

数理方程 B 第一章参考答案

1. 解:

(1) 首先极坐标形式下的拉普方程为

$$\Delta_2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

由于球形如 $u = u(r)$, ($r = \sqrt{x^2 + y^2} \neq 0$) 的解, 因此 u 的自变量只有 r , 则有

$$\Delta_2 u = \frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} = 0, (r \neq 0)$$

$$\text{亦即 } r^2 \frac{d^2 u}{dr^2} + r \frac{du}{dr} = 0$$

该方程为一欧拉方程.

做变量替换 $r = e^t$, 则有

$$\frac{du}{dr} = \frac{du}{dt} \frac{dt}{dr} = \frac{1}{r} \frac{du}{dt} \Rightarrow r \frac{du}{dr} = \frac{du}{dt}$$

$$\begin{aligned} \frac{d^2 u}{dr^2} &= \frac{d\left(\frac{1}{r} \frac{du}{dt}\right)}{dr} = -\frac{1}{r^2} \frac{du}{dt} + \frac{1}{r} \frac{d\left(\frac{du}{dt}\right)}{dr} = -\frac{1}{r^2} \frac{du}{dt} + \frac{1}{r} \frac{d\left(\frac{du}{dt}\right)}{dt} \frac{dt}{dr} \\ &= -\frac{1}{r^2} \frac{du}{dt} + \frac{1}{r^2} \frac{d^2 u}{dt^2} \Rightarrow r^2 \frac{d^2 u}{dr^2} = -\frac{du}{dt} + \frac{d^2 u}{dt^2} \end{aligned}$$

所以有

$$\frac{d^2 u}{dt^2} = 0$$

故其通解为

$$u(t) = C_1 + C_2 t \Rightarrow u(r) = C_1 + C_2 \ln r, C_1, C_2 \text{ 为常数}, r > 0$$

(2) 依题, 首先将方程化成如下形式:

$$\frac{d^2 u}{dr^2} + \frac{2}{r} \frac{du}{dr} + k^2 u = 0, (k \text{ 为正常数})$$

两边同乘以 r , 得到

$$r \frac{d^2 u}{dr^2} + \frac{2du}{dr} + k^2 ur = 0$$

设

$$f(r) = ur$$

则

$$\begin{aligned} \frac{df}{dr} &= r \frac{du}{dr} + u \\ \frac{d^2 f}{dr^2} &= \frac{du}{dr} + r \frac{d^2 u}{dr^2} + \frac{du}{dr} = r \frac{d^2 u}{dr^2} + \frac{2du}{dr} \end{aligned}$$

所以

$$\frac{d^2 f}{dr^2} + k^2 ur = 0$$

其特征方程为

$$\lambda^2 + k^2 = 0$$

k 为正常数, 故其有两个纯虚的根

$$\lambda_1 = -ki, \lambda_2 = ki$$

方程通解为

$$f = C_1 \cos kr + C_2 \sin kr \Rightarrow ur = C_1 \cos kr + C_2 \sin kr, C_1, C_2 \text{ 为常数}$$

故原方程的通解为

$$u = \frac{1}{r} (C_1 \cos kr + C_2 \sin kr), C_1, C_2 \text{ 为常数}, r > 0$$

2. 解:

$$u_x = F' + G'$$

$$u_y = \lambda_1 F' + \lambda_2 G'$$

$$u_{xx} = F'' + G''$$

$$u_{yy} = \lambda_1^2 F'' + \lambda_2^2 G''$$

$$u_{xy} = \lambda_1 F'' + \lambda_2 G''$$

所以, 带入题目中方程左边, 有

$$\lambda_1^2 F'' + \lambda_2^2 G'' - (\lambda_1 + \lambda_2)(\lambda_1 F'' + \lambda_2 G'') + \lambda_1 \lambda_2 (F'' + G'')$$

展开得到上式为 0, 满足方程.

3. 解:

$$u_t = -\frac{1}{2} t^{-\frac{3}{2}} \exp \left\{ -\frac{(x-\xi)^2}{4a^2 t} \right\} + t^{-\frac{1}{2}} \frac{(x-\xi)^2}{4a^2 t^2} \exp \left\{ -\frac{(x-\xi)^2}{4a^2 t} \right\}$$

化简得

$$u_t = -\frac{1}{2} t^{-\frac{3}{2}} \exp \left\{ -\frac{(x-\xi)^2}{4a^2 t} \right\} + t^{-\frac{5}{2}} \frac{(x-\xi)^2}{4a^2} \exp \left\{ -\frac{(x-\xi)^2}{4a^2 t} \right\}$$

$$u_x = t^{-\frac{1}{2}} \left(-\frac{x-\xi}{2a^2 t} \right) \exp \left\{ -\frac{(x-\xi)^2}{4a^2 t} \right\} = -\frac{1}{2} t^{-\frac{3}{2}} \frac{x-\xi}{a^2} \exp \left\{ -\frac{(x-\xi)^2}{4a^2 t} \right\}$$

$$u_{xx} = -\frac{1}{2a^2} t^{-\frac{3}{2}} \exp \left\{ -\frac{(x-\xi)^2}{4a^2 t} \right\} + \left(-\frac{1}{2} t^{-\frac{3}{2}} \frac{x-\xi}{a^2} \right) \left(-\frac{x-\xi}{2a^2 t} \right) \exp \left\{ -\frac{(x-\xi)^2}{4a^2 t} \right\}$$

化简得

$$u_{xx} = -\frac{1}{2a^2} t^{-\frac{3}{2}} \exp \left\{ -\frac{(x-\xi)^2}{4a^2 t} \right\} + t^{-\frac{5}{2}} \frac{(x-\xi)^2}{4a^4} \exp \left\{ -\frac{(x-\xi)^2}{4a^2 t} \right\}$$

将其带入, 可知其满足方程.

$$\lim_{t \rightarrow 0} u(t, x) = \lim_{t \rightarrow 0} \frac{1}{\sqrt{t}} \exp \left\{ -\frac{(x-\xi)^2}{4a^2 t} \right\}, (x \neq \xi)$$

该极限只与 t 有关，故其他变量可视为常量，则设

$$\frac{(x-\xi)^2}{4a^2} = k > 0, k \text{ 与 } t \text{ 无关}$$

极限可化为

$$\lim_{t \rightarrow 0} \frac{1}{e^{\frac{k}{t}} \sqrt{t}} \xrightarrow{\frac{1}{t}=h, t>0} \lim_{h \rightarrow +\infty} \frac{\sqrt{h}}{e^{kh}} = 0$$

4. 解:

$$u = axe^{2x+y}$$

则

$$u_x = ae^{2x+y} + 2axe^{2x+y}$$
$$u_{xx} = 2ae^{2x+y} + 2ae^{2x+y} + 4axe^{2x+y} = 4ae^{2x+y}(1+x)$$

$$u_y = axe^{2x+y}$$

$$u_{yy} = axe^{2x+y}$$

所以

$$u_{xx} - 4u_{yy} = 4ae^{2x+y} = e^{2x+y} \Rightarrow a = \frac{1}{4}$$

故所求特解为

$$u = \frac{1}{4}xe^{2x+y}$$

5. 解:

$$u_x = yf'$$
$$u_y = xf'$$

带入方程即可知满足

6. 解:

(1) 先考虑一阶线性微分方程通解问题, 即

$$\frac{dy}{dx} + P(x)y = Q(x)$$

的通解为

$$y = e^{-\int P(x)dx} \left(\int Q(x)e^{\int P(x)dx} dx + C \right), C \text{ 为常数}$$

则此题可看成 u 关于 y 的一阶线性微分方程, 故

$$u = C(x)e^{-\int a(x,y)dy}$$

注: 此题为二变量方程和题设中 u 有 x, y, z 三个变量有些不符, 故“积分常数”含 x 或 x, z , 写哪一种都行。

(2) 设

$$u_y = h(x, y)$$

则

$$u_{xy} = u_{yx} = h_x$$

注: 本门课程中涉及到的混合偏导在没有特殊说明的情况下都是可交换的

所以方程化为

$$h_x + h = 0$$

解得

$$h = C_1(y)e^{-x} \Rightarrow u_y = C_1(y)e^{-x} \Rightarrow u = e^{-x}C(y) + D(x)$$

(3) 此题用到了叠加原理

书中提示让我们先求一个形如 $v(x)$ 的特解, 设其通解 $u(x, t) = v(x) + w(x, t)$ 则将方程分解为

$$\begin{aligned} 0 &= v_{tt} = a^2 v_{xx} + 3x^2 \\ w_{tt} &= a^2 w_{xx} \end{aligned}$$

其中第一个方程通解为

$$v(x) = -\frac{1}{4a^2}x^4 + Cx$$

其中第二个方程通解

$$w(x, t) = f(x + at) + g(x - at)$$

故原方程通解为

$$u(x, t) = -\frac{1}{4a^2}x^4 + Cx + f(x + at) + g(x - at)$$

注: 所求特解的形式不唯一, 选取一种即可, 本题中亦可直接写 $v(x) = -\frac{1}{4a^2}x^4$.

7. 解:

先确定总的方程, 与热有关的有热传导和场势方程, 此题中有热的变化故选取热传导方程, 物体内部没有热源所以选择其次形式。题干也给出了一个边界条件. 再接着“翻译”物理过程, 特别要注意傅里叶定律当中方向导数的求法。

$$(1) \quad \begin{cases} u_t = a^2 u_{xx}, (t \geq 0, 0 \leq x \leq l) \\ u(0, x) = \varphi(x) \\ u_x(t, 0) = 0 \\ u(t, l) = u_0 \end{cases}$$

(2) 不妨考虑左端流进 q_1 , 右端流进 q_2 , 则所得方程及边界条件为:
傅里叶定律:

$$\begin{aligned} & -k \frac{\partial u}{\partial \vec{n}} (\text{流出}) + q (\text{流入}) = 0 \\ & \begin{cases} u_t = a^2 u_{xx}, (t \geq 0, 0 \leq x \leq l) \\ u(0, x) = \varphi(x) \\ -k \frac{\partial u}{\partial \vec{n}}|_{x=0} = -k[-u_x(t, 0)] = ku_x(t, 0) \Rightarrow ku_x(t, 0) + q_1 = 0 \Rightarrow -ku_x(t, 0) = q_1 \\ -k \frac{\partial u}{\partial \vec{n}}|_{x=l} = -ku_x(t, l) \Rightarrow -ku_x(t, l) + q_2 = 0 \Rightarrow ku_x(t, l) = q_2 \end{cases} \end{aligned}$$

(3) 由牛顿冷却定律, 得到方程及边界条件为:

$$\begin{cases} u_t = a^2 u_{xx}, (t \geq 0, 0 \leq x \leq l) \\ u(0, x) = \varphi(x) \\ u(t, 0) = \mu(t) \\ -k \frac{\partial u}{\partial n} \big|_{x=l} = -k u_x(t, l) = h(l)[u(t, l) - \theta(t)] \Rightarrow (k u_x + h u) \big|_{x=l} = h(l) \theta(t) \end{cases}$$

8. 解:

先考虑无条件的一维弦振动, 列出方程

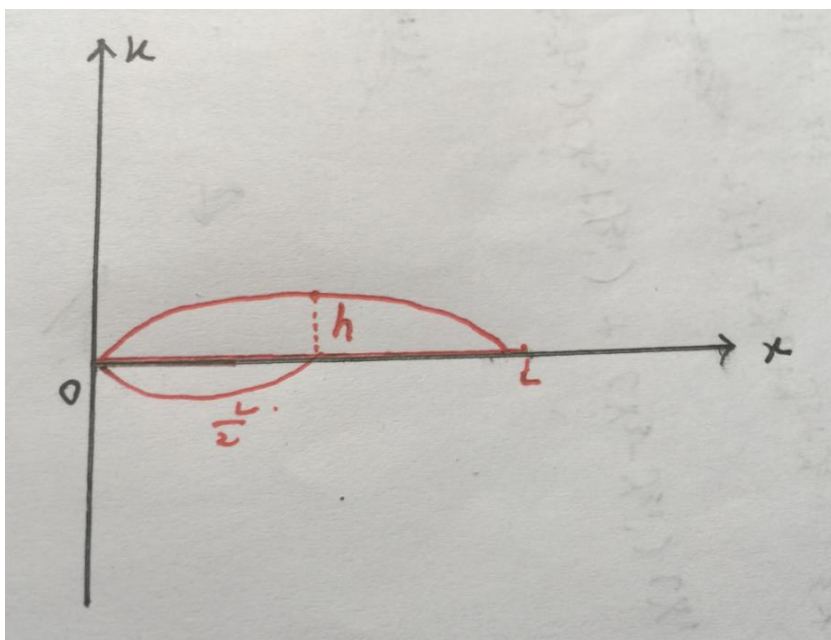
$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \quad (t \geq 0, 0 \leq l \leq x)$$

考虑第一类边界条件, 得

$$u(t, 0) = 0$$

$$u(t, l) = 0$$

如图所示



由于弦做微小横振动, 故可将左右两个曲边三角形看成三角形, 由三角形相似, 得

$$u(0, x) = \frac{2h}{l} x \quad 0 \leq x \leq \frac{l}{2}$$

$$u(0, x) = \frac{2h}{l} (l - x) \quad \frac{l}{2} < x \leq l$$

考虑第二类边界条件, 得

$$u_t(0, x) = 0$$

所以所得定解问题为:

$$\left\{ \begin{array}{l} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \quad (t > 0, 0 \leq l \leq x) \\ u(0, x) = \frac{2h}{l}x \quad 0 \leq x \leq \frac{l}{2} \\ u(0, x) = \frac{2h}{l}(l-x) \quad \frac{l}{2} < x \leq l \\ u(t, 0) = 0 \\ u(t, l) = 0 \\ u_t(0, x) = 0 \end{array} \right.$$

9. 解:

(1) 由

$$u_t = x^2 \Rightarrow u = x^2 t + C(x) \xrightarrow{u(0,x)=x^2} C(x) = x^2$$

所以

$$u = x^2(t+1)$$

(2) 由球对称可知, u 的空间分布只与半径 r 有关, 故可得方程为

$$\left\{ \begin{array}{l} u_{tt} = a^2 \left(u_{rr} + \frac{2}{r} u_r \right) \\ u(0, r) = \varphi(r) \\ u_t(0, r) = \psi(r) \end{array} \right.$$

根据提升, 设

$$v(t, r) = ru$$

则方程可化为

$$\left\{ \begin{array}{l} v_{tt} = a^2 v_{rr} \\ v(0, r) = r\varphi(r) \\ v_t(0, r) = r\psi(r) \end{array} \right.$$

由达朗贝尔公式, 得

$$v(t, r) = \frac{(r+at)\varphi(r+at) + (r-at)\varphi(r-at)}{2} + \frac{1}{2a} \int_{r-at}^{r+at} \xi \psi(\xi) d\xi$$

所以

$$u(t, r) = \frac{(r+at)\varphi(r+at) + (r-at)\varphi(r-at)}{2r} + \frac{1}{2ar} \int_{r-at}^{r+at} \xi \psi(\xi) d\xi$$

(3) 根据提示, 直接研究边界条件

$$\begin{aligned} u &= \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} \\ &\xrightarrow{x^2+y^2+z^2=1} u = \frac{1}{\sqrt{1-2xx_0-2yy_0-2zz_0+x_0^2+y_0^2+z_0^2}} \\ &= \frac{1}{\sqrt{5+4y}} \end{aligned}$$

通过对比系数,得到

$$\begin{cases} x_0 = 0 \\ y_0 = -2 \\ z_0 = 0 \end{cases}$$

满足

$$x_0^2 + y_0^2 + z_0^2 > 1$$

故

$$u = \frac{1}{\sqrt{x^2 + (y+2)^2 + z^2}}$$

(4) 设

$$\begin{cases} \xi = x + t \\ \eta = x - t \\ \begin{cases} u_t = u_\xi - u_\eta \\ u_{tt} = u_{\xi\xi} + u_{\eta\eta} \end{cases} \\ \begin{cases} u_x = u_\xi + u_\eta \\ u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \end{cases} \end{cases}$$

则原方程化为:

$$\begin{cases} u_{\xi\eta} = 0 \\ u(0, \eta) = \varphi(x) \\ u(\xi, 0) = \psi(x) \\ \varphi(0) = \psi(0) \end{cases}$$

由

$$u_{\xi\eta} = 0 \Rightarrow u = f(\xi) + g(\eta)$$

$$\begin{aligned} u(0, \eta) &= f(0) + g(\eta) \xrightarrow{\xi=x+t=0} f(0) + g(2x) = \varphi(2x) \Rightarrow g(x) \\ &= \varphi\left(\frac{x}{2}\right) - f(0) \end{aligned}$$

$$\begin{aligned} u(\xi, 0) &= f(\xi) + g(0) \xrightarrow{\eta=x-t=0} f(\xi) + g(0) = \psi(2x) \Rightarrow f(x) \\ &= \psi\left(\frac{x}{2}\right) - g(0) \end{aligned}$$

另由

$$\varphi(0) = \psi(0)$$

可得

$$\varphi(0) = \psi(0) = f(0) + g(0)$$

所以

$$\begin{aligned} u &= f(\xi) + g(\eta) = \psi\left(\frac{x+t}{2}\right) + \varphi\left(\frac{x-t}{2}\right) - [f(0) + g(0)] \\ &= \psi\left(\frac{x+t}{2}\right) + \varphi\left(\frac{x-t}{2}\right) - \varphi(0) \end{aligned}$$

10. 解: 提示有误, 应该是设

$$\begin{cases} \xi = x - at \\ \eta = t \end{cases}$$

依题, 方程可化为

$$\begin{cases} \frac{\partial u}{\partial t} = -a \frac{\partial u}{\partial x} + f(t, x) (t > 0, -\infty < x < +\infty) \\ u(0, x) = \varphi(x) (a \neq 0, a \text{ 为常数}) \end{cases}$$

设

$$u(t, x) = v(t, x) + w(t, x)$$

$v(t, x), w(t, x)$ 分别满足

$$\begin{cases} \frac{\partial v}{\partial t} = -a \frac{\partial v}{\partial x} \\ v(0, x) = \varphi(x) \end{cases}$$

$$\begin{cases} \frac{\partial w}{\partial t} = -a \frac{\partial w}{\partial x} + f(t, x) \\ w(0, x) = 0 \end{cases}$$

关于 $v(t, x)$ 为齐次方程, 根据变量替换有

$$\frac{\partial v}{\partial t} = -a \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \xi}$$

则方程可化为

$$\begin{cases} \frac{\partial v}{\partial \eta} = 0 \\ v(0, x) = \varphi(x) \end{cases}$$

则有

$$\begin{cases} v = f(\xi) = f(x - at) \\ v(x) = \varphi(x) \end{cases}$$

所以

$$v(t, x) = \varphi(x - at)$$

关于 $w(t, x)$ 为齐次边界条件方程, 则利用齐次化原理求解

则设有函数 $g(t, x, \tau)$ 满足

$$\begin{cases} \frac{\partial g}{\partial t} = -a \frac{\partial g}{\partial x} \\ g(t, x, \tau)|_{t=\tau} = f(\tau, x) \end{cases}$$

根据提示的变量代换, 设

$$\begin{cases} \xi = x - at \\ \eta = t \\ \zeta = \tau \end{cases}$$

则得

$$\begin{cases} \frac{\partial g}{\partial \eta} = 0 \\ g(t, x, \tau)|_{t=\tau} = f(\tau, x) \end{cases}$$

则有

$$\begin{cases} g = p(\xi, \zeta) = p(x - at, \tau) \\ p(x - at, \tau)|_{t=\tau} = f(\tau, x) \end{cases}$$

$$p(x - at, \tau)|_{t=\tau} = p(x - a\tau, \tau) = f(\tau, x) \xrightarrow{x=x-a(t-\tau)} g(t, x, \tau) = p(x - at, \tau) \\ = f[\tau, x - a(t - \tau)]$$

根据齐次化原理

$$w(t, x) = \int_0^t g(t, x, \tau) d\tau = \int_0^t f[\tau, x - a(t - \tau)] d\tau$$

所以原方程通解为

$$u(t, x) = \varphi(x - at) + \int_0^t f[\tau, x - a(t - \tau)] d\tau$$

11. 解: 根据达朗贝尔公式

$$u(t, x) = \frac{\varphi(x + at) + \varphi(x - at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi(\xi) d\xi$$

(1) 依题

$$u(t, 0) = \frac{\varphi(at) + \varphi(-at)}{2} + \frac{1}{2a} \int_{-at}^{at} \psi(\xi) d\xi = 0$$

(2) 依题

$$u_x(t, x) = \frac{\varphi'(x + at) + \varphi'(x - at)}{2} + \frac{1}{2a} [\psi(x + at) - \psi(x - at)]$$

$\varphi'(x)$ 为奇函数

$$u_x(t, 0) = \frac{\varphi'(at) + \varphi'(-at)}{2} + \frac{1}{2a} [\psi(at) - \psi(-at)] = 0$$

12. 解:

作奇延拓, 则有

$$\begin{cases} \tilde{u}_{tt} = a^2 \tilde{u}_{xx} (-\infty < x < +\infty, t > 0) \\ \tilde{u}(0, x) = \sin x, \tilde{u}_t(0, x) = kx \\ \tilde{u}(t, 0) = 0 \end{cases}$$

则由达朗贝尔公式

$$\tilde{u}(t, x) = \frac{\sin(x + at) + \sin(x - at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} k\xi d\xi = \sin x \cos at + ktx$$

所以

$$u(t, x) = \sin x \cos at + ktx (x > 0, t > 0)$$

第二章作业参考解答

刘炜昊

2021 年 4 月 6 日

目录

目录	1
1 课本 $P_{252} T_1$	2
2 课本 $P_{252} T_2$	4
3 课本 $P_{252} T_3$	6
4 课本 $P_{252} T_4$	8
5 课本 $P_{252} T_5$	9
6 课本 $P_{253} T_6$	16
7 课本 $P_{253} T_7$	18
8 课本 $P_{254} T_8$	21
9 课本 $P_{254} T_9$	23
10 课本 $P_{254} T_{10}$	24
11 课本 $P_{255} T_{11}$	32
12 课本 $P_{255} T_{12}$	33
13 写在最后	37

1 课本 P_{252} T_1

求方程 $y'' + \lambda y = 0 (0 < x < l)$ 在下列边界条件下的固有值和固有函数:

$$(1) y'(0) = 0, y(l) = 0;$$

$$(2) y'(0) = 0, y'(l) + hy(l) = 0;$$

$$(3) y'(0) - ky(0) = 0, y'(l) + hy(l) = 0 (k, h > 0).$$

Sol:

由 S-L 定理可知: $\lambda \geq 0$, $\lambda = 0$ 当且仅当两端均为第 II 类边界条件. (或分类讨论得出 $\lambda \geq 0$ 且仅第 II 类边界条件时 $\lambda = 0$, 否则为零解)

(1) 对于问题

$$\begin{cases} y''(x) + \lambda y(x) = 0 \\ y'(0) = 0, y(l) = 0 \end{cases}$$

可令 $\lambda \stackrel{\text{def}}{=} k^2 > 0$, 则得到 $y(x) = A \cos kx + B \sin kx, 0 \leq x \leq l$.

