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Differential games with mixed leadership: The open-loop solution ☆

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ABSTRACT

This paper introduces the notion of mixed leadership in nonzero-sum differential games, where there is no fixed hierarchy in decision making with respect to the players. Whether a particular player is leader or follower depends on the instrument variable s/he is controlling, and it is possible for a player to be both leader and follower, depending on the control variable. The paper studies two-player open-loop differential games in this framework, and obtains a complete set of equations (differential and algebraic) which yield the controls in the mixed-leadership Stackelberg solution. The underlying differential equations are coupled and have mixed-boundary conditions. The paper also discusses the special case of linear-quadratic differential games, in which case solutions to the coupled differential equations can be expressed in terms of solutions to coupled Riccati differential equations which are independent of the state trajectory.

1. Introduction

In 1934, von Stackelberg introduced a concept of a hierarchical solution for markets where some firms have power of domination over others [11]. This solution concept is now known as the *Stackelberg equilibrium* or the *Stackelberg solution* which, in the context of two-person nonzero-sum games, involves players with asymmetric roles, one *leading* (accordingly called *the leader*) and the other one *following* (called *the follower*). The game proceeds with the leader announcing his policy first (which would be his *action* if the information pattern is not dynamic), and the follower reacting to it by optimizing his performance index under the leader's announced policy. Of course, the leader has to anticipate this response (assuming that he knows the utility or cost function of the follower) and pick that policy which will optimize his performance index given the follower's rational response. Assuming that the follower's optimum (rational) response is unique to each announced policy of the leader (that is, he has a unique rational response curve), then the best policy of the leader is the one that optimizes his performance index on the rational reaction curve of the follower, which together with the corresponding unique policy/action of the follower is known as the Stackelberg solution. If the follower's response is not unique, however, then the rational response curve is replaced with a rational reaction set, in which case taking a pessimistic approach on the part of the leader, his optimization problem is to find the best policy under worst choices by the follower (*worst* from the point of view of the leader) from the rational response set; such a solution is known as the *generalized Stackelberg solution* [8,4].

The notion of the Stackelberg solution was later extended to multi-period settings in the early 1970's by Simaan and Cruz [9,10], who also introduced the notion of a *feedback Stackelberg* solution where the leader dictates his policy choices on the

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follower only stage-wise, and not globally. Such a solution concept requires (in a dynamic game setting) that the players know the current state of the game at every point in time, and its derivation involves a backward recursion (as in dynamic programming), where at every step of the recursion the Stackelberg solution of a static game is obtained. When the leader has dynamic information, and is able to announce his policy for the entire duration of the dynamic game ahead of time (and not stage-wise), then the Stackelberg solution, even though well defined as a concept, is generally very difficult to obtain, because the underlying optimization problems are then on the policy spaces of the two players, with the reaction sets or functions generally being infinite-dimensional. Derivation of such *global Stackelberg* solutions for dynamic games with dynamic information patterns also has connections to incentive design or mechanism design problems, and is still an active research area; see the text by Başar and Olsder [4] for these connections, and for an overview of various types of Stackelberg solutions.

It is possible to introduce the global and the feedback Stackelberg solutions also for dynamic games defined in continuous time, the so-called *differential games*. The latter, that is the feedback Stackelberg solution, is introduced in differential games as the limit of a sequence of feedback Stackelberg solutions of discretized (in time) versions of the original differential game, which become the collection of point-wise Stackelberg solutions of coupled Hamilton–Jacobi–Bellman systems, as first demonstrated by Başar and Haurie [2]. They also showed (for a class of dynamic and differential games) that the Stackelberg solution is indeed an equilibrium solution, in the sense of Nash, where the leader and the follower enter the game symmetrically, albeit with some additional information on the actions of the follower provided to the leader as part of the information pattern (that is, even though the play is symmetric, the information pattern is not). This setting has been applied in Başar et al. [3] to a differential game model of capitalism. Several other applications exist in economics and management science [6]. Another recent application area has been supply chains and marketing channels; see [5] and [7].

Most of the literature on Stackelberg games, with a few exceptions, has assumed that the roles of the players are fixed at the outset, and the leader remains a leader for the entire duration of the game; likewise follower remains as follower. Perhaps the first paper that brought up the issue of whether being a leader is always advantageous to a player is [1]. It turns out that leadership is not always a preferred option for the players: for some classes of games, leadership of one specific player is preferred by both (in the sense that both players collect highest utilities when compared with what they would collect under other combinations), which is a *stable* situation, whereas there are other classes of games where either both players prefer their own leadership or neither do, which leads to a *stalemate* situation. Sometimes, the players do not have the option of leadership open to them, but leadership is governed by an exogenous process (say a Markov chain), which determines who should be the leader and who the follower at each stage of the game, perhaps based on the history of the game or the current state of the game. Başar and Haurie [2] have shown that feedback Stackelberg solution is a viable concept for such (stochastic) dynamic games also, and have obtained recursions for the solution.

