Memoryless Gaussian Sources and Channels 11

In this lecture, armed with the quantities of information for continuous random variables developed in the previous lecture, we study memoryless Gaussian sources and channels.

11.1 Rate-distortion Function of Memoryless Gaussian Sources

We consider a discrete-time information source which emits i.i.d. source symbols, obeying Gaussian distribution of mean zero and variance σ^2. We let the distortion measure be the squared error, i.e., $d(s, \hat{s}) = (s - \hat{s})^2$.

The problem formulation described in Section 4.1 of Lecture 4 still applies, centering around the Markov chain $\underline{S} \leftrightarrow W \leftrightarrow \underline{\hat{S}}$. Although $\underline{S}$ and $\underline{\hat{S}}$ are now continuous in general,* the index W is still integer-valued, drawn from a finite set $\{1, 2, \dots, M_n\}$. So the challenge is to represent a continuous random vector $\underline{S}$ in a finite resolution determined by the rate of a source encoder/decoder pair.

Checking the proof of the converse part of Shannon's fundamental theorem for source coding in Section 4.3, we see that by replacing the entropies with differential entropies for $\underline{S}$ and $\underline{\hat{S}}$, the entire proof carries over verbatim. Therefore, for the considered memoryless Gaussian source, its rate-distortion function $R(D)$ satisfies

$$R(D) \geq R_{\mathrm{I}}(D) = \min_{f_{\hat{S}|S}} I(S; \hat{S}), \quad \text{s.t. } \mathbf{E}[(S - \hat{S})^2] \leq D. \tag{11.1}$$

Note that here we generally need to optimize over the conditional probability density function $f_{\hat{S}|S}$.

Let us minimize $I(S; \hat{S})$ in (11.1) to evaluate $R_{\mathrm{I}}(D)$. First, it is immediate to see that (11.1) is non-trivial only over $D \in [0, \sigma^2]$: if $D > \sigma^2$, we simply set $\hat{S} = 0$ with probability one, leading to $I(S; \hat{S}) = I(S; 0) = 0$ and $\mathbf{E}[(S - \hat{S})^2] = \mathbf{E}[S^2] = \sigma^2 < D$.

* As reproduction, $\hat{S}$ can be either continous or discrete. As will be shown, for the specific setup considered in this section, the optimal choice of $\hat{S}$ that achieves the rate-distortion function is in fact continuous. When the source S is not Gaussian, however, the corresponding rate-distortion function may be achieved when $\hat{S}$ is discrete.

Then, for $D \in [0, \sigma^2]$, we have

$$
\begin{aligned}
I(S;\hat{S}) &= h(S) - h(S|\hat{S}) \\
&\overset{(a)}{=} \frac{1}{2}\log(2\pi e\sigma^2) - h(S - \hat{S}|\hat{S}) \\
&\overset{(b)}{\geq} \frac{1}{2}\log(2\pi e\sigma^2) - h(S - \hat{S}) \\
&\overset{(c)}{\geq} \frac{1}{2}\log(2\pi e\sigma^2) - \frac{1}{2}\log(2\pi e D) \\
&= \frac{1}{2}\log\frac{\sigma^2}{D},
\end{aligned}
\tag{11.2}
$$

where, (a) is obtained by applying the first item of Proposition 10.3, for each $S = s$, (b) is obtained by applying Proposition 10.2, i.e., conditioning reduces differential entropy, and (c) is obtained by applying the fact that Gaussian is the maximum entropy probability density function subject to a variance constraint, i.e., Example 10.6.[†]

It remains to identify some $f_{\hat{S}|S}$ to achieve the lower bound in (11.2). For this, it suffices to set $\hat{S} \sim \mathcal{N}(0, \sigma^2 - D)$, and $Z \sim \mathcal{N}(0, D)$ independent of $\hat{S}$, such that $S = \hat{S} + Z$. This way, the lower bound in (11.2) is achieved.

We still need to check the proof of the achievability part. Going through the proof steps in Section 4.4, we see that we simply need to consider a random codebook $\mathbf{C}$ consisting of i.i.d. $\mathcal{N}(0, \sigma^2 - D)$ random variables, and the proof can be directly adapted.

In summary, we have the following rate-distortion function for a discrete-time memoryless Gaussian source subject to squared error distortion measure:

$$
R(D) = \frac{1}{2}\log\frac{\sigma^2}{D},
\tag{11.3}
$$

if $0 < D < \sigma^2$, and zero otherwise.