由 $y'(0) = kB = 0$ 得到 $B = 0$, 则 $y(x) = A \cos kx, 0 \leq x \leq l$.

由 $y(l) = A \cos kl = 0$, 欲求非零解则 $A \neq 0$, 故 $kl = \frac{\pi}{2} + n\pi, n = 0, 1, 2, \dots$

因此 $k_n = \frac{2n+1}{2l}\pi$, 则固有值为 $\lambda_n = \left(\frac{2n+1}{2l}\pi\right)^2$, 固有函数为 $y_n(x) = \cos \frac{(2n+1)\pi x}{2l}, n \in \mathbb{N}$.

(2) 对于问题

$$\begin{cases} y''(x) + \lambda y(x) = 0 \\ y'(0) = 0, y'(l) + hy(l) = 0 \end{cases}$$

可令 $\lambda \stackrel{\text{def}}{=} k^2 > 0$, 则得到 $y(x) = A \cos kx + B \sin kx, 0 \leq x \leq l$.

由 $y'(0) = kB = 0$ 得到 $B = 0$, 则 $y(x) = A \cos kx, 0 \leq x \leq l$.

由 $y'(l) + hy(l) = -kA \sin kl + A \cos kl = 0 \Rightarrow -A(k \sin kl - h \cos kl) = -A\sqrt{k^2 + h^2} \sin(kl - \varphi) = 0$, 其中 $\tan \varphi = \frac{h}{k}, \varphi = \arctan \frac{h}{k}$

欲求非零解则 $A\sqrt{k^2 + h^2} \neq 0$, 故 $k_n l - \arctan \frac{h}{k_n} = n\pi$, 固有值为 $\lambda_n = k_n^2$, 固有函数为 $y_n(x) = \cos k_n x, n = 0, 1, 2, \dots$

(3) 对于问题

$$\begin{cases} y''(x) + \lambda y(x) = 0 \\ y'(0) - ky(0) = 0, y'(l) + hy(l) = 0 (k, h > 0) \end{cases}$$

可令 $\lambda \stackrel{\text{def}}{=} \mu^2 > 0$, 则得到 $y(x) = A \cos \mu x + B \sin \mu x, 0 \leq x \leq l$.

根据题目条件可得

$$\begin{cases} y'(0) - ky(0) = \mu B - kA = 0 \\ y'(l) + hy(l) = -\mu A \sin \mu l + \mu B \cos \mu l + hA \cos \mu l + hB \sin \mu l = 0 \end{cases}$$

$$\Rightarrow \begin{pmatrix} k & -\mu \\ h \cos \mu l - \mu \sin \mu l & h \sin \mu l + \mu \cos \mu l \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

欲使 A 和 B 有非零解, 应有系数行列式为 0:

$$\begin{vmatrix} k & -\mu \\ h \cos \mu l - \mu \sin \mu l & h \sin \mu l + \mu \cos \mu l \end{vmatrix} = (kh - \mu^2) \sin \mu l + (k\mu + \mu h) \cos \mu l = 0$$

得到 μ_n 满足:

$$(kh - \mu_n^2) \tan \mu_n l + \mu_n(k + h) = 0, n = 1, 2, \dots$$

因此固有值为 $\lambda_n = \mu_n^2$, 固有函数为 $y_n(x) = \cos \mu_n x + \frac{k}{\mu_n} \sin \mu_n x, n = 1, 2, \dots$.

2 课本 $P_{252} T_2$

解下列固有值问题:

- (1) $\begin{cases} y'' - 2ay' + \lambda y = 0 & (0 < x < 1, a \text{ 为常数}), \\ y(0) = y(1) = 0; \end{cases}$
- (2) $\begin{cases} (r^2 R')' + \lambda r^2 R = 0 & (0 < r < a), \\ |R(0)| < +\infty, R(a) = 0; \end{cases}$ [提示: 令 $y = rR$.]
- (3) $\begin{cases} y^{(4)} + \lambda y = 0 & (0 < x < l), \\ y(0) = y(l) = y''(0) = y''(l) = 0 \end{cases}$

Sol:

(1) 解特征方程: $u^2 - 2au + \lambda = 0$, 记 $\Delta \stackrel{\text{def}}{=} a^2 - \lambda$.

若 $\lambda < a^2$, 则 $\Delta > 0, u_{1,2} = a \pm \sqrt{\Delta} \in \mathbb{R}, y = Ae^{u_1 x} + Be^{u_2 x}$, 故

$$\begin{cases} y(0) = A + B = 0 \\ y(1) = Ae^{u_1} + Be^{u_2} = 0 \end{cases} \Rightarrow A = B = 0.$$

若 $\lambda = a^2$, 则 $\Delta = 0, u_{1,2} = a \in \mathbb{R}, y = Ae^{ax} + Bxe^{ax}$, 故

$$\begin{cases} y(0) = A = 0 \\ y(1) = Ae^a + Be^a = 0 \end{cases} \Rightarrow A = B = 0.$$

故 $\lambda > a^2$, 则 $\Delta < 0, u_{1,2} = a \pm i\sqrt{-\Delta}, y = e^{ax}(A \cos \sqrt{\lambda - a^2}x + B \sin \sqrt{\lambda - a^2}x)$, 故

$$\begin{cases} y(0) = A = 0 \\ y(1) = e^a(A \cos \sqrt{\lambda - a^2} + B \sin \sqrt{\lambda - a^2}) = 0 \end{cases} \Rightarrow A = 0, \lambda_n - a^2 = (n\pi)^2, n = 1, 2, \dots$$

故固有值为 $\lambda_n = (n\pi)^2 + a^2$, 固有值函数为 $y_n(x) = e^{ax} \sin n\pi x, n \in \mathbb{N}_+$.

(2) 令 $y(r) = rR(r)$, 则 $R(r) = \frac{y(r)}{r}$, 那么 $R'(r) = \frac{y'(r)}{r} - \frac{y(r)}{r^2}, R''(r) = \frac{y''(r)}{r} - \frac{2y'(r)}{r^2} + \frac{2y(r)}{r^3}$.

代入泛定方程得: $rR''(r) + 2R'(r) + \lambda rR(r) = y''(r) + \lambda y(r) = 0, 0 < r < a$, 即

$$\begin{cases} y''(r) + \lambda y(r) = 0, 0 < r < a \\ y(0) = y(a) = 0 \end{cases}$$

由 Sturm-Liouville 定理可知 $\lambda > 0$, 可令 $\lambda \stackrel{\text{def}}{=} k^2 > 0$, 则 $y(r) = A \cos kr + B \sin kr$.

由 $y(0) = A = 0$ 可得 $y(r) = B \sin kr$; 由 $y(a) = B \sin ka = 0$, 欲求得非零解, 则 $B \neq 0, k_n a = n\pi, n = 1, 2, \dots$

因此固有值为 $\lambda_n = \left(\frac{n\pi}{a}\right)^2$, 固有函数为 $R_n(r) = \frac{1}{r} \sin \frac{n\pi r}{a}, n \in \mathbb{N}_+$.

(3) 若 $\lambda > 0$, 令 $\lambda \stackrel{\text{def}}{=} \omega^4, (\omega > 0)$, 这时特征根为两对共轭复根:

$$k_{1,2} = \omega \left(\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i \right), k_{3,4} = \omega \left(\frac{-\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i \right)$$

因此相应的泛定方程解为

$$y(x) = C_1 e^{\frac{\sqrt{2}}{2}\omega x} \cos \frac{\sqrt{2}}{2}\omega x + C_2 e^{\frac{\sqrt{2}}{2}\omega x} \sin \frac{\sqrt{2}}{2}\omega x + C_3 e^{-\frac{\sqrt{2}}{2}\omega x} \cos \frac{\sqrt{2}}{2}\omega x + C_4 e^{-\frac{\sqrt{2}}{2}\omega x} \sin \frac{\sqrt{2}}{2}\omega x$$

代入边界条件可得: $C_1 = C_2 = C_3 = C_4 = 0$, 因此当 $\lambda > 0$ 时无相应的固有值.

若 $\lambda = 0$, 则 $y(x) = C_3 x^3 + C_2 x^2 + C_1 x + C_0$, 则

$$\begin{cases} y(0) = C_0 = 0 \\ y(l) = l^3 C_3 + l^2 C_2 + l C_1 + C_0 = 0 \\ y''(0) = 2C_2 = 0 \\ y''(l) = 6l C_3 + 2C_2 = 0 \end{cases} \Rightarrow C_1 = C_2 = C_3 = C_4 = 0 \text{ (得到零解)}$$

故 $\lambda < 0$, 令 $\lambda \stackrel{\text{def}}{=} -\omega^4$, 这样特征方程存在 4 个特征根: $k_{1,2} = \pm i\omega, k_{3,4} = \pm \omega$

因此可得到: $y(x) = C_1 e^{\omega x} + C_2 e^{-\omega x} + C_3 \cos \omega x + C_4 \sin \omega x$.

代入边界条件 $y(0) = y(l) = y''(0) = y''(l) = 0$, 分别得到:

$$\begin{aligned} C_1 + C_2 + C_3 &= 0, C_1 + C_2 - C_3 = 0 \\ C_1 e^{\omega l} + C_2 e^{-\omega l} + C_3 \cos \omega l + C_4 \sin \omega l &= 0 \\ C_1 e^{\omega l} + C_2 e^{-\omega l} - C_3 \cos \omega l - C_4 \sin \omega l &= 0 \end{aligned}$$

解得 $C_1 = C_2 = C_3 = 0, C_4 \sin \omega l = 0$

所以欲求非零解, 只有 $C_4 \neq 0$, 因此 $\sin \omega l = 0 \Rightarrow \omega l = n\pi \Rightarrow \omega = \frac{n\pi}{l}, n = 1, 2, \dots$

因此可得到固有值为 $\lambda_n = -\left(\frac{n\pi}{l}\right)^4$, 固有函数为 $y_n(x) = \sin \frac{n\pi x}{l}, n \in \mathbb{N}_+$.

3 课本 $P_{252} T_3$

一条均匀的弦固定于 $x = 0$ 及 $x = l$, 在开始的一瞬间, 它的形状是一条以 $(\frac{l}{2}, h)$ 为顶点的抛物线, 初速度为零, 且没有外力作用, 求弦作横振动的位移函数.

Sol:

设位移函数为 $u(t, x)$, 其满足的方程为

$$\begin{cases} u_{tt} = a^2 u_{xx} \quad (t > 0, 0 < x < l). \\ u(t, 0) = u(t, l) = 0 \\ u(0, x) = \frac{4h}{l^2} x(l-x), u_t(0, x) = 0 \end{cases}$$

令 $u(t, x) = T(t)X(x)$, 分离变量得: $\frac{1}{a^2} \frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$, 得到固有值问题:

$$\begin{cases} x'' + \lambda x = 0 \\ X(0) = X(l) = 0 \end{cases}$$

由 Sturm-Liouville 定理可知 $\lambda > 0$, 令 $\lambda \stackrel{\text{def}}{=} k^2 > 0$, 得到

$$X(x) = A \cos kx + B \sin kx.$$

由 $X(0) = A = 0$, 此时 $X(x) = B \sin kx$; 由 $X(l) = B \sin kl = 0$, 欲求得非零解则 $B \neq 0$, 故 $k_n l = n\pi, n = 1, 2, \dots$

因此固有值为 $\lambda_n = k_n^2 = \left(\frac{n\pi}{l}\right)^2$, 固有函数为 $X_n(x) = \sin \frac{n\pi x}{l}, n \in \mathbb{N}_+$.

随后解关于 $T(t)$ 的 ODE: $T''(t) + a^2 \lambda_n T(t) = 0$, 得到 $T_n(t) = A_n \cos \frac{n\pi a t}{l} + B_n \sin \frac{n\pi a t}{l}$.

叠加得到: $u(t, x) = \sum_{n=1}^{+\infty} \left(A_n \cos \frac{n\pi a t}{l} + B_n \sin \frac{n\pi a t}{l} \right) \sin \frac{n\pi x}{l}, t \geq 0, 0 \leq x \leq l$.

代入初值条件:

$$\begin{cases} u(0, x) = \sum_{n=1}^{+\infty} A_n \sin \frac{n\pi x}{l} = \frac{4h}{l^2} x(l-x) \stackrel{\text{def}}{=} f(x) \\ u_t(0, x) = \sum_{n=1}^{+\infty} \frac{n\pi a}{l} B_n \sin \frac{n\pi x}{l} = 0 \end{cases} \Rightarrow B_n = 0 (n = 1, 2, \dots)$$

以下计算 $A_n = \frac{[f(x), X_n(x)]}{\|X_n(x)\|^2}, n = 1, 2, \dots$, 其中

$$\|X_n(x)\|^2 = \int_0^l \sin^2 \frac{n\pi x}{l} dx = \int_0^l \frac{1}{2} \left(1 - \cos \frac{2n\pi x}{l} \right) dx = \frac{l}{2}$$

$$[f(x), X_n(x)] = \int_0^l f(x) X_n(x) dx = \int_0^l \frac{4h}{l^2} x(l-x) \sin \frac{n\pi x}{l} dx = 4hl \int_0^1 (t \sin n\pi t - t^2 \sin n\pi t) dt$$

其中

$$\begin{aligned}
 \int_0^1 t \sin n\pi t dt &= \frac{-1}{n\pi} \int_0^1 t d(\cos n\pi t) = -\frac{1}{n\pi} t \cos n\pi t \Big|_0^1 + \frac{1}{n\pi} \int_0^1 \cos n\pi t dt = -\frac{1}{n\pi} (-1)^n \\
 \int_0^1 t^2 \sin n\pi t dt &= \frac{-1}{n\pi} \int_0^1 t^2 d(\cos n\pi t) \\
 &= -\frac{1}{n\pi} t^2 \cos n\pi t \Big|_0^1 + \frac{1}{n\pi} \int_0^1 2t \cos n\pi t dt \\
 &= -\frac{1}{n\pi} (-1)^n + \frac{2}{(n\pi)^2} \int_0^1 t d(\sin n\pi t) \\
 &= -\frac{1}{n\pi} (-1)^n + \frac{2}{(n\pi)^2} t \sin n\pi t \Big|_0^1 - \frac{2}{(n\pi)^2} \int_0^1 \sin n\pi t dt \\
 &= -\frac{1}{n\pi} (-1)^n + \frac{2}{(n\pi)^3} \cos n\pi t \Big|_0^1 \\
 &= -\frac{1}{n\pi} (-1)^n + \frac{2}{(n\pi)^3} [(-1)^n - 1]
 \end{aligned}$$

因此得到

$$A_n = \frac{[f(x), X_n(x)]}{||X_n(x)||^2} = \frac{2}{l} \cdot 4hl \cdot \frac{2}{(n\pi)^3} [1 - (-1)^n] = \frac{16h}{(n\pi)^3} [1 - (-1)^n] = \begin{cases} \frac{32h}{(n\pi)^3}, & n = 2k + 1 \\ 0, & n = 2k \quad (k \in \mathbb{N}) \end{cases}$$

$$\text{故 } u(t, x) = \frac{32h}{\pi^3} \sum_{k=0}^{+\infty} \frac{1}{(2k+1)^3} \cos \frac{(2k+1)\pi at}{l} \sin \frac{(2k+1)\pi x}{l}, \quad t > 0, 0 < x < l.$$

4 课本 $P_{252} T_4$

利用圆内狄氏问题的一般解式, 解边值问题 $\begin{cases} \Delta_2 u = 0 & (r < a) \\ u|_{r=a} = f \end{cases}$, 其中 f 分别为

(1) $f = A$ (常数);

(2) $f = A \cos \theta$;

(3) $f = Axy$;

(4) $f = \cos \theta \sin 2\theta$;

(5) $f = A \sin^2 \theta + B \cos^2 \theta$.

Sol:

可知在圆内该泛定方程的通解为 $u(r, \theta) = C_0 + \sum_{k=1}^{+\infty} r^k (C_k \cos k\theta + D_k \sin k\theta), r < a$.

(1) 对于 $u|_{r=a} = f = A$, 可知 $u(a, \theta) = C_0 + \sum_{k=1}^{+\infty} r^k (C_k \cos k\theta + D_k \sin k\theta) = A$

因而 $C_k = D_k = 0$ ($k \in \mathbb{N}_+$), $C_0 = A$, 故 $u(r, \theta) = A, r < a$.

(2) 对于 $u|_{r=a} = f = A \cos \theta$, 可知 $u(a, \theta) = C_0 + \sum_{k=1}^{+\infty} r^k (C_k \cos k\theta + D_k \sin k\theta) = A \cos \theta$

因而 $C_1 = \frac{A}{a}, C_k = 0$ ($k = 0, 2, 3, 4, \dots$), $D_k = 0$ ($k \in \mathbb{N}_+$), 故 $u(r, \theta) = \frac{A}{a} r \cos \theta, r < a$.

(3) 对于 $u|_{r=a} = f = Axy = Ar^2 \sin \theta \cos \theta = \frac{A}{2} a^2 \sin 2\theta$

可知 $u(a, \theta) = C_0 + \sum_{k=1}^{+\infty} r^k (C_k \cos k\theta + D_k \sin k\theta) = \frac{A}{2} a^2 \sin 2\theta$

因而 $D_2 = \frac{A}{2}, D_k = 0$ ($k = 0, 1, 3, 4, 5, \dots$), $C_k = 0$ ($k \in \mathbb{N}$), 故 $u(r, \theta) = \frac{A}{2} r^2 \cos \theta, r < a$.

(4) 对于 $u|_{r=a} = f = \cos \theta \sin 2\theta = \frac{1}{2} (\sin 3\theta + \sin \theta)$

可知 $u(a, \theta) = C_0 + \sum_{k=1}^{+\infty} r^k (C_k \cos k\theta + D_k \sin k\theta) = \frac{1}{2} (\sin 3\theta + \sin \theta)$

因而 $D_1 = \frac{1}{2a}, D_3 = \frac{1}{2a^3}$, 其余为 0, 故 $u(r, \theta) = \frac{1}{2} \left[\frac{r}{a} \sin \theta + \left(\frac{r}{a} \right)^3 \sin 3\theta \right], r < a$.