In this paper we again consider the variable leadership dynamic game, but with one additional twist which is that the same player can act as leader in some decisions and as follower in others, which we call the *mixed-leadership game*. We could view this situation also as one where there are no clear leaders and followers. For example, in the context of a manufacturer selling a product through a retailer, the manufacturer could act as a leader for advertising decisions, while the retailer could be the price leader. In this case, the manufacturer acts as a follower for making his wholesale price decision and the retailer acts as a follower for making his decision relating to local advertising of the product. Closer to home, it is possible to visualize a household scenario where the wife acts as leader for decisions relating to domestic activities, whereas the husband acts as the leader for external decisions.

Accordingly, in this paper we formulate and analyze dynamic/differential games where some decisions are announced by player 1 and some by player 2, simultaneously. Then, each player must obtain his optimal response for those decisions that are to be made by him in the role of the follower. Given these responses, the remaining decisions are made simultaneously. For convenience in exposition, we will partition all decisions into *lead* and *follow-up* decisions. The follow-up decisions are obtained simultaneously as the optimal responses to the lead decisions, where in the sprit of parallel play we will adopt the Nash equilibrium solution [4]. Thus, the procedure requires the solutions of two-player Nash games (the parallel play) and one Stackelberg game (hierarchical play) in a nested fashion. We will restrict the study to the open-loop information structure, with the study of mixed-leadership game under feedback information pattern left for a future paper.

The next section formulates a general class of two-player mixed leadership open-loop differential games and derives a set of sufficient conditions for its equilibrium. Section 3 applies the general results to a linear-quadratic game. Section 4 concludes the paper.

2. The mixed-leadership differential game model

Consider two players P1 and P2, responsible for making decisions (u_1, v_1) and (u_2, v_2), respectively, with these decision variables being continuous functions on the interval $[0, T]$. We have a two-player differential game with state equation

$$\dot{x} = f(x, u_1, v_1, u_2, v_2, t), \quad x(0) = x_0, \tag{1}$$

and with player i having the cost functional

$$J_i(u_1, v_1, u_2, v_2) = \int_0^T l_i(x, u_1, v_1, u_2, v_2, t)dt + S_i(x(T)), \quad i = 1, 2. \tag{2}$$

Here, x_0 is the initial state known by both players, $x(t) \in R^n$, and $U^i = (u_i, v_i) \in \Omega^i := \Omega^i_u \times \Omega^i_v \subseteq R^{m_i} \times R^{n_i}$, $i = 1, 2$, $\forall t \in [0, T]$, where $[0, T]$ denotes the fixed duration of the game. As hinted earlier, the underlying information structure is open-loop for both players, so that the controls $U^1(\cdot)$ and $U^2(\cdot)$ depend only on the time variable t and the initial state x_0. Further, we allow only for controls that are continuous in t, and this determines the admissible control function set $(\Gamma^1_u \times \Gamma^1_v, \Gamma^2_u \times \Gamma^2_v)$. For the differential game dynamics to be well defined, we assume that f satisfies the following regularity conditions:

(i) f is jointly continuous in $t \in [0, T]$ and $U^i \in \Omega^i$, $i = 1, 2$, for each $x \in R^n$.
(ii) f is uniformly Lipschitz in x.

These conditions guarantee that the state Eq. (1) yields a unique continuously differentiable state trajectory for each admissible control $(U^1(\cdot), U^2(\cdot))$.

In the mixed-leadership game considered here, P1 controls $U^1 = (u_1, v_1)$ and P2 controls $U^2 = (u_2, v_2)$. P1 and P2 both act as leader in their u controls, and act as follower in their v controls. In other words, first P1 picks u_1 and P2 picks u_2, simultaneously. Then P1 and P2 follow with their decisions v_1 and v_2, respectively, and again simultaneously. Now, for every admissible $(u_1(\cdot), u_2(\cdot))$, the followers problem is a two-player non-cooperative differential game, for which we look for an open-loop Nash equilibrium [4], that is a pair $(\hat{v}_1, \hat{v}_2)$ such that

$$J_1(u_1, \hat{v}_1, u_2, \hat{v}_2) = \min_{v_1 \in \Gamma^1_v} {}_1(u_1, v_1, u_2, \hat{v}_2)$$

and

$$J_2(u_1, \hat{v}_1, u_2, \hat{v}_2) = \min_{v_2 \in \Gamma^2_v} J_2(u_1, \hat{v}_1, u_2, v_2)$$

for all $u_1(\cdot) \in \Gamma^1_u$ and $u_2(\cdot) \in \Gamma^2_u$). Note that $\hat{v}_1$ and $\hat{v}_2$ are in general functions of u_1 and u_2, which we write as $\hat{v}_i(u^1, u^2)$, $i = 1, 2$, to capture this functional dependence. The following theorem now provides a characterization for these optimal Nash responses.

