Solutions to some sample problems

Problem 1

$$(DE)\quad \begin{cases} -\varepsilon u'' + u' = 0, & x \in (0,1), \\ u(0) = g, \\ u(1) = 0. \end{cases}$$

b) (WF) Find $u \in V_g = \{v : \|v\|^2 + \|v'\|^2 < 0,\ v(0) = g,\ v(1) = 0\}$ such that

$$(\varepsilon u', v') + (u', v) = 0 \qquad \forall v \in V_0,$$

where

$$V_0 = \{v : \|v\|^2 + \|v'\|^2 < 0,\ v(0) = 0,\ v(1) = 0\}.$$

Now let us construct the following subspaces:

$$V_{g,h} = \{v : v \in C^0(0,1),\ v|_{I_i} \in P_1(I_i),\ i = 0,1,\dots,n,\ v(0) = g, v(1) = 0\},$$

$$V_{0,h} = \{v : v \in C^0(0,1),\ v|_{I_i} \in P_1(I_i),\ i = 0,1,\dots,n,\ v(0) = 0, v(1) = 0\},$$

$$I = [0,1] = \bigcup_{i=0}^{n} I_i.$$

(GFEM) Find $u_h \in V_{h,g}$ such that

$$(\varepsilon u_h', v') + (u_h', v) = 0, \quad \forall v \in V_{h,0}.$$

Now let $\varepsilon = 1$, $g = 0 \Rightarrow V_{g,h} = V_{0,h}$, and subtract (GFEM) from (WF):

$$(GO) \qquad ((u - u_h)', v') + ((u - u_h)', v) = 0 \qquad \forall v \in V_{h,0}.$$

Energy norm: $\quad \|v\|_E^2 = a(v,v) = (v', v') + (v', v).$

On the other hand

$$(v',v) = -(v,v') + v'v\Big|_0^1 = -(v',v)$$

$$\Rightarrow \quad 2(v',v)=0 \quad \Rightarrow \quad (v',v)=0 .$$

Therefore,

$$(*) \qquad \|v\|_E^2 = \|v'\|^2$$

A priori estimate: Denote $e = u - u_h$, then

$$\|e\|_E^2 = \left((u-u_h)', (u-u_h)'\right) + \left((u-u_h)', u-u_h\right)$$

$$= \left((u-u_h)', (u - v + v - u_h)'\right) + \left((u-u_h)', u - v + v + u_h\right)$$

$$= \left((u-u_h)', (u-v)'\right) + \underbrace{\left((u-u_h)', (v+u_h)'\right)} + \left((u-u_h)', u-v\right) + \underbrace{\left((u-u_h)', v+u_h\right)}$$

$$(GO) = 0, \quad v+u_h \in V_{h,0}$$

$$= \left((u-u_h)', (u-v)'\right) + \left((u-u_h)', u-v\right)$$

$$\le \underbrace{\|(u-u_h)'\|}_{=\|e\|_E} \|(u-v)'\| + \underbrace{\|(u-u_h)'\|}_{=\|e\|_E} \|u-v\|$$

$$= \|e\|_E \left(\|(u-v)'\| + \|u-v\| \right)$$

Divide both sides by $\|e\|_E$:

$$\|e\|_E \le \|(u-v)'\| + \|u-v\|. \qquad \forall v \in V_{h,0}$$

Now set $v = \pi u$, where πu is an interpolant of u in $V_{h,0}$ and using the standard interpolation estimates:

$$\|e\|_E \le \|(u-\Pi u)'\| + \|u-\Pi u\|$$
$$\le c_1 h \|u''\| + c_2 h^2 \|u''\|$$
$$= (c_1+c_2 h) h \|u''\| ,$$

where h is the mesh-size.

$$\|e\|_E \le (c_1+c_2 h) h \|u''\|$$

A posteriori estimates.