(5) 对于 $u|_{r=a} = f = A \sin^2 \theta + B \cos^2 \theta = \frac{A+B}{2} + \frac{B-A}{2} \cos 2\theta$

可知 $u(a, \theta) = C_0 + \sum_{k=1}^{+\infty} r^k (C_k \cos k\theta + D_k \sin k\theta) = \frac{A+B}{2} + \frac{B-A}{2} \cos 2\theta$

因而 $C_0 = \frac{A+B}{2}, C_2 = \frac{B-A}{2a^2}$, 其余为 0, 故 $u(r, \theta) = \frac{A+B}{2} + \frac{B-A}{2} \frac{r^2}{a^2} \cos 2\theta, r < a$.

5 课本 $P_{252} T_5$

解下列定解问题:

$$(1) \begin{cases} u_{tt} = a^2 u_{xx} (0 < x < l, t > 0), \\ u(t, 0) = u_x(t, l) = 0, \\ u(0, x) = 0, u_t(0, x) = x; \end{cases}$$

$$(2) \begin{cases} u_t = a^2 u_{xx} (0 < x < l, t > 0), \\ u(t, 0) = u(t, l) = 0, \\ u(0, x) = x(l - x); \end{cases}$$

$$(3) \begin{cases} u_{tt} = a^2 u_{xx} - 2hu_t \quad (0 < x < l, t > 0, 0 < h < \frac{\pi a}{l}, h \text{ 为常数}), \\ u(t, 0) = u(t, l) = 0, \\ u(0, x) = \varphi(x), u_t(0, x) = \psi(x); \end{cases}$$

$$(4) \begin{cases} u_{tt} = a^2 u_{xx} (0 < x < l, t > 0), \\ u_x(t, 0) = 0, u_x(t, l) + hu(t, l) = 0 (h > 0, h \text{ 为常数}), \\ u(0, x) = \varphi(x), u_t(0, x) = \psi(x); \end{cases}$$

$$(5) \begin{cases} \Delta_2 u = 0 (r < a), \\ u_r(a, \theta) - hu(a, \theta) = f(\theta) (h > 0), \end{cases} \quad \text{特别地, 计算 } f(\theta) = \cos^2 \theta \text{ 时 } u \text{ 的值;}$$

(6) 环域内的狄氏问题

$$\begin{cases} \Delta_2 u = 0 (a < r < b) \\ u(a, \theta) = 1, u(b, \theta) = 0 \end{cases}$$

(7) 扇形域内的狄氏问题

$$\begin{cases} \Delta_2 u = 0 (r < a, 0 < \theta < \alpha), \\ u(r, 0) = u(r, \alpha) = 0 \\ u(a, \theta) = f(\theta). \end{cases}$$

Sol:

(1) 令 $u(t, x) = T(t)X(x)$, 分离变量得: $\frac{1}{a^2} \frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$, 得到固有值问题:

$$\begin{cases} x'' + \lambda x = 0 \\ X(0) = X'(l) = 0 \end{cases}$$

由 Sturm-Liouville 定理可知 $\lambda > 0$, 令 $\lambda \stackrel{\text{def}}{=} k^2 > 0$, 得到 $X(x) = A \cos kx + B \sin kx$.

由 $X(0) = A = 0$, 此时 $X(x) = B \sin kx$; 由 $X'(l) = kB \cos kl = 0$, 欲求得非零解则 $kB \neq 0$, 故 $k_n l = \frac{2n+1}{2} \pi, n = 0, 1, 2, \dots$

因此固有值为 $\lambda_n = k_n^2 = \left(\frac{2n+1}{2l} \pi\right)^2$, 固有函数为 $X_n(x) = \sin \frac{2n+1}{2l} \pi x, n \in \mathbb{N}$.

随后解关于 $T(t)$ 的 ODE: $T''(t) + a^2 \lambda_n T(t) = 0$, 得到

$$T_n(t) = A_n \cos \frac{2n+1}{2l} \pi at + B_n \sin \frac{2n+1}{2l} \pi at.$$

叠加得到: $u(t, x) = \sum_{n=0}^{+\infty} \left(A_n \cos \frac{2n+1}{2l} \pi a t + B_n \sin \frac{2n+1}{2l} \pi a t \right) \sin \frac{2n+1}{2l} \pi x$.

代入初值条件:

$$\begin{cases} u(0, x) = \sum_{n=0}^{+\infty} A_n \sin \frac{2n+1}{2l} \pi x = 0 \\ u_t(0, x) = \sum_{n=0}^{+\infty} \frac{(2n+1)\pi a}{2l} B_n \sin \frac{2n+1}{2l} \pi x = x \end{cases} \Rightarrow A_n = 0 (n = 0, 1, 2, \dots)$$

以下计算 $B_n = \frac{2l}{(2n+1)\pi a} \frac{[x, X_n(x)]}{\|X_n(x)\|^2} (n = 0, 1, 2, \dots)$, 其中

$$\|X_n(x)\|^2 = \int_0^l \sin^2 \frac{2n+1}{2l} \pi x dx = \int_0^l \frac{1}{2} \left(1 - \cos \frac{2n+1}{l} \pi x \right) dx = \frac{l}{2}$$

$$\begin{aligned} [x, X_n(x)] &= \int_0^l x \sin \frac{2n+1}{2l} \pi x dx = \frac{-2l}{(2n+1)\pi} \int_0^l x d\left(\cos \frac{2n+1}{2l} \pi x\right) \\ &= \frac{-2l}{(2n+1)\pi} x \cos \frac{2n+1}{2l} \pi x \Big|_0^l + \frac{2l}{(2n+1)\pi} \int_0^l \cos \frac{2n+1}{2l} \pi x dx \\ &= \frac{4l^2}{(2n+1)^2 \pi^2} \sin \frac{2n+1}{2l} \pi x \Big|_0^l \\ &= \frac{4l^2}{(2n+1)^2 \pi^2} \cdot (-1)^n \end{aligned}$$

因此得到

$$B_n = \frac{2l}{(2n+1)\pi a} \frac{[x, X_n(x)]}{\|X_n(x)\|^2} = \frac{2l}{(2n+1)\pi a} \cdot \frac{2}{l} \cdot \frac{4l^2}{(2n+1)^2 \pi^2} \cdot (-1)^n = \frac{16l^2}{(2n+1)^3 \pi^3 a} (-1)^n$$

故 $u(t, x) = \frac{16l^2}{\pi^3 a} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^3} \cdot \sin \frac{(2n+1)}{2l} \pi a t \cdot \sin \frac{2n+1}{2l} \pi x, t > 0, 0 < x < l$.

(2) 令 $u(t, x) = T(t)X(x)$, 分离变量得: $\frac{1}{a^2} \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$, 得到固有值问题:

$$\begin{cases} x'' + \lambda x = 0 \\ X(0) = X(l) = 0 \end{cases}$$

由 Sturm-Liouville 定理可知 $\lambda > 0$, 令 $\lambda \stackrel{\text{def}}{=} k^2 > 0$, 得到 $X(x) = A \cos kx + B \sin kx$.

由 $X(0) = A = 0$, 此时 $X(x) = B \sin kx$; 由 $X(l) = B \sin kl = 0$, 欲求得非零解则 $B \neq 0$,

故 $k_n l = n\pi, n = 1, 2, \dots$

因此固有值为 $\lambda_n = k_n^2 = \left(\frac{n\pi}{l}\right)^2$, 固有函数为 $X_n(x) = \sin \frac{n\pi x}{l}, n \in \mathbb{N}_+$.

随后解关于 $T(t)$ 的 ODE: $T'(t) + a^2 \lambda_n T(t) = 0$, 得到 $T_n(t) = e^{-a^2 \lambda_n t} = e^{-(\frac{n\pi a}{l})^2 t}$.

叠加得到: $u(t, x) = \sum_{n=1}^{+\infty} C_n e^{-(\frac{n\pi a}{l})^2 t} \sin \frac{n\pi x}{l}$.

代入初值条件: $u(0, x) = \sum_{n=1}^{+\infty} C_n \sin \frac{n\pi x}{l} = x(l-x)$.

以下计算 $C_n = \frac{[x(l-x), X_n(x)]}{\|X_n(x)\|^2} (n = 0, 1, 2, \dots)$, 其中

$$\|X_n(x)\|^2 = \int_0^l \sin^2 \frac{n\pi x}{l} dx = \int_0^l \frac{1}{2} \left(1 - \cos \frac{2n\pi x}{l} \right) dx = \frac{l}{2}$$

$$[x(l-x), X_n(x)] = \int_0^l x(l-x) \sin \frac{n\pi x}{l} dx = \int_0^1 (t \sin n\pi t - t^2 \sin n\pi t) dt$$

其中

$$\int_0^1 t \sin n\pi t dt = \frac{-1}{n\pi} \int_0^1 t d(\cos n\pi t) = -\frac{1}{n\pi} t \cos n\pi t \Big|_0^1 + \frac{1}{n\pi} \int_0^1 \cos n\pi t dt = -\frac{1}{n\pi} (-1)^n$$

$$\begin{aligned} \int_0^1 t^2 \sin n\pi t dt &= \frac{-1}{n\pi} \int_0^1 t^2 d(\cos n\pi t) \\ &= -\frac{1}{n\pi} t^2 \cos n\pi t \Big|_0^1 + \frac{1}{n\pi} \int_0^1 2t \cos n\pi t dt \\ &= -\frac{1}{n\pi} (-1)^n + \frac{2}{(n\pi)^2} \int_0^1 t d(\sin n\pi t) \\ &= -\frac{1}{n\pi} (-1)^n + \frac{2}{(n\pi)^2} t \sin n\pi t \Big|_0^1 - \frac{2}{(n\pi)^2} \int_0^1 \sin n\pi t dt \\ &= -\frac{1}{n\pi} (-1)^n + \frac{2}{(n\pi)^3} \cos n\pi t \Big|_0^1 \\ &= -\frac{1}{n\pi} (-1)^n + \frac{2}{(n\pi)^3} [(-1)^n - 1] \end{aligned}$$

因此得到

$$C_n = \frac{[x(l-x), X_n(x)]}{\|X_n(x)\|^2} = \frac{2}{l} \cdot \frac{2l^3}{n^3\pi^3} \cdot [1 - (-1)^n] = \frac{4l^2}{n^3\pi^3} \cdot [1 - (-1)^n] = \begin{cases} \frac{8l^2}{n^3\pi^3}, & n = 2k+1 \\ 0, & n = 2k \quad (k \in \mathbb{N}) \end{cases}$$

$$\text{故 } u(t, x) = \frac{8l^2}{\pi^3} \sum_{k=0}^{+\infty} \frac{1}{(2k+1)^3} \cdot e^{-(\frac{2k+1}{l}\pi a)^2 t} \cdot \sin \frac{2k+1}{l} \pi x, \quad t > 0, 0 < x < l.$$

(3) 令 $u(t, x) = T(t)X(x)$, 分离变量得: $\frac{1}{a^2} \frac{T''(t)}{T(t)} + \frac{2h}{a^2} \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$, 得到固有值问题:

$$\begin{cases} x'' + \lambda x = 0 \\ X(0) = X(l) = 0 \end{cases}$$

由 Sturm-Liouville 定理可知 $\lambda > 0$, 令 $\lambda \stackrel{\text{def}}{=} k^2 > 0$, 得到 $X(x) = A \cos kx + B \sin kx$.

由 $X(0) = A = 0$, 此时 $X(x) = B \sin kx$; 由 $X(l) = B \sin kl = 0$, 欲求得非零解则 $B \neq 0$, 故 $k_n l = n\pi, n = 1, 2, \dots$

因此固有值为 $\lambda_n = k_n^2 = \left(\frac{n\pi}{l}\right)^2$, 固有函数为 $X_n(x) = \sin \frac{n\pi x}{l}, n \in \mathbb{N}_+$.

随后解关于 $T(t)$ 的 ODE: $T''(t) + 2hT'(t) + a^2\lambda_n T(t) = 0$, 其中 $0 < h < \frac{\pi a}{l}$, 其特征方程为 $k^2 + 2hk + (\frac{n\pi a}{l})^2 = 0$, 得到特征根 $k_{1,2} = -h \pm i\sqrt{(\frac{n\pi a}{l})^2 - h^2}$, 可记 $\omega_n = \sqrt{(\frac{n\pi a}{l})^2 - h^2}$, 由此得到 $T_n(t) = e^{-ht}(A_n \cos \omega t + B_n \sin \omega t)$.

叠加得到: $u(t, x) = \sum_{n=1}^{+\infty} e^{-ht}(A_n \cos \omega t + B_n \sin \omega t) \sin \frac{n\pi x}{l}$.

代入初值条件: $u(0, x) = \sum_{n=1}^{+\infty} A_n \sin \frac{n\pi x}{l} = \varphi(x)$, 得到系数为

$$A_n = \frac{[\varphi(x), X_n(x)]}{\|X_n(x)\|^2} = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi x}{l} dx$$

又 $u_t(0, x) = \sum_{n=1}^{+\infty} (\omega_n B_n - hA_n) \sin \frac{n\pi x}{l} = \psi(x)$, 得到

$$\omega_n B_n - hA_n = \frac{[\psi(x), X_n(x)]}{\|X_n(x)\|^2} = \frac{2}{l} \int_0^l \psi(x) \sin \frac{n\pi x}{l} dx$$

因此 $B_n = \frac{hA_n}{\omega_n} + \frac{2}{\omega_n l} \int_0^l \psi(x) \sin \frac{n\pi x}{l} dx$

故 $u(t, x) = \sum_{n=1}^{+\infty} e^{-ht}(A_n \cos \omega t + B_n \sin \omega t) \sin \frac{n\pi x}{l}, t > 0, 0 < x < l$, 其中

$$\omega_n = \sqrt{(\frac{n\pi a}{l})^2 - h^2}, A_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi x}{l} dx, B_n = \frac{hA_n}{\omega_n} + \frac{2}{\omega_n l} \int_0^l \psi(x) \sin \frac{n\pi x}{l} dx.$$

(4) 令 $u(t, x) = T(t)X(x)$, 分离变量得: $\frac{1}{a^2} \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$, 得到固有值问题:

$$\begin{cases} x'' + \lambda x = 0 \\ X'(0) = 0, X'(l) + hX(l) = 0 \end{cases}$$

由 Sturm-Liouville 定理可知 $\lambda > 0$, 令 $\lambda \stackrel{\text{def}}{=} k^2 > 0$, 得到 $X(x) = A \cos kx + B \sin kx$.

由 $X'(0) = kB = 0 \Rightarrow B = 0$, 此时 $X(x) = A \cos kx$;

又由 $X'(l) + hX(l) = -kA \sin kl + hA \cos kl = 0 \Rightarrow A\sqrt{h^2 + k^2} \sin(\arctan \frac{h}{k} - kl)$, 欲求得非零解则 $A\sqrt{h^2 + k^2} \neq 0$, 故 $k_n l - \arctan \frac{h}{k_n} = n\pi \Rightarrow \tan k_n l = \frac{h}{k_n}, n = 1, 2, \dots$

因此固有值为 $\lambda_n = k_n^2$, k_n 满足 $k_n \tan k_n l = h$, 固有函数为 $X_n(x) = \cos k_n x, n \in \mathbb{N}_+$.

随后解关于 $T(t)$ 的 ODE: $T''(t) + a^2\lambda_n T(t) = 0$, 得到 $T_n(t) = A_n \cos k_n at + B_n \sin k_n at$.

叠加得到: $u(t, x) = \sum_{n=1}^{+\infty} (A_n \cos k_n at + B_n \sin k_n at) \cos k_n x$.

代入初值条件:

$$\begin{cases} u(0, x) = \sum_{n=1}^{+\infty} A_n \cos k_n x = \varphi(x) \\ u_t(0, x) = \sum_{n=1}^{+\infty} k_n a B_n \cos k_n x = \psi(x) \end{cases}$$

那么

$$A_n = \frac{[\varphi(x), X_n(x)]}{\|X_n(x)\|^2} = \frac{1}{\|\cos k_n x\|^2} \int_0^l \varphi(x) \cos k_n x dx, n \in \mathbb{N}_+$$

$$B_n = \frac{1}{k_n a} \frac{[\psi(x), X_n(x)]}{\|X_n(x)\|^2} = \frac{1}{k_n a \|\cos k_n x\|^2} \int_0^l \psi(x) \cos k_n x dx, n \in \mathbb{N}_+$$

其中

$$\begin{aligned} \|X_n(x)\|^2 &= \int_0^l \cos^2 k_n x dx = \int_0^l \frac{1}{2} (1 + \cos 2k_n x) dx = \frac{l}{2} + \frac{1}{4k_n} \sin 2k_n l \\ &= \frac{l}{2} + \frac{1}{2k_n} \cdot \frac{\sin k_n l \cos k_n l}{\sin^2 k_n l + \cos^2 k_n l} = \frac{l}{2} + \frac{1}{2k_n} \cdot \frac{\tan k_n l}{\tan^2 k_n l + 1} \\ &= \frac{l}{2} + \frac{1}{2k_n} \cdot \frac{hk_n}{h^2 + k_n^2} = \frac{l}{2} + \frac{h}{2(h^2 + k_n^2)}. \end{aligned}$$

故 $u(t, x) = \sum_{n=1}^{+\infty} (a_n \cos k_n at + b_n \sin k_n at) \cos k_n x, t > 0, 0 < x < l$, 其中, $\lambda_n = k_n^2$ ($n = 1, 2, \dots$), k_n 是 $k_n \tan k_n l = h$ 的正实根, 且

$$A_n = \frac{1}{\|X_n\|^2} \int_0^l \varphi(x) \cos k_n x dx, B_n = \frac{1}{k_n a \|X_n\|^2} \int_0^l \psi(x) \cos k_n x dx, \|X_n\|^2 = \frac{l}{2} + \frac{h}{2(k_n^2 + h^2)}.$$

(5) 可知在圆内该泛定方程的通解为 $u(r, \theta) = C_0 + \sum_{k=1}^{+\infty} r^k (C_k \cos k\theta + D_k \sin k\theta), r < a$.

代入边界条件:

$$\begin{aligned} f(\theta) &= u_r(a, \theta) - hu(a, \theta) \\ &= \sum_{k=1}^{+\infty} k a^{k-1} (C_k \cos k\theta + D_k \sin k\theta) - \left[h C_0 + h \sum_{k=1}^{+\infty} a^k (C_k \cos k\theta + D_k \sin k\theta) \right] \\ &= -h C_0 + \sum_{k=1}^{+\infty} (k - ha) a^{k-1} (C_k \cos k\theta + D_k \sin k\theta) \end{aligned}$$

其中

$$\begin{aligned} \|\cos k\theta\|^2 &= \int_0^{2\pi} \cos^2 k\theta d\theta = \int_0^{2\pi} \frac{1 + \cos 2k\theta}{2} d\theta = \pi, k \in \mathbb{N}_+; \\ \|\sin k\theta\|^2 &= \int_0^{2\pi} \sin^2 k\theta d\theta = \int_0^{2\pi} \frac{1 - \cos 2k\theta}{2} d\theta = \pi, k \in \mathbb{N}_+. \end{aligned}$$

$$\Rightarrow \begin{cases} C_0 = -\frac{1}{2\pi h} \int_0^{2\pi} f(\theta) d\theta \\ C_k = \frac{1}{(ka^{k-1} - ha^k) \pi} \int_0^{2\pi} f(\theta) \cos k\theta d\theta, k \in \mathbb{N}_+ \\ D_k = \frac{1}{(ka^{k-1} - ha^k) \pi} \int_0^{2\pi} f(\theta) \sin k\theta d\theta, k \in \mathbb{N}_+ \end{cases}$$

由此得知通解 $u(r, \theta) = C_0 + \sum_{k=1}^{+\infty} r^k (C_k \cos k\theta + D_k \sin k\theta), r < a$, 系数如上所示.

特别地, 当 $f(\theta) = \cos^2 \theta = \frac{1}{2} + \frac{1}{2} \cos 2\theta$ 时, 有

$$\frac{1}{2} + \frac{1}{2} \cos 2\theta = -hC_0 + \sum_{k=1}^{\infty} (k - ha)a^{k-1} (C_k \cos k\theta + D_k \sin k\theta)$$

得到: $C_0 = -\frac{1}{2h}, C_2 = \frac{1}{2a(2-ha)}$, 其余为 0. 因而, $u(r, \theta) = -\frac{1}{2h} + \frac{r^2 \cos 2\theta}{2a(2-ha)}, r < a$.

(6) 可知在环中该泛定方程的通解为 $u(r, \theta) = C_0 + D_0 \ln r + \sum_{k=1}^{+\infty} (C_k r^k + D_k r^{-k})(A_k \cos k\theta + B_k \sin k\theta), a < r < b$.

