Harmonic Functions in Lie Sphere Geometry

Mica Li

Graciously sponsored by the Ledell Family Research Scholarship

Contents

1 Introduction

Harmonic functions have been an area of study in analysis, stochastic processes, physics, engineering, complex analysis, and many other fields. Harmonic functions are defined to be twice continuously differentiable functions that satisfy the Laplace equation $\Delta f = 0$. Tristan Needham's 1994 paper *The Geometry of Harmonic Functions* brought attention to the lesser known geometric properties of harmonic functions, especially over circles. Needham's results reveal a deeper meaning of the Mean Value Property of harmonic functions. While the results are interesting, they are limited to only 2-dimensions as they are studied under complex analysis.

We hope to generalize the results that Needham summarizes to more general dimensions and we expect to obtain results similar to that of the Kelvin transform – non-trivial symmetry of harmonic functions in general dimensions. Since harmonic functions seem to naturally work over spheres and circles, the general theory can also be formulated in terms of language such as Lie sphere geometry, which has greatly simplified problems involving circles and spheres in the past.

2 Harmonic Functions

Harmonic functions come from the *Laplace equation*

$$\begin{aligned} u_{xx} &= 0 && \text{in one dimension} \\ \nabla \cdot \nabla u = \Delta u = u_{xx} + u_{yy} &= 0 && \text{in two dimensions} \\ \nabla \cdot \nabla u = \Delta u = u_{xx} + u_{yy} + u_{zz} &= 0 && \text{in three dimension} \end{aligned}$$

The Laplace equation comes from the diffusion or wave processes being independent of time, reducing the diffusion and wave equations to the Laplace equation.

Definition 1. *A solution to the Laplace equation is called a* ***harmonic function***

Poisson's equation arises from the inhomogeneous version of Laplace's equation

$$\Delta u = f$$

where f is a given function.

Laplace and Poisson equations arise in many different fields, meaning that harmonic functions also appear in many cases. Some examples include Maxwell's equations, steady fluid flow, and Brownian motion. One of the basic mathematical problems is to solve the Laplace and Poisson equations on some domain D where there are conditions imposed on the boundary D. These conditions give rise to the **Maximum Principle** which dictates where the maximum and minimum of a harmonic function will appear on some domain D.

Theorem 1. ***Maximum Principle*** *Let D be a connected bounded open set(in $\mathbb{R}^2$ or $\mathbb{R}^3$). Let $u(x,y)$ be a harmonic function in D that is continuous on $\bar{D} = D \cup bdy\, D\,(D')$ (commonly called the closure of D). Then the maximum and minimum values of u are attained on D' and nowhere on the interior.*

Proof. Let $\epsilon > 0$. Without loss of generality, we assume that we are in $\mathbb{R}^2$. We define a new function $v(x) = u(x) + \epsilon|x|^2$. Then:

$$\Delta v = \Delta u + \epsilon \Delta(x^2 + y^2) = 0 + 4\epsilon > 0 \qquad \text{on the interior of D}$$

However, by the second derivative test, $\Delta v = v_{xx} + v_{yy} \leq 0$ to have an interior maximum point [Wei]. Thus, no interior maximum exists for $v(x)$ on the interior of D.

We now need to show that $v(x)$ achieves a maximum somewhere on the boundary of D. Given that $\bar{D}$ is closed and bounded(so that $\bar{D}$ is compact by Heine-Borel) , and $v(x)$ is continuous, this means that we are guaranteed that $v(x)$ has a maximum somewhere on $\bar{D}$ (See *Theorem 4.16 in* [Rud53]).

Given that $v(x)$ cannot have a maximum on the interior of D, this means that $v(x)$ must have a maximum on the *boundary of* D. Let x_0 be the point at which the maximum of $v(x)$ is attained on D, meaning that for all $x \in D$:

$$u(x) \leq v(x) \leq v(x_0) = u(x+0) + \epsilon|x_0|^2 \leq max_{\text{bdy D}} u + \epsilon l^2$$

where l is the greatest possible distance from the origin to the bdy D. Give the fact that ϵ is arbitrary, we have:

$$u(x) \leq max_{\text{bdy D}} u$$

Since the maximum of u is achieved at some x_M on the boundary, we have that $u(x) \leq u)x_M$. □

The existence of the minimum on the boundary is proved in a similar way. To prove the *strong form of the Maximum Principle*, where we show that x_M cannot be in the interior of D unless $u \equiv$ constant, we need some further prerequisites [Str07].

2.1 Poisson's Formula

One important property of the Laplace equation is that it is *invariant under* ***all*** *rigid motions*. A rigid motion is classified as translations and rotations in $\mathbb{R}^2$ and $\mathbb{R}^3$. This invariance implies that it is easier to formulate the Laplace equation under polar and spherical coordinates.

This natural tendency to formulate the Laplace equation under polar and spherical coordinates leads us to consider the **Dirichlet's Problem** on a circle [Str07].