Theorem 1. *In addition to conditions* (i)–(ii), assume that

(iii) *$f(\cdot, U^1, U^2, t)$ is continuously differentiable on $R^n, \forall t \in [0, T]$, and $\forall U^i \in \Omega^i$, $i = 1, 2$,*
(iv) *$l_i(\cdot, U^1, U^2, t)$ is continuously differentiable on $R^n, \forall t \in [0, T]$, and $\forall U^i \in \Omega^i$, and $S_i(\cdot)$ is continuously differentiable on $R^n, i = 1, 2$.*

Then, if $(u_1, u_2) \in \Gamma^1_u \times \Gamma^2_u$ is a given pair of open-loop controls, and the resulting followers game admits an open-loop Nash equilibrium, say $(\hat{v}_1(u_1, u_2), \hat{v}_2(u_1, u_2)) \in \Gamma^1_v \times \Gamma^2_v$, there exist functions $q_1(\cdot) : [0, T] \to R^n$ and $q_2(\cdot) : [0, T] \to R^n$ such that the following relations hold:

$$\dot{x} = f(x, u_1, \hat{v}_1, u_2, \hat{v}_2, t), \quad x(0) = x_0, \tag{3}$$
$$\hat{v}_1(t) = \operatorname*{argmin}_{v_1 \in \Omega^1_v} H_1(x, u_1, v_1, u_2, \hat{v}_2, q_1, t), \tag{4}$$
$$\hat{v}_2(t) = \operatorname*{argmin}_{v_2 \in \Omega^2_v} H_2(x, u_1, \hat{v}_1, u_2, v_2, q_2, t), \tag{5}$$
$$\dot{q}'_1(t) = -\frac{\partial}{\partial x} H_1(x, u_1(t), \hat{v}_1(t), u_2(t), \hat{v}_2(t), q_1(t), t), \quad q'_1(T) = \frac{\partial}{\partial x} S_1(x(T)), \tag{6}$$
$$\dot{q}'_2(t) = -\frac{\partial}{\partial x} H_2(x, u_1(t), \hat{v}_1(t), u_2(t), \hat{v}_2(t), q_2(t), t), \quad q'_2(T) = \frac{\partial}{\partial x} S_2(x(T)), \tag{7}$$

where H_i is the Hamiltonian function for player i,

$$H_i(x, U^1, U^2, q_i, t) = l_i(x, U^1, U^2, t) + q'_i f_i(x, U^1, U^2, t), \quad t \in [0, T], \quad i = 1, 2, \tag{8}$$

and prime denotes transpose.

(v) *Furthermore, let (4) and (5) admit inner solutions, and $H_1(x, u_1, u_2, \cdot, v_2, q_1, t)$ and $H_2(x, u_1, u_2, v_1, \cdot, q_2, t)$ be continuously differentiable and strictly convex on Ω^1_v* and *Ω^2_v, respectively.*

Then, under the additional condition (v), (4) and (5) can be replaced by the stationarity conditions:

$$\nabla_{v_1} H_1(x, u_1, \hat{v}_1, u_2, \hat{v}_2, q_1, t) = 0, \tag{9}$$
$$\nabla_{v_2} H_2(x, u_1, \hat{v}_1, u_2, \hat{v}_2, q_2, t) = 0, \tag{10}$$

respectively.

Proof. This is a standard open-loop Nash differential game in (v_1, v_2) for any given (u_1, u_2). The proof follows from [4, p. 310]. □

Now, to solve for the *leading* decisions, (u_1, u_2), we follow an argument similar to that in Başar and Olsder [4, Chapter 7], in the derivation of the open-loop Stackelberg solution. The difference is that while in the standard two-player Stackelberg differential game the leader's problem is a constrained optimal control problem, in the present case the two leaders are faced with another differential game, constrained by the optimal (Nash) response functions covered by the previous theorem. Now, assuming that $\hat{v}_1$ and $\hat{v}_2$ as characterized in Theorem 1 are unique, the differential game at this upper-level of decision hierarchy would be characterized by the cost functions

$$J_1(u_1, \hat{v}_1, u_2, \hat{v}_2) \quad \text{and} \quad J_2(u_1, \hat{v}_1, u_2, \hat{v}_2) \tag{11}$$

and dynamics

$$\dot{x} = f(x, u_1, \hat{v}_1, u_2, \hat{v}_2, t), \quad x(0) = x_0, \tag{12}$$

where $\hat{v}_1$ and $\hat{v}_2$ are functions of (u_1, u_2). What we are seeking (in abstract form description, at this point) is a pair $(u_1^* \in \Gamma_u^1, u_2^* \in \Gamma_u^2)$ satisfying

$$J_1(u_1^*, \hat{v}_1(u_1^*, u_2^*), u_2^*, \hat{v}_2(u_1^*, u_2^*)) = \min_{u_1 \in \Gamma_u^1} J_1(u_1, \hat{v}_1(u_1, u_2^*), u_2^*, \hat{v}_2(u_1, u_2^*)) \tag{13}$$

and

$$J_2(u_1^*, \hat{v}_1(u_1^*, u_2^*), u_2^*, \hat{v}_2(u_1^*, u_2^*)) = \min_{u_2 \in \Gamma_u^2} J_2(u_1^*, \hat{v}_1(u_1^*, u_2), u_2, \hat{v}_2(u_1^*, u_2)). \tag{14}$$

Such a pair (u_1^*, u_2^*) clearly constitutes an open-loop Nash equilibrium for the upper-level differential game, which together with $v_1^* = \hat{v}_1(u_1^*, u_2^*)$, $v_2^* = \hat{v}_2(u_1^*, u_2^*)$ constitute the Stackelberg solution to the original mixed-leadership game.