Remark 11.1 By considering the inverse of $R(D)$, we obtain the distortion-rate function

$$
D(R) = \sigma^2 2^{-2R},
\tag{11.4}
$$

where R is in bits/source symbol. This indicates the minimum distortion at a given code rate. With $R = 1$ bit/source symbol, the minimum distortion is $\sigma^2/4$, and every additional bit can further reduce the distortion by a factor of 4 (i.e., approximately 6dB).

[†] In fact, Example 10.6 assumes that the considered random variable has zero mean, but here no such constraint is imposed. The reader is therefore invited to slightly modify the solution of Example 10.6 to establish the needed result here.

Remark 11.2 There is a geometric view of $R(D)$ which is heuristically insightful. Consider each codeword $\underline{\hat{s}}$ as a point in $\mathbb{R}^n$. If we let all source vectors $\underline{s}$ that lie within an Euclidean distance $\sqrt{nD}$ to $\underline{\hat{s}}$ be reproduced by $\underline{\hat{s}}$, then the resulting distortion will be upper bounded by D.

Since $S \sim \mathcal{N}(0, \sigma^2)$, from the WLLN, for any $\epsilon > 0$, as $n \to \infty$, $\underline{S}$ lies within two sphere shells: $\mathsf{O}(o, \sqrt{n(\sigma^2 - \epsilon)})$ and $\mathsf{O}(o, \sqrt{n(\sigma^2 + \epsilon)})$, with high probability. Here we use $\mathsf{O}(o, r)$ to denote a sphere centered at the origin with radius r.

So we need at least

$$\begin{aligned} & \frac{\left|\mathsf{O}(o, \sqrt{n(\sigma^2 + \epsilon)})\right| - \left|\mathsf{O}(o, \sqrt{n(\sigma^2 - \epsilon)})\right|}{\left|\mathsf{O}(o, \sqrt{nD})\right|} \\ = \; & \frac{(\sigma^2 + \epsilon)^{n/2} - (\sigma^2 - \epsilon)^{n/2}}{D^{n/2}} \end{aligned} \tag{11.5}$$

codewords to "cover" the region where $\underline{S}$ occurs, with high probability. This ratio can be further written as

$$\begin{aligned} & \left(\frac{\sigma^2 + \epsilon}{D}\right)^{n/2} \cdot \left[1 - \left(\frac{\sigma^2 - \epsilon}{\sigma^2 + \epsilon}\right)^{n/2}\right] \\ = \; & \left(\frac{\sigma^2 + \epsilon}{D}\right)^{n/2} (1 + o(1)), \end{aligned} \tag{11.6}$$

where $o(1) \to 0$ as $n \to \infty$. This hence translates to a code rate of

$$\frac{1}{n} \log \left(\frac{\sigma^2 + \epsilon}{D}\right)^{n/2} (1 + o(1)) = \frac{1}{2} \log \frac{\sigma^2 + \epsilon}{D} + o(1/n), \tag{11.7}$$

which approaches $R(D)$ as $\epsilon \to 0$ and $n \to \infty$.

11.2 Capacity-cost Function of Memoryless Gaussian Channels

We consider a discrete-time channel which adds i.i.d. Gaussian random variables (i.e., "noise") to its input symbols; that is, the channel input-output relationship is

$$Y_i = X_i + Z_i, \tag{11.8}$$

where $Z_i \sim \mathcal{N}(0, \sigma^2)$ and $Z_1, Z_2, \ldots, Z_n$ are mutually independent and independent of $X_1, X_2, \ldots, X_n$. This channel model is called the additive white Gaussian noise (AWGN) channel.

In practical systems, the amplitude of X_i usually corresponds to the signal strength, e.g., current or voltage, and therefore it is reasonable to impose an average power constraint on $X_1, X_2, \ldots, X_n$ as

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \leq P. \tag{11.9}$$

Of course, other forms of constraints may also exist; for example, one may require the maximum magnitude of X_i to be no greater than a peak constraint, and this corresponds to a constraint on the support of X, — the alphabet $\mathcal{X}$ is no longer the entire $\mathbb{R}$ but a bounded subset of it. The resulting channel capacity would usually be very difficult to evaluate, and in this lecture we focus on the average power constraint only.