代入边界条件, 得:

$$\begin{cases} u(a, \theta) = C_0 + D_0 \ln a + \sum_{k=1}^{+\infty} (C_k a^k + D_k a^{-k})(A_k \cos k\theta + B_k \sin k\theta) = 1 \\ u(b, \theta) = C_0 + D_0 \ln b + \sum_{k=1}^{\infty} (C_k b^k + D_k b^{-k})(A_k \cos k\theta + B_k \sin k\theta) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} C_k = D_k = 0, k > 0 \\ C_0 + D_0 \ln a = 1 \\ C_0 + D_0 \ln b = 0 \end{cases} \Rightarrow \begin{cases} C_0 = \frac{\ln b}{\ln b - \ln a} \\ D_0 = \frac{-1}{\ln b - \ln a} \end{cases}$$

因此得到: $u(r, \theta) = \frac{\ln b - \ln r}{\ln b - \ln a}, a < r < b$.

(7) 分离变量, 令 $u(r, \theta) = R(r)\Theta(\theta)$, 得到 $r^2 \frac{R''}{R} + r \frac{R'}{R} = \frac{\Theta''}{\Theta} \stackrel{\text{def}}{=} -\lambda$

由此得到固有值问题:

$$\begin{cases} \Theta''(\theta) + \lambda \Theta(\theta) = 0 \\ \Theta(0) = \Theta(\alpha) = 0 \end{cases}$$

由 Sturm-Liouville 定理可知 $\lambda > 0$, 可设 $\lambda \stackrel{\text{def}}{=} k^2 > 0$, 则 $\Theta(\theta) = A \cos k\theta + B \sin k\theta, 0 < \theta < \alpha$.

由 $\Theta(0) = A = 0$, 得到 $\Theta(\theta) = B \sin k\theta$; 由 $\Theta(\alpha) = B \sin k\alpha = 0$, 欲求非零解则 $B \neq 0$, 那么 $k_n = \frac{n\pi}{\alpha}$, $n = 1, 2, \dots$, 得到固有值为 $\lambda_n = \left(\frac{n\pi}{\alpha}\right)^2$, 固有值函数为 $\Theta_n(\theta) = \sin \frac{n\pi\theta}{\alpha}$, $n \in \mathbb{N}_+$.

另外, 关于径向部分得到 Euler 方程: $r^2 R''(r) + rR'(r) + \left(\frac{n\pi}{\alpha}\right)^2 R(r) = 0$, 解得 $R_n(r) = C_n r^{\frac{n\pi}{\alpha}} + D_n r^{-\frac{n\pi}{\alpha}}$.

由此得到通解为 $u(r, \theta) = \sum_{n=1}^{+\infty} (C_n r^{\frac{n\pi}{\alpha}} + D_n r^{-\frac{n\pi}{\alpha}}) \sin \frac{n\pi\theta}{\alpha}$.

代入边界条件得:

$$\begin{cases} |u(0, \theta)| < +\infty \Rightarrow D_n = 0, n \in \mathbb{N}_+ \\ u(a, \theta) = \sum_{n=1}^{+\infty} C_n a^{\frac{n\pi}{\alpha}} \sin \frac{n\pi\theta}{\alpha} = f(\theta) \end{cases}$$

因此系数 $C_n = \frac{1}{a^{\frac{n\pi}{\alpha}}} \cdot \frac{[f(\theta), \Theta_n(\theta)]}{\|\Theta_n(\theta)\|^2}$, $n \in \mathbb{N}_+$, 其中

$$\|\Theta_n(\theta)\|^2 = \int_0^\alpha \sin^2 \frac{n\pi\theta}{\alpha} d\theta = \int_0^\alpha \frac{1 - \cos \frac{2n\pi\theta}{\alpha}}{2} d\theta = \frac{\alpha}{2}$$

$$[f(\theta), \Theta_n(\theta)] = \int_0^\alpha f(\theta) \sin \frac{n\pi\theta}{\alpha} d\theta$$

因此 $C_n = \frac{2}{\alpha a^{\frac{n\pi}{\alpha}}} \int_0^\alpha f(\theta) \sin \frac{n\pi\theta}{\alpha} d\theta$, $n \in \mathbb{N}_+$, 故得到解:

$$u(r, \theta) = \frac{2}{\alpha} \sum_{n=1}^{+\infty} \left(\frac{r}{a}\right)^{\frac{n\pi}{\alpha}} \int_0^\alpha f(\xi) \sin \frac{n\pi\xi}{\alpha} d\xi \sin \frac{n\pi\theta}{\alpha}, \quad r < a, 0 < \theta < \alpha.$$

6 课本 $P_{253} T_6$

长为 $2l$ 的均匀杆, 两端与侧面均绝热, 若初始温度为

$$\varphi(x) = \begin{cases} \frac{1}{2A} & (|x-l| < A < l) \\ 0 & (\text{其余的 } x), \end{cases}$$

求 $u(x, t)$ 及 $t \rightarrow +\infty$ 时的情况. 又当 $A \rightarrow 0$ 时, 解的极限如何?

Sol:

列出边界问题如下:

$$\begin{cases} u_t = a^2 u_{xx}, & t > 0, 0 < x < 2l \\ u_x(t, 0) = u_x(t, 2l) = 0 \\ u(0, x) = \varphi(x) = \begin{cases} \frac{1}{2A} & , |x-l| < A < l \\ 0 & , \text{其余的 } x \end{cases} \end{cases}$$

令 $u(t, x) = T(t)X(x)$, 分离变量得: $\frac{1}{a^2} \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$, 得到固有值问题:

$$\begin{cases} x'' + \lambda x = 0 \\ X'(0) = X'(2l) = 0 \end{cases}$$

由 Sturm-Liouville 定理可知 $\lambda \geq 0$, 当 $\lambda = 0$ 时得到固有函数可取 $X_0(x) = 1$; 当 $\lambda > 0$ 时, 令 $\lambda \stackrel{\text{def}}{=} k^2 > 0$, 得到 $X(x) = A \cos kx + B \sin kx$, $k \neq 0$.

由 $X'(0) = kB = 0$, 此时 $X(x) = A \sin kx$; 由 $X'(2l) = -kA \sin 2kl = 0$, 欲求得非零解则 $kB \neq 0$, 故 $2k_n l = n\pi, n = 1, 2, \dots$

因此固有值为 $\lambda_n = k_n^2 = \left(\frac{n\pi}{2l}\right)^2$, 固有函数为 $X_n(x) = \cos \frac{n\pi x}{2l}, n \in \mathbb{N}$.

随后解关于 $T(t)$ 的 ODE: $T'(t) + a^2 \lambda_n T(t) = 0$, 得到 $T_n(t) = C_n e^{-(\frac{n\pi a}{2l})^2 t}, n \in \mathbb{N}$.

叠加得到: $u(t, x) = \sum_{n=0}^{+\infty} C_n e^{-(\frac{n\pi a}{2l})^2 t} \cos \frac{n\pi x}{l}$.

代入初值条件: $u(0, x) = \sum_{n=0}^{+\infty} C_n \cos \frac{n\pi x}{2l} = \varphi(x)$.

那么

$$\begin{aligned} C_0 &= \frac{\int_0^{2l} \varphi(x) dx}{\int_0^{2l} 1 \cdot dx} = \frac{1}{2l} \cdot \frac{2A}{2A} = \frac{1}{2l}; \\ C_n &= \frac{\int_0^{2l} \cos \frac{n\pi x}{2l} \varphi(x) dx}{\int_0^{2l} \cos^2 \frac{n\pi x}{2l} dx} = \frac{1}{l} \int_{l-A}^{l+A} \frac{1}{2A} \cos \frac{n\pi x}{2l} dx \\ &= \frac{l}{n\pi A} \left[\sin \frac{n\pi(l+A)}{2l} - \sin \frac{n\pi(l-A)}{2l} \right] = \frac{2}{A n \pi} \sin \frac{n\pi A}{2l} \cos \frac{n\pi}{2}, n \in \mathbb{N}_+ \end{aligned}$$

故

$$\begin{aligned}
 u(t, x) &= \frac{1}{2l} + \frac{1}{\pi A} \sum_{n=1}^{+\infty} \frac{2}{n} \sin \frac{n\pi A}{2l} \cos n\pi 2 \cdot e^{-(\frac{n\pi a}{2l})^2 t} \cdot \cos \frac{n\pi x}{2l} \\
 &= \frac{1}{2l} + \frac{1}{\pi A} \sum_{k=1}^{+\infty} \frac{1}{k} \sin \frac{k\pi A}{l} \cos k\pi \cdot e^{-(\frac{k\pi a}{l})^2 t} \cdot \cos \frac{k\pi x}{l} \\
 &= \frac{1}{2l} + \frac{1}{\pi A} \sum_{k=1}^{+\infty} \frac{(-1)^k}{k} \sin \frac{k\pi A}{l} \cdot e^{-(\frac{k\pi a}{l})^2 t} \cdot \cos \frac{k\pi x}{l}, \quad t > 0, 0 < x < 2l.
 \end{aligned}$$

对于 $A \rightarrow 0$ 及 $t \rightarrow +\infty$, 由于 $u(t, x) \in C^2$, 由 Dirichlet 收敛定理可知其 Fourier 级数一致收敛, 那么

$$\begin{aligned}
 \lim_{t \rightarrow +\infty} u(t, x) &= \frac{1}{2l} + \frac{1}{\pi A} \sum_{k=1}^{+\infty} \frac{(-1)^k}{k} \sin \frac{k\pi A}{l} \cdot \left[\lim_{t \rightarrow +\infty} e^{-(\frac{k\pi a}{l})^2 t} \right] \cdot \cos \frac{k\pi x}{l} \\
 &= \frac{1}{2l} \\
 \lim_{A \rightarrow 0} u(t, x) &= \frac{1}{2l} + \sum_{k=1}^{+\infty} \left[\lim_{A \rightarrow 0} \frac{1}{\pi A} \sin \frac{k\pi A}{l} \right] \cdot \frac{(-1)^k}{k} \cdot e^{-(\frac{k\pi a}{l})^2 t} \cdot \cos \frac{k\pi x}{l} \\
 &= \frac{1}{2l} + \frac{1}{l} \sum_{k=1}^{+\infty} (-1)^k e^{-(\frac{k\pi a}{l})^2 t} \cdot \cos \frac{k\pi x}{l}.
 \end{aligned}$$

7 课本 $P_{253} T_7$

解下列定解问题:

$$(1) \begin{cases} u_t = a^2 \Delta_3 u \\ u|_{r=R} = 0, u(t, 0) \text{ 有限}, \\ u|_{t=0} = f(r); \end{cases}$$

[提示: 采用球坐标系, 由定解条件可知 $u = u(t, r)$.]

$$(2) \begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^4 u}{\partial x^4} (t > 0, 0 < x < l) \\ u(0, x) = x(l-x), u_t(0, x) = 0, \\ u(t, 0) = u(t, l) = 0, \\ u_{xx}(t, 0) = u_{xx}(t, l) = 0. \end{cases}$$

Sol:

(1) 在球坐标系下沿径向展开: $\frac{\partial u}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right)$

分离变量, 令 $u(t, r) = T(t)R(r)$, 得到 $\frac{T'(t)}{T(t)} = \frac{1}{r^2} \frac{1}{R} (r^2 R')' \stackrel{\text{def}}{=} -\lambda$.

解以下固有值问题:

$$\begin{cases} (r^2 R')' + \lambda r^2 R = 0 \\ |R(0)| < +\infty, R(r) = 0 \end{cases}$$

由 Sturm-Liouville 定理可知 $\lambda > 0$, 令 $R(r) = \frac{v(r)}{r}$, 则 $R'(r) = \frac{v'(r)}{r} - \frac{v(r)}{r^2}$, $r^2 R'(r) = rv'(r) - v(r)$, $(r^2 R'(r))' = r^2 v''(r)$.

因此, 泛定方程化为 $v''(r) + \lambda v(r) = 0$, 令 $\lambda = k^2 > 0, k \neq 0$, 则 $v(r) = A \cos kr + B \sin kr$.

由 $v(0) = rR(r)|_{r=0} = A = 0$, 得到 $v(r) = B \sin kr$; 由 $v(R) = B \sin kR$, 欲求非零解则有 $kR = n\pi$, 故 $k = \frac{n\pi}{R}, n = 1, 2, \dots$

因此得到固有值为 $\lambda_n = \left(\frac{n\pi}{R}\right)^2$, 固有函数为 $R_n(r) = \frac{1}{r} \sin \frac{n\pi r}{R}, n \in \mathbb{N}_+$.

此时由关于 $T(t)$ 的 ODE: $T'(t) + \lambda T(t) = 0$ 得到 $T_n(t) = C_n e^{-(\frac{n\pi}{R})^2 t}$.

叠加得到: $u(t, r) = \sum_{n=1}^{+\infty} C_n e^{-(\frac{n\pi}{R})^2 t} \frac{1}{r} \sin \frac{n\pi r}{R}, t > 0, r \geq 0$.

代入初值条件: $u(0, x) = \sum_{n=0}^{+\infty} C_n \frac{1}{r} \sin \frac{n\pi r}{R} = f(r)$.

由泛定方程可知内积权重为 $\rho(x) = x^2$, 则

$$\begin{aligned} \left\| \frac{1}{r} \sin \frac{n\pi r}{R} \right\|^2 &= \int_0^R r^2 \left(\frac{1}{r} \sin \frac{n\pi r}{R} \right)^2 dr = \frac{R}{2}; \\ \Rightarrow C_n &= \frac{[f(r), \frac{1}{r} \sin \frac{n\pi r}{R}]}{\left\| \frac{1}{r} \sin \frac{n\pi r}{R} \right\|^2} = \frac{2}{R} \int_0^R r f(r) \sin \frac{n\pi r}{R} dr \end{aligned}$$

因此得: $u(t, r) = \frac{2}{Rr} \sum_{n=1}^{+\infty} \left(\int_0^R t f(t) \sin \frac{n\pi t}{R} dt \right) \cdot e^{-(\frac{n\pi}{R})^2 t} \cdot \sin \frac{n\pi r}{R}, t > 0, r \geq 0.$

(2) 分离变量, 令 $u(t, r) = T(t)X(x)$, 得到 $\frac{1}{a^2} \frac{T''(t)}{T(t)} = \frac{X^{(4)}(x)}{X(x)} \stackrel{\text{def}}{=} -\lambda.$

解以下固有值问题:

$$\begin{cases} X^{(4)}(x) + \lambda X(x) = 0 \\ X(0) = X(l) = X''(0) = X''(l) = 0 \end{cases}$$

由于 $\int_0^l \lambda X^2 dx = -\int_0^l X^{(4)} X dx = -\int_0^l X dX^{(3)} = \int_0^l X^{(3)} X' dx = \int_0^l X' dX'' = -\int_0^l (X'')^2 dx \leq 0$, 即 $\lambda \int_0^l X^2 dx \leq 0$, 而 $\int_0^l X^2 dx \geq 0$, 因此 $\lambda \leq 0$.

当 $\lambda = 0$ 时, $X(x) = Ax + B$, 由边界条件可得 $A = B = 0$, 得到零解, 故 $\lambda < 0$, 令 $\lambda \stackrel{\text{def}}{=} -\omega^4$, 这样特征方程存在 4 个特征根: $k_{1,2} = \pm\omega, k_{3,4} = \pm i\omega$

因此可得到: $y(x) = C_1 e^{\omega x} + C_2 e^{-\omega x} + C_3 \cos \omega x + C_4 \sin \omega x.$

代入边界条件 $y(0) = y(l) = y''(0) = y''(l) = 0$, 分别得到:

$$\begin{aligned} C_1 + C_2 + C_3 &= 0, C_1 + C_2 - C_3 = 0 \\ C_1 e^{\omega l} + C_2 e^{-\omega l} + C_3 \cos \omega l + C_4 \sin \omega l &= 0 \\ C_1 e^{\omega l} + C_2 e^{-\omega l} - C_3 \cos \omega l - C_4 \sin \omega l &= 0 \end{aligned}$$

解得 $C_1 = C_2 = C_3 = 0, C_4 \sin \omega l = 0$

所以欲求非零解, 只有 $C_4 \neq 0$, 因此 $\sin \omega l = 0 \Rightarrow \omega l = n\pi \Rightarrow \omega = \frac{n\pi}{l}, n = 1, 2, \dots$

因此可得到固有值为 $\lambda_n = -\left(\frac{n\pi}{l}\right)^4$, 固有函数为 $y_n(x) = \sin \frac{n\pi x}{l}, n \in \mathbb{N}_+.$

随后解关于 $T(t)$ 的 ODE: $T''(t) + a^2 \lambda_n T(t) = 0$, 得到 $T_n(t) = A_n \cosh \left[\left(\frac{n\pi}{l} \right) at \right] + B_n \sinh \left[\left(\frac{n\pi}{l} \right) at \right].$

叠加得到: $u(t, x) = \sum_{n=1}^{+\infty} \left\{ A_n \cosh \left[\left(\frac{n\pi}{l} \right) at \right] + B_n \sinh \left[\left(\frac{n\pi}{l} \right) at \right] \right\} \sin \frac{n\pi x}{l}, t \geq 0, 0 \leq x \leq l.$

代入初值条件:

$$\begin{cases} u(0, x) = \sum_{n=1}^{+\infty} A_n \sin \frac{n\pi x}{l} = x(l-x) \\ u_t(0, x) = \sum_{n=1}^{+\infty} \left(\frac{n\pi}{l} \right)^2 a B_n \sin \frac{n\pi x}{l} = 0 \end{cases} \Rightarrow B_n = 0 (n = 1, 2, \dots)$$

以下计算 $A_n = \frac{[x(l-x), X_n(x)]}{\|X_n(x)\|^2}, n = 1, 2, \dots$, 其中

$$\|X_n(x)\|^2 = \int_0^l \sin^2 \frac{n\pi x}{l} dx = \int_0^l \frac{1}{2} \left(1 - \cos \frac{2n\pi x}{l} \right) dx = \frac{l}{2}$$

$$[x(l-x), X_n(x)] = \int_0^l x(l-x) \sin \frac{n\pi x}{l} dx = \int_0^1 (t \sin n\pi t - t^2 \sin n\pi t) dt$$

其中

$$\begin{aligned} \int_0^1 t \sin n\pi t dt &= \frac{-1}{n\pi} \int_0^1 t d(\cos n\pi t) = -\frac{1}{n\pi} t \cos n\pi t \Big|_0^1 + \frac{1}{n\pi} \int_0^1 \cos n\pi t dt = -\frac{1}{n\pi} (-1)^n \\ \int_0^1 t^2 \sin n\pi t dt &= \frac{-1}{n\pi} \int_0^1 t^2 d(\cos n\pi t) \\ &= -\frac{1}{n\pi} t^2 \cos n\pi t \Big|_0^1 + \frac{1}{n\pi} \int_0^1 2t \cos n\pi t dt \\ &= -\frac{1}{n\pi} (-1)^n + \frac{2}{(n\pi)^2} \int_0^1 t d(\sin n\pi t) \\ &= -\frac{1}{n\pi} (-1)^n + \frac{2}{(n\pi)^2} t \sin n\pi t \Big|_0^1 - \frac{2}{(n\pi)^2} \int_0^1 \sin n\pi t dt \\ &= -\frac{1}{n\pi} (-1)^n + \frac{2}{(n\pi)^3} \cos n\pi t \Big|_0^1 \\ &= -\frac{1}{n\pi} (-1)^n + \frac{2}{(n\pi)^3} [(-1)^n - 1] \end{aligned}$$

因此得到

$$C_n = \frac{[x(l-x), X_n(x)]}{||X_n(x)||^2} = \frac{2}{l} \cdot \frac{2l^3}{n^3\pi^3} \cdot [1 - (-1)^n] = \frac{4l^2}{n^3\pi^3} \cdot [1 - (-1)^n] = \begin{cases} \frac{8l^2}{n^3\pi^3}, & n = 2k+1 \\ 0, & n = 2k \quad (k \in \mathbb{N}) \end{cases}$$

$$\text{故 } u(t, x) = \frac{8l^2}{\pi^3} \sum_{k=0}^{+\infty} \frac{1}{(2k+1)^3} \cdot \cosh \left[\left(\frac{n\pi}{l} \right) at \right] \cdot \sin \frac{2k+1}{l} \pi x, \quad t > 0, 0 < x < l.$$

8 课本 $P_{254} T_8$

一半径为 a 的半圆形平板, 其圆周边界上的温度保持 $u(a, \theta) = T\theta(\pi - \theta)$, 而直径边界上的温度为零度, 板的侧面绝缘, 试求板内的稳定温度分布.