Now, the upper-level nonzero-sum differential game introduced above is not a standard one, since $\hat{v}_1$ and $\hat{v}_2$ are defined through the equality constraints (9), (10), and the differential constraints (6), (7). But in view of (13) and (14), which transform computation of the Nash equilibrium into solution of two optimal control problems, with respectively $u_2 = u_2^*$ and $u_1 = u_1^*$ fixed, we now have two non-standard optimal control problems. Let us focus on the first one, defined by (13). Here, the open-loop control variable is u_1, and we have two equality constraints, (9) and (10) with $u_2 = u_2^*$, and three differential constraints, (12)–(14) with again $u_2 = u_2^*$. Note also that the differential constraints have mixed-boundary conditions, with that of first one specified at the initial time and those of the other two specified at the terminal time. This is similar to the mixed-boundary-value optimal control problem encountered in the derivation of the standard open-loop Stackelberg game (see, for instance, [4, pp. 408–410]), with the difference being that here we have one additional equality constraint and one additional differential constraint. By viewing v_1 and v_2 also as free variables, in view of the equality constraints (9) and (10) that define them, we can now rewrite all five constraints (for the problem at hand) as follows in terms of the three control variables (u_1, v_1, v_2):

$$\dot{x} = f(x, u_1, v_1, u_2^*, v_2, t), \quad x(0) = x_0, \tag{15}$$

$$\dot{q}_1' = -\frac{\partial}{\partial x} H_1(x, u_1, v_1, u_2^*, v_2, q_1, t), \quad q_1'(T) = \frac{\partial}{\partial x} S_1(x(T)), \tag{16}$$

$$\dot{q}_2' = -\frac{\partial}{\partial x} H_2(x, u_1, v_1, u_2^*, v_2, q_2, t), \quad q_2'(T) = \frac{\partial}{\partial x} S_2(x(T)), \tag{17}$$

$$\nabla_{v_1} H_1(x, u_1, v_1, u_2^*, v_2, q_1, t) = 0, \tag{18}$$

$$\nabla_{v_2} H_2(x, u_1, v_1, u_2^*, v_2, q_2, t) = 0. \tag{19}$$

The Hamiltonian function associated with this optimal control problem, as the counterpart of expression (7.62) in [4], is

$$\begin{aligned} &L_1(x, u_1, v_1, u_2^*, v_2, q_1, q_2, \phi, \psi_1, \psi_2, \mu_1, \mu_2, t) \\ &\quad = l_1(x, u_1, v_1, u_2^*, v_2, t) + \phi' f(x, u_1, v_1, u_2^*, v_2, t) + \psi_1' \left(-\frac{\partial H_1(x, u_1, v_1, u_2^*, v_2, q_1, t)}{\partial x} \right)' \\ &\quad\quad + \psi_2' \left(-\frac{\partial H_2(x, u_1, v_1, u_2^*, v_2, q_2, t)}{\partial x} \right)' + \nabla_{v_1} H_1(x, u_1, u_2^*, v_1, v_2, q_1, t) \cdot \mu_1 + \nabla_{v_2} H_2(x, u_1, u_2^*, v_1, v_2, q_2, t) \cdot \mu_2, \end{aligned} \tag{20}$$

where $\phi(\cdot) : [0, T] \to R^n, \psi_1(\cdot) : [0, T] \to R^n$, and $\psi_2(\cdot) : [0, T] \to R^n$ are the co-state variables corresponding to the state Eqs. (15)–(17), respectively, and $\mu_1(\cdot) : [0, T] \to R^{n_1}$, and $\mu_2(\cdot) : [0, T] \to R^{n_2}$ are the Lagrange multipliers corresponding to the constraints (18) and (19), respectively. It again follows from [4, p. 409], that under suitable differentiability conditions (to be made precise in Theorem 2 to follow), the co-states satisfy the following set of differential equations and the boundary conditions:

$$\begin{aligned} &\dot{\phi}' = -\frac{\partial}{\partial x} L_1, \quad \phi'(T) = \frac{\partial S_1(x(T))}{\partial x} - \frac{\partial^2}{\partial x^2} S_1(x(T)) \psi_1(T) - \frac{\partial^2}{\partial x^2} S_2(x(T)) \psi_2(T), \\ &\dot{\psi}_1' = -\frac{\partial}{\partial q_1} L_1, \quad \psi_1(0) = 0, \\ &\dot{\psi}_2' = -\frac{\partial}{\partial q_2} L_1, \quad \psi_2(0) = 0. \end{aligned}$$

Further, assuming inner solutions with respect to the three independent control variables (u_1, v_1, v_2), there is the additional set of three stationarity conditions (as in Başar and Olsder [4, p. 409]):

$$\nabla_{u_1} L_1 = 0, \quad \nabla_{v_1} L_1 = 0, \quad \nabla_{v_2} L_1 = 0. \tag{21}$$