The problem formulation described in Section 6.1 of Lecture 6 still applies, centering around the Markov chain $W \leftrightarrow \underline{X} \leftrightarrow \underline{Y} \leftrightarrow \hat{W}$, where W and $\hat{W}$ are still integer-valued, drawn from a finite set $\{1, 2, \ldots, M_n\}$.

Checking the proof of the converse part of Shannon's fundamental theorem for channel coding in Section 6.3, we see that by replacing entropies with differential entropies for $\underline{X}$ and $\underline{Y}$, the entire proof carries over verbatim. Therefore, for the considered memoryless Gaussian channel, its capacity-cost function $C(P)$ satisfies

$$C(P) \leq C_{\mathrm{I}}(P) = \max_{f_X} I(X;Y), \quad \text{s.t. } \mathbf{E}[X^2] \leq P. \tag{11.10}$$

Note that here we generally need to optimize over the probability density function f_X.

Let us maximize $I(X;Y)$ in (11.10) to evaluate $C_{\mathrm{I}}(P)$. We start with

$$\begin{aligned} I(X;Y) &= h(Y) - h(Y|X) \\ &\overset{(a)}{=} h(Y) - h(X+Z|X) \\ &\overset{(b)}{=} h(Y) - h(Z|X) \\ &\overset{(c)}{=} h(Y) - h(Z) \\ &= h(Y) - \frac{1}{2}\log(2\pi e \sigma^2), \end{aligned} \tag{11.11}$$

where, (a) is from the channel law $Y = X + Z$, (b) is obtained by applying the first item of Proposition 10.3 of Lecture 10, for each $X = x$, and (c) is because X and Z are independent.

At this point, note that $Y = X + Z$ has a second moment constraint as

$$\begin{aligned}\mathbf{E}[Y^2] &= \mathbf{E}[(X+Z)^2] \\ &= \mathbf{E}[X^2] + 2\mathbf{E}[X]\mathbf{E}[Z] + \mathbf{E}[Z^2] \\ &= \mathbf{E}[X^2] + \mathbf{E}[Z^2] \\ &\leq P + \sigma^2. \end{aligned} \tag{11.12}$$

Therefore, $h(Y)$ is maximized when $Y \sim \mathcal{N}(0, P+\sigma^2)$, following the same argument for the derivation of (11.2), based upon Example 10.6, leading to

$$\begin{aligned} I(X;Y) &= h(Y) - \frac{1}{2}\log(2\pi e\sigma^2) \\ &\leq \frac{1}{2}\log(2\pi e(P+\sigma^2)) - \frac{1}{2}\log(2\pi e\sigma^2) \\ &= \frac{1}{2}\log\left(1+\frac{P}{\sigma^2}\right). \end{aligned} \tag{11.13}$$

Furthermore, equality holds in (11.13) if $X \sim \mathcal{N}(0, P)$.

We still need to check the proof of the achievability part. Going through the proof steps in Section 6.4, we see that we simply need to consider a random codebook **C** whose elements are i.i.d. $\mathcal{N}(0, P)$ random variables, and the proof can be directly adapted.

In summary, we have the following capacity-cost function for a discrete-time memoryless channel subject to average power constraint:

$$C(P) = \frac{1}{2}\log\left(1+\frac{P}{\sigma^2}\right). \tag{11.14}$$

Next we briefly discuss the case where the noise is non-Gaussian. Going through our bounding steps for solving (11.10), we see that, if Z is not necessarily Gaussian, then we can still arrive at

$$I(X;Y) = h(X+Z) - h(Z). \tag{11.15}$$

Now, for any X with differential entropy $h(X)$, we can apply the EPI (Theorem 10.2) to obtain

$$e^{2h(X+Z)} \geq e^{2h(X)} + e^{2h(Z)}, \tag{11.16}$$

and consequently,

$$I(X;Y) \geq \frac{1}{2}\ln\left(1+e^{2(h(X)-h(Z))}\right). \tag{11.17}$$

If Z has mean zero and variance σ^2, then $h(Z) \leq \frac{1}{2}\ln(2\pi e\sigma^2)$ nats, and the lower bound (11.17) becomes

$$I(X;Y) \geq \frac{1}{2}\ln\left(1+\frac{N(X)}{\sigma^2}\right), \tag{11.18}$$

where $N(X)$ is the entropy power of X. If we let X be Gaussian with variance P, then we further obtain

$$I(X;Y) \geq \frac{1}{2}\ln\left(1+\frac{P}{\sigma^2}\right). \tag{11.19}$$

That is, by deviating the additive noise from Gaussian to any non-Gaussian distribution with the same noise power, the channel capacity will not be decreased. In other words, Gaussian noise is the worst-case noise under a noise power constraint.