Sol:

达到稳态后 $u = u(r, \theta)$, 得到定解问题:

$$\begin{cases} \Delta_2 u = 0 \\ u(a, \theta) = T\theta(\pi - \theta) \\ u(a, 0) = u(a, \pi) = 0 \end{cases}$$

分离变量: 令 $u(r, \theta) = R(r)\Theta(\theta)$, 得到 $r^2 \frac{R''}{R} + r \frac{R'}{R} = \frac{\Theta''}{\Theta} \stackrel{\text{def}}{=} -\lambda$, 解以下固有值问题:

$$\begin{cases} \Theta''(\theta) + \Theta(\theta) = 0 \\ \Theta(0) = \Theta(\pi) = 0 \end{cases}$$

由 Sturm-Liouville 定理可知 $\lambda > 0$, 可令 $\lambda \stackrel{\text{def}}{=} k^2 > 0$, 得到 $\Theta(\theta) = A \cos k\theta + B \sin k\theta$, $k \neq 0$.

由 $\Theta'(0) = A = 0$, 此时 $\Theta(\theta) = B \sin k\theta$; 由 $\Theta(\pi) = B \sin k\pi = 0$, 欲求得非零解则 $B \neq 0$, 故 $k_n = n\pi, n = 1, 2, \dots$

因此固有值为 $\lambda_n = k_n^2 = n^2$, 固有函数为 $\Theta_n(\theta) = \sin n\theta, n \in \mathbb{N}_+$.

随后解关于 $R(r)$ 的 ODE: $r^2 R''(r) + r R'(r) + \lambda_n R(r) = 0$, 得到 $R_n(r) = C_n r^n + D_n r^{-n}, n \in \mathbb{N}_+$.

叠加得到: $u(t, x) = \sum_{n=1}^{+\infty} (C_n r^n + D_n r^{-n}) \sin n\theta$.

代入边界条件得:

$$\begin{cases} |u(0, \theta)| < +\infty \Rightarrow D_n = 0, n \in \mathbb{N}_+ \\ u(a, \theta) = \sum_{n=1}^{+\infty} C_n a^n \sin n\theta = T\theta(\pi - \theta) \end{cases}$$

以下计算 $C_n = \frac{1}{a^n} \cdot \frac{[T\theta(\pi - \theta), \Theta_n(\theta)]}{\|\Theta_n(\theta)\|^2}$, $n = 1, 2, \dots$, 其中

$$\|\Theta_n(\theta)\|^2 = \int_0^\pi \sin^2 n\theta d\theta = \int_0^\pi \frac{1}{2} (1 - \cos 2n\theta) d\theta = \frac{\pi}{2}$$

$$[T\theta(\pi - \theta), \Theta_n(\theta)] = \int_0^\pi T\theta(\pi - \theta) \sin n\theta d\theta = \pi^3 T \int_0^1 (t \sin n\pi t - t^2 \sin n\pi t) dt$$

其中

$$\int_0^1 t \sin n\pi t dt = \frac{-1}{n\pi} \int_0^1 t d(\cos n\pi t) = -\frac{1}{n\pi} t \cos n\pi t \Big|_0^1 + \frac{1}{n\pi} \int_0^1 \cos n\pi t dt = -\frac{1}{n\pi} (-1)^n$$

$$\begin{aligned}
\int_0^1 t^2 \sin n\pi t dt &= \frac{-1}{n\pi} \int_0^1 t^2 d(\cos n\pi t) \\
&= -\frac{1}{n\pi} t^2 \cos n\pi t \Big|_0^1 + \frac{1}{n\pi} \int_0^1 2t \cos n\pi t dt \\
&= -\frac{1}{n\pi} (-1)^n + \frac{2}{(n\pi)^2} \int_0^1 t d(\sin n\pi t) \\
&= -\frac{1}{n\pi} (-1)^n + \frac{2}{(n\pi)^2} t \sin n\pi t \Big|_0^1 - \frac{2}{(n\pi)^2} \int_0^1 \sin n\pi t dt \\
&= -\frac{1}{n\pi} (-1)^n + \frac{2}{(n\pi)^3} \cos n\pi t \Big|_0^1 \\
&= -\frac{1}{n\pi} (-1)^n + \frac{2}{(n\pi)^3} [(-1)^n - 1]
\end{aligned}$$

因此得到

$$\begin{aligned}
C_n &= \frac{1}{a^n} \cdot \frac{[T\theta(\pi - \theta), \Theta_n(\theta)]}{\|\Theta_n(\theta)\|^2} \\
&= \frac{1}{a^n} \cdot \frac{2}{\pi} \cdot \pi^3 T \cdot \frac{2}{n^3 \pi^3} \cdot [1 - (-1)^n] \\
&= \frac{4T}{\pi n^3 a^n} \cdot [1 - (-1)^n] = \begin{cases} \frac{8T}{\pi n^3 a^n}, & n = 2k + 1 \\ 0, & n = 2k \quad (k \in \mathbb{N}) \end{cases}
\end{aligned}$$

$$\text{故 } u(r, \theta) = \frac{8T}{\pi} \sum_{k=1}^{+\infty} \frac{1}{(2k+1)^3} \cdot \left(\frac{r}{a}\right)^{2k+1} \cdot \sin(2k+1)\theta, \quad t > 0, 0 < r < a, 0 < \theta < \pi.$$

9 课本 $P_{254} T_9$

求方程 $u_{xx} - u_y = 0$ 满足条件

$$\lim_{x \rightarrow +\infty} u(x, y) = 0$$

的解 $u = X(x)Y(y)$.

Sol:

由于 $u(x, y) = X(x)Y(y)$, 因此 $u_{xx} = X''(x)Y(y) = u_y = X(x)Y'(y)$, 得到

$$\frac{X''(x)}{X(x)} = \frac{Y'(y)}{Y(y)} \stackrel{\text{def}}{=} k$$

故可得到关于 $X(x)$ 和 $Y(y)$ 的两个 ODE:

$$X''(x) + kX(x) = 0, \quad Y'(y) + kY(y) = 0$$

因此 $Y(y) = C_0 e^{-ky}$.

若 $k < 0$, 则 $X(x) = A \cos \sqrt{-k}x + B \sin \sqrt{-k}x$, 显然不满足 $\lim_{x \rightarrow +\infty} u(x, y) = 0$; 若 $k = 0$, 则 $X(x) = Ax + B$, 显然也不满足 $\lim_{x \rightarrow +\infty} u(x, y) = 0$, 故 $k > 0$, 得到 $X(x) = C_1 e^{-\sqrt{k}x} + C_2 e^{\sqrt{k}x}$.

由于 $\lim_{x \rightarrow +\infty} u(x, y) = 0$, 故 $\lim_{x \rightarrow +\infty} X(x) = 0$, 因此 $C_2 = 0$, $X(x) = C_1 e^{-\sqrt{k}x}$

所以得到 $u(x, y) = X(x)Y(y) = C e^{-\sqrt{k}x + ky}$, 此处 $k > 0$, C 为任意常数.

10 课本 P_{254} T_{10}

$$(1) \begin{cases} u_t = a^2 u_{xx} \\ u(t, 0) = u_0, u_x(t, l) = 0 \\ u(0, x) = \varphi(x) \end{cases}$$

$$(2) \begin{cases} u_t = a^2 u_{xx} \\ u(t, 0) = 0, u(t, l) = -\frac{q}{k} \\ u(0, x) = u_0 \end{cases}$$

并求 $\lim_{t \rightarrow +\infty} u(t, x)$;

$$(3) \begin{cases} \frac{\partial^2 u}{\partial x^2} - a^2 \frac{\partial u}{\partial t} + Ae^{-2x} = 0 \\ u(t, 0) = u(t, l) = 0 \\ u(0, x) = T_0 \end{cases}$$

$$(4) \begin{cases} u_{tt} = a^2 u_{xx} + b \sinh x \\ u(t, 0) = u(t, l) = 0 \\ u(0, x) = u_t(0, x) = 0 \end{cases}$$

$$(5) \begin{cases} u_{tt} = u_{xx} + g (g \text{ 为常数}), \\ u(t, 0) = 0, u_x(t, l) = E (E \text{ 为常数}), \\ u(0, x) = Ex, u_t(0, x) = 0; \end{cases}$$

[提示: 先求一个满足泛定方程和边界条件的 $v(x)$, 再令 $u(t, x) = w(t, x) + v(x)$.]

$$(6) \begin{cases} \Delta_2 u = a + b(x^2 - y^2) (a, b \text{ 为常数}, r < R), \\ u(R, \theta) = c (c \text{ 为常数}). \end{cases}$$

Sol:

(1) 令 $u(t, x) = v(t, x) + u_0$, 则 $v(t, x)$ 满足:

$$\begin{cases} v_t = a^2 v_{xx} \\ v(t, 0) = v_x(t, l) = 0 \\ v(0, x) = \varphi(x) - u_0 \end{cases}$$

分离变量: 令 $v(t, x) = T(t)X(x)$, 得到 $\frac{1}{a^2} \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} \stackrel{\text{def}}{=} -\lambda$, 解以下固有值问题:

$$\begin{cases} X''(x) + X(x) = 0 \\ X(0) = X'(l) = 0 \end{cases}$$

由 Sturm-Liouville 定理可知 $\lambda > 0$, 可令 $\lambda \stackrel{\text{def}}{=} k^2 > 0$, 得到 $X(x) = A \cos kx + B \sin kx$, $k \neq 0$.

由 $X(0) = A = 0$, 此时 $X(x) = B \sin kx$; 由 $X'(l) = kB \cos kl = 0$, 欲求得非零解则 $kB \neq 0$, 故 $k_n = \frac{(2n+1)\pi}{2l}$, $n = 0, 1, 2, \dots$

因此固有值为 $\lambda_n = k_n^2 = \left(\frac{2n+1}{2l}\pi\right)^2$, 固有函数为 $X_n(x) = \sin \frac{2n+1}{2l}\pi x, n \in \mathbb{N}_+$.

设 $v(t, x) = \sum_{n=0}^{+\infty} T_n(t) \sin \frac{2n+1}{2l}\pi x$.

随后解关于 $T(t)$ 的 ODE: $T'(t) + a^2\lambda T(t) = 0$, 得到 $T_n(t) = e^{-a^2\lambda_n t} = e^{-(\frac{2n+1}{2l}a)^2 t}, n \in \mathbb{N}$.

叠加得到: $v(t, x) = \sum_{n=0}^{+\infty} C_n e^{-(\frac{2n+1}{2l}a)^2 t} \sin \frac{2n+1}{2l}\pi x$.

代入初始条件得: $v(0, x) = \sum_{n=0}^{+\infty} C_n \sin \frac{2n+1}{2l}\pi x = \varphi(x) - u_0$.

那么系数为:

$$\begin{aligned} C_n &= \frac{[\varphi(x) - u_0, \sin \frac{2n+1}{2l}\pi x]}{\|\sin \frac{2n+1}{2l}\pi x\|^2} \\ &= \frac{2}{l} \int_0^l (\varphi(x) - u_0) \sin \frac{2n+1}{2l}\pi x dx \\ &= \frac{2}{l} \int_0^l \varphi(x) \sin \frac{2n+1}{2l}\pi x dx - \frac{4u_0}{(2n+1)\pi} \end{aligned}$$

故解为:

$$u(t, x) = u_0 + \sum_{n=0}^{+\infty} \left[\frac{2}{l} \int_0^l \varphi(x) \sin \frac{2n+1}{2l}\pi x dx - \frac{4u_0}{(2n+1)\pi} \right] e^{-(\frac{2n+1}{2l}a)^2 t} \sin \frac{2n+1}{2l}\pi x.$$

(2) 令 $u(t, x) = v(t, x) - \frac{q}{k}x$, 则 $v(t, x)$ 满足

$$\begin{cases} v_t = a^2 v_{xx} \\ v(t, 0) = v_x(t, l) = 0 \\ v(0, x) = u_0 + \frac{q}{k}x \end{cases}$$

分离变量: 令 $v(t, x) = T(t)X(x)$, 得到 $\frac{1}{a^2} \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} \stackrel{\text{def}}{=} -\lambda$, 解以下固有值问题:

$$\begin{cases} X''(x) + X(x) = 0 \\ X(0) = X'(l) = 0 \end{cases}$$

由 Sturm-Liouville 定理可知 $\lambda > 0$, 可令 $\lambda \stackrel{\text{def}}{=} k^2 > 0$, 得到 $X(x) = A \cos kx + B \sin kx, k \neq 0$.

由 $X(0) = A = 0$, 此时 $X(x) = B \sin kx$; 由 $X'(l) = kB \cos kl = 0$, 欲求得非零解则 $kB \neq 0$, 故 $k_n = \frac{2n+1}{2l}\pi, n = 0, 1, 2, \dots$

因此固有值为 $\lambda_n = k_n^2 = \left(\frac{2n+1}{2l}\pi\right)^2$, 固有函数为 $X_n(x) = \sin \frac{2n+1}{2l}\pi x, n \in \mathbb{N}$.

随后解关于 $T(t)$ 的 ODE: $T'(t) + a^2\lambda T(t) = 0$, 得到 $T_n(t) = e^{-a^2\lambda_n t} = e^{-(\frac{2n+1}{2l}a)^2 t}, n \in \mathbb{N}$.

叠加得到: $v(t, x) = \sum_{n=0}^{+\infty} C_n e^{-(\frac{2n+1}{2l}a)^2 t} \sin \frac{2n+1}{2l}\pi x$.

代入初始条件得: $v(0, x) = \sum_{n=0}^{+\infty} C_n \sin \frac{2n+1}{2l}\pi x = u_0 + \frac{q}{k}x$.

那么系数为:

$$\begin{aligned} C_n &= \frac{[u_0 + \frac{q}{k}x, \sin \frac{2n+1}{2l}\pi x]}{\|\sin \frac{2n+1}{2l}\pi x\|^2} \\ &= \frac{2}{l} \int_0^l \left(u_0 + \frac{q}{k}x\right) \sin \frac{2n+1}{2l}\pi x dx \end{aligned}$$

其中

$$\begin{aligned} \int_0^l \sin \frac{2n+1}{2l}\pi x dx &= -\frac{2l}{(2n+1)\pi} \cos \frac{2n+1}{2l}\pi x \Big|_0^l = \frac{2l}{(2n+1)\pi} \\ \int_0^l x \sin \frac{2n+1}{2l}\pi x dx &= -\frac{2l}{(2n+1)\pi} x \cos \frac{2n+1}{2l}\pi x \Big|_0^l + \frac{2l}{(2n+1)\pi} \int_0^l \cos \frac{2n+1}{2l}\pi x dx \\ &= \frac{4l^2}{(2n+1)^2\pi^2} \sin \frac{2n+1}{2l}\pi x \Big|_0^l \\ &= \frac{4l^2}{(2n+1)^2\pi^2} (-1)^n \end{aligned}$$

因此系数为: $C_n = \frac{2u_0}{l} \cdot \frac{2l}{(2n+1)\pi} + \frac{2q}{kl} \cdot \frac{4l^2}{(2n+1)^2\pi^2} (-1)^n = \frac{4u_0}{(2n+1)\pi} + \frac{8ql}{k(2n+1)^2\pi^2} (-1)^n$.

故解为: $u(t, x) = -\frac{q}{k}x + \sum_{n=0}^{+\infty} \left[\frac{4u_0}{(2n+1)\pi} + \frac{8ql}{k(2n+1)^2\pi^2} (-1)^n \right] e^{-(\frac{2n+1}{2l}a)^2 t} \sin \frac{2n+1}{2l}\pi x$, 得到 $\lim_{t \rightarrow +\infty} u(t, x) = -\frac{q}{k}x$.

(3) 先求解对应的齐次问题:

$$\begin{cases} a^2 u_t = u_{xx} \\ u(t, 0) = u(t, l) = 0 \end{cases}$$

分离变量: 令 $u(t, x) = T(t)X(x)$, 得到 $a^2 \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} \stackrel{\text{def}}{=} -\lambda$, 解以下固有值问题:

$$\begin{cases} X''(x) + X(x) = 0 \\ X(0) = X(l) = 0 \end{cases}$$

由 Sturm-Liouville 定理可知 $\lambda > 0$, 可令 $\lambda \stackrel{\text{def}}{=} k^2 > 0$, 得到 $X(x) = A \cos kx + B \sin kx, k \neq$

0.

由 $X(0) = A = 0$, 此时 $X(x) = B \sin kx$; 由 $X(l) = B \sin kl = 0$, 欲求得非零解则 $B \neq 0$, 故 $k_n = \frac{n\pi}{l}$, $n = 1, 2, \dots$

因此固有值为 $\lambda_n = k_n^2 = \left(\frac{n\pi}{l}\right)^2$, 固有函数为 $X_n(x) = \sin \frac{n\pi x}{l}$, $n \in \mathbb{N}_+$.

设 $u(t, x) = \sum_{n=1}^{+\infty} C_n T_n(t) \sin \frac{n\pi x}{l}$.

令 $f(x) = Ae^{-2x} = \sum_{n=1}^{+\infty} f_n \sin \frac{n\pi x}{l}$, $T_0 = \sum_{n=1}^{+\infty} t_n \sin \frac{n\pi x}{l}$, 则展开系数为:

$$f_n = \frac{2}{l} \int_0^l Ae^{-2x} \sin \frac{n\pi x}{l} dx = \frac{2n\pi A}{4l^2 + (n\pi)^2} [1 - (-1)^n e^{-2l}]$$

$$t_n = \frac{2}{l} \int_0^l T_0 \sin \frac{n\pi x}{l} dx = \frac{2T_0}{n\pi} [1 - (-1)^n]$$

代入原问题中得:

$$\begin{cases} a^2 \sum_{n=1}^{+\infty} T'_n(t) \sin \frac{n\pi x}{l} = - \sum_{n=1}^{+\infty} T_n(t) \left(\frac{n\pi}{l}\right)^2 \sin \frac{n\pi x}{l} + \sum_{n=1}^{+\infty} f_n \sin \frac{n\pi x}{l} \\ \sum_{n=1}^{+\infty} T_n(0) \sin \frac{n\pi x}{l} = \sum_{n=1}^{+\infty} t_n \sin \frac{n\pi x}{l} \end{cases}$$

对比系数得:

$$\begin{cases} a^2 T'_n(t) + \left(\frac{n\pi}{l}\right)^2 T_n(t) = f_n \\ T_n(0) = t_n \end{cases}$$

因此得到:

$$\begin{aligned} & \left[T_n e^{\left(\frac{n\pi}{la}\right)^2 t} \right]' = \frac{f_n}{a^2} e^{\left(\frac{n\pi}{la}\right)^2 t} \\ \Rightarrow T_n(t) &= \left(\frac{l}{n\pi}\right)^2 f_n + C_n e^{-\left(\frac{n\pi}{la}\right)^2 t} \\ \text{又 } T_n(0) &= \left(\frac{l}{n\pi}\right)^2 f_n + C_n = t_n \\ \Rightarrow C_n &= t_n - \left(\frac{l}{n\pi}\right)^2 f_n \\ \Rightarrow T_n(t) &= \left[t_n - \left(\frac{l}{n\pi}\right)^2 f_n \right] e^{-\left(\frac{n\pi}{la}\right)^2 t} + \left(\frac{l}{n\pi}\right)^2 f_n \\ \Rightarrow u(t, x) &= \sum_{n=1}^{+\infty} T_n(t) \sin \frac{n\pi x}{l} \end{aligned}$$

(4) 先求解对应的齐次问题:

$$\begin{cases} u_{tt} = a^2 u_{xx} \\ u(t, 0) = u(t, l) = 0 \end{cases}$$

分离变量: 令 $u(t, x) = T(t)X(x)$, 得到 $\frac{1}{a^2} \frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} \stackrel{\text{def}}{=} -\lambda$, 解以下固有值问题:

$$\begin{cases} X''(x) + X(x) = 0 \\ X(0) = X(l) = 0 \end{cases}$$

由 Sturm-Liouville 定理可知 $\lambda > 0$, 可令 $\lambda \stackrel{\text{def}}{=} k^2 > 0$, 得到 $X(x) = A \cos kx + B \sin kx$, $k \neq 0$.

由 $X(0) = A = 0$, 此时 $X(x) = B \sin kx$; 由 $X(l) = B \sin kl = 0$, 欲求得非零解则 $B \neq 0$, 故 $k_n = \frac{n\pi}{l}$, $n = 1, 2, \dots$

因此固有值为 $\lambda_n = k_n^2 = \left(\frac{n\pi}{l}\right)^2$, 固有函数为 $X_n(x) = \sin \frac{n\pi x}{l}$, $n \in \mathbb{N}_+$.

设 $u(t, x) = \sum_{n=1}^{+\infty} T_n(t) \sin \frac{n\pi x}{l}$.