$$\|e\|_E^2 = (e',e') + (e',e)$$

$$= (e',e') + (e',e) - \underbrace{\left[(e',(\Pi e)') + (e',\Pi e)\right]}_{=0\ ,\ (GO)\ ,\ \Pi e \in V_{h,0}}$$

$$= (e',(e-\Pi e)') + (e', e-\Pi e)$$

$$= \sum_{i=0}^{n} \int_{I_i} e'(e-\Pi e)'\,dx + (e', e-\Pi e)$$

$$= \sum_{i=0}^{n} \left[\int_{I_i} -e''(e-\Pi e)\,dx + \underbrace{e'(e-\Pi e)\Big|_{x_i}^{x_{i+1}}}_{=0,\ \text{since } e(x_i)=\Pi e(x_i),\ e(x_{i+1})=\Pi e(x_{i+1})} \right] + (e', e-\Pi e)$$

$$= \sum_{i=0}^{n} \int_{I_i} (-e''+e')(e-\Pi e)\,dx$$

$$-e''+e' = -(u-u_h)'' + (u-u_h)'$$
$$= \underbrace{-u''+u'}_{=0} - (-u_h''+u_h')$$
$$= -u_h''+u_h = R(u_h)$$

$$= \sum_{i=0}^{n} \int_{I_i} R(u_h)(e-\Pi e)\,dx$$

$$\le \sum_{i=0}^{n} c\,h\,\|R(u_h)\|_{L^2(I_i)} \|e'\|_{L^2(I_i)}$$

$$\leq C \sum_{i=0}^{n} h \|R(u_h)\|_{L^2(I_i)} \sum_{i=0}^{n} \|e'\|_{L^2(I_i)}$$

Note that
$$\sum_{i=0}^{n} \|e'\|_{L^2(I_i)} \leq C_1 \|e'\|_{L^2[0,1]}$$

Prove it!

We obtain:

$$\|e\|_E^2 \leq \underbrace{\|e'\|}_{=\|e\|_E} \, C_2 \sum_{i=0}^{n} h \|R(u_h)\|_{L^2(I_i)}$$

$$\Rightarrow \|e\|_E \leq C_2 \sum_{i=0}^{n} h \|R(u_h)\|_{L^2(I_i)}.$$

c) Assume $g \neq 0$.

A posteriori error estimates using goal oriented argument.

$$\begin{cases} \overbrace{-\varepsilon u'' + u'}^{Lu} = 0 & \text{in } (0,1) \\ u(0) = g, \quad u(1) = 0 \end{cases}$$

primal problem.

Let, the aim is to estimate error in the following goal functional:

$$J(u) = \int_0^1 u \, \varphi_\Omega \, dx$$

The error in the goal functional is

$$J(e) = J(u - u_h) = J(u) - J(u_h) = \int_0^1 e \, \varphi_\Omega \, dx + e \, \varphi_T \Big|_0^1$$

Next, we derive the adjoint operator for L:

$$(Lu,\varphi) = (-\varepsilon u'' + u', \varphi) =$$

$$= (u', \varepsilon\varphi') - \varepsilon u'\varphi\big|_0^1 - (u,\varphi') + u\varphi\big|_0^1$$

$$= -(u, \varepsilon\varphi'') + \varepsilon u\varphi'\big|_0^1 - \varepsilon u'\varphi\big|_0^1 - (u,\varphi') + u\varphi\big|_0^1$$

$$= (u, \underbrace{-\varepsilon\varphi'' - \varphi'}_{L^*\varphi}) + \underbrace{\varepsilon u\varphi'\big|_0^1 - \varepsilon u'\varphi\big|_0^1 + u\varphi\big|_0^1}_{\text{boundary terms.}}$$

Note that the first and last boundary terms include u. Since we have Dirichlet boundary condition error in these two term are 0. The second boundary term involves u', therefore we chose φ to be zero on the boundary.

We introduce:

$$\begin{cases} \varepsilon\varphi'' - \varphi' = \psi_\Omega & \text{in } (0,1) \\ \varphi(0) = \varphi(1) = 0 \end{cases}$$

dual problem

Then: $J(e) = (e, \psi_\Omega) = (e, -\varepsilon\varphi'' - \varphi')$

$$= (-\varepsilon e'' + e', \varphi) + \varepsilon \underset{\overset{\shortparallel}{0}}{\varphi' e}\big|_0^1 - \varepsilon \underset{\overset{\shortparallel}{0}}{\varphi e'}\big|_0^1 + \underset{\overset{\shortparallel}{0}}{\varphi e}\big|_0^1$$

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$$= (\underbrace{-\varepsilon u'' + u'}_{=0} - \varepsilon u_h'' + u_h', \varphi)$$

$$= (-R(u_h), \varphi)$$

So, we obtain the first estimate:

$$(1) \qquad J(u-u_h) = \left(-R(u_h), \varphi\right).$$

Furthermore using the fact that $R(u_h) \perp V_{h,0}$:

$$J(u-u_h) = \left(-R(u_h), \varphi - \pi\varphi\right)$$
$$\leq c \sum_{i=0}^{n} h \, \|R(u_h)\|_{L^2(I_i)} \, \|\varphi''\|_{L^2(I_i)}$$