Definition 2. ***Dirichlet's Problem*** *We let f be a continuous function on the boundary of some domain $D(\partial D)$. Does there exist a continuous function u on $\bar{D}$ such that u is* ***harmonic on D*** *and $u = f$ on ∂D? [Kra99]*

We consider a circle with radius R and some circumference C centered at the origin. Suppose that there is some sort of steady heat flow on the interior of this circle and the temperature at some point a is some definite value $T(a)$ where T is a function that models the heat flow. Poisson was able to explicitly write out a formula for $T(a)$ in terms of $T(C)$ and it is known as **Poisson's Formula**. Poisson's formula is able to determine the value of $T(a)$ at any interior point a using the values of R and C. This is done through expressing the temperature as a function of the angle $T = T(\theta)$, since $z = Re^{i\theta}$ traces out C.

$$\boxed{T(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{R^2 - |a|^2}{|z-a|^2}\right] T(\theta)\, d\theta} \tag{1}$$

The expression

$$\left[\frac{R^2 - |a|^2}{|z-a|^2}\right] \tag{2}$$

is called the **Poisson Kernel** and is often denoted as $\mathscr{P}_a(z)$.

In relation to Dirichlet's problem on a circle, we have from Schwarz that Poisson's formula explicitly solves Dirichlet's problem, meaning that $T(a)$ is harmonic [Nee94]. For more geometric intuition and the derivation of Poisson's formula, see [Nee94].

Poisson's formula has many important consequences. The **Mean Value Property** is a key consequence that comes from Poisson's formula.

Theorem 2. ***Mean Value Property.*** *Let u be a harmonic function in a disk D and continuous on $\bar{D}$. Then the value of u at the center of D equals to the average of u on its circumference.*

This property, along with the invariance of the Laplace equation suggests that the more natural formulation of harmonic function lies in some non-Euclidean geometry.

We now have all we need to prove the strong version of the maximum principle.

Theorem 3. ***Maximum Principle(Strong form)*** *Let u(x) be harmonic on some domain D. If u(x) attains its maximum on the interior of D, u(x) is a constant function.*

Proof. Let u be harmonic on D and let it be a continuous function on $\bar{D}$. Then the maximum of u is obtained somewhere on $\bar{D}$, $x_M \in \bar{D}$. We want to show that if $x_M \in D$, then $u \equiv$ constant.

Since x_M is the point where the maximum of u is achieved we know that

$$u(x) \leq u(x_M) = M \qquad \forall x \in D$$

Then, we let $B_r(x_M)$ be a ball centered at x_M such that the radius r is small enough so that $B_r(x_M)$ is a proper subset of D. Then, by the mean value property, the value of $u(x_M)$ is equal to the average of u on the circumference of $B_r(x_M)$ (A_c). The average of a function cannot be greater than its maximum, so we have the following inequality:

$$\begin{aligned} M = u(x_M) = A_c \leq M \\ \Rightarrow M = A_c \end{aligned}$$

This means that $u(x) = M$ $\forall x$ on the circumference of $B_r(x_M)$. We repeat this process for a ball centered at another point in D, filling the entire domain D with circles. Given that D is connected, this means that $u(x) \equiv M$ $\forall x \in D$, meaning that u is a constant function [Str07] $\square$

2.2 The Kelvin Transform

The Kelvin Transform is an important tool in the study of harmonic functions as it is able to extend the idea of harmonic functions to infinity. It accomplishes this by transforming a function that is harmonic inside a unit sphere to a function that is harmonic *outside* the unit sphere.

We define a map $x \mapsto x*$ on $\mathbb{R}^n \cup \{\infty\}$(one-point compactification of $\mathbb{R}^n$) as follows:

$$x^* = \begin{cases} \frac{x}{|x|^2} & \text{if } x \neq 0, \infty \\ 0 & \text{if } x = \infty \\ \infty & \text{if } x = 0 \end{cases}$$

This mapping is known as the *inversion* of $\mathbb{R}^n \cup \{\infty\}$ relative to the unit sphere. However, this transform has one flaw: it does not preserve harmonic functions when $n \geq 2$. Thus, the Kelvin transform, discovered by Lord Kelvin in the 1840s, extends this inversion so that the inversion preserves harmonic functions $\forall n \geq 2$ [ABR01].

The Kelvin Transformation is defined as follows. We first define an inversion mapping in $\mathbb{R}^n$ of x on a sphere $S(0, R)$ that has a center of 0 and some fixed radius R. Then, the inversion mapping of $x \mapsto x^*$ is as follows:

$$x^* = \frac{R^2}{|x|^2} x$$

Then the Kelvin transform with respect to the sphere $S(0, R)$ on some harmonic function u is defined on a domain $D \subset \mathbb{R}^n$ [kel]

$$\boxed{u^*(x^*) = \left(\frac{R}{|x^*|}\right)^{n-2} u\left(\frac{R^2}{|x^*|^2} x^*\right)} \tag{3}$$

Some useful properties of the Kelvin transform are as follows:

- the Kelvin transform is linear
- it preserves uniform convergence on compact sets
- it is its own inverse
- **the Kelvin transform of every harmonic function is harmonic**

The Kelvin transform also makes it possible to discuss harmonic functions at infinity. This is illustrated with the following theorem [ABR01]

Theorem 4. *Assume $n > 2$. Let u be harmonic on $\mathbb{R}^n \backslash D$ and let $D \subset \mathbb{R}^n$ be compact. Then u is harmonic at ∞ iff $\lim_{x\to\infty} u(x) = 0$.*

3 Non-Euclidean Geometries

3.1 Projective Geometry

The main setting of Lie sphere geometry is projective space, $\mathbb{P}^n$ and will also require some Möbius geometry [Cec92]. We will first discuss projective space and projective geometry.