Now, instead of focusing on (13), if we had started with the optimal control problem defined by (14), as a result of symmetric arguments we would have arrived at the Hamiltonian (for P2):

$$\begin{aligned} &L_2(x, u_1^*, v_1, u_2, v_2, q_1, q_2, \alpha, \beta_1, \beta_2, \gamma_1, \gamma_2, t) \\ &\quad = l_2(x, u_1^*, v_1, u_2, v_2, t) + \alpha' f(x, u_1^*, v_1, u_2, v_2, t) + \beta_1' \left(-\frac{\partial H_1(x, u_1^*, v_1, u_2, v_2, q_1, t)}{\partial x} \right)' \\ &\qquad + \beta_2' \left(-\frac{\partial H_2(x, u_1^*, v_1, u_2, v_2, q_2, t)}{\partial x} \right)' + \nabla_{v_1} H_1(x, u_1^*, u_2, v_1, v_2, q_1, t) \cdot \gamma_1 + \nabla_{v_2} H_2(x, u_1^*, u_2, v_1, v_2, q_2, t) \cdot \gamma_2 \end{aligned} \tag{22}$$

where $\alpha(\cdot): [0,T] \to R^n, \beta_1(\cdot): [0,T] \to R^n, \beta_2(\cdot): [0,T] \to R^n, \gamma_1(\cdot): [0,T] \to R^{n_1}$, and $\gamma_2(\cdot): [0,T] \to R^{n_2}$ are the corresponding co-state variables and Lagrange multipliers (for P2, where the co-states satisfy the following set of differential equations and the boundary conditions (again under suitable differentiability conditions, to be made precise in Theorem 2):

$$\begin{aligned} \dot{\alpha}' &= -\frac{\partial}{\partial x} L_2, \quad \alpha'(T) = \frac{\partial S_2(x(T))}{\partial x} - \frac{\partial^2}{\partial x^2} S_1(x(T))\beta_1(T) - \frac{\partial^2}{\partial x^2} S_2(x(T))\beta_2(T), \\ \dot{\beta}_1' &= -\frac{\partial}{\partial q_1} L_2, \quad \beta_1(0) = 0, \\ \dot{\beta}_2' &= -\frac{\partial}{\partial q_2} L_2, \quad \beta_2(0) = 0, \end{aligned}$$

and, if again u_2, v_1, v_2 are inner solutions, we have

$$\nabla_{u_2} L_2 = 0, \quad \nabla_{v_1} L_2 = 0, \quad \nabla_{v_2} L_2 = 0. \tag{23}$$

We now collect together the results above in the form of the following Theorem 2.

Theorem 2. *For the mixed-leadership differential game considered here, assume, in addition to* (iii)–(v) of Theorem 1, *that*

(vi) $f(\cdot, U^1, U^2, t)$ *is twice continuously differentiable on* $R^n, \forall t \in [0,T]$,
(vii) $l_1(\cdot, U^1, U^2, t)$ *and* $l_2(\cdot, U^1, U^2, t)$ *are twice continuously differentiable on* $R^n, \forall t \in [0,T]$,
(viii) $f(x, u_1, u_2, \cdot, \cdot, t), l_i(x, u_1, u_2, \cdot, \cdot, t), i = 1, 2,$ *are continuously differentiable on* $R^{n_1} \times R^{n_2}, \forall t \in [0,T]$.

Then if $U^{1*}(t)$ *and* $U^{2*}(t), t \in [0,T]$, *provide an optimal open-loop Stackelberg solution, and* $x^*(t), t \in [0,T]$, *denotes the corresponding state trajectory, and*

(ix) $(u_1^*(t), v_1^*(t), u_2^*(t), v_2^*(t))$ *is in the interior of* $\Omega_u^1 \times \Omega_v^1 \times \Omega_u^2 \times \Omega_v^2$,

there exist continuously differential functions $\phi(\cdot), \psi_1(\cdot), \psi_2(\cdot), \alpha(\cdot), \beta_1(\cdot), \beta_2(\cdot): [0,T] \to R^n$ *and continuous functions* $\mu_1(\cdot), \gamma_1(\cdot): [0,T] \to R^{n_1}$ *and* $\mu_2(\cdot), \gamma_2(\cdot): [0,T] \to R^{n_2}$, *such that the following conditions are satisfied:*

$$\dot{x}^* = f(x^*, U^{1*}, U^{2*}, t), \quad x^*(0) = x_0, \tag{24}$$

$$\dot{q}_i' = -\frac{\partial}{\partial x} H_i(x^*, U^{1*}, U^{2*}, q_i, t), \quad q_i'(T) = \frac{\partial S_i(x^*(T))}{\partial x}, \quad i = 1, 2, \tag{25}$$

$$\dot{\phi}' = -\frac{\partial}{\partial x} L_1(x^*, U^{1*}, U^{2*}, q_1, q_2, \phi, \psi_1, \psi_2, \mu_1, \mu_2, t), \quad \phi'(T) = \frac{\partial S_1(x^*(T))}{\partial x} - \frac{\partial^2}{\partial x^2} S_1(x^*(T))\psi_1(T) - \frac{\partial^2}{\partial x^2} S_2(x^*(T))\psi_2(T), \tag{26}$$

$$\dot{\psi}_1' = -\frac{\partial}{\partial q_1} L_1(x^*, U^{1*}, U^{2*}, q_1, q_2, \phi, \psi_1, \psi_2, \mu_1, \mu_2, t), \quad \psi_1(0) = 0, \tag{27}$$

$$\dot{\psi}_2' = -\frac{\partial}{\partial q_2} L_1(x^*, U^{1*}, U^{2*}, q_1, q_2, \phi, \psi_1, \psi_2, \mu_1, \mu_2, t), \quad \psi_2(0) = 0, \tag{28}$$