Note: Recall the error: $e = u - u_h$, $u \in V_0$, $u_h \in V_{h,0}$. Hence u_h does not have derivative at the nodal points. The integration by parts in ($\star$) should be done in each element:

$$(-\varepsilon e'', \varphi) = \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} -\varepsilon e'' \varphi \, dx$$
$$= \sum_{i=0}^{n} \left(\int_{x_i}^{x_{i+1}} -\varepsilon e \varphi'' dx + \underbrace{\varepsilon e' \varphi \Big|_{x_i}^{x_{i+1}}}_{T_i^1} - \underbrace{\varepsilon e \varphi' \Big|_{x_i}^{x_{i+1}}}_{T_i^2} \right)$$

$$\sum_{i=0}^{n} T_i^1 = \varepsilon \Big(\cancel{(e'\varphi)(x_1)} - (e'\varphi)(x_0) + \cancel{(e'\varphi)(x_2)} - \cancel{(e'\varphi)(x_1)} + $$
$$+ (e'\varphi)(x_3) - \cancel{(e'\varphi)(x_2)} + \dots + (e'\varphi)(x_{n+1}) - \cancel{(e'\varphi)(x_n)} \Big)$$
$$= \varepsilon \left((e'\varphi)(x_{n+1}) - (e'\varphi)(x_0) \right)$$
$$= \varepsilon \, e'\varphi \Big|_0^1.$$

Similarly $\sum_{i=0}^{n} T_i^2 = -\varepsilon e'\varphi \Big|_0^1$, and therefore the discussion above is true.

Problem #3 f)

The SD method applied to the convection-reaction-diffusion is:

(SD) Find $u \in V_{h,g} = \{ v : v \in C^0(\Omega), v|_K \in P_1(K)\ \forall K \in \mathcal{T}_h, v = g_1 \text{ on } \Gamma_i \}$ such that:

$$(\bar{\beta}\cdot\nabla u + \alpha u, v) + (\bar{\beta}\cdot\nabla u + \alpha u, \delta(\bar{\beta}\cdot\nabla v + \alpha v)) + (\hat{\varepsilon}\nabla u, \nabla v) =$$

$$= (f, v + \delta(\bar{\beta}\cdot\nabla v + \alpha v)) + \int_{\Gamma_0} g_2 v\, ds ,$$

$\forall v \in V_{h,0} = \{ v : v \in C^0(\Omega), v|_K \in P_1(K)\ \forall K \in \mathcal{T}_h, v = 0 \text{ on } \Gamma_i \}$.

Let us denote $Lu = \bar{\beta}\cdot\nabla u + \alpha u$, then choose $v = u$:

$$(\bar{\beta}\cdot\nabla u + \alpha u, u) + (\delta\, Lu, Lu) + (\hat{\varepsilon}\nabla u, \nabla u) = (f, u) + (f, \delta Lu) + (g_2, v)_{\Gamma_0}$$

For the advection term we have

$$(\bar{\beta}\cdot\nabla u, u) = -(u, (\nabla\cdot\bar{\beta})u) - (u, \bar{\beta}\cdot\nabla u) + \int_{\Gamma_-} \underbrace{\bar{\beta}\cdot\bar{n}}_{\leq 0}\, u^2\, dx + \int_{\Gamma_0} \underbrace{\bar{\beta}\cdot\bar{n}}_{\geq 0}\, u^2\, dx$$

Note: $\bar{\beta}\cdot\bar{n} > 0$ on Γ_0 therefore the second integral is non-negative. Similarly $\bar{\beta}\cdot\bar{n} < 0$ on Γ_-, that gives the first integral is nonpositive. We choose u to be zero on Γ_-.

Related question: Why can we choose $u = 0$ on Γ_-, while we have nonzero Dirichlet boundary condition?