To build up Projective Geometry, we first follow Artin's axiomatic approach [Art57] and then lead more into the geometric intuition of what projective geometry is. We begin with two sets: a set of points P and a set of lines l. From here, we build up our axioms. We first define what it means for two lines to be considered *parallel*.

Definition 3. ***Parallelism.*** *Two lines l and m are considered to be* ***parallel*** *if $l = m$ or no point P lies on both l and m. We denote that two lines are parallel by writing $l \parallel m$. If l and m are not parallel, we write $l \nparallel m$. If we have that $l \nparallel m$, $\exists$ at least one point P such that P lies on both l and m.*

Axiom 1. *Given two distinct points P and Q, $\exists$ a unique line l such that P lies on l and Q lies on l. We write $l = P + Q$. If $l \nparallel m$, then there is exactly one point P that lies on both l and m.*

Axiom 2. *Given a point P and a line l, $\exists$ one and only one line m such that P lies on m and such that $m \parallel l$.*

Axiom 3. *$\exists$ three distinct points A, B, C such that C does not lie on the line $A + B$. In other words, $\exists$ three non-collinear points. This also means that $A + B$ and $A + C$ are not parallel.*

Axiom 4. *Given two points P and Q, $\exists$ a translation τ_{PQ} which moves P into Q*

$$\tau_{PQ}(P) = Q$$

Axiom 5. *If τ_1 and τ_2 are translations with the same traces and if $\tau_1 \neq 1$, $\tau_2 \neq 1$, $\tau_1 \neq \tau_2$, then $\exists$ $\alpha \in k$ such that $\tau_2 = \tau_1^\alpha$.*

Axiom 4 and Axiom 5 are somewhat related and are written as Axiom 4(a) and Axiom 4(b) in Artin. For a more in-depth idea about building up projective geometry axiomatically, see [Art57]. We now turn our sights to a more geometric approach that is more intuitive.

The main idea of projective geometry is to be able to properly handle points at infinity. In traditional Euclidean space, points at infinity are tricky to deal with and will often end up invalidating certain theorems. Projective geometry is useful in this way as it makes no distinction between "regular" points and points at infinity meaning that many concepts such as conics and quadrics become simpler.

Definition 4. ***Projective Space.*** *Suppose that we have a vector field E over some field K. The* ***projective space $\boldsymbol{P}$**(E)* *induced by E is the set of equivalence classes of nonzero vectors in E under the equivalence relation $\sim$ defined $\forall u, v \in E - \{0\}$ such that*

$$u \sim v \quad \textit{iff} \quad v = \lambda u, \ \textit{for some } \lambda \in K - \{0\} \tag{4}$$

*The **canonical projection** $p : (E - \{0\}) \to \boldsymbol{P}(E)$ is the function that associates the equivalence class $[u]_\sim$ to $u \neq 0$.*

*The **dimension** of **P**(E)(dim(**P**(E))) is defined in the following way. If E has infinite dimension then **P**(E) = dim(E). If E has finite dimension where dim(E) = $n \geq 1$ then **P**(E) = $n - 1$.*

Essentially a projective space **P**(E) is a set of equivalence classes of vectors in E. One thing to note is that the equivalence class $[u]_\sim$ is usually called a *point* where the entire equivalence class is viewed as a singular object.

The topology of projective space is defined through the *quotient topology.*

Definition 5. ***Quotient Map.*** *Let X, Y be topological spaces and $p : X \to Y$ is a surjective map. p is a **quotient map** provided that a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.*

Definition 6. ***Quotient Topology.*** *Let X be a space and A to be a set. If $p : X \to A$ is a surjective map, then $\exists$ exactly one topology $\mathscr{T}$ on A relative to which p is a quotient map. This is called the quotient topology induced by p.*

Definition 7. *A subset C of X is **saturated** if C contains every set $p^{-1}(\{y\})$ that it intersects. In other words C is saturated if it equals the complete inverse image of a subset of Y.*

Using this definition we can see that saying that p is a quotient map is equivalent to saying that p is continuous and that p maps saturated open sets of X to open sets of Y. These definitions are taken from [Mun74].

Definition 8. *If a topological space satisfies the following statements, then the space is called a **Hausdorff** space. We let X be a topological space, then the following statements are all equivalent*

1. *Any two distinct points of X have disjoint neighborhoods*
2. *The intersection of the closed neighborhoods of any point of X consists of that point alone*
3. *The diagonal of the product space $X \times X$ ($D := \{(x, x) \in X \times X | x \in X\}$) is a closed set*

For the proof of this statement, reference [Bou66].

Theorem 5. *The projective space $P(E)$ endowed with the quotient topology induced from $E\backslash\{0\}$ is Hausdorff and compact. It is also the quotient $S\backslash U$ of the unit sphere by the subgroup U of elements of K having absolute value 1.*

Proof. We first show that $P(E)$ is Hausdorff.

Lemma 1. *The quotient of a topological space V by an equivalence relation R is Hausdorff if and only if the saturation of every open set U is open and the graph of the relation R in $V \times V$ is closed.*

We let V be the space $E\backslash\{0\}$ and the equivalence relation R to be collinearity. The saturation of an open set U is the union of its homothetic images aU for some $a \in K^*$. This is a union of open sets so the union itself is open. We define the coordinates $(x_1, x_2, \ldots, x_n$ in E and $(y_1, y_2, \ldots, y_n)$ in the second factor E in $E \times E$. Then, the graph of R is defined by the equation $x_i y_i - x_j y_j = 0$ where $i \neq j$, which is closed. Thus, by *Lemma 1*, $P(E)$ is Hausdorff.

