Generator Expansion Example

In this example, we try to solve a generator expansion problem based on example 6.5 in the Introduction to Linear Programming book by Dimitris Bertsimas and John N. Tsitsiklis [1].

1 Problem Statement

An electric utility is installing two generators (indexed by $j = 1, 2$) with different fixed and operating costs, in order to meet the demand within its service region for a day. The day is divided into three parts of equal duration, indexed by $i = 1, 2, 3$. These correspond to parts of the day during which demand takes a base, medium, or peak value, respectively. The fixed cost per unit capacity of generator j is amortized over its lifetime and amounts to c_j per day. The operating cost of generator j during the i^{th} part of the day is $f_{i,j}$. If the demand during the i^{th} part of the day cannot be served due to lack of capacity, additional capacity must be purchased at a cost of g_i per unit capacity purchased. Finally, the capacity of each generator j is required to be at least b_j.

There are two sources of uncertainty, namely, the exact value of the demand d_i during each part of the day, and the availability a_j of generator j. The demand d_i can take one of four values $d_{i,1}, ..., d_{i,4}$, with probability $p_{i,1}, , p_{i,4}$, respectively. The availability of generator 1 is $a_{1,1}, ..., a_{1,4}$, with probability $q_{1,1}, , q_{1,4}$, respectively. Similarly, the availability of generator 2 is $a_{2,1}, ..., a_{2,5}$, with probability $q_{2,1}, , q_{2,5}$, respectively. If we enumerate all the possible events, we see that there is a total of $4^3 \times 4 \times 5 = 1280$ scenarios ω. Let us use d_i^ω and a_j^ω to denote the demands and availabilities, respectively, under scenario ω. To optimize the cost, we need to optimize installed capacity of the generator as well as the operating levels of the generators and the amount of power purchased to meet the unmet demand at different times of the day under different scenarios. For this, we formulate a multi-time scale model with decisions in 2 time scales: decisions for the day, and decisions for the different parts of the day. The integrated model is formally defined in the next sub-section

2 Integrated model

2.1 Indices and Sets

$j \in \mathcal{J}$ set of generators

$i \in \mathcal{I}$ set of parts of the day

$\omega \in \Omega$ set of scenarios

2.2 Variables

x_j installed capacity of generator j for the day

$y_{i,j}^\omega$ operating levels of generator j during the i^{th} part of the day under scenario ω.

$\tilde{y}_i^\omega$ power purchased under scenario ω, during the i^{th} part of the day

2.3 Parameters

a_j^ω availability of generator j in scenario ω

b_j minimum capacity of generator j to be installed

c_j fixed cost per unit capacity of generator j per day amortized over its lifetime

d_i^ω demand of power during the i^{th} part of the day in scenario ω

$f_{i,j}$ operating cost of generator j during the i^{th} part of the day

g_i cost of additional capacity purchased per unit capacity in the i^{th} part of the day

2.4 Constraints

We add a constraint to ensure that the capacity of each generator j is required to be at least b_j.

$$x_j \geq b_j \quad \forall\, j \in \mathcal{J} \tag{1}$$

The operating level of generators are constrained by $a_j^\omega x_j$.

$$y_{i,j}^\omega \leq a_j^\omega x_j \quad \forall\, i \in \mathcal{I}, j \in \mathcal{J}, \omega \in \Omega \tag{2}$$

The demand for power needs to be satisfied and is ensured by the following constraint.

$$\sum_{j \in \mathcal{J}} y_{i,j}^\omega + \tilde{y}_i^\omega \geq d_i^\omega \quad \forall\, i \in \mathcal{I}, \omega \in \Omega \tag{3}$$

2.5 Objective

The objective of the model is to minimize the net cost consisting of the fixed cost of setting up the generators $\sum_{j \in \mathcal{J}} c_j x_j$, as well as the expected operating cost $E_{\omega \in \Omega}\left[\sum_{i \in \mathcal{I}}\left(\sum_{j \in \mathcal{J}} f_{i,j} y_{i,j}^\omega + g_i \tilde{y}_i^\omega\right)\right]$. The objective is the sum of these components

