Introduction to Riemannian Manifolds

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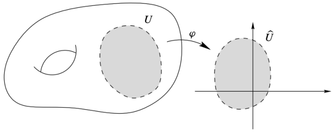
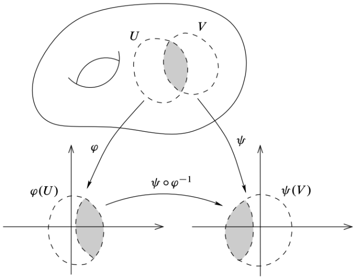
Smooth Manifolds

Definition: Smooth Manifold

A **smooth manifold** is a manifold that is endowed with some "smooth structure"

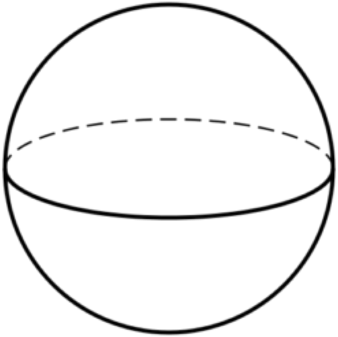


(a) A coordinate chart

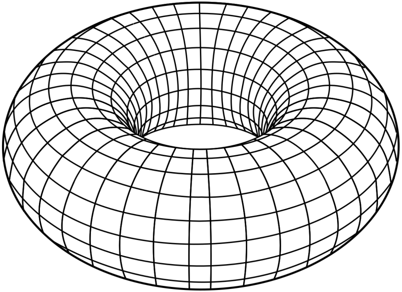


(b) A transition map

Examples of Smooth Manifolds



(c) Sphere



(d) Torus

Figure: Examples of smooth manifolds

Geometry on a Manifold?

Question

How do we do geometry on manifolds?

- Length is well-defined on $\mathbb{R}^n$

Length in $\mathbb{R}^n$

$$\int_a^b \sqrt{|v(t)|}\, dt$$

- The issue is that transition maps do not preserve length or angles in general

Idea

We need a new structure so that we can measure length and angles on a manifold

Length on a Manifold?

Question

How do we develop a notion of distance on a manifold?

- In general length is velocity multiplied by time(i.e. vt)
- We need to be able to find the velocity of a curve on a manifold
- Need to define length of a tangent vector

Tangent Spaces

Definition: Tangent Space

The **tangent space** at a point $p \in M$, T_pM, is the space of all tangent vectors at that point p.

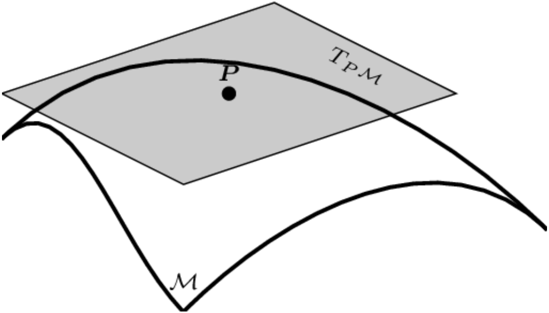


Figure: Tangent space at a point p

Riemannian Metrics

Definition: Riemannian Metrics

A **Riemannian metric** is a "smoothly" chosen inner product on T_pM.

An example of the Riemannian metric is

- the standard inner product on $\mathbb{R}^n$

Length and Distance on a Manifold

Now that we are equipped with the Riemannian metric we can define length and distance

Definition

Let $\gamma : [a, b] \to M$ be a curve. Then the length of γ is

$$L_g(\gamma) = \int_a^b |\gamma'(t)|_g \, dt$$

Equipped with this length we can now define the distance metric on a Riemannian manifold.

Distance on a Riemannian Manifold

Let $P = \{\gamma : \gamma \text{a curve from p to q}, p, q \in M\}$. Then,

$$d_g(p, q) = \inf_{\gamma \in P} L_g(\gamma)$$

Riemannian Manifolds

Definition: Riemannian Manifold

A Riemannian manifold is a pair (M, g) where M is a smooth manifold and g is a Riemannian metric defined on it

An immediate result from this is that

Proposition

Every smooth manifold will admit a Riemannian metric

Where do we go from here?

Using these ideas, one can also generalize

- angles
- volume
- integration and differentiation

Thus, we are now well-equipped to do analysis on manifolds

Acknowledgements

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