Remark 11.3 Dual to $R(D)$ in source representation, there is a geometric view of $C(P)$. Consider length-n vectors as points in $\mathbb{R}^n$. The AWGN channel yields $\underline{Y} = \mathsf{C}(w) + \underline{Z}$, for $w \in \{1, 2, \ldots, M_n\}$.

Due to the WLLN, the "length" of an AWGN noise vector $\underline{Z}$ is

$$\begin{aligned}\sqrt{\sum_{i=1}^{n} Z_i^2} &= \sqrt{n}\sqrt{\frac{1}{n}\sum_{i=1}^{n} Z_i^2} \\ &= \sqrt{n}\sqrt{\sigma^2 + o(1)}\end{aligned} \tag{11.20}$$

with high probability, where $o(1) \to 0$ as $n \to \infty$. In other words, for any $\epsilon > 0$, as $n \to \infty$, the vector $\underline{Z}$ lies within a sphere of radius $\sqrt{n(\sigma^2+\epsilon)}$ with high probability.

It is reasonable to let the decoder decode any received signal within the sphere $\mathsf{O}(\mathsf{C}(w), \sqrt{n(\sigma^2+\epsilon)})$ as w. In fact, when transmitting the codeword $\mathsf{C}(w)$, the received signal $\underline{Y}$ is, with high probability, contained between two sphere shells: $\mathsf{O}(\mathsf{C}(w), \sqrt{n(\sigma^2+\epsilon)})$ and $\mathsf{O}(\mathsf{C}(w), \sqrt{n(\sigma^2-\epsilon)})$. Such a decoder will yield reliable decoding result if all the spheres $\mathsf{O}(\mathsf{C}(w), \sqrt{n(\sigma^2+\epsilon)})$, $w \in \{1, 2, \ldots, M_n\}$, do not intersect each other.

On the other hand, due to the average power constraint on $\underline{X}$, the received signal $\underline{Y}$ will be, with high probability, contained within a sphere centered at the origin, of radius $\sqrt{n(P+\sigma^2)}$. The situation is thus like packing radius-$\sqrt{n(\sigma^2+\epsilon)}$ noise spheres within a radius-$\sqrt{n(P+\sigma^2)}$ signal sphere.

Instead of solving the extremely difficult problem of calculating the exact number of spheres that can be packed, we provide a simple estimate of its upper bound:

$$\frac{|\mathsf{O}(o, \sqrt{n(P+\sigma^2)})|}{|\mathsf{O}(o, \sqrt{n(\sigma^2+\epsilon)})|} = \left(1+\frac{P-\epsilon}{\sigma^2+\epsilon}\right)^{n/2}. \tag{11.21}$$

This immediately corresponds to an upper bound on the achievable code rate, as

$$\frac{1}{n}\log\left(1+\frac{P-\epsilon}{\sigma^2+\epsilon}\right)^{n/2}=\frac{1}{2}\log\left(1+\frac{P-\epsilon}{\sigma^2+\epsilon}\right), \tag{11.22}$$

which, as $\epsilon \longrightarrow 0$, is exactly $C(P)$.

This sphere packing argument can be made rigorous by delicate geometric analysis, yileding both achievability and converse proofs of AWGN channel capacity-cost functions; see, e.g., [38].

11.3 Heuristic Discussion of Waveform Gaussian Channels

In the previous section, we have studied discrete-time memoryless Gaussian channels. But in reality, communication takes place in a continuous-time setting. So it remains to be answered how the discrete-time memoryless Gaussian channel model is related to waveform channels.

Our discussion in this section is largely heuristic and mathematically non-rigorous. Consider a waveform channel,

$$Y(t)=X(t)+Z(t), \tag{11.23}$$

where $Z(t)$ is a white noise process of power spectrum density $N_0/2$ watts/Hz, which may be understood as the "derivative" of a Brownian motion, so that at any two time instants t_1 and t_2, $Z(t_1)$ and $Z(t_2)$ are independent, and the autocorrelation function of $Z(t)$ is an impulsive function. Such a process is, of course, pathological and physically unrealizable, but it may be used as an approximation for describing a noise process that has an almost flat power spectrum over a very wide bandwidth.