2.6 Optimization model

The entire model is given below: (4)

$$\min \quad \phi = \sum_{j \in \mathcal{J}} c_j x_j + E_{\omega \in \Omega} \left[\sum_{i \in \mathcal{I}} \left(\sum_{j \in \mathcal{J}} f_{i,j} y_{i,j}^{\omega} + g_i \tilde{y}_i^{\omega} \right) \right] \tag{4a}$$

$$\text{s.t.} \quad x_j \geq b_j \quad \forall\, j \in \mathcal{J} \tag{4b}$$

$$y_{i,j}^{\omega} \leq a_j^{\omega} x_j \quad \forall\, i \in \mathcal{I}, j \in \mathcal{J}, \omega \in \Omega \tag{4c}$$

$$\sum_{j \in \mathcal{J}} y_{i,j}^{\omega} + \tilde{y}_i^{\omega} \geq d_i^{\omega} \quad \forall\, i \in \mathcal{I}, \omega \in \Omega \tag{4d}$$

3 High-level model

The high-level model is adapted from the full space model with the operating decisions for the three parts of the day aggregated. The high-level model takes in only one-time scale, i.e., for the day. We further average all the scenarios to reduce the complexity of the problem further. With a slight abuse in notation, the operating variables of the aggregated model are summarized below: From

Table 1: Variables in high-level model

Variable in full-space	Physical meaning	Variable in high-level model	Physical meaning
$y_{i,j}^{\omega}$	Operating levels of generator j during the i^{th} part of the day under scenario ω	y_j	Net operating levels of generator j for the day
$\tilde{y}_i^{\omega}$	Power purchased under scenario ω, during the i^{th} part of the day	$\tilde{y}$	Net power purchased for the day

the high-level model we obtain the installed capacity of the generators x_j. The optimization model is given below:

$$\min \quad \phi = \sum_{j \in \mathcal{J}} c_j x_j + \sum_{j \in \mathcal{J}} \left(\sum_{i \in \mathcal{I}} f_{i,j} \right) y_j + \left(\sum_{i \in \mathcal{I}} g_i \right) \tilde{y} \tag{5a}$$

$$\text{s.t.} \quad x_j \geq b_j \quad \forall\, j \in \mathcal{J} \tag{5b}$$

$$y_j \leq 3 E_{\omega \in \Omega} \left[a_j^{\omega} \right] x_j \quad \forall\, j \in \mathcal{J} \tag{5c}$$

$$\sum_{j \in \mathcal{J}} y_j + \tilde{y} \geq \sum_{i \in \mathcal{I}} E_{\omega \in \Omega} \left[d_i^{\omega} \right] \tag{5d}$$

4 Parametrized high-level model

Based on the data, we can either underestimate or overestimate the effect of the demand in the aggregated model. Therefore we add a prefactor to the net demand to compensate for the mismatch. The same can be said for the availability of the generators. Hence, we add a prefactor to the availability of the generators to compensate for the mismatch. Furthermore, to better capture the differences in the operation of the generators, we add a third parameter which signifies the minimum capacity of each generator i.e., the capacity of each generator is greater than this parameter. This can help in cases when an optimum configuration consists of having both generators installed with a reasonable capacity. Without this parameter, there is a high chance of installing only one generator for cases where there are some intricate differences between the generators. Therefore, the parameters we add to the high-level model are

1. ρ_1 which represents the prefactor to the net demand,

2. ρ_2 which represents the prefactor to the availability of generators,

3. ρ_3 which represents the minimum capacity installed for each of the generators.

The parameterized model is as follows:

$$\begin{aligned}
\min \quad & \phi = \sum_{j \in \mathcal{J}} c_j x_j + \sum_{j \in \mathcal{J}} \left(\sum_{i \in \mathcal{I}} f_{i,j} \right) y_j + \left(\sum_{i \in \mathcal{I}} g_i \right) \tilde{y} && \text{(6a)} \\
\text{s.t.} \quad & x_j \geq b_j \quad \forall\, j \in \mathcal{J} && \text{(6b)} \\
& y_j \leq 3\rho_2 E_{\omega \in \Omega} \left[a_j^\omega \right] x_j \quad \forall\, j \in \mathcal{J} && \text{(6c)} \\
& \sum_{j \in \mathcal{J}} y_j + \tilde{y} \geq \rho_1 \sum_{i \in \mathcal{I}} E_{\omega \in \Omega} \left[d_i^\omega \right] && \text{(6d)} \\
& x_j \geq \rho_3 \quad \forall\, j \in \mathcal{J} && \text{(6e)} \\
& && \text{(6f)}
\end{aligned}$$

5 Low-level model

The low-level model is the full-space model (4) with the installed capacity x_j of the generators fixed from the parametrized high-level model (6). The low-level model takes decisions for the parts of the day under different scenarios.

References

[1] Dimitris Bertsimas and John Tsitsiklis. *Introduction to Linear Optimization*. 1st. Athena Scientific, 1997. ISBN: 1886529191.