To show the second result of the theorem, we recall that the canonical map $p : E\backslash\{0\} \to P(E)$ is continuous so $p(S) = P(E)$ meaning that $P(E)$ is compact.$P(E)$ is also the quotient of the sphere S because of the equivalence relation induced on S by R. If $N(x) = N(y) = 1$ we have a relation in the form $x = ay$ for some $a \in K^*$ iff $v(a) = 1$. Then, $P(E) = S\backslash U$. □

We now turn our attention to *Lorentz Space* to better understand projective space.

Definition 9. *Given f a measurable function on a measure space (X, μ) and $0 < p, q \leq \infty$ we define*

$$\|f\|_{L^{p,q}} = \begin{cases} \left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{dt}{t}\right)^{1/q} & \text{if } q < \infty \\ \sup_{t>0} t^{1/p} f^*(t) & \text{if } q = \infty \end{cases} \tag{5}$$

The set of all f with $\|f\|_{L^{p,q}} < \infty$ is denoted by $L^{p,q}(X, \mu)$ and is the ***Lorentz space*** *with indices p and q [Gra14].*

Lorentz space in [Cec92] is written as $\mathbb{R}_1^{n+1}$ which has signature $(n, 1)$, meaning that the space has rank n and index 1.

We then define an indefinite scalar product on the Lorentz space, called the *Lorentz metric.*

$$(x, y) = -x_1 y_1 + x_2 y_2 + \cdots + x_{n+1} y_{n+1} \tag{6}$$

A vector is called

- spacelike if (x, x) is positive
- timelike if (x, x) is negative
- lightlike if $(x, x) = 0$

In Lorentz space, the set of all lightlike vectors written in the following equation

$$x_1^2 = x_2^2 + \cdots + x_{n+1}^2 \tag{7}$$

forms a cone of revolution. More often, lightlike vectors are called isotropic and this cone is called an *isotropy cone*. We also have that timelike vectors are inside the cone while spacelike vectors are outside the cone.

Letting x be a lightlike vector in Lorentz space means that the equivalence class of x, $[x]$ can be represented by a vector of the form $(1, u)$ for some $u \in \mathbb{R}^n$. Then the equation of the lightcone $(x, x) = 0$ in Lorentz space becomes $u \cdot u = 1$ in $\mathbb{R}^n$, which is the equation for the unit sphere in $\mathbb{R}^n$. Thus, the set of points in the projective space, $\mathbb{P}^n$ can be determined by lightlike vectors in Lorentz space and is also diffeomorphic to the sphere S^{n-1} [Cec92].

From here, we can discuss Mobius geometries and transformations as a second step to defining Lie sphere geometry.

3.2 Mobius Geometry

Mobius geometry describes the geometry of *unoriented* spheres in $\mathbb{R}^n$. First, we consider the stereographic projection $\sigma : \mathbb{R}^n \to S^n - \{P\}$ where S^n is the unit sphere in $\mathbb{R}^{n+1}$ and $P = (-1, 0, \ldots, 0)$ is the south pole of S^n. The formula for $\sigma(u)$ is given as follows:

$$\sigma(u) = \left(\frac{1 - u \cdot u}{1 + u \cdot u}, \frac{2u}{1 + u \cdot u}\right) \tag{8}$$

We then embed $\mathbb{R}^{n+1} \to \mathbb{P}^{n+1}$ with the embedding $\phi(u) = [(1, u)]$. Then the map $\phi\sigma : \mathbb{R}^n \to \mathbb{P}^{n+1}$ is given by

$$\phi\sigma(u) = \left[\left(1, \frac{1 - u \cdot u}{1 + u \cdot u}, \frac{2u}{1 + u \cdot u}\right)\right] \tag{9}$$

$$= \left[\left(\frac{1 + u \cdot u}{2}, \frac{1 - u \cdot u}{2}, u\right)\right] \tag{10}$$

We let $(z_1, \ldots, z_{n+2})$ be homogeneous coordinates on $\mathbb{P}^{n+1}$ and denote $(,)$ to be the Lorentz metric on the Lorentz space $\mathbb{R}_1^{n+2}$. This means that $\phi\sigma(\mathbb{R}^n)$ is the set of points in P^{n+1} that are on the n-sphere Σ given

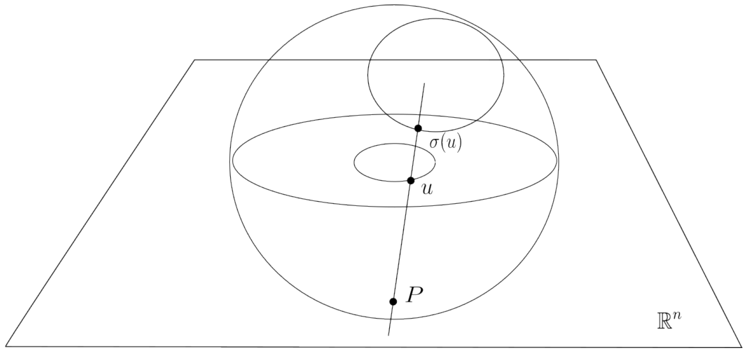

Figure 1: *Stereographic projection*

by the equations $(z, z) = 0$. The only exception is the point P, called the *improper point*, whereas all other points are called *proper points*. Σ is known as the **Mobius sphere** or as **Mobius space**.