$$\dot{\alpha}'(t) = -\frac{\partial}{\partial x} L_2(x^*, U^{1*}, U^{2*}, q_1, q_2, \alpha, \beta_1, \beta_2, \gamma_1, \gamma_2, t), \quad \alpha'(T) = \frac{\partial S_2(x^*(T))}{\partial x} - \frac{\partial^2}{\partial x^2} S_1(x^*(T))\beta_1(T) - \frac{\partial^2}{\partial x^2} S_2(x^*(T))\beta_2(T), \tag{29}$$

$$\dot{\beta}_1'(t) = -\frac{\partial}{\partial q_1} L_2(x^*, U^{1*}, U^{2*}, q_1, q_2, \alpha, \beta_1, \beta_2, \gamma_1, \gamma_2, t), \quad \beta_1(0) = 0, \tag{30}$$

$$\dot{\beta}_2'(t) = -\frac{\partial}{\partial q_2} L_2(x^*, U^{1*}, U^{2*}, q_1, q_2, \alpha, \beta_1, \beta_2, \gamma_1, \gamma_2, t), \quad \beta_2(0) = 0, \tag{31}$$

$$\nabla_{v_1} H_1(x^*, U^{1*}, U^{2*}, q_1, t) = 0, \tag{32}$$

$$\nabla_{v_2} H_2(x^*, U^{1*}, U^{2*}, q_2, t) = 0, \tag{33}$$

$$\nabla_{u_1} L_1(x^*, U^{1*}, U^{2*}, q_1, q_2, \phi, \psi_1, \psi_2, \mu_1, \mu_2, t) = 0, \tag{34}$$

$$\nabla_{v_1} L_1(x^*, U^{1*}, U^{2*}, q_1, q_2, \phi, \psi_1, \psi_2, \mu_1, \mu_2, t) = 0, \tag{35}$$
$$\nabla_{u_2} L_2(x^*, U^{1*}, U^{2*}, q_1, q_2, \alpha, \beta_1, \beta_2, \gamma_1, \gamma_2, t) = 0, \tag{36}$$
$$\nabla_{v_2} L_2(x^*, U^{1*}, U^{2*}, q_1, q_2, \alpha, \beta_1, \beta_2, \gamma_1, \gamma_2, t) = 0, \tag{37}$$
$$\nabla_{v_1} L_2(x^*, U^{1*}, U^{2*}, q_1, q_2, \phi, \psi_1, \psi_2, \mu_1, \mu_2, t) = 0, \tag{38}$$
$$\nabla_{v_2} L_1(x^*, U^{1*}, U^{2*}, q_1, q_2, \alpha, \beta_1, \beta_2, \gamma_1, \gamma_2, t) = 0, \tag{39}$$

where H_1 and H_2 are as defined in Theorem 1, and L_1 and L_2 are as defined in (20) and (22), respectively.

Proof. The result follows directly from the discussion preceding the statement of the theorem, in view of Theorem 7.5 of [4] applied to each of the optimal control problems defined by (13) and (14). □

Remark 1. Note that we have a total of 17 unknown quantities (1 state vector, 4 control vectors, 2 co-state vectors associated with the lower-level game, 6 co-state vectors associated with the upper-level game, and 4 vector-valued Lagrange multiplier functions associated with the upper-level game), and we have exactly 17 compatible vector equations in (24)–(39) to solve for these variables. Hence, there are no missing equations, and everything is accounted for.

Remark 2. If we consider a game with two leaders $L1$ and $L2$ with the objective functions J_1 and J_2 respectively, and two followers $F1$ and $F2$ with their objective functions denoted as J_{F1} and J_{F2}, respectively, then the problem treated in this paper can be viewed as a special case of this two-leaders, two-followers game with $J_{F1} = J_1$ and $J_{F2} = J_2$. Theorem 2 can be modified for this general game by replacing H_1 and H_2 by H_{F1} and H_{F2}, respectively. It should be noted that in the general case, the followers play a Nash differential game to obtain v_1 and v_2 given u_1 and u_2, and the leaders play a Nash differential game to obtain u_1 and u_2 in view of the optimal responses of the followers.

In the next section, we apply Theorem 2 to a scalar linear-quadratic mixed-leadership differential game.

3. A linear-quadratic problem

Consider the scalar version of the differential game discussed in the previous section, with linear dynamics and quadratic cost functions. Namely, we have $U^1 = (u_1, v_1) \in R^2, U^2 = (u_2, v_2) \in R^2$, and

$$\dot{x} = ax + b_1u_1 + c_1v_1 + b_2u_2 + c_2v_2, \quad x(0) = x_0,$$
$$l_1(x, u, v_1, u_2, v_2, t) = \frac{1}{2}(Q_1x^2 + B_1u_1^2 + v_1^2 - 2r_1u_1v_2 - 2s_1v_1u_2)$$
$$l_2(x, u, v_1, u_2, v_2, t) = \frac{1}{2}(Q_2x^2 + B_2u_2^2 + v_2^2 - 2r_2u_1v_2 - 2s_2v_1u_2)$$
$$S_1(x) = \frac{1}{2}S_1x^2, \quad S_2(x) = \frac{1}{2}S_2x^2,$$

where the parameters $Q_1, Q_2, S_1, S_2, \beta_1, \beta_2$ are all positive.