There are two constraints on the channel input process $X(t)$: first, it has an average power no greater than P watts, and second, it has a bandwidth no greater than W Hz.

According to the Whittaker-Nyquist-Kotelnikov-Shannon sampling theorem, a signal bandlimited to W Hz can be completely reconstructed by sampling the signal every $\frac{1}{2W}$ seconds. Therefore, over a time interval $[0, T]$, after bandlimiting $Y(t)$ within W Hz, we can approximate $Y(t) = X(t) + Z(t)$ as $2WT$ discrete-time memoryless Gaussian channels, with mutually independent Gaussian noises.

Remark 11.4 A delicate issue is that since a signal cannot be simultaneously limited in time and frequency, a signal within bandwidth W Hz has to "leak" beyond the finite time interval $[0, T]$, and vice versa. A rigorous treatment is to carefully define an approximate time-limited or bandwidth-limited signal, and use more sophisticated basis functions than the sinc function used in the preceding sampling theorem argument. Mathematically, there are approximately $2WT$ orthonormal basis functions $\{\phi_1(t), \phi_2(t), \ldots, \phi_{2WT}(t)\}$ that have approximate time span T seconds and frequency span W Hz, such that any $X(t)$ can be approximately written as $X(t) \approx \sum_{i=1}^{2WT} X_i \phi_i(t)$ where $X_i = \int_0^T \phi_i(t) X(t) \mathrm{d}t$, and such that the noise corresponding to X_i is $Z_i = \int_0^T \phi_i(t) Z(t) \mathrm{d}t$, which is Gaussian and is independent of any other Z_j, $j \neq i$. This approximation tends to be accurate as T grows large, and can be made rigorous in the limit of $T \to \infty$ [39] [40].

In the $2WT$ decomposed discrete-time memoryless Gaussian channels $\{Y_i = X_i + Z_i\}_{i=1,\ldots,2WT}$, each channel has its noise obey $\mathcal{N}(0, N_0/2)$, and all the channels have their noises mutually independent. On the other hand, the average input power per decomposed channel is $(PT)/(2WT) = P/(2W)$. So from the previous section, each such decomposed discrete-time memoryless Gaussian channel has a capacity-cost function of

$$\frac{1}{2} \log \left(1 + \frac{P/(2W)}{N_0/2}\right) = \frac{1}{2} \log \left(1 + \frac{P}{N_0 W}\right). \tag{11.24}$$

Now putting together all the $2WT$ decomposed discrete-time memoryless Gaussian channels, and normalizing by time interval length T, the waveform channel has a capacity-cost function as

$$\frac{2WT}{T} \frac{1}{2} \log \left(1 + \frac{P}{N_0 W}\right) = W \log \left(1 + \frac{P}{N_0 W}\right) \quad \text{bits/second.} \tag{11.25}$$

Remark 11.5 From (11.25), we may roughly divide the operation of a waveform channel into two regimes. When $P/N_0 \gg W$,

$$W \log \left(1 + \frac{P}{N_0 W}\right) \approx W \log \frac{P}{N_0 W} \approx W \log \frac{P}{N_0} \quad \text{bits/second,} \tag{11.26}$$

which grows linearly with W. Hence the bandwidth is the bottleneck of performance, and this is called the bandwidth-limited regime. When $W \gg P/N_0$,

$$W \log \left(1 + \frac{P}{N_0 W}\right) \approx \frac{P \log_2 e}{N_0} \quad \text{bits/second,} \tag{11.27}$$

which grows linearly with P but is insensitive to any further increase of W. This is called the power-limited regime.

11.4 Parallel Gaussian Sources or Channels

In this section, we study parallel Gaussian sources or channels; that is, a collection of mutually independent Gaussian sources or channels.

