor a_1 \rfloor x_1 + \lceil a_2 \rceil x_2 \leq b + y + (1 - f_2)\, x_2$ as $x_1 \geq 0$, and
$a_2 = \lceil a_2 \rceil - (1 - f_2)$. Now the Corollary gives

$$\lfloor a_1 \rfloor x_1 + \lceil a_2 \rceil x_2 \leq \lfloor b \rfloor + \left[y + (1 - f_2)\, x_2\right] / (1 - f),$$

which is the required inequality.

Example

Consider the set $X = \left\{(x,y) \in \mathbb{Z}_+^3 \times \mathbb{R}_+^1 : \frac{10}{3}x_1 + 1x_2 + \frac{11}{4}x_3 \leq \frac{21}{2} + y\right\}$. Using the MIR, we have
$f = 1/2, f_1 = 1/3, f_2 = 0, f_3 = 3/4$, and thus

$$3x_1 + x_2 + \frac{5}{2}x_3 \leq 10 + 2y$$

is valid for X.

Gomory Mixed Integer Cut

Here we continue to consider mixed integer programs. As for all integer programs, any row of an optimal linear programming tableau, in which an integer variable is basic but fractional, can be used to generate a cut removing the optimal linear programming solution. Specifically, such a row leads to a set of the form:

$$X^G = \left\{ (x_{B_u}, x, y) \in \mathbb{Z}^1 \times \mathbb{Z}_+^{n_1} \times \mathbb{R}_+^{n_2} : x_{B_u} + \sum_{j \in N_1} \bar{a}_{uj} x_j + \sum_{j \in N_2} \bar{a}_{uj} y_j = \bar{a}_{u0} \right.$$

where $n_i = |N_i|$ for $i = 1, 2$. If $\bar{a}_{u0} \notin \mathbb{Z}^1, f_j = \bar{a}_{uj} - \lfloor \bar{a}_{uj} \rfloor$ for $j \in N_1$, and $f_0 = \bar{a}_{u0} - \lfloor \bar{a}_{u0} \rfloor$, the Gomory mixed integer cut

$$\sum_{f_j \leq f_0} f_j x_j + \sum_{f_j > f_0} \frac{f_0 \left(1 - f_j\right)}{1 - f_0} x_j + \sum_{\bar{a}_{uj} > 0} \bar{a}_{uj} y_j - \sum_{\bar{a}_{uj} < 0} \frac{f_0}{1 - f_0} \bar{a}_{uj} y_j \geq f_0$$

is valid for X^G.

Proof. The mixed integer rounding inequality for X^G is

$$x_{B_u}+\sum_{f_j\le f_0}\lfloor \bar{a}_{uj}\rfloor x_j+\sum_{f_j>f_0}\left(\lfloor \bar{a}_{uj}\rfloor+\frac{f_j-f_0}{1-f_0}\right)x_j+\sum_{\bar{a}_{uj}<0}\frac{\bar{a}_{uj}}{1-f_0}y_j\le \lfloor \bar{a}_{u0}\rfloor .$$

Substituting for x_{B_u} proves the claim.

Example: GMI

$$Z = \min 4x_1 + 7x_2 + y_1 - y_2$$
$$5x_1 + 4x_2 - y_1 - 2y_2 = 5$$
$$3x_2 + y_1 - y_2 = 6$$
$$x_1 \in \mathbb{Z}^2_+, y \in \mathbb{R}^2_+.$$

Example: GMI

Solving as a linear program gives

$$
\begin{aligned}
&Z = \min \frac{86}{7} + \frac{8}{7}x_1 \quad + \frac{12}{7}y_2 \\
&\frac{5}{7}x_1 + x_2 \quad - \frac{3}{7}y_2 = \frac{11}{7} \\
&\frac{-15}{7}x_1 \quad + y_1 + \frac{2}{7}y_2 = \frac{9}{7} \\
&x_1 \in \mathbb{Z}_+^2, y \in \mathbb{R}_+^2.
\end{aligned}
$$

The basic variable x_2 is fractional and the first row gives the mixed integer rounding (MIR) cut

$$\frac{1}{3}x_1 + x_2 - y_2 \leq 1$$

which after elimination of x_2 becomes the Gomory mixed integer cut:

$$\frac{8}{21}x_1 + \frac{4}{7}y_2 \geq \frac{4}{7}$$

Adding this cut and reoptimizing leads to the solution $x = (0, 2), y = (1, 1)$ which is feasible and hence optimal.

References

- Wolsey, L. A. (2020). Integer programming. John Wiley & Sons.
- Conforti, M., Cornuéjols, G., Zambelli, G (2014). Integer programming. Graduate Texts in Mathematics