Then the system (24)–(39) of Theorem 2 can be written as follows:

$$\dot{x}^* = ax^* + b_1u_1^* + c_1v_1^* + b_2u_2^* + c_2v_2^*, \quad x^*(0) = x_0, \tag{40}$$
$$\dot{q}_1 = -Q_1x^* - aq_1, \quad q_1(T) = S_1x^*(T) \tag{41}$$
$$\dot{q}_2 = -Q_2x^* - aq_2, \quad q_2(T) = S_2x^*(T) \tag{42}$$
$$\dot{\phi} = -Q_1x^* - a\phi + Q_1\psi_1 + Q_2\psi_2, \quad \phi(T) = S_1x^*(T) - S_1\psi_1(T) - S_2\psi_2(T) \tag{43}$$
$$\dot{\psi}_1 = a\psi_1 - c_1\mu_1, \quad \psi_1(0) = 0, \tag{44}$$
$$\dot{\psi}_2 = a\psi_2 - c_2\mu_2, \quad \psi_2(0) = 0, \tag{45}$$
$$\dot{\alpha} = -Q_2x^* - a\alpha + Q_1\beta_1 + Q_2\beta_2, \quad \alpha(T) = S_2x^*(T) - S_1\beta_1(T) - S_2\beta_2(T) \tag{46}$$
$$\dot{\beta}_1 = a\beta_1 - c_1\gamma_1, \quad \beta_1(0) = 0, \tag{47}$$
$$\dot{\beta}_2 = a\beta_2 - c_2\gamma_2, \quad \beta_2(0) = 0, \tag{48}$$
$$v_1^* - s_1u_2^* + c_1q_1 = 0, \tag{49}$$
$$v_2^* - r_2u_1^* + c_2q_2 = 0, \tag{50}$$
$$B_1u_1^* - r_1v_2^* + b_1\phi - r_2\mu_2 = 0, \tag{51}$$
$$v_1^* - s_1u_2^* + c_1\phi + \mu_1 = 0, \tag{52}$$
$$B_2u_2^* - s_2v_1^* + b_2\alpha - s_1\gamma_1 = 0, \tag{53}$$
$$v_2^* - r_2u_1^* + c_2\alpha + \gamma_2 = 0, \tag{54}$$
$$-s_2u_2^* + c_1\alpha + \gamma_1 = 0, \tag{55}$$
$$-r_1u_1^* + c_2\phi + \mu_2 = 0. \tag{56}$$

We next solve the linear system of algebraic Eqs. (49)–(56) to obtain $u_1^*, v_1^*, u_2^*, v_2^*, \mu_1, \mu_2, \gamma_1$, and γ_2. By substituting these in (40)–(48), we obtain the following two-point boundary-value system of linear differential equations:

$$\begin{aligned}
\dot{x}^* &= ax^* + \frac{s_1s_2c_1^2 - s_2c_1b_2 - B_2c_1^2}{B_2 - 2s_1s_2}q_1 + \frac{r_1r_2c_2^2 - r_1c_2b_1 - B_1c_2^2}{B_1 - 2r_1r_2}q_2 - \frac{(b_1+c_2r_2)^2}{B_1 - 2r_1r_2}\phi - \frac{(b_2+c_1s_1)^2}{B_2 - 2s_1s_2}\alpha, \quad x^*(0) = x_0,\\
\dot{q}_1 &= -Q_1x^* - aq_1, \quad q_1(T) = S_1x^*(T),\\
\dot{q}_2 &= -Q_2x^* - aq_2, \quad q_2(T) = S_2x^*(T),\\
\dot{\phi} &= -Q_1x^* - a\phi + Q_1\psi_1 + Q_2\psi_2, \quad \phi(T) = S_1x^*(T) - S_1\psi_1(T) - S_2\psi_2(T),\\
\dot{\psi}_1 &= -c_1^2q_1 + c_1^2\phi + a\psi_1, \quad \psi_1(0) = 0,\\
\dot{\psi}_2 &= \frac{r_1^2c_2^2}{B_1 - 2r_1r_2}q_2 + \frac{r_1c_2b_1 - r_1r_2c_2^2 + c_2^2B_1}{B_1 - 2r_1r_2}\phi + a\psi_2, \quad \psi_2(0) = 0,\\
\dot{\alpha} &= -Q_2x^* - a\alpha + Q_1\beta_1 + Q_2\beta_2, \quad \alpha(T) = S_2x^*(T) - S_1\beta_1(T) - S_2\beta_2(T),\\
\dot{\beta}_1 &= \frac{s_2^2c_1^2}{B_2 - 2s_1s_2}q_1 + \frac{s_2c_1b_2 - s_1s_2c_1^2 + c_2^2B_2}{B_2 - 2s_1s_2}\alpha + a\beta_1, \quad \beta_1(0) = 0,\\
\dot{\beta}_2 &= -c_2^2q_2 + c_2^2\alpha + a\beta_2, \quad \beta_2(0) = 0.
\end{aligned}$$