Consider a k-dimensional parallel Gaussian source $\underline{S}$, in which each component is Gaussian and all the components are mutually independent. So $\underline{S}$ is a Gaussian random vector with zero mean and diagonal covariance matrix:

$$\mathbf{K}_S = \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_k^2 \end{bmatrix}. \tag{11.28}$$

We assume that all the diagonal elements of $\mathbf{K}_S$ are strictly positive. We adopt the following squared error distortion measure:

$$d(\underline{s}, \underline{\hat{s}}) = \|\underline{s} - \underline{\hat{s}}\|^2 = \sum_{i=1}^{k} (s_i - \hat{s}_i)^2. \tag{11.29}$$

It can be shown that the rate-distortion function is given by

$$R(D) = \min_{f_{\underline{\hat{S}}|\underline{S}}} I(\underline{S}; \underline{\hat{S}}), \quad \text{s.t.} \sum_{i=1}^{k} \mathbf{E}\left[(S_i - \hat{S}_i)^2\right] \le D. \tag{11.30}$$

When $D \ge \sum_{i=1}^{k} \sigma_i^2$, $R(D) = 0$, and hence we only need to focus on the region of $D < \sum_{i=1}^{k} \sigma_i^2$, over which the distortion constraint is an equality, $\sum_{i=1}^{k} \mathbf{E}\left[(S_i - \hat{S}_i)^2\right] = D$. Similar to the derivation in Section 11.1, we have

$$\begin{aligned} I(\underline{S}; \underline{\hat{S}}) &= h(\underline{S}) - h(\underline{S}|\underline{\hat{S}}) \\ &= \frac{1}{2} \sum_{i=1}^{k} \ln(2\pi e \sigma_i^2) - h(\underline{S} - \underline{\hat{S}}|\underline{\hat{S}}) \\ &\ge \frac{1}{2} \sum_{i=1}^{k} \ln(2\pi e \sigma_i^2) - h(\underline{S} - \underline{\hat{S}}) \\ &\ge \frac{1}{2} \sum_{i=1}^{k} \ln(2\pi e \sigma_i^2) - h\left(\mathcal{N}\left(0, \mathbf{K}_{S-\hat{S}}\right)\right), \end{aligned} \tag{11.31}$$

where $\mathbf{K}_{S-\hat{S}}$ denotes the covariance matrix of $\underline{S} - \underline{\hat{S}}$, and the last inequality is due to Exercise 4 of Lecture 10, i.e., vector Gaussian is the maximum entropy density subject to a covariance constraint, a generalization of Example 10.6 of Lecture 10.

Then,

$$\begin{aligned} I(\underline{S};\underline{\hat{S}}) &\geq \frac{1}{2}\sum_{i=1}^{k}\ln(\sigma_i^2) - \frac{1}{2}\ln|\mathbf{K}_{S-\hat{S}}| \\ &\geq \frac{1}{2}\sum_{i=1}^{k}\ln(\sigma_i^2) - \frac{1}{2}\sum_{i=1}^{k}\ln(\mathbf{E}[(S_i-\hat{S}_i)^2]), \end{aligned} \quad (11.32)$$

where we have used Hadamard's inequality (Exercise 4 of Lecture 10),

$$|\mathbf{K}_{S-\hat{S}}| \leq \prod_{i=1}^{k}\mathbf{K}_{S-\hat{S},ii} = \prod_{i=1}^{k}\mathbf{E}[(S_i-\hat{S}_i)^2]. \quad (11.33)$$

Denoting $D_i = \mathbf{E}[(S_i - \hat{S}_i)^2]$, $i = 1, \ldots, k$, and letting $(S_i, \hat{S}_i)$ be jointly distributed such that $S_i = \hat{S}_i + Z_i$ where Z_i is independent of $\hat{S}_i$ and is Gaussian with mean zero and variance D_i, all the inequalities throughout the preceding derivation hold equal. We thus have the following optimization problem for solving $R(D)$:

$$\min_{\underline{D}} \frac{1}{2}\sum_{i=1}^{k}\ln\frac{\sigma_i^2}{D_i}, \quad \text{s.t.} \sum_{i=1}^{k} D_i = D, 0 < D_i \leq \sigma_i^2, \forall i = 1, \ldots, k. \quad (11.34)$$

This is a standard convex optimization problem, and we may start with the Lagrangian to find its solution:

$$J(\underline{D}, \lambda, \underline{\mu}) = \frac{1}{2}\sum_{i=1}^{k}\ln\frac{\sigma_i^2}{D_i} + \lambda\sum_{i=1}^{k} D_i + \sum_{i=1}^{k}\mu_i(D_i - \sigma_i^2). \quad (11.35)$$