We let ξ be a spacelike vector in $\mathbb{R}_1^{n+2}$ and let $\xi^\perp$ be the polar hyperplane to $[\xi]$ in $\mathbb{P}^{n+1}$. There exists a polar relationship between points and hyperplanes in $\mathbb{P}$ due to the scalar product(e.g. Lorentz metric) on $\mathbb{R}^{n+1}$. $x^\perp$ denotes the polar hyperplane of $[x]$ in $\mathbb{P}^n$. $[x]$ is the *pole* of $x^\perp$.

The polar hyperplane $\xi^\perp$ intersects the sphere Σ in an $(n-1)-$sphere S^{n-1}. S^{n-1} is the image under $\phi\sigma$ of an $(n-1)-$sphere in $\mathbb{R}^n$. However, it the sphere contains the **improper point**, the $(n-1)-$sphere is the image under $\phi\sigma$ of a hyperplane in $\mathbb{R}^n$. Thus, $\exists$ a bijective correspondence between all hyperspheres and hyperplanes in $\mathbb{R}^n$.

Definition 10. ***Mobius Transformation and Mobius Group.*** *A projective transformation of $\mathbb{P}^{n+1}$ which preserves the condition $(\eta, \eta) = 0$ where η is spacelike vector. It maps spacelike points to spacelike points and preserves orthogonality between spheres and planes in $\mathbb{R}^n$. Additionally, the transformation takes lightlike vectors to lightlike vectors, which induces a conformal diffeomorphism of the sphere Σ to itself. Thus, the group of conformal diffeomorphisms of the sphere is the* ***Mobius group*** *[Cec92].*

3.3 Lie Sphere Geometry

We now have the basis to construct Lie's geometry in $\mathbb{R}^n$. Lie sphere geometry is the study of the **Lie quadric** and how it interacts with the **Lie transformation**. We will first discuss the *Lie quadric*.

Definition 11. ***Hypersurface.*** *A manifold or algebraic variety that is one dimension less than the ambient space it is embedded in [Sam88].*

Definition 12. ***(Hyper)quadric.*** *A hypersurface in an affine or projective space defined by a quadratic equation. in $\mathbb{R}^2$ a quadric is called a* ***conic*** *[Sam88].*

The Lie quadric is the quadric Q^{n+1} that lies in the projective plane $\mathbb{P}^{n+2}$ (i.e a quadric that lies in the projective plane). The **Lie metric** is

$$< x, x >= -x_1^2 + x_2^2 + \cdots + x_{n+2}^2 - x_{n+3}^2 = 0 \tag{11}$$

There exists a bijective correspondence between objects in Euclidean space and points in the Lie quadric. They are given in the following table

Euclidean	Lie
$u \in \mathbb{R}^n$	$\begin{bmatrix} \frac{1+u\cdot u}{2} \\ \frac{1-u\cdot u}{2} \\ u \\ 0 \end{bmatrix}$
∞	$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$
sphere with center p and signed radius r	$\begin{bmatrix} \frac{1+p\cdot p-r^2}{2} \\ \frac{1-p\cdot p+r^2}{2} \\ p \\ r \end{bmatrix}$
planes where $u \cdot N = h$, N is the unit normal	$\begin{bmatrix} h \\ -h \\ N \\ 1 \end{bmatrix}$

In Lie sphere geometry points are considered to be spheres of zero radius and they are called **point spheres**. Although useful to have the correspondences between Lie sphere geometry and Euclidean space $\mathbb{R}^n$, it is often more convenient to consider Lie sphere geometry on the sphere S^n [Cec92].

S^n is considered to be the unit sphere in $\mathbb{R}^{n+1}$ that is embedded into $\mathbb{P}^{n+1}$ with the embedding ϕ. This turns S^n into $\phi(S^n)$ which is the Mobius sphere Σ. By considering this space, S^n, the correspondences drawn up in the table can be replaced with a single equation. Given an oriented sphere S with signed radius $\rho \neq 0$, $-\pi < \rho < \pi$ with center p corresponds to a point on the Lie quadric Q^{n+1} in the following way

$$S \longleftrightarrow \begin{bmatrix} cos(\rho) \\ \rho \\ sin(\rho) \end{bmatrix} \tag{12}$$

We now shift our attention to the **Lie sphere transformation**, the second piece of Lie sphere geometry.

Definition 13. ***Lie Transformation.*** *Any transformation from $\mathbb{P}^{n+2} \to \mathbb{P}^{n+2}$ that preserves the Lie quadric Q is called a Lie transformation. In other words, a Lie transformation will bijectively map an oriented sphere to itself. Lie transformations will also preserve oriented contact of spheres [Pin81].*

Lie sphere geometry considers lines, circles, and points on equal footing, meaning that there is a singular object in this geometry called the "Lie cycle"(i.e an oriented circle or sphere).

To make the distinction between Mobius geometry, Euclidean geometry, and Lie sphere geometry, we compare one of the basic axioms from each.

- Euclidean geometry: Given a line and a point where the point does not lie on the line, $\exists$ a unique line that goes through the point and is parallel to the line.
- Mobius geometry: Given a unique circle and a point that is not on the circle, $\exists$ a circle that contains the point and is tangent to the original circle.
- Lie sphere geometry: Given three distinct oriented circles(cycles), some of which touch, $\exists$ a unique cycle that touches all three [FS90].