令 $f(x) = b \sinh x = \sum_{n=1}^{+\infty} f_n \sin \frac{n\pi x}{l}$, $T_0 = \sum_{n=1}^{+\infty} t_n \sin \frac{n\pi x}{l}$, 则展开系数为:

$$\begin{aligned} f_n &= \frac{2b}{l} \int_0^l b \sinh x \sin \frac{n\pi x}{l} dx \\ &= \frac{2b}{l} \left[\cosh x \sin \frac{n\pi x}{l} \Big|_0^l - \int_0^l \cosh x \cdot \frac{n\pi}{l} \cos \frac{n\pi x}{l} dx \right] \\ &= \frac{2b}{l} \left[-\frac{n\pi}{l} \int_0^l \cos \frac{n\pi x}{l} d \sinh x \right] \\ &= \frac{2b}{l} \left[-\frac{n\pi}{l} \cos \frac{n\pi x}{l} \sinh x \Big|_0^l - \left(\frac{n\pi}{l}\right)^2 \int_0^l \sinh x \sin \frac{n\pi x}{l} dx \right] \\ &= \frac{2b}{l} \left[\frac{1}{1 + \left(\frac{n\pi}{l}\right)^2} \cdot \left(-\frac{n\pi}{l}\right) \cdot (-1)^n \sinh l \right] \\ \Rightarrow f_n &= \frac{2b}{l} \cdot \frac{\left(\frac{n\pi}{l}\right)}{1 + \left(\frac{n\pi}{l}\right)^2} \cdot (-1)^{n+1} \sinh l \end{aligned}$$

代入原问题中得:

$$\begin{cases} \sum_{n=1}^{+\infty} T_n''(t) \sin \frac{n\pi x}{l} = -\sum_{n=1}^{+\infty} T_n(t) \left(\frac{n\pi a}{l}\right)^2 \sin \frac{n\pi x}{l} + \sum_{n=1}^{+\infty} f_n \sin \frac{n\pi x}{l} \\ \sum_{n=1}^{+\infty} T_n(0) \sin \frac{n\pi x}{l} = \sum_{n=1}^{+\infty} T_n'(0) \sin \frac{n\pi x}{l} = 0 \end{cases}$$

对比系数得:

$$\begin{cases} T_n''(t) + \left(\frac{n\pi a}{l}\right)^2 T_n(t) = f_n \\ T_n(0) = T_n'(0) = 0 \end{cases}$$

因此得到:

$$T_n(t) = A_n \sin \frac{n\pi at}{l} + B_n \cos \frac{n\pi at}{l} + \left(\frac{l}{n\pi a}\right)^2 f_n$$

由 $T_n(0) = B_n + \left(\frac{l}{n\pi a}\right)^2 f_n = 0$, 得到 $B_n = -\left(\frac{l}{n\pi a}\right)^2 f_n$; 由 $T_n'(0) = \frac{n\pi a}{l} \cdot A_n = 0$, 得到 $A_n = 0$, 所以解为:

$$\begin{aligned} u(t, x) &= \sum_{n=1}^{+\infty} \left(\frac{l}{n\pi a}\right)^2 f_n \left(1 - \cos \frac{n\pi at}{l}\right) \sin \frac{n\pi x}{l} \\ &= \sum_{n=1}^{+\infty} \left(\frac{l}{n\pi a}\right)^2 \frac{2b}{l} \cdot \frac{\left(\frac{n\pi}{l}\right)}{1 + \left(\frac{n\pi}{l}\right)^2} \cdot (-1)^{n+1} \sinh l \left(1 - \cos \frac{n\pi at}{l}\right) \sin \frac{n\pi x}{l} \\ &= \frac{2bl^2 \sinh l}{\pi a^2} \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n(l^2 + n^2 \pi^2)} \left(1 - \cos \frac{n\pi at}{l}\right) \sin \frac{n\pi x}{l} \end{aligned}$$

(5) 令 $u(t, x) = v(t, x) + g$, 则 $v(t, x)$ 满足:

$$\begin{cases} v_{tt} = v_{xx} + g \\ v(t, 0) = v_x(t, l) = 0 \\ v(0, x) = v_t(0, x) = 0 \end{cases}$$

分离变量: 令 $v(t, x) = T(t)X(x)$, 得到 $\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} \stackrel{\text{def}}{=} -\lambda$, 解以下固有值问题:

$$\begin{cases} X''(x) + X(x) = 0 \\ X(0) = X'(l) = 0 \end{cases}$$

由 Sturm-Liouville 定理可知 $\lambda > 0$, 可令 $\lambda \stackrel{\text{def}}{=} k^2 > 0$, 得到 $X(x) = A \cos kx + B \sin kx$, $k \neq 0$.

由 $X(0) = A = 0$, 此时 $X(x) = B \sin kx$; 由 $X'(l) = kB \cos kl = 0$, 欲求得非零解则 $kB \neq 0$, 故 $k_n = \frac{(2n+1)\pi}{2l}$, $n = 0, 1, 2, \dots$

因此固有值为 $\lambda_n = k_n^2 = \left(\frac{2n+1}{2l}\pi\right)^2$, 固有函数为 $X_n(x) = \sin \frac{2n+1}{2l}\pi x$, $n \in \mathbb{N}_+$.

设 $v(t, x) = \sum_{n=0}^{+\infty} C_n T_n(t) \sin \frac{2n+1}{2l} \pi x$.

令 $g = \sum_{n=0}^{+\infty} g_n \sin \frac{2n+1}{2l} \pi x$, 则展开系数为:

$$g_n = \frac{2}{l} \int_0^l g \sin \frac{2n+1}{2l} \pi x dx = \frac{4g}{(2n+1)\pi}.$$

代入原问题中得:

$$\begin{cases} \sum_{n=0}^{+\infty} T_n''(t) \sin \frac{2n+1}{2l} \pi x + \sum_{n=0}^{+\infty} T_n(t) \left(\frac{2n+1}{2l} \pi \right)^2 \sin \frac{2n+1}{2l} \pi x = \sum_{n=0}^{+\infty} g_n \sin \frac{2n+1}{2l} \pi x \\ \sum_{n=0}^{+\infty} T_n'(0) \sin \frac{2n+1}{2l} \pi x = \sum_{n=0}^{+\infty} T_n(0) \sin \frac{2n+1}{2l} \pi x = 0 \end{cases}$$

对比系数得:

$$\begin{cases} T_n''(t) + \left(\frac{2n+1}{2l} \pi \right)^2 T_n(t) = g_n \\ T_n'(0) = T_n(0) = 0 \end{cases}$$

因此得到:

$$T_n(t) = A_n \cos \frac{2n+1}{2l} \pi t + B_n \sin \frac{2n+1}{2l} \pi t + \frac{4l^2 g_n}{(2n+1)^2 \pi^2}$$

由 $T_n(0) = A_n + \frac{4l^2 g_n}{(2n+1)^2 \pi^2} = 0$, 得到 $A_n = -\frac{4l^2 g_n}{(2n+1)^2 \pi^2}$; 由 $T_n'(0) = \frac{2n+1}{2l} \pi \cdot B_n = 0$, 得到 $B_n = 0$, 所以得到:

$$\begin{aligned} v(t, x) &= \sum_{n=0}^{+\infty} \frac{4l^2 g_n}{(2n+1)^2 \pi^2} \left(1 - \cos \frac{2n+1}{2l} \pi t \right) \sin \frac{2n+1}{2l} \pi x \\ &= \frac{16gl^2}{\pi^3} \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^3} \left(1 - \cos \frac{2n+1}{2l} \pi t \right) \sin \frac{2n+1}{2l} \pi x \end{aligned}$$

因此, 本题的解为:

$$u(t, x) = v(t, x) + Ex = Ex + \frac{16gl^2}{\pi^3} \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^3} \left(1 - \cos \frac{2n+1}{2l} \pi t \right) \sin \frac{2n+1}{2l} \pi x.$$

(6) 由于 $\Delta_2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$, 所以可以观察出方程显然有特解 (对称):

$$u_1 = \frac{a}{4} (x^2 + y^2) + \frac{b}{12} (x^4 - y^4) = \frac{a}{4} r^2 + \frac{b}{12} r^4 \cos 2\theta$$

作变换: $u = v + u_1 = V + \frac{a}{4}r^2 + \frac{b}{12}r^4 \cos 2\theta$, 则有:

$$\begin{cases} \Delta_2 v = 0, (a, b \text{ 为常数}, r < R) \\ v(R, \theta) = C - \frac{a}{4}R^2 - \frac{b}{12}R^4 \cos 2\theta \end{cases}$$

由齐次 Laplace 方程在圆内解的一般公式, 可设

$$v(r, \theta) = A_0 + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^n (A_n \cos n\theta + B_n \sin n\theta)$$

依据边界条件有:

$$v|_{r=R} = A_0 + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) = C - \frac{a}{4}R^2 - \frac{b}{12}R^4 \cos 2\theta$$

此式比较系数得到: $A_0 = C - \frac{a}{4}R^2$, $A_2 = -\frac{b}{12}R^4$, 其余的 A_n, B_n 都为 0, 这样 $v(r, \theta) = C - \frac{a}{4}R^2 - \frac{b}{12}R^4 \frac{r^2}{R^2} \cos 2\theta$, 最后得到:

$$u(r, \theta) = v(r, \theta) + u_1 = C + \frac{a}{4}(r^2 - R^2) + \frac{b}{12}r^2(r^2 - R^2) \cos 2\theta$$

11 课本 P_{255} T_{11}

在下列条件下, 求环域 $a < r < b$ 内泊松方程 $\Delta_2 u = A$ (A 为常数) 的解:

$$(1) u(a, \theta) = u_1, u(b, \theta) = u_2 \quad (u_1, u_2 \text{ 为常数});$$

$$(2) u(a, \theta) = u_1, \frac{\partial u(b, \theta)}{\partial n} = u_2.$$

Sol:

由于边界条件不含 θ , 可知: $u = u(r)$, 故得到

$$\begin{aligned}\Delta_2 u &= u''(r) + \frac{1}{r}u'(r) = A \\ \Rightarrow (ru'(r))' &= Ar \\ \Rightarrow ru'(r) &= \frac{A}{2}r^2 + C_1 \\ \Rightarrow u'(r) &= \frac{A}{2}r + \frac{C_1}{r} \\ \Rightarrow u(r) &= \frac{A}{4}r^2 + C_1 \ln r + C_2, \quad a < r < b\end{aligned}$$

(1) 代入边界条件得到:

$$\begin{cases} u|_{r=a} = \frac{A}{4}a^2 + C_1 \ln a + C_2 = u_1 \\ u|_{r=b} = \frac{A}{4}b^2 + C_1 \ln b + C_2 = u_2 \end{cases} \Rightarrow \begin{cases} C_1 = \frac{u_1 - u_2 + \frac{A}{4}(b^2 - a^2)}{\ln a - \ln b} \\ C_2 = u_2 - \frac{A}{4}b^2 - \frac{u_1 - u_2 + \frac{A}{4}(b^2 - a^2)}{\ln a - \ln b} \ln b \end{cases}$$

因此得到解为: $u(r, \theta) = u_2 + \frac{A}{4}(r^2 - b^2) + \frac{u_1 - u_2 + \frac{A}{4}(b^2 - a^2)}{\ln a - \ln b} \cdot (\ln r - \ln b), \quad a < r < b.$

(2) 代入边界条件得到:

$$\begin{cases} u|_{r=a} = \frac{A}{4}a^2 + C_1 \ln a + C_2 = u_1 \\ u'|_{r=b} = \frac{A}{2}b + \frac{C_1}{b} = u_2 \end{cases} \Rightarrow \begin{cases} C_1 = u_2 b - \frac{A}{2}b^2 \\ C_2 = u_1 - \frac{A}{4}a^2 - \left(u_2 b - \frac{A}{2}b^2\right) \ln a \end{cases}$$

因此得到解为: $u(r, \theta) = u_1 + \frac{A}{4}(r^2 - a^2) + b \left(u_2 - \frac{Ab}{2}\right) \ln \frac{r}{a}, \quad a < r < b.$

12 课本 P_{255} T_{12}

解下列矩形区域内的定解问题:

$$(1) \begin{cases} \Delta_2 u = f(x, y) & (0 < x < a, 0 < y < b) \\ u(0, y) = \varphi_1(y), u(a, y) = \varphi_2(y) \\ u(x, 0) = \psi_1(x), u(x, b) = \psi_2(x) \end{cases}$$

[提示: 先求一个满足以上边界条件得函数 $Ax + B$, 然后用固有函数方法求解.]

$$(2) \begin{cases} u_{tt} = a^2 \Delta_2 u & (t > 0, 0 < x < l_1, 0 < y < l_2), \\ u|_{x=0} = u|_{x=l_1} = u|_{y=0} = u|_{y=l_2} = 0, \\ u|_{t=0} = Axy(l_1 - x)(l_2 - y), \\ u_t|_{t=0} = 0; \end{cases}$$

$$(3) \begin{cases} u_t = a^2 \Delta_2 u & (t > 0, 0 < x < l_1, 0 < y < l_2) \\ u|_{x=0} = u|_{x=l_1} = u|_{y=0} = u|_{y=l_2} = 0, \\ u|_{t=0} = \varphi(x, y) \end{cases}$$

Sol:

(1) 显然, 满足以上关于 x 的边界条件的一个特解为: $u_1 = \varphi_1(y) + \frac{\varphi_2(y) - \varphi_1(y)}{a}x$
此时, 记 $u(x, y) = u_1 + v(x, y)$, 可知 $v(x, y)$ 满足:

$$\begin{cases} v_{xx} + v_{yy} = f(x, y) - \varphi_1''(y) - \frac{\varphi_2''(y) - \varphi_1''(y)}{a}x \\ v(0, y) = v(a, y) = 0 \\ v(x, 0) = \psi_1(x) - \varphi_1(0) - \frac{\varphi_2(0) - \varphi_1(0)}{a}x \\ v(x, b) = \psi_2(x) - \varphi_1(b) - \frac{\varphi_2(b) - \varphi_1(b)}{a}x \end{cases}$$

先求解对应的齐次问题:

$$\begin{cases} v_{xx} + v_{yy} = 0 \\ v(0, y) = v(a, y) = 0 \end{cases}$$

分离变量: 令 $v(x, y) = X(x)Y(y)$, 得到 $\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} \stackrel{\text{def}}{=} -\lambda$, 解以下固有值问题:

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(a) = 0 \end{cases}$$

由 Sturm-Liouville 定理可知 $\lambda > 0$, 可令 $\lambda \stackrel{\text{def}}{=} k^2 > 0$, 得到 $X(x) = A \cos kx + B \sin kx$, $k \neq 0$.

由 $X(0) = A = 0$, 此时 $X(x) = B \sin kx$; 由 $X(a) = B \sin ka = 0$, 欲求得非零解则 $B \neq 0$, 故 $k_n = \frac{n\pi}{a}$, $n = 1, 2, \dots$

因此固有值为 $\lambda_n = k_n^2 = \left(\frac{n\pi}{a}\right)^2$, 固有函数为 $X_n(x) = \sin \frac{n\pi x}{a}, n \in \mathbb{N}$.

设 $v(x, y) = \sum_{n=1}^{+\infty} f_n(y) \sin \frac{n\pi x}{a}$, $f(x, y) - \varphi_1''(y) - \frac{\varphi_2''(y) - \varphi_1''(y)}{a}x = \sum_{n=1}^{+\infty} C_n \sin \frac{n\pi x}{a}$, 其中

$$C_n = \frac{2}{a} \int_0^a \left[f(x, y) - \varphi_1''(y) - \frac{\varphi_2''(y) - \varphi_1''(y)}{a}x \right] \sin \frac{n\pi x}{a} dx$$

代入原问题中得:

$$\begin{cases} -\sum_{n=1}^{+\infty} \left(\frac{n\pi}{a}\right)^2 f_n(y) \sin \frac{n\pi x}{a} + \sum_{n=1}^{+\infty} f_n''(y) \sin \frac{n\pi x}{a} = \sum_{n=0}^{+\infty} C_n \sin \frac{n\pi x}{a} \\ \sum_{n=1}^{+\infty} f_n(0) \sin \frac{n\pi x}{a} = \psi_1(x) - \varphi_1(0) - \frac{\varphi_2(0) - \varphi_1(0)}{a}x \\ \sum_{n=1}^{+\infty} f_n(b) \sin \frac{n\pi x}{a} = \psi_2(x) - \varphi_1(b) - \frac{\varphi_2(b) - \varphi_1(b)}{a}x \end{cases}$$

对比系数得:

$$\begin{cases} f_n''(y) - \left(\frac{n\pi}{a}\right)^2 f_n(y) = \frac{2}{a} \int_0^a \left[f(x, y) - \varphi_1''(y) - \frac{\varphi_2''(y) - \varphi_1''(y)}{a}x \right] \sin \frac{n\pi x}{a} dx \\ f_n(0) = \frac{2}{a} \int_0^a \left[\psi_1(x) - \varphi_1(0) - \frac{\varphi_2(0) - \varphi_1(0)}{a}x \right] \sin \frac{n\pi x}{a} dx \\ f_n(b) = \frac{2}{a} \int_0^a \left[\psi_2(x) - \varphi_1(b) - \frac{\varphi_2(b) - \varphi_1(b)}{a}x \right] \sin \frac{n\pi x}{a} dx \end{cases} \quad (*)$$

因此得到:

$$u(x, y) = \varphi_1(y) + \frac{\varphi_2(y) - \varphi_1(y)}{a}x + \sum_{n=1}^{+\infty} f_n(y) \sin \frac{n\pi x}{a},$$

其中 $f_n(y)$ 由 (*) ODE 边值问题所确定.

(2) 分离变量: 令 $u(t, x, y) = T(t)X(x)Y(y)$, 得到

$$\frac{1}{a^2} \frac{T''(t)}{T(t)} - \frac{Y''(y)}{Y(y)} = \frac{X''(t)}{X(t)} \stackrel{\text{def}}{=} -\lambda$$

$$\frac{1}{a^2} \frac{T''(t)}{T(t)} + \lambda = \frac{Y''(y)}{Y(y)} \stackrel{\text{def}}{=} -\mu$$

得到以下两个固有值问题:

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(l_1) = 0 \end{cases} \quad \begin{cases} Y''(y) + \lambda Y(y) = 0 \\ Y(0) = Y(l_2) = 0 \end{cases}$$

由 Sturm-Liouville 定理可知 $\lambda, \mu > 0$, 可令 $\lambda \stackrel{\text{def}}{=} k^2 > 0$, 得到 $X(x) = \tilde{A} \cos kx + B \sin kx$, $k \neq 0$.

由 $X(0) = \tilde{A} = 0$, 此时 $X(x) = B \sin kx$; 由 $X(l_1) = B \sin kl_1 = 0$, 欲求得非零解则 $B \neq 0$, 故 $k_n = \frac{n\pi}{l_1}$, $n = 1, 2, \dots$

因此关于 $X(x)$ 的固有值为 $\lambda_n = k_n^2 = \left(\frac{n\pi}{l_1}\right)^2$, 固有函数为 $X_n(x) = \sin \frac{n\pi x}{l_1}$, $n \in \mathbb{N}$.

同理可得, 关于 $Y(y)$ 的固有值为 $\mu_n = \left(\frac{n\pi}{l_2}\right)^2$, 固有函数为 $Y_n(y) = \sin \frac{n\pi y}{l_2}$, $n \in \mathbb{N}$.

随后解关于 $T(t)$ 的 ODE: $T''(t) + a^2(\lambda + \mu)T(t) = 0$, 得到 $T_{mn}(t) = C_{mn} \cos \omega_{mn}t + D_{mn} \sin \omega_{mn}t$, $m, n \in \mathbb{N}$, 其中 $\omega_{mn} = \sqrt{\left(\frac{m\pi}{l_1}\right)^2 + \left(\frac{n\pi}{l_2}\right)^2}$.

叠加得到: $u(t, x, y) = \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} (A_{mn} \cos \omega_{mn}t + B_{mn} \sin \omega_{mn}t) \sin \frac{m\pi x}{l_1} \sin \frac{n\pi y}{l_2}$.