Letting $\partial J/\partial D_i = 0$, we obtain $\lambda + \mu_i = \frac{1}{2D_i}$ if $D_i < \sigma_i^2$; letting $\partial J/\partial \mu_i = 0$, we obtain $D_i = \sigma_i^2$ if $\mu_i > 0$. Therefore, we have $D_i = \frac{1}{2\lambda}$ if $\frac{1}{2\lambda} < \sigma_i^2$, and $D_i = \sigma_i^2$ otherwise; that is,

$$D_i = \min\left\{\frac{1}{2\lambda}, \sigma_i^2\right\}, \quad i = 1, \ldots, k, \quad (11.36)$$

where λ is chosen to satisfy

$$\sum_{i=1}^{k}\min\left\{\frac{1}{2\lambda}, \sigma_i^2\right\} = D. \quad (11.37)$$

This is the optimal solution achieving the rate-distortion function $R(D)$.

Inspecting the optimal solution, we see that it corresponds to a resource allocation scheme with a total budget of D: for those source components with small variances, we do not need to spend resources (i.e., rates) to represent them; for those source components with large variances, we spend resources to represent them, leaving the same amount of "residual errors" (as quantified by $D_i = \frac{1}{2\lambda}$) across them. Such a scheme is called "reverse water-filling".

In a similar fashion, we consider a k-dimensional parallel Gaussian channel $f_{\underline{Y}|\underline{X}}$, whose i-th component is given by $Y_i = X_i + Z_i$, where $Z_i \sim \mathcal{N}(0, \sigma_i^2)$ and all $\{Z_i\}_{i=1,\dots,k}$ are mutually independent. We assume that all the noise variances are strictly positive. The average power constraint P is across these k component channels. It can be shown that the capacity-cost function is given by

$$C(P) = \max_{f_{\underline{X}}} I(\underline{X}; \underline{Y}), \quad \text{s.t.} \sum_{i=1}^{k} \mathbf{E}[X_i^2] \le P. \tag{11.38}$$

We leave it as an exercise to derive that the optimal solution achieving the capacity-cost function $C(P)$ has the following form: $\underline{X}$ is a Gaussian random vector with mean zero and covariance matrix

$$\mathbf{K}_X = \begin{bmatrix} P_1 & & \\ & \ddots & \\ & & P_k \end{bmatrix}. \tag{11.39}$$

where $\underline{P}$ is given by

$$P_i = \max\left\{\frac{1}{2\lambda} - \sigma_i^2, 0\right\}, \quad i = 1, \dots, k, \tag{11.40}$$

and λ is chosen to satisfy

$$\sum_{i=1}^{k} \max\left\{\frac{1}{2\lambda} - \sigma_i^2, 0\right\} = P. \tag{11.41}$$

This optimal solution also corresponds to a resource allocation scheme, with a total budget of P: for those channel components with strong noises, we simply set the transmit power $P_i = 0$ and thus give up transmitting information over them; for those channel components with weak noises, we allocate powers to them, rendering the received powers to have the same level $\frac{1}{2\lambda}$ across these channel components. Such a scheme is called "water-filling".

Notes

Gaussian sources and Gaussian channels have been canonical continuous source and channel models in information theory. The rate-distortion function of the memoryless Gaussian source was established by Shannon in his original paper on the rate-distortion theory [16]. The capacity-cost function of the memoryless Gaussian channel was established by Shannon in his original 1948 article [1]. In his subsequent work [41], the capacity-cost function was studied from a geometric perspective, and was extended to waveform Gaussian channels, whose rigorous treatment appeared later in, e.g., [40] [39].

The water-filling solution of parallel Gaussian channels was studied also by Shannon in [41], and the reverse water-filling solution of parallel Gaussian sources was originally derived in [42].