The set of all Lie cycles, form a three dimensional space in four dimensional real projective space and that space is the **Lie quadric**.

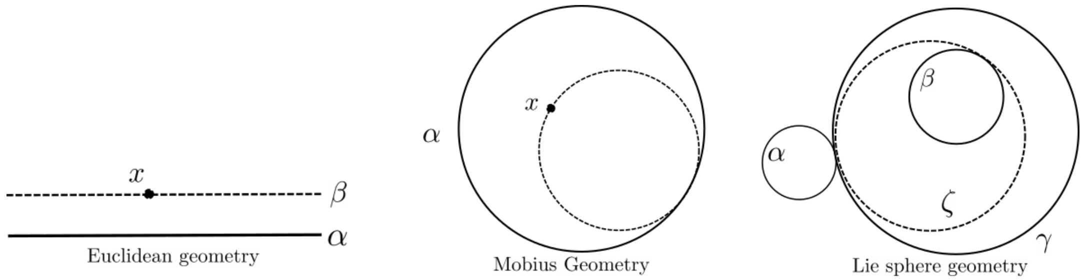

Figure 2: *Illustration of the differences between Euclidean, Mobius, and Lie sphere geometries*

4 The Kelvin Transform and the Mean Value Property

We let u be a harmonic function on the disk D, where u is continuous on the closure of D, $\bar{D}$. The mean value property of harmonic functions states that the value of u at a, is equal to the average of u on the circumference on a disk D centered at a with radius r.

$$\boxed{u(a) = \frac{1}{2\pi r}\int_{\partial D} u(x)\,dx} \tag{13}$$

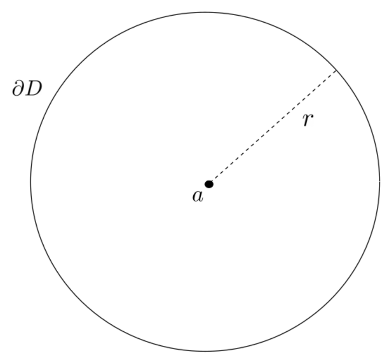

Figure 3: *Disk D centered on a with radius r*

We want to somehow apply the *Kelvin transform* the point a, in an attempt to combine the transform with the mean value property. However, because the Kelvin transform is defined in the following way [AG01]

$$x^* = \frac{r^2}{\|x-a\|^2}(x-a) + a$$
$$f \to f^* : f^*(x) = \left(\frac{r}{\|x-a\|}\right)^{n-2} f(x^*)$$

the transform is only defined on the space $\mathbb{R}^n \backslash \{a\}$. We need to somehow transform a so that it is no longer the center of a disk. This means that we need to make another point in D the center of inversion and draw a circle of radius r^* around it such that a is contained in the new circle.
We let c be the new center of inversion and the Kelvin transform on a can be defined as the following

$$a^* = \frac{(r^*)^2}{\|a-c\|^2}(a-c) + c \tag{14a}$$
$$u \to u^* : u^*(a) = \left(\frac{r^*}{\|a-c\|}\right)^{n-2} u(a^*) \tag{14b}$$

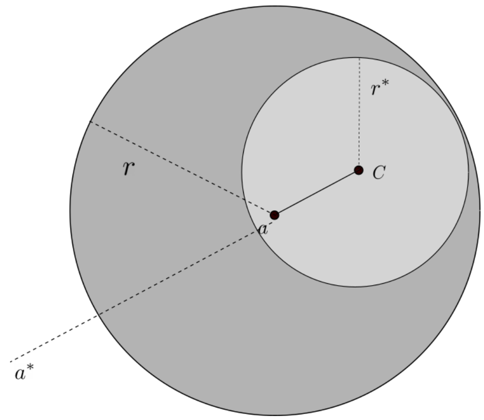

Figure 4: *Defining a new center of inversion*

Given that u is defined and continuous on the closure of D, u is also defined and continuous on the new circle(which we will call D^*). This is because the distance between a and c will always be less than r if we select a point c that is on the interior of D. This means that there will always exist some circle D^* that has a radius r^* such that $r^* < r$ where D^* is contained in D.