代入初始条件得:

$$\begin{cases} u_t(0, x, y) = \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} \omega_{mn} a D_{mn} \sin \frac{m\pi x}{l_1} \sin \frac{n\pi y}{l_2} = 0 \Rightarrow D_{mn} = 0, m, n \in \mathbb{N}_+ \\ u(0, x, y) = \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} C_{mn} \sin \frac{m\pi x}{l_1} \sin \frac{n\pi y}{l_2} = Axy(l_1 - x)(l_2 - y) \end{cases}$$

可知系数 C_{mn} 为:

$$\begin{aligned} C_{mn} &= \frac{4}{l_1 l_2} \int_0^{l_1} \int_0^{l_2} Axy(l_1 - x)(l_2 - y) \sin \frac{m\pi x}{l_1} \sin \frac{n\pi y}{l_2} dy dx \\ &= 4Al_1^2 l_2^2 \left[\int_0^1 x(1-x) \sin m\pi x dx \right] \cdot \left[\int_0^1 y(1-y) \sin n\pi y dy \right] \end{aligned}$$

其中

$$\begin{aligned} \int_0^1 x \sin n\pi x dx &= \frac{-1}{n\pi} \int_0^1 t d(\cos n\pi x) = -\frac{1}{n\pi} x \cos n\pi x \Big|_0^1 + \frac{1}{n\pi} \int_0^1 \cos n\pi x dx = -\frac{1}{n\pi} (-1)^n \\ \int_0^1 x^2 \sin n\pi x dx &= \frac{-1}{n\pi} \int_0^1 x^2 d(\cos n\pi x) \\ &= -\frac{1}{n\pi} x^2 \cos n\pi x \Big|_0^1 + \frac{1}{n\pi} \int_0^1 2x \cos n\pi x dx \\ &= -\frac{1}{n\pi} (-1)^n + \frac{2}{(n\pi)^2} \int_0^1 x d(\sin n\pi x) \\ &= -\frac{1}{n\pi} (-1)^n + \frac{2}{(n\pi)^2} x \sin n\pi x \Big|_0^1 - \frac{2}{(n\pi)^2} \int_0^1 \sin n\pi x dx \\ &= -\frac{1}{n\pi} (-1)^n + \frac{2}{(n\pi)^3} \cos n\pi x \Big|_0^1 \\ &= -\frac{1}{n\pi} (-1)^n + \frac{2}{(n\pi)^3} [(-1)^n - 1] \end{aligned}$$

因此得到

$$C_{mn} = 4Al_1^2l_2^2 \cdot \frac{2}{(m\pi)^3}[1 - (-1)^m] \cdot \frac{2}{(n\pi)^3}[1 - (-1)^n] = \frac{16Al_1^2l_2^2}{m^3n^3\pi^6}[1 - (-1)^m][1 - (-1)^n]$$

那么本题解为: $u(t, x, y) = \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} C_{mn} \cos \left[\sqrt{\left(\frac{m\pi}{l_1}\right)^2 + \left(\frac{n\pi}{l_2}\right)^2} at \right] \sin \frac{m\pi x}{l_1} \sin \frac{n\pi y}{l_2}$, 系数 C_{mn} 如上所述.

(3) 分离变量: 令 $u(t, x, y) = T(t)X(x)Y(y)$, 得到

$$\begin{aligned} \frac{1}{a^2} \frac{T'(t)}{T(t)} - \frac{Y''(y)}{Y(y)} &= \frac{X''(t)}{X(t)} \stackrel{\text{def}}{=} -\lambda \\ \frac{1}{a^2} \frac{T'(t)}{T(t)} + \lambda &= \frac{Y''(y)}{Y(y)} \stackrel{\text{def}}{=} -\mu \end{aligned}$$

得到以下两个固有值问题:

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(l_1) = 0 \end{cases} \quad \begin{cases} Y''(y) + \lambda Y(y) = 0 \\ Y(0) = Y(l_2) = 0 \end{cases}$$

由 Sturm-Liouville 定理可知 $\lambda, \mu > 0$, 可令 $\lambda \stackrel{\text{def}}{=} k^2 > 0$, 得到 $X(x) = A \cos kx + B \sin kx$, $k \neq 0$.

由 $X(0) = A = 0$, 此时 $X(x) = B \sin kx$; 由 $X(l_1) = B \sin kl_1 = 0$, 欲求得非零解则 $B \neq 0$, 故 $k_n = \frac{n\pi}{l_1}$, $n = 1, 2, \dots$

因此关于 $X(x)$ 的固有值为 $\lambda_n = k_n^2 = \left(\frac{n\pi}{l_1}\right)^2$, 固有函数为 $X_n(x) = \sin \frac{n\pi x}{l_1}$, $n \in \mathbb{N}$.

同理可得, 关于 $Y(y)$ 的固有值为 $\mu_n = \left(\frac{n\pi}{l_2}\right)^2$, 固有函数为 $Y_n(y) = \sin \frac{n\pi y}{l_2}$, $n \in \mathbb{N}$.

随后解关于 $T(t)$ 的 ODE: $T'(t) + a^2(\lambda + \mu)T(t) = 0$, 得到 $T_{mn}(t) = e^{-a^2(\lambda_m + \mu_n)t} = e^{-\left[\left(\frac{m\pi}{l_1}\right)^2 + \left(\frac{n\pi}{l_2}\right)^2\right]a^2t}$, $m, n \in \mathbb{N}$.

叠加得到: $u(t, x, y) = \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} C_{mn} e^{-\left[\left(\frac{m\pi}{l_1}\right)^2 + \left(\frac{n\pi}{l_2}\right)^2\right]a^2t} \sin \frac{m\pi x}{l_1} \sin \frac{n\pi y}{l_2}$.

代入初始条件得: $u(0, x, y) = \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} C_{mn} \sin \frac{m\pi x}{l_1} \sin \frac{n\pi y}{l_2} = \varphi(x, y)$, 可知系数 C_{mn} 为:

$$C_{mn} = \frac{4}{l_1 l_2} \int_0^{l_1} \int_0^{l_2} \varphi(x, y) \sin \frac{m\pi x}{l_1} \sin \frac{n\pi y}{l_2} dy dx$$

那么本题解为: $u(t, x, y) = \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} C_{mn} e^{-\left[\left(\frac{m\pi}{l_1}\right)^2 + \left(\frac{n\pi}{l_2}\right)^2\right]a^2t} \sin \frac{m\pi x}{l_1} \sin \frac{n\pi y}{l_2}$, 系数 C_{mn} 如上所述.

13 写在最后

本答案仅基于本人对数理方程的粗略理解，为方便同学们复习及纠正自己的答案而编写，很多题目也仅用了一种方法，仅提供参考，具体到对每个同学的意见，在批改作业的过程中也已经写了批注。

此外，考虑到本参考答案可能会存在一定的错误，后期可能会有修正版，可以[点击此处](#)查看最新版，对这些可能存在的错误，还请同学们海涵。

2020-2021 春季学期数理方程 B 助教
本科 18 级 地球和空间科学学院 刘炜昊
2021 年 4 月 于合肥

数理方程 B 第三章 参考答案

1. 解:

由

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

设

$$u(r, \theta, z) = R(r)\theta(\theta)Z(z)$$

则

$$R''(r)\theta(\theta)Z(z) + \frac{1}{r} R'(r)\theta(\theta)Z(z) + \frac{1}{r^2} R(r)\theta''(\theta)Z(z) + R(r)\theta(\theta)Z''(z) = 0$$

两边同时除以 $R(r)\theta(\theta)Z(z)$ 得

$$\frac{R''(r)}{R(r)} + \frac{1}{r} \frac{R'(r)}{R(r)} + \frac{1}{r^2} \frac{\theta''(\theta)}{\theta(\theta)} + \frac{Z''(z)}{Z(z)} = 0$$

设

$$\begin{cases} \frac{Z''(z)}{Z(z)} = \lambda \\ \frac{\theta''(\theta)}{\theta(\theta)} = -m^2 \end{cases}$$

则

$$\frac{R''(r)}{R(r)} + r \frac{R'(r)}{R(r)} = -\lambda + \frac{m^2}{r^2}$$

所以

$$\begin{cases} Z''(z) - \lambda Z(z) = 0 \\ \theta''(\theta) + m^2 \theta(\theta) = 0 \\ r^2 R''(r) + r R'(r) + (\lambda r^2 - m^2) R(r) = 0 \end{cases}$$

2. 解:

(1)

$$\frac{d}{dx} J_0(ax) \stackrel{ax=t}{\Rightarrow} \frac{d}{dx} J_0(ax) = \frac{d}{dt} J_0(t) \frac{dt}{dx} = a \frac{d}{dt} J_0(t) = a J_{-1}(t) = -a J_1(t) = -a J_1(ax)$$

(2)

$$\frac{d}{dx} [x J_1(ax)] \stackrel{ax=t}{\Rightarrow} \frac{d}{dt} \left[\frac{t}{a} J_1(t) \right] \frac{dt}{dx} = \frac{d}{dt} [t J_1(t)] = t J_0(t) = ax J_0(ax)$$

3. 解:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} J_0(x \cos \theta) \cos \theta d\theta &= \int_0^{\frac{\pi}{2}} \sum_{k=0}^{+\infty} (-1)^k \frac{1}{k! \Gamma(k+1)} \left(\frac{x \cos \theta}{2} \right)^{2k} \cos \theta d\theta \\ &= \int_0^{\frac{\pi}{2}} \sum_{k=0}^{+\infty} (-1)^k \frac{1}{k!^2} \left(\frac{x}{2} \right)^{2k} \cos^{2k+1} \theta d\theta \\ &= \sum_{k=0}^{+\infty} (-1)^k \frac{1}{k!^2} \left(\frac{x}{2} \right)^{2k} \int_0^{\frac{\pi}{2}} \cos^{2k+1} \theta d\theta = \sum_{k=0}^{+\infty} (-1)^k \frac{1}{k!^2} \left(\frac{x}{2} \right)^{2k} \frac{(2k)!!}{(2k+1)!!} = \sum_{k=0}^{+\infty} (-1)^k \frac{1}{k!^2} \left(\frac{x}{2} \right)^{2k} \frac{2^k k!}{(2k+1)!!} \\ &= \sum_{k=0}^{+\infty} (-1)^k \frac{1}{k!} \frac{x^{2k}}{2^k} \frac{1}{(2k+1)!!} = \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k}}{(2k)!! (2k+1)!!} \\ &= \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k}}{(2k+1)!} = \frac{1}{x} \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = \frac{\sin x}{x} \end{aligned}$$

4. 解

$$\frac{d}{dx} \sqrt{x} J_{\frac{3}{2}}(x) = \frac{d}{dx} x^{-1} \left[x^{\frac{3}{2}} J_{\frac{3}{2}}(x) \right] = -x^{-2} \left[x^{\frac{3}{2}} J_{\frac{3}{2}}(x) \right] + x^{-1} \frac{d}{dx} x^{\frac{3}{2}} J_{\frac{3}{2}}(x) = -x^{-\frac{1}{2}} J_{\frac{3}{2}}(x) + x^{-1} \left[x^{\frac{3}{2}} J_{\frac{1}{2}}(x) \right] = -x^{-\frac{1}{2}} J_{\frac{3}{2}}(x) + x^{\frac{1}{2}} J_{\frac{1}{2}}(x)$$

$$\begin{aligned}\frac{d^2}{dx^2}\sqrt{x}J_{\frac{3}{2}}(x) &= \frac{d}{dx}x^{-\frac{1}{2}}J_{\frac{3}{2}}(x) + \frac{d}{dx}x^{\frac{1}{2}}J_{\frac{1}{2}}(x) = \frac{d}{dx}x^{-2}\left[x^{\frac{3}{2}}J_{\frac{3}{2}}(x)\right] + x^{\frac{1}{2}}J_{-\frac{1}{2}}(x) \\ \frac{d}{dx}x^{-2}\left[x^{\frac{3}{2}}J_{\frac{3}{2}}(x)\right] &= 2x^{-3}\left[x^{\frac{3}{2}}J_{\frac{3}{2}}(x)\right] + (-x^{-2})\frac{d}{dx}\left[x^{\frac{3}{2}}J_{\frac{3}{2}}(x)\right] = 2x^{-\frac{3}{2}}J_{\frac{3}{2}}(x) - x^{-\frac{1}{2}}J_{\frac{1}{2}}(x)\end{aligned}$$

所以

$$\begin{aligned}\frac{d^2}{dx^2}\sqrt{x}J_{\frac{3}{2}}(x) &= 2x^{-\frac{3}{2}}J_{\frac{3}{2}}(x) - x^{-\frac{1}{2}}J_{\frac{1}{2}}(x) + x^{\frac{1}{2}}J_{-\frac{1}{2}}(x) \\ x^2\frac{d^2}{dx^2}\sqrt{x}J_{\frac{3}{2}}(x) &= 2x^{\frac{1}{2}}J_{\frac{3}{2}}(x) - x^{\frac{3}{2}}J_{\frac{1}{2}}(x) + x^{\frac{5}{2}}J_{-\frac{1}{2}}(x) \\ (x^2 - 2)\sqrt{x}J_{\frac{3}{2}}(x) &= x^{\frac{5}{2}}J_{\frac{3}{2}}(x) - 2x^{\frac{1}{2}}J_{\frac{1}{2}}(x) \\ x^2\frac{d^2}{dx^2}\sqrt{x}J_{\frac{3}{2}}(x) + (x^2 - 2)\sqrt{x}J_{\frac{3}{2}}(x) &= x^{\frac{5}{2}}J_{\frac{3}{2}}(x) - x^{\frac{3}{2}}J_{\frac{1}{2}}(x) + x^{\frac{5}{2}}J_{-\frac{1}{2}}(x)\end{aligned}$$

由递推公式

$$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x}J_v(x)$$

取

$$v = \frac{1}{2}$$

则有

$$J_{-\frac{1}{2}}(x) + J_{\frac{3}{2}}(x) = \frac{1}{x}J_{\frac{1}{2}}(x)$$

两边同乘以 $x^{\frac{5}{2}}$, 得

$$x^{\frac{5}{2}}J_{-\frac{1}{2}}(x) + x^{\frac{5}{2}}J_{\frac{3}{2}}(x) = x^{\frac{3}{2}}J_{\frac{1}{2}}(x) \Rightarrow x^{\frac{5}{2}}J_{-\frac{1}{2}}(x) - x^{\frac{3}{2}}J_{\frac{1}{2}}(x) = -x^{\frac{5}{2}}J_{\frac{3}{2}}(x)$$

故

$$x^2\frac{d^2}{dx^2}\sqrt{x}J_{\frac{3}{2}}(x) + (x^2 - 2)\sqrt{x}J_{\frac{3}{2}}(x) = 0$$

5. 解:

由例 1 可得等式

$$e^{ix\sin\theta} = J_0(x) + 2\sum_{k=1}^{+\infty}J_{2k}(x)\cos 2k\theta + 2i\sum_{k=1}^{+\infty}J_{2k-1}(x)\sin(2k-1)\theta$$

(1) 令 $\theta = 0$, 则

$$1 = J_0(x) + 2\sum_{k=1}^{+\infty}J_{2k}(x)$$

(2) 令 $\theta = \frac{\pi}{2}$, 则

$$e^{ix} = J_0(x) + 2\sum_{k=1}^{+\infty}(-1)^kJ_{2k}(x) + 2i\sum_{k=1}^{+\infty}(-1)^{k-1}J_{2k-1}(x)$$

令 $\theta = -\frac{\pi}{2}$, 则

$$e^{-ix} = J_0(x) + 2\sum_{k=1}^{+\infty}(-1)^kJ_{2k}(x) - 2i\sum_{k=1}^{+\infty}(-1)^{k-1}J_{2k-1}(x)$$

所以

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} = 2\sum_{k=1}^{+\infty}(-1)^{k-1}J_{2k-1}(x) = 2\sum_{k=0}^{+\infty}(-1)^kJ_{2k+1}(x)$$

(3) 由(2)得

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} = J_0(x) + 2 \sum_{k=1}^{+\infty} (-1)^k J_{2k}(x)$$

6. 解:

递推公式为

$$\begin{aligned} J'_v(x) &= J_{v-1}(x) - \frac{v}{x} J_v(x) \\ J'_v(x) &= \frac{v}{x} J_v(x) - J_{v+1}(x) \\ J_{v-1}(x) + J_{v+1}(x) &= \frac{2v}{x} J_v(x) \\ J_{v-1}(x) - J_{v+1}(x) &= 2J'_v(x) \end{aligned}$$

-

(1) 在第二个递推公式中取 $v=0$, 则

$$J'_0(x) = -J_1(x)$$

$$J''_0(x) = -J'_1(x) \xrightarrow{\text{第二个递推公式}} J''_0(x) = J_2(x) - \frac{1}{x} J_1(x) = J_2(x) + \frac{1}{x} J'_0(x) \Rightarrow J_2(x) = J''_0(x) - \frac{1}{x} J'_0(x)$$

(2) 直接证明比较困难, 我们可以考虑 **执果索因**

由要证的

$$J_3(x) + 3J'_0(x) + 4J_0^{(3)}(x) = 0$$

观察这个等式我们发现其各项系数均为常数, 对比四个递推公式我们首先从第四个入手, 取 $v=2$, 得

$$J_1(x) - J_3(x) = 2J'_2(x)$$

即

$$\begin{aligned} J_1(x) - J_3(x) &= 2J'_2(x) \\ J'_0(x) &= -J_1(x) \end{aligned}$$

得

$$J_3(x) + J'_0(x) + 2J'_2(x) = 0$$

对比其与我们要证的式子, 即可知需证

$$J'_2(x) - J'_0(x) = 2J_0^{(3)}(x)$$

再从第三个式子入手, 取 $v=1$, 得

$$J_0(x) - J_2(x) = 2J'_1(x) = -2J''_0(x) \Rightarrow J_2(x) - J_0(x) = 2J''_0(x)$$

两边求导, 得

$$J'_2(x) - J'_0(x) = 2J_0^{(3)}(x)$$

综上可知原式得证

7. 解: (1)

$$\begin{aligned} \frac{d}{dx} [J_v^2(x)] &= 2J_v(x)J'_v(x) \xrightarrow{\text{递推公式 4}} 2J_v(x)[J_{v-1}(x) - J_{v+1}(x)] \xrightarrow{\text{递推公式 3}} \frac{x}{2v} [J_{v-1}(x) + J_{v+1}(x)] [J_{v-1}(x) - J_{v+1}(x)] \\ &= \frac{x}{2v} [J_{v-1}^2(x) - J_{v+1}^2(x)] \end{aligned}$$

(2)

$$\frac{d}{dx} [xJ_0(x)J_1(x)] = J'_0(x)xJ_1(x) + J_0(x)\frac{d}{dx} [xJ_1(x)] = -xJ_1^2(x) + J_0(x)xJ_0(x) = x[J_0^2(x) - J_1^2(x)]$$

8. 解:

$$\begin{aligned} \int_0^x x^n J_0(x) dx &= \int_0^x x^{n-1} x J_0(x) dx = \int_0^x x^{n-1} dx J_1(x) = x^n J_1(x) - (n-1) \int_0^x x^{n-1} J_1(x) dx = x^n J_1(x) + (n-1) \int_0^x x^{n-1} J'_0(x) dx \\ &= x^n J_1(x) + (n-1) \int_0^x x^{n-1} dJ_0(x) = x^n J_1(x) + (n-1)x^{n-1} J_0(x) - (n-1)^2 \int_0^x x^{n-2} J_0(x) dx \end{aligned}$$

(1)

$$\int_0^x x^3 J_0(x) dx = x^3 J_1(x) + 2x^2 J_0(x) - 4 \int_0^x x J_0(x) dx = x^3 J_1(x) + 2x^2 J_0(x) - 4x J_1(x) = 2x^2 J_0(x) + (x^3 - 4x) J_1(x)$$

(2)

$$\int_0^x x^4 J_1(x) dx = - \int_0^x x^4 J_0'(x) dx = - \int_0^x x^4 dJ_0(x) = 4 \int_0^x x^3 J_0(x) dx - x^4 J_0(x) = (8x^2 - x^4) J_0(x) + 4(x^3 - 4x) J_1(x)$$

9. 解：由递推公式

$$J_{v-1}(x) - J_{v+1}(x) = 2J_v'(x)$$

$$\int J_3(x) dx = \int J_1(x) - 2J_2'(x) dx = \int J_1(x) dx - J_2(x) = \int -J_0'(x) dx - J_2(x)$$

$$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x} J_v(x) \Rightarrow J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$$

所以

$$\int J_3(x) dx = -J_0(x) - 2 \left[\frac{2}{x} J_1(x) - J_0(x) \right] + C = J_0(x) - 4x^{-1} J_1(x) + C$$

10. 解：(1)

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$$

$$\int_0^x x^2 J_2(x) dx = \int_0^x 2x J_1(x) - x^2 J_0(x) dx$$

$$\int_0^x x J_1(x) dx = - \int_0^x x J_0'(x) dx = - \int_0^x x dJ_0(x) = \int_0^x J_0(x) dx - x J_0(x)$$

$$\int_0^x x^2 J_0(x) dx = \int_0^x x dx J_1(x) = x^2 J_1(x) - \int_0^x x J_1(x) dx$$

所以

$$\int_0^x x^2 J_2(x) dx = \int_0^x 2x J_1(x) - x^2 J_0(x) dx = -x^2 J_1(x) + 3 \int_0^x x J_1(x) dx$$