We now introduce scaling coefficients, $P_j,\ j = 1,\ldots,8$, continuously differentiable on $[0,T]$, which when multiplied by x^* generate $q_1, q_2, \phi, \psi_1, \psi_2, \alpha, \beta_1, \beta_2$, respectively, that is

$$q_1 = P_1x^*, \quad q_2 = P_2x^*, \quad \phi = P_3x^*, \quad \psi_1 = P_4x^*, \quad \psi_2 = P_5x^*, \quad \alpha = P_6x^*, \quad \beta_1 = P_7x^*, \quad \beta_2 = P_8x^*.$$

A simple substitution into the above set of differential equations, and some manipulations lead to the following set of 8 coupled differential equations with mixed-boundary conditions for these scaling coefficients, which can be solved independently of the state trajectory $x^*(\cdot)$:

$$\begin{aligned}
\dot{P}_1 &= -2aP_1 - FP_1 - Q_1, \quad P_1(T) = S_1,\\
\dot{P}_2 &= -2aP_2 - FP_2 - Q_2, \quad P_2(T) = S_2,\\
\dot{P}_3 &= -2aP_3 - FP_3 - Q_1 + Q_1P_4 + Q_2P_5, \quad P_3(T) = S_1 - S_1P_4(T) - S_2P_5(T),\\
\dot{P}_4 &= -FP_4 - c_1^2P_1 + c_1^2P_3, \quad P_4(0) = 0,\\
\dot{P}_5 &= -FP_5 + \frac{r_1^2c_2^2}{B_1 - 2r_1r_2}P_2 + \frac{r_1c_2b_1 - r_1r_2c_2^2 + c_2^2B_1}{B_1 - 2r_1r_2}P_3, \quad P_5(0) = 0,\\
\dot{P}_6 &= -2aP_6 - FP_6 - Q_2 + Q_1P_7 + Q_2P_8, \quad P_6(T) = S_2 - S_1P_7(T) - S_2P_8(T),\\
\dot{P}_7 &= -FP_7 + \frac{s_2^2c_1^2}{B_2 - 2s_1s_2}P_1 + \frac{s_2c_1b_2 - s_1s_2c_1^2 + c_2^2B_2}{B_2 - 2s_1s_2}P_6, \quad P_7(0) = 0,\\
\dot{P}_8 &= -FP_8 - c_2^2P_2 + c_2^2P_6, \quad P_8(0) = 0,
\end{aligned}$$

where

$$\begin{aligned}
F := {} & \frac{s_1s_2c_1^2 - s_2c_1b_2 - B_2c_1^2}{B_2 - 2s_1s_2}P_1 + \frac{r_1r_2c_2^2 - r_1c_2b_1 - B_1c_2^2}{B_1 - 2r_1r_2}P_2\\
& - \frac{(b_1+c_2r_2)^2}{B_1 - 2r_1r_2}P_3 - \frac{(b_2+c_1s_1)^2}{B_2 - 2s_1s_2}P_6,
\end{aligned}$$

and the corresponding state trajectory is generated by

$$\dot{x}^* = (a+F)x^*, \quad x^*(0) = x_0.$$

The set of equations above that determine the P_i's are coupled Riccati equations with mixed-boundary conditions. No general existence and uniqueness results exist that can be directly applied to this set of equations. One way to obtain some conditions for existence and uniqueness of P_i's would be to formulate the differential game as a static mixed-leadership game in the infinite-dimensional $L^2[0,T]$ space (as it was done in [4, p. 411], for the regular Stackelberg differential game), obtain conditions that involve linear operators, and then convert them into checkable conditions on the system parameters and the interval $[0,T]$. Under such conditions, which would lead to the existence of a unique set of solutions $\{P_j(\cdot),\ j = 1,\ldots,8\}$ to the coupled Riccati equations over a given interval $[0,T]$, the Stackelberg solution to the linear-quadratic mixed-leadership differential game is given by

$$\begin{aligned}
u_1^*(t) &= \frac{r_1c_2P_2(t) + (b_1 + r_2c_2)P_3(t)}{2r_1r_2 - B_1}x^*(t),\\
u_2^*(t) &= \frac{s_2c_1P_1(t) + (b_2 + s_1c_1)P_6(t)}{2s_1s_2 - B_2}x^*(t),\\
v_1^*(t) &= s_1u_2^*(t) - c_1P_1(t)x^*(t),\\
v_2^*(t) &= r_2u_1^*(t) - c_2P_2(t)x^*(t).
\end{aligned}$$

4. Concluding remarks

In this paper, we have introduced the notion of *mixed leadership* in two-player Stackelberg differential games, where each player controls several variables some of which s/he uses as leader and others as follower. We have adopted the open-loop information structure where the players' controls are functions of the initial state and time, and for this class of games we have obtained a complete set of equations satisfied by the control laws. These equations are coupled differential and algebraic equations where some of the differential equations have specified initial conditions while others have their terminal conditions specified. Obtaining effective algorithms for solving such equations is still a challenge which we leave for future research. Another challenge is to obtain the counterpart of the results in this paper under the closed-loop information structure and using the feedback Stackelberg concept (again within the mixed-leadership framework); this extension is currently under study.

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