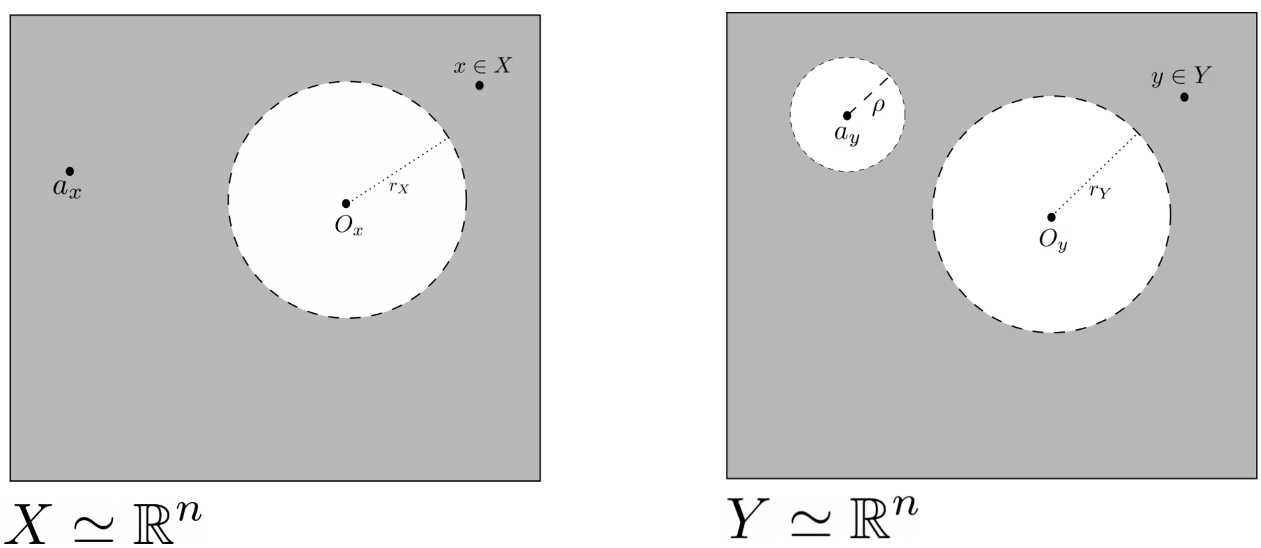

Figure 5: *I'll come up with a caption eventually*

We are given two spaces X and Y, which are locally similar to n-dimensional Euclidean space. The spaces X and Y have a functional relationship where

$$r_x \coloneqq |x|_X \tag{15}$$

$$y = \frac{x}{r_x^2} \tag{16}$$

This relationship is an involution since

$$\begin{aligned} r_y \coloneqq |y|_Y &= \frac{1}{r_x} \\ x &= \frac{y}{r_y^2} \end{aligned}$$

Thus, it is obvious to see that x depends on y and that y depends on x.

We define a function $f_x : X \to \mathbb{R}$. By substitution we have that $f_y : Y \to \mathbb{R}$ is defined by $f_y(y) = f_x(x)$. We define the Kelvin transform as follows

$$f^*(x^*) = \frac{1}{|x^*|^{n-2}} f(\frac{x^*}{|x^*|^2}) \tag{17}$$

The Kelvin transform $\mathscr{K}$ is a function that maps

$$\mathscr{K} : (X \to \mathbb{R}) \to (Y \to \mathbb{R})$$

Then, this means that when we combine $\mathscr{K}$ with f_x and y we have

$$(\mathscr{K} f_x)(y) := \frac{1}{|y|_Y^{n-2}} f_y(y) = \frac{1}{r_y^{n-2}} f_y(y) = r_x^{n-2} f_y(y) \tag{18}$$

The Kelvin transform maps harmonic functions to harmonic functions. Assuming that f_x is harmonic, we have that $(\mathscr{K} f) : Y \to \mathbb{R}$ is also harmonic. Given that $(\mathscr{K} f) : Y \to \mathbb{R}$ is harmonic, the mean value property can be applied to it. Then, from Figure 5 and the mean value property we have that

$$(\mathscr{K} f)(a_y) = \frac{1}{2\pi\rho} \int_{S_y} (\mathscr{K} f) dA_y \tag{19}$$

The mean value property gives us equation 19 and substituting the previous equations into equation 19 we have:

$$(\mathscr{K} f)(a_y) = |a_x|_X^{n-2} f_x(a_x) = |a_x|_X^{n-2} f_y(\frac{a_x}{|a_x|_X^2}) \tag{20}$$

The integral becomes

$$\frac{1}{2\pi\rho} \int_{S_y} (\mathscr{K} f) dA_y = \frac{1}{2\pi\rho} \int_{S_x} |x|^{n-2} f_x(x) dA_y = \frac{1}{2\pi\rho} \int_{S_x} |x|^{n-2} f_x(x) |x|^{2(n-1)} dA_x \tag{21}$$

Thus, we have the following equality

$$|a_x|_X^{n-2} f_y(\frac{a_x}{|a_x|_X^2}) = \frac{1}{2\pi\rho} \int_{S_x} |x|^{n-2} f_x(x) |x|^{2(n-1)} dA_x \tag{22}$$

Changing the center of inversion(see Figure 4), we end up with

$$f_x(a_x) = \frac{1}{|a_x|_X^{n-2} 2\pi\rho} \int_{S_x} |x|^{3n-4} f_x(x) dA_x \tag{23}$$

This is the Poisson formula on a disk, meaning that it can be derived from the Kelvin transform and the mean value property.

Going from the Poisson Formula to the Kelvin Transform(?). We set $x = x_0$ where x_0 is the center of the disk. Then the Poisson Kernel becomes

$$P(x, \xi) = \frac{r^2}{\omega_n r |x - \xi|^n} = \frac{r}{\omega_n |x - \xi|^n}$$

Then the Poisson formula becomes

$$f(x) = \int_{S_x} \frac{r}{\omega_n |x - \xi|^n} f(\xi) d\sigma(\xi) \tag{24}$$

When $x = x_0$ the Poisson formula becomes the mean value property.

5 Defining Functions on Non-Euclidean Geometries

Suppose that X and Y are two different realizations of a common space M as $\mathbb{R}^n \cup \{\infty\}$ spaces. In these spaces, X and Y are equipped with two distinct metrics ds_X and ds_Y. These two metrics are conformal since

$$ds_X(x) = \varphi(y) ds_Y(y)$$

Where $\varphi(y)$ is some scalar function.