Exercises

1. Consider a channel with additive exponential noise, $Y = X + Z$, where the noise Z obeys an exponential distribution with mean λ, independent of X, and the input X has support $[0, \infty)$ and a mean constraint $\mathbf{E}X \le \mu$. Calculate the information capacity-cost function $C_{\mathrm{I}}(\mu)$ of this channel.
2. Show that for a Gaussian signal observed via an additive noise channel, when the noise is also Gaussian, the estimation quality is the worst. Let the signal be $X \sim \mathcal{N}(0, P)$, and the noise Z be independent of X with mean zero and variance N. Prove the following inequality:
$$\mathbf{E}[(X - \mathbf{E}[X|X+Z])^2] \le \frac{PN}{P+N}, \tag{11.42}$$
where the equality holds when $Z \sim \mathcal{N}(0, N)$.
3. For a continuous random variable S with mean zero and variance σ^2, consider its information rate-distortion function under the squared error distortion measure, $d(s, \hat{s}) = (s - \hat{s})^2$.
   a) Show that $R_{\mathrm{I}}(D) = 0$ when $D \ge \sigma^2$.
   b) Show that $R_{\mathrm{I}}(D) \ge h(S) - \frac{1}{2}\log(2\pi e D)$ when $D < \sigma^2$.
   c) Show that $R_{\mathrm{I}}(D) \le \frac{1}{2}\log\frac{\sigma^2}{D}$ when $D < \sigma^2$.
4. We have seen that a useful trick for calculating rate-distortion functions is to construct suitable test channels from $\hat{S}$ to S. But in the optimization problem for solving rate-distortion functions, we need to characterize the forward channel from S to $\hat{S}$.
   a) What is the forward channel $P_{\hat{S}|S}$ for a Bernoulli source under Hamming distortion?
   b) What is the forward channel $f_{\hat{S}|S}$ for a Gaussian source under squared error distortion?
   c) Calculate the information rate-distortion function for a Laplace source under absoluate error distortion, i.e., $f_S(s) = \frac{1}{2b}e^{-|s|/b}$, and $d(s, \hat{s}) = |s - \hat{s}|$.
5. Consider a memoryless additive noise channel $Y = X + Z$ where X has support $[-1/2, 1/2]$, and the noise Z is uniform

over $[-1, 1]$, independent of X. Calculate the information capacity of the channel, $C_{\mathrm{I}} = \max_{f_X} I(X; Y)$.

6. Consider a memoryless Gaussian channel $Y = X + Z$, where X has an average power constraint P, and the noise is $Z \sim \mathcal{N}(0, \sigma_Z^2)$. Suppose that, besides Y, the decoder also observes a noisy version of Z, $V = Z + W$ where $W \sim \mathcal{N}(0, \sigma_W^2)$ is independent of Z and X. What is the information capacity-cost function of this channel model?
7. Consider independent Gaussian random variables $X \sim \mathcal{N}(0, \sigma_X^2)$ and $Z \sim \mathcal{N}(0, \sigma_Z^2)$. If there is another random variable V which is only uncorrelated with X, satisfying $\mathbf{E}[XV] = 0$ and $\mathbf{E}[V^2] \leq \sigma_Z^2$, prove that $I(X; X + V) \geq I(X; X + Z)$, and discuss the condition under which equality holds.
8. Consider the parallel Gaussian channel model in Section 11.4.

   a) Derive the water-filling optimal solution.
   b) Show that as $P \longrightarrow \infty$, the rate loss due to using uniform power allocation $P_i = P/k$, $i = 1, \ldots, k$, instead of the water-filling optimal solution, asymptotically vanishes.

9. Consider a channel with two inputs (X_1, X_2) and two outputs (Y_1, Y_2) obeying the following channel law:

$$Y_1 = X_1 + Z_1, \tag{11.43}$$
$$Y_2 = h(X_1) + X_2 + Z_2, \tag{11.44}$$

   where $h(\cdot)$ is a given function, and (Z_1, Z_2) are independent noises. A decoding scheme is as follows: first decode X_1 from Y_1, and then decode X_2 from $Y_2 - h(X_1)$. Use a reasoning based on mutual information analysis to argue that this decoding scheme is suboptimal in general.
10. Consider a joint source channel coding setup where the source is a memoryless Gaussian source with mean zero and variance Q, and the channel is a memoryless Gaussian channel whose Gaussian noise has mean zero and variance σ^2. Let the conversion ratio between source and channel be $r = 1$. Consider an average squared error distortion D for source reproduction and an average input power constraint P for channel transmission.

    a) Identify the fundamental performance limit between D and P.
    b) Show that it is possible to design simple symbol-level mappings to achieve the fundamental performance limit.
    c) Verify that the "double matching" conditions in Section 6.5 hold for the designed symbol-level mappings.