# ChE 597 Computational Optimization

## Practice Exam 1

This is a closed book examination. Only a calculator and a single piece of note-book paper with hand-written notes are allowed. **Please print your name on the upper right corner of each page and on the last page where indicated. Your signature is required on the bottom of this page where indicated.** Read all questions carefully; answer them concisely and **neatly**. Show all work on these pages. Point values in the right-hand margins are based on 100points total for the examination.

Read everything in the following paragraphs, print your name, and sign your name prior to starting the exam. Failure to comply or complete the steps outlined below could result in a score of 0% on the exam.

During this exam, I (the undersigned) agree that I will:

1. Write my name in the upper right hand corner of every page that is to be graded for the exam. If your name is not placed on a page, that page will not be graded.
2. Circle or box the final answer. Any answer that is not circled or boxed will not be graded.
3. Show all work and state all assumptions that are made.
4. Not cheat in any manner whatsoever. I understand that if I am caught cheating by verbal, written, or any other method that I will receive an F for the course. Furthermore, my actions will be reported to the Dean of Students.
5. Submit the completed exam to one of the instructors or teaching assistants at the end of the exam.
6. Not carry the exam out of the examination room.

I understand all of these conditions and will abide by them.

Print Name:

Signature:

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1. Take the dual of the following linear program

$a^T x \le b \quad p \le 0.$

$\underbrace{p(b - a^T x)}_{\le 0}$ with $p \le 0$, $b - a^T x \ge 0$

$$
\begin{aligned}
\min \quad & 2x_1 - x_2 \\
& 2x_1 + 3x_2 - x_3 + x_4 \le 0 \quad (p_1 \le 0) \\
& 3x_1 + x_2 + 4x_3 - 2x_4 \ge 3 \quad (p_2 \ge 0) \\
& -x_1 - x_2 + 2x_3 + x_4 = 6 \quad (p_3) \\
& x_2 \ge 0 \\
& x_4 \le 0
\end{aligned}
$$

**[15 pts]**

$$
\begin{aligned}
g(p_1, p_2, p_3) = \min_x \; & 2x_1 - x_2 + p_1(-2x_1 - 3x_2 + x_3 - x_4) \\
& + p_2(3 - 3x_1 - x_2 - 4x_3 + 2x_4) \\
& + p_3(6 + x_1 + x_2 - 2x_3 - x_4)
\end{aligned}
$$

$$
\begin{aligned}
= \min_x \; & (2 - 2p_1 - 3p_2 + p_3)x_1 + \\
& (-1 - 3p_1 - p_2 + p_3)x_2 \\
& + (p_1 - 4p_2 - 2p_3)x_3 + \\
& (-p_1 + 2p_2 - p_3)x_4 \\
& + 3p_2 + 6p_3
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & 3p_2 + 6p_3 \\
\text{s.t.} \quad & 2 - 2p_1 - 3p_2 + p_3 = 0 \\
& -1 - 3p_1 - p_2 + p_3 \ge 0 \\
& p_1 - 4p_2 - 2p_3 = 0 \\
& -p_1 + 2p_2 - p_3 \le 0
\end{aligned}
$$

2. Consider the following logic conditions

   (a) If $x$ is true and $y$ is true, then $z$ is true

   (b) If $x$ is true ~~and/~~or $y$ is false, then $w$ is false
   or

   (c) If $z$ and $w$ are not true, then either $x$ ~~and/~~or $y$ are true.
   or

   Formulate linear inequalities in terms of $0-1$ variables to model the above condition.

**[20 pts]**

(a)

$(x \wedge y) \Rightarrow z$

$\neg(x \wedge y) \vee z$

$\neg x \vee \neg y \vee z$

$1 - x + (1 - y) + z \geq 1$

(c)

$(\neg z \wedge \neg w) \Rightarrow (x \vee y)$

$\neg(\neg z \wedge \neg w) \vee (x \vee y)$

$(z \vee w) \vee (x \vee y)$

$z + w + x + y \geq 1$

(b)

$(x \vee \neg y) \Rightarrow \neg w$

$\neg(x \vee \neg y) \vee \neg w$

$(\neg x \wedge y) \vee \neg w$

$(\neg x \vee \neg w) \wedge (y \vee \neg w)$

$1 - x + 1 - w \geq 1, \quad y + (1 - w) \geq 1$

3. Formulate the following disjunctive program using the big-M reformulation

$$\min Z = 2u + 3v + c$$
$$\text{s.t. } -u + v \le 0$$
$$\begin{bmatrix} u + v \le 2 \\ u + 2v \ge 2 \\ c = 3 \end{bmatrix} \vee \begin{bmatrix} u + v \le 6 \\ 2u + v \ge 6 \\ c = 1 \end{bmatrix}$$
$$0 \le u \le 6 \quad 0 \le v \le 6$$

$[u \geq 2] \vee [u \leq 1]$

$\min \ Z = 2u + 3v + c$

$-u + v \leq 0$

$0 \leq u \leq 6, \quad 0 \leq v \leq 6.$

$y=1$ to represent the first disjunct is true

$u + v \leq 2 + 4(1-y)$ (*) (Because the other disjunct have $u+v \leq 6$)

$u + 2v \geq 2 - 2(1-y)$

$-2(1-y) + 3 \leq c \leq 3$

$\cancel{u+v \leq 6}$ (redundant because we have (*))

$2u + v \geq 6 - 6y$

$1 \leq c \leq 1 + 2y$

4. Consider the following quadratic program with linear constraints

$$
\begin{aligned}
\min f &= x_1^2 + x_2^2 - 8x_1 - x_2 + 10 \\
\text{st} \quad & x_1 - x_2 \leq 0 \\
& x_1 + x_2 \leq 4 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

(a) Write down the KKT conditions

(b) Derive the Langrangian dual function.

**[25 pts]**

5. *Short Questions*

   (a) Show that the conic hull of a finite number of points $x_1, x_2, \ldots, x_m \in \mathbb{R}^n$ is a convex set.

   (b) Explain when the least squares loss $f(\beta) = \|y - X\beta\|_2^2$ is strongly convex and when it is not. Hint: the Hessian of $f(\beta)$ is $X^T X$.

   (c) Suppose we have two equivalent formulations of an MILP problem $K^1 = \{(x,y) : A_1 x + G_1 y \leq b_1, y$ integral $\}$ and $K^2 = \{(x,y) : A_2 x + \; G_2 y \leq b_2, y$ integral $\}$ where $K^1 = K^2$. Explain what "$K_1$ is tighter than $K_2$" mean.

   (d) Explain why SOCP can be represented by SDP.

**[20 pts]**