We restrict ourselves so that X and Y are always Mobius related. Thus, we let M be a "Mobius space" where $M \simeq S^n \simeq \mathbb{R}^n \cup \{\infty\}$. X, $Y \simeq \mathbb{R}^n$ are projections of M to $\mathbb{R}^n$ and are equipped with the Euclidean metric. The projections are two different stereographic projections with two different north poles. If X and Y are a single inversion of each other then $ds_X = \frac{1}{|x|_X^2} ds_Y$. Generally, between any two realizations X, Y

$$ds_Y = \varphi(x) ds_X$$
$$dV_Y = \varphi(x)^n dV_X$$

Where dV_X is the density measure.

Theorem 6. *We let $\mathscr{D}(M)$ be the space of density fields on M. Let μ, $\upsilon \in \mathscr{D}(M)$ then $\exists!\ g : M \to \mathbb{R} \cup \{\infty\}$ such that $\upsilon = g\mu$.*

Consider the space of α-power densities $\mathscr{D}^\alpha(M)$ (fractional density). We then have the following

- Given F, $G \in \mathscr{D}^\alpha(M)$, $\exists!\ h : M \to \mathbb{R}$ such that $F = hG$
- Given $\mu \in \mathscr{D}(M)$ density, there corresponds an α-density denoted by $\mu^\alpha \in \mathscr{D}^\alpha$
- $\mathscr{D}^\alpha(M) \times \mathscr{D}^\beta(M) \to \mathscr{D}^{\alpha+\beta}(M)$
- $F \cdot G = FG$

We let X, Y be two different $\mathbb{R}^n$ realizations (Euclideanization) of M. Then, changing the metric under a Mobius transformation we have

$$dV_Y(y) = \varphi(x)^n dV_x(x)$$
$$[dV_Y]^\alpha = \varphi(x)^{n\alpha} (dV_X)^\alpha \in \mathscr{D}^\alpha(M)$$

Given any $F \in \mathscr{D}^\alpha(M)$ we have the following two realizations of F as a function $F : \mathbb{R}^n \to \mathbb{R}$

$$\text{Using } dV_X\text{: } F = f_x dV_x^\alpha \tag{25}$$
$$\text{Using } dV_Y\text{: } F = \tilde{f} dV_Y^\alpha = \tilde{f} \varphi^{n\alpha} dV_x^\alpha \tag{26}$$
$$\Rightarrow \tilde{f} = f \varphi^{n\alpha} \tag{27}$$

We consider

$$\mathscr{L} : \mathscr{D}^{\frac{1}{n}(\frac{n}{2}-1}(M) \xrightarrow{\mathscr{L}} \mathscr{D}^{\frac{1}{n}(\frac{n}{2}+1} \tag{28}$$
$$\mathscr{L}^* : \mathscr{D}^{\frac{1}{n}(\frac{n}{2}+1} \xleftarrow{L^*} \mathscr{D}^{\frac{1}{n}(\frac{n}{2}-1}(M) \tag{29}$$

Then

$$F \in \mathscr{D}^{\frac{1}{n}(\frac{n}{2}-1}(M)$$
$$\mathscr{L}(F) = \mathscr{L}(f_X dV_X^{\frac{1}{n}(\frac{n}{2}-1}) := (\Delta f_X) dV_X^{\frac{1}{n}(\frac{n}{2}+1}$$

If $\mathscr{L}(F) = 0$ then we say that F is harmonic.

References

[ABR01] Sheldon Axler, Paul Bourdon, and Wade Ramey. *Harmonic Function Theory.* Springer-Verlag New York, 2001.

[AG01] David H. Armitage and Stephen J. Gardiner. *Classical Potential Theory.* Springer Monographs in Mathematics, 2001.

[Art57] Emil Artin. *Geometric Algebra.* Interscience Publishers, 1957.

[Bou66] Nicolas Bourbaki. *General Topology.* Addison-Wesley, Reading, Mass., 1966.

[Cec92] Thomas E. Cecil. *Lie Sphere Geometry.* Universitext, Springer-Verlag, New York, 1992.

[FS90] Jay P. Fillmore and Arthur Springer. Lecture notes for math 151 lie's higher sphere geometry, Spring 1990.

[Gra14] Loukas Grafakos. *Classical Fourier Analysis.* Springer New York, 2014.

[kel] Kelvin transformation. *Encyclopedia of Mathematics.*

[Kra99] Steven G. Krantz. *Handbook of Complex Variables.* Springer Science + Business Media New York, 1999.

[Mun74] James R. Munkres. *Topology.* Prentice Hall, 1974.

[Nee94] Tristan Needham. The geometry of harmonic functions. *Mathematics Magazine*, 67(2):92–108, 1994.

[Pin81] Ulrich Pinkall. Dupinsche hyperflächen. *Doctoral Disstertation. Universität, Freiburg*, 1981.

[Rud53] Walter Rudin. *Principles of Mathematical Analysis.* McGraw-Hill, 1953.

[Sam88] Pierre Samuel. *Projective Geometry.* Springer-Verlag, 1988.

[Str07] Walter A Strauss. *Partial differential equations: An introduction.* John Wiley & Sons, 2007.

[Wei] Eric W Weisstein. Second derivative test. *MathWorld - A Wolfram Web Resource.*