So we get that $(\bar{\beta}\cdot\nabla u, u) \geq -\frac{1}{2}(\nabla\cdot\bar{\beta}\, u, u)$ and

$$\Big(\underbrace{\big(\alpha - \tfrac{1}{2}\bar{\beta}\cdot u\big)}_{= c.} u, u\Big) + \delta\|Lu\|^2 + \hat{\varepsilon}\|\nabla u\|^2 \leq \|f\|\|u\| + \delta\|f\|\|Lu\| + \|g_2\|_{\Gamma_0}\|u\|_{\Gamma_0}$$

$$(*)\quad c\|u\|^2 + \delta\|Lu\|^2 + \hat{\varepsilon}\|\nabla u\|^2 \leq \|f\|\|u\| + \delta\|f\|\|Lu\| + \|g_2\|_\Gamma\|u\|_\Gamma$$

Now, we use Young's inequality: $\boxed{ab \leq \frac{1}{2\gamma}a^2 + \frac{\gamma}{2}b^2, \quad \gamma>0.}$

$$\|f\|\|u\| \leq \frac{1}{2c_0}\|f\|^2 + \frac{c_0}{2}\|u\|^2,$$

$$\|f\|\|Lu\| \leq \frac{1}{2c_1}\|f\|^2 + \frac{c_1}{2}\|Lu\|^2,$$

$$\|g_2\|_\Gamma\|u\|_\Gamma \leq \frac{1}{2c_2}\|g_2\|_\Gamma^2 + \frac{c_2}{2}\|u\|_\Gamma^2.$$

And Trace inequality: $\boxed{\|g\|_\Gamma^2 \leq \gamma\big(\|g\|^2 + \|\nabla g\|^2\big), \quad \gamma>0.}$

$$\|u\|_\Gamma^2 \leq c_3\big(\|u\|^2 + \|\nabla u\|^2\big).$$

Now getting back to $(*)$:

$$c\|u\|^2 + \delta\|Lu\|^2 + \hat{\varepsilon}\|\nabla u\|^2 \leq \frac{1}{2c_0}\|f\|^2 + \frac{c_0}{2}\|u\|^2 + \delta\Big(\frac{1}{2c_1}\|f\|^2 + \frac{c_1}{2}\|Lu\|^2\Big)$$
$$+ \frac{1}{2c_2}\|g_2\|_\Gamma^2 + \frac{c_2}{2}c_3\big(\|u\|^2 + \|\nabla u\|^2\big)$$

$$\Rightarrow \underbrace{\Big(c - \frac{c_0}{2} - \frac{c_2c_3}{2}\Big)}_{\gamma_1}\|u\|^2 + \underbrace{\delta\Big(1 - \frac{c_1}{2}\Big)}_{\delta\gamma_2(\delta)}\|Lu\|^2 + \underbrace{\Big(\hat{\varepsilon} - \frac{c_2c_3}{2}\Big)}_{\hat{\varepsilon}\gamma_3(\hat{\varepsilon})}\|\nabla u\|^2 \leq$$

$$\underbrace{\Big(\frac{1}{2c_0} + \frac{\delta}{2c_1}\Big)}_{\gamma_4(\delta)}\|f\|^2 + \underbrace{\frac{1}{2c_2}}_{\gamma_5}\|g_2\|_\Gamma^2$$

We can choose c_0, c_1, c_2 and c_3 such that the coefficients are non negative (Q: why?). So:

$$\gamma_1 \|u\|^2 + \delta \gamma_2(\delta) \|Lu\|^2 + \hat{\varepsilon} \gamma_3(\hat{\varepsilon}) \|\nabla u\|^2 \leq \gamma_4(\delta)\|f\|^2 + \gamma_5 \|g_2\|_\Gamma^2 .$$

The right-hand-side can be written as:

$$\gamma_4(\delta)\|f\|^2 + \|f\|^2 \frac{\gamma_5}{\|f\|^2} \|g_2\|_\Gamma^2 = \left(\gamma_4(\delta) + \gamma_5 \frac{\|g_2\|_\Gamma^2}{\|f\|^2}\right)\|f\|^2$$

Next, denote $c_{min} = \min\left(\gamma_1, \gamma_2(\delta), \gamma_3(\hat{\varepsilon})\right)$. Then

$$\|u\|^2 + \delta\|Lu\|^2 + \hat{\varepsilon}\|\nabla u\|^2 \leq$$

$$\leq \frac{1}{c_{min}}\left(\gamma_1 \|u\|^2 + \delta\gamma_2(\delta)\|Lu\|^2 + \hat{\varepsilon}\gamma_3(\hat{\varepsilon})\|\nabla u\|^2\right)$$

$$\leq \underbrace{\frac{1}{c_{min}}\left(\gamma_4(\delta) + \gamma_5 \frac{\|g_2\|_\Gamma^2}{\|f\|^2}\right)}_{C}\|f\|^2$$

Finally:

$$\|u\|^2 + \delta\|Lu\|^2 + \hat{\varepsilon}\|\nabla u\|^2 \leq C\|f\|^2 .$$