(2)

$$J_1(x) = -J_0'(x)$$

$$\int_0^x x J_1(x) dx = - \int_0^x x J_0'(x) dx = - \int_0^x x dJ_0(x) = -x J_0(x) + \int_0^x J_0(x) dx$$

11. 解：(1)

$$\int J_0(x) \sin x dx = x J_0(x) \sin x - \int x dJ_0(x) \sin x$$

$$\int x dJ_0(x) \sin x = \int x [J_0'(x) \sin x + J_0(x) \cos x] dx$$

$$= \int x J_0'(x) \sin x dx + \int x J_0(x) \cos x dx = - \int x J_1(x) \sin x dx + \int \cos x dx J_1(x) = x J_1(x) \cos x + C$$

$$\int x J_1(x) \sin x dx = - \int x J_1(x) d \cos x = -x J_1(x) \cos x + \int \cos x dx J_1(x)$$

所以

$$\int J_0(x) \sin x dx = x J_0(x) \sin x - x J_1(x) \cos x + C$$

(2)

$$\int J_0(x) \cos x dx = x J_0(x) \cos x - \int x dJ_0(x) \cos x$$

$$\int x dJ_0(x) \cos x = \int x [J_0'(x) \cos x - J_0(x) \sin x] dx$$

$$= \int x J_0'(x) \cos x dx - \int x J_0(x) \sin x dx = - \int x J_1(x) \cos x dx - \int \sin x dx J_1(x) = -x J_1(x) \sin x + C$$

$$\int x J_1(x) \cos x dx = \int x J_1(x) d \sin x = x J_1(x) \sin x - \int \sin x dx J_1(x)$$

12. 解:

$$f(x) = \sum_{n=1}^{+\infty} A_n J_0(\omega_n x)$$

其中

$$A_n = \frac{\langle f(x), J_0(\omega_n x) \rangle}{\langle J_0(\omega_n x), J_0(\omega_n x) \rangle} = \frac{\int_0^2 x f(x) J_0(\omega_n x) dx}{N_{01}^2} = \frac{\int_0^1 x J_0(\omega_n x) dx}{\frac{2^2}{2} J_1^2(2\omega_n)}$$

$$\int_0^1 x J_0(\omega_n x) dx \xrightarrow{\omega_n x=t} \frac{1}{\omega_n^2} \int_0^{\omega_n} t J_0(t) dt = \frac{1}{\omega_n^2} \omega_n J_1(\omega_n) = \frac{J_1(\omega_n)}{\omega_n}$$

$$A_n = \frac{J_1(\omega_n)}{2\omega_n J_1^2(2\omega_n)}$$

所以

$$f(x) = \sum_{n=1}^{+\infty} A_n J_0(\omega_n x) = \sum_{n=1}^{+\infty} \frac{J_1(\omega_n)}{2\omega_n J_1^2(2\omega_n)} (\omega_n x)$$

13. 解:

$$f(x) = \sum_{n=1}^{+\infty} A_n J_1(\omega_n x)$$

其中

$$A_n = \frac{\langle f(x), J_1(\omega_n x) \rangle}{\langle J_1(\omega_n x), J_1(\omega_n x) \rangle} = \frac{\int_0^1 x f(x) J_1(\omega_n x) dx}{N_{11}^2} = \frac{\int_0^1 x^2 J_1(\omega_n x) dx}{\frac{1^2}{2} J_2^2(\omega_n)}$$

$$\int_0^1 x^2 J_1(\omega_n x) dx \xrightarrow{\omega_n x=t} \frac{1}{\omega_n^3} \int_0^{\omega_n} t^2 J_1(t) dt = \frac{1}{\omega_n^3} \omega_n^2 J_2(\omega_n) = \frac{J_2(\omega_n)}{\omega_n}$$

$$A_n = \frac{2}{\omega_n J_2(\omega_n)}$$

所以

$$f(x) = \sum_{n=1}^{+\infty} A_n J_1(\omega_n x) = f(x) = \sum_{n=1}^{+\infty} \frac{2}{\omega_n J_2(\omega_n)} J_1(\omega_n x)$$

14. 解: 依题

$$\begin{aligned}
\int_0^1 x f^2(x) dx &= \int_0^1 x \left[\sum_{n=1}^{+\infty} A_n J_0(\omega_n x) \right]^2 dx \\
&= \int_0^1 x \left[\sum_{n=1}^{+\infty} A_n^2 J_0^2(\omega_n x) + 2 \sum_{n=2}^{+\infty} A_1(A_2 + \cdots A_n) J_0(\omega_1 x) J_0(\omega_n x) \right. \\
&\quad \left. + 2 \sum_{n=3}^{+\infty} A_2(A_3 + \cdots A_n) J_0(\omega_2 x) J_0(\omega_n x) + \cdots 2A_{n-1}A_n J_0(\omega_{n-1} x) J_0(\omega_n x) \right] dx \\
&= \sum_{n=1}^{+\infty} A_n^2 \int_0^1 x J_0^2(\omega_n x) dx + 2 \sum_{n=2}^{+\infty} A_1(A_2 + \cdots A_n) \int_0^1 x J_0(\omega_1 x) J_0(\omega_n x) dx \\
&\quad + \cdots 2A_{n-1}A_n \int_0^1 J_0(\omega_{n-1} x) J_0(\omega_n x) dx
\end{aligned}$$

依题利用内积的性质

$$\langle J_0(\omega_n x), J_0(\omega_n x) \rangle = N_{01}^2 = \frac{1^2}{2} J_1^2(\omega_n) = \frac{1}{2} J_1^2(\omega_n)$$

$$\langle J_0(\omega_n x), J_0(\omega_m x) \rangle = 0, n \neq m$$

故

$$\int_0^1 x f^2(x) dx = \sum_{n=1}^{+\infty} A_n^2 \int_0^1 x J_0^2(\omega_n x) dx = \frac{1}{2} \sum_{n=1}^{+\infty} A_n^2 J_1^2(\omega_n)$$

15. 解:

$$1 = \sum_{n=1}^{+\infty} \frac{2}{\omega_n J_1(\omega_n)} J_0(\omega_n x) = \sum_{n=1}^{+\infty} A_n J_0(\omega_n x)$$

对比系数得

$$A_n = \frac{2}{\omega_n J_1(\omega_n)}, n = 1, 2, \cdots$$

令 $f(x) = 1$

$$\frac{1}{2} = \int_0^1 x dx = \frac{1}{2} \sum_{n=1}^{+\infty} \frac{4}{\omega_n^2 J_1^2(\omega_n)} J_1^2(\omega_n) = 2 \sum_{n=1}^{+\infty} \frac{1}{\omega_n^2} \Rightarrow \sum_{n=1}^{+\infty} \frac{1}{\omega_n^2} = \frac{1}{4}$$

16. 解: 先说分析思路

依题, 设温度函数为 $u(t, r)$

所以

$$\begin{cases} u_t = u_{rr} + \frac{1}{r} u_r & (t > 0, 0 < r < R) \\ u(t, R) = u_0, u(0, r) = 0 \end{cases}$$

解决此类问题, 当然要选择适当的边界条件, 因此如何选择适当的边界条件成为至关重要的问题。首先先无脑进行分离变量

$$u(t, r) = T(t)R(r)$$

$$\begin{cases} T'(t)R(r) = T(t)R''(r) + \frac{1}{r} T(t)R'(r) \\ T(t)R(R) = u_0, T(0)R(r) = 0 \end{cases}$$

即

$$\begin{cases} \frac{T'(t)}{T(t)} = \frac{R''(r)}{R(r)} + \frac{1}{rR(r)} R'(r) = \frac{rR''(r) + R'(r)}{rR(r)} = -\lambda \\ T(t)R(R) = 0, T(0)R(r) = 0 \end{cases}$$

则有

$$\begin{cases} T'(t) + \lambda T(t) = 0 \\ r^2 R''(r) + rR'(r) + \lambda r^2 = 0 \\ T(t)R(R) = 0, T(0)R(r) = 0 \end{cases}$$

得到了一个 0 阶贝塞尔方程, 考虑贝塞尔方程的边界条件

$$\begin{cases} \{x^2 y'' + xy' + (\lambda x^2 - \nu^2)y = 0, (0 < x < a, \nu \geq 0) \\ \alpha y(a) + \beta y'(a) = 0 \\ y(0) \text{有界} \end{cases}$$

考虑到本题中的条件

$$T(t)R(R) = u_0, T(0)R(r) = 0$$

则需要“调整”为, 此题中 $R(0)$ 有界 (物理意义上为初始温度不可能为无穷大

$$R(R) = 0$$

设

$$u(t, r) = v(t, r) + w(t, r)$$

$v(t, r)$ 满足分离变量之后的贝塞尔方程

$$\begin{cases} v_t = v_{rr} + \frac{1}{r}v_r & (t > 0, 0 < r < R) \\ v(t, R) = 0, u(0, r) = ? \end{cases}$$

$$\begin{cases} w_t = w_{rr} + \frac{1}{r}w_r & (t > 0, 0 < r < R) \\ w(t, R) = u_0, w(0, r) = ? \end{cases}$$

那么如何解决两个? 条件的匹配呢, 可以从找特解要尽可能往简单找的思路考虑。之前分析的 $v(t, r)$ 满足分离变量之后的贝塞尔方程, 故其对应齐次方程。则 $w(t, r)$ 为特解。

$$w(t, R) = u_0$$

则可取 $w(t, r) = u_0$,

$$\begin{cases} w_t = 0 & (t > 0, 0 < r < R) \\ w(t, R) = u_0, w(0, r) = u_0 \end{cases}$$

那么

$$\begin{cases} v_t = v_{rr} + \frac{1}{r}v_r & (t > 0, 0 < r < R) \\ v(t, R) = 0, v(0, r) = -u_0 \end{cases}$$

$v(t, r)$ 的解决:

$$v(t, r) = T(t)R(r)$$

$$\begin{cases} T'(t)R(r) = T(t)R''(r) + \frac{1}{r}T(t)R'(r) \\ T(t)R(R) = 0, T(0)R(r) = -u_0 \end{cases}$$

即

$$\begin{cases} \frac{T'(t)}{T(t)} = \frac{R''(r)}{R(r)} + \frac{1}{rR(r)}R'(r) = \frac{rR''(r) + R'(r)}{rR(r)} = -\lambda \\ T(t)R(R) = 0, T(0)R(r) = 0 \end{cases}$$

则有

$$\begin{cases} T'(t) + \lambda T(t) = 0 \\ r^2 R''(r) + rR(r) + \lambda r^2 = 0 \\ R(R) = 0, T(0)R(r) = -u_0 \end{cases}$$

下面来解决 $R(r)$ 对应的贝塞尔方程:

$$\begin{cases} r^2 R''(r) + rR(r) + \lambda r^2 = 0 \\ R(R) = 0, R(0) \text{有界} \end{cases}$$

此为一零阶贝塞尔方程, 且边界条件为第一类

则其固有值为

$$\lambda_n = \omega_n^2, n = 1, 2, \dots$$

固有函数为

$$R_n(x) = J_0(\omega_n r), n = 1, 2, \dots$$

其中 ω_n 为 $J_0(\omega R) = 0$ 的正实根

故

$$T_n'(t) + \lambda_n T_n(t) = 0$$

得

$$T_n(t) = e^{-\omega_n^2 t}$$

所以

$$v(t, r) = \sum_{n=1}^{+\infty} A_n e^{-\omega_n^2 t} J_0(\omega_n r)$$

又

$$v(0, r) = \sum_{n=1}^{+\infty} A_n J_0(\omega_n r) = -u_0$$

所以

$$A_n = \frac{\langle J_0(\omega_n r), -u_0 \rangle}{\langle J_0(\omega_n r), J_0(\omega_n r) \rangle} = \frac{-u_0 \int_0^R r J_0(\omega_n r) dx}{N_{01}^2} = -u_0 \frac{\frac{1}{\omega_n^2} \int_0^{\omega_n R} t J_0(t) dt}{\frac{R^2}{2} J_1^2(\omega_n R)} = -u_0 \frac{2}{R \omega_n J_1(\omega_n R)}$$

所以

$$v(t, r) = -2u_0 \frac{1}{R} \sum_{n=1}^{+\infty} \frac{1}{\omega_n J_1(\omega_n R)} e^{-\omega_n^2 t} J_0(\omega_n r)$$

故

$$u(t, r) = v(t, r) + w(t, r) = u_0 - 2u_0 \frac{1}{R} \sum_{n=1}^{+\infty} \frac{1}{\omega_n J_1(\omega_n R)} e^{-\omega_n^2 t} J_0(\omega_n r)$$

参考答案实际是对 $v(t, r)$ 作了一个处理

设

$$x = \frac{r}{R}$$

则 $v(t, r) = v(t, x)$

$$\begin{cases} v_t = v_{rr} + \frac{1}{r} v_r & (t > 0, 0 < r < R) \\ v(t, R) = 0, u(0, r) = -u_0 \end{cases}$$

变为

$$\begin{cases} v_t = \frac{1}{R^2} v_{xx} + \frac{1}{R^2 x} v_x & (t > 0, 0 < x < 1) \\ v(t, 1) = 0, v(0, x) = -u_0 \end{cases}$$

$v(t, r)$ 的解决:

$$v(t, x) = T(t)R(x)$$

$$\begin{cases} T'(t)R(x) = \frac{1}{R^2} T(t)R''(x) + \frac{1}{R^2 x} T(t)R'(x) \\ T(t)R(1) = 0, T(0)R(x) = -u_0 \end{cases}$$

即

$$\begin{cases} R^2 \frac{T'(t)}{T(t)} = \frac{R''(x)}{R(x)} + \frac{1}{xR(x)} R'(x) = \frac{xR''(x) + R'(x)}{xR(x)} = -\lambda \\ T(t)R(1) = 0, T(0)R(x) = 0 \end{cases}$$

则有

$$\begin{cases} T'(t) + \frac{\lambda}{R^2} T(t) = 0 \\ x^2 R''(x) + xR'(x) + \lambda x^2 = 0 \\ R(1) = 0, T(0)R(x) = -u_0 \end{cases}$$

下面来解决 $R(r)$ 对应的贝塞尔方程:

$$\begin{cases} x^2 R''(x) + xR'(x) + \lambda x^2 = 0 \\ R(1) = 0, R(0) \text{有界} \end{cases}$$

此为一零阶贝塞尔方程, 且边界条件为第一类

则其固有值为

$$\lambda_n = \omega_n^2, n = 1, 2, \dots$$

固有函数为

$$R_n(x) = J_0(\omega_n x), n = 1, 2, \dots$$

其中 ω_n 为 $J_0(\omega) = 0$ 的正实根

故

$$T_n'(t) + \frac{\lambda_n}{R^2} T_n(t) = 0$$

得

$$T_n(t) = e^{-\frac{\omega_n^2}{R^2} t}$$

所以

$$v(t, r) = \sum_{n=1}^{+\infty} A_n e^{-\frac{\omega_n^2}{R^2} t} J_0(\omega_n x)$$

又

$$v(0, x) = \sum_{n=1}^{+\infty} A_n J_0(\omega_n x) = -u_0$$

所以

$$A_n = \frac{\langle J_0(\omega_n x), -u_0 \rangle}{\langle J_0(\omega_n), J_0(\omega_n x) \rangle} = \frac{-u_0 \int_0^1 x J_0(\omega_n x) dx}{N_{01}^2} = -u_0 \frac{\frac{1}{\omega_n^2} \int_0^{\omega_n} t J_0(t) dt}{\frac{1}{2} J_1^2(\omega_n)} = -u_0 \frac{2}{\omega_n J_1(\omega_n)}$$

所以

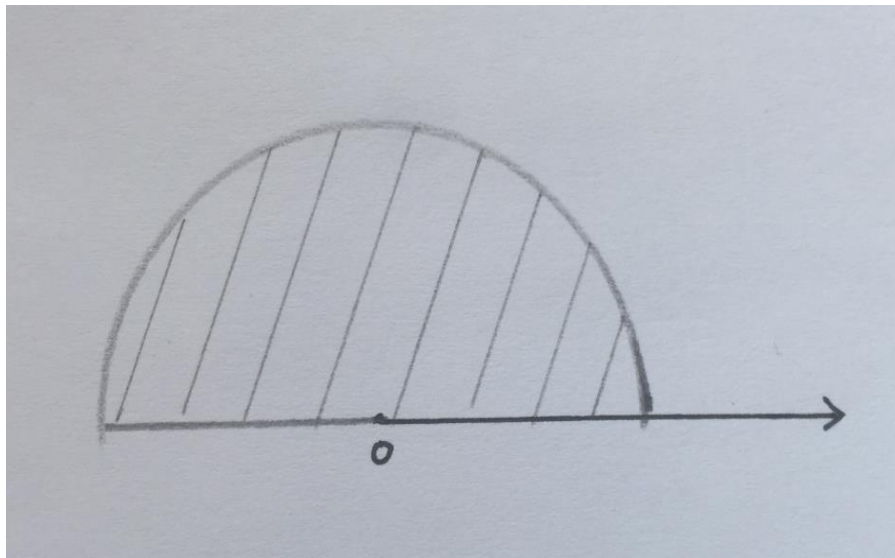
$$v(t, r) = -2u_0 \sum_{n=1}^{+\infty} \frac{1}{\omega_n J_1(\omega_n)} e^{-\omega_n^2 t} J_0(\omega_n x)$$

故

$$u(t, r) = v(t, r) + w(t, r) = u_0 - 2u_0 \sum_{n=1}^{+\infty} \frac{1}{\omega_n J_1(\omega_n)} e^{-\frac{\omega_n^2}{R^2} t} J_0(\omega_n x) = u_0 - 2u_0 \sum_{n=1}^{+\infty} \frac{1}{\omega_n J_1(\omega_n)} e^{-\frac{\omega_n^2}{R^2} t} J_0\left(\omega_n \frac{r}{R}\right)$$

17. 解: 边缘固定指的是边缘的曲线部分和直线部分都无横向振动

考虑如图极坐标系



设该薄膜的横向振动函数为 $u(t, r, \varphi)$

则根据题意列出方程及边界条件如下:

$$\begin{cases} u_{tt} = a^2 \Delta_2 u = a^2 (u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi}) \\ u(t, r, 0) = u(t, r, \pi) = 0 \\ u(t, 0, \varphi) = u(t, R, \varphi) = 0 \\ (0 < r < R, t > 0, 0 \leq \varphi \leq \pi) \end{cases}$$

由分离变量法, 设 $u(t, r, \varphi) = T(t)R(r)\Phi(\varphi)$

则有

$$\begin{cases} T''(t)R(r)\Phi(\varphi) = a^2\Delta_2 u = a^2(T(t)R''(r)\Phi(\varphi) + \frac{1}{r}T(t)R'(r)\Phi(\varphi) + \frac{1}{r^2}T(t)R(r)\Phi''(\varphi)) \\ \Phi(0) = \Phi(\pi) = 0 \\ (0 < r < R, t > 0, 0 \leq \varphi \leq \pi) \end{cases}$$

即

$$\begin{cases} \frac{T''(t)}{a^2T(t)} = \frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Phi''(\varphi)}{\Phi(\varphi)} \\ \Phi(0) = \Phi(\pi) = 0 \\ R(0) = R(b) = 0 \\ (0 < r < R, t > 0, 0 \leq \varphi \leq \pi) \end{cases}$$

设

$$\frac{\Phi''(\varphi)}{\Phi(\varphi)} = -m^2$$

由

$$\Phi(0) = \Phi(\pi) = 0$$

由第一类边界条件知其特征值大于 0

其特征值为

$$m, m = 1, 2, \dots$$

其特征函数为

$$\Phi_m(\varphi) = \sin m\varphi, m = 1, 2, \dots$$

设

$$\frac{T''(t)}{a^2T(t)} = -k^2$$

则有

$$\begin{cases} r^2R''(r) + rR'(r) + (k^2r^2 - m^2)R(r) = 0 \\ R(R) = 0, R(0) = 0 \text{(有界)} \end{cases}$$

则该方程的固有值为

$$k_n^2 = \omega_{mn}^2, m = 1, 2, \dots, n = 1, 2, \dots$$

则该方程的固有函数为

$$R_n(r) = J_m(\omega_{mn}r), m = 1, 2, \dots, n = 1, 2, \dots$$

其中 ω_{mn} 为方程 $J_m(\omega b)$ 的所有正根 $m = 1, 2, \dots, n = 1, 2, \dots$

所以

$$\frac{T_n''(t)}{a^2T_n(t)} = -\omega_{mn}^2$$

$$T_n(t) = A_{mn}\cos(\omega_{mn}t) + B_{mn}\sin(\omega_{mn}t)$$

所以

$$u(t, r, \varphi) = \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} J_m(\omega_{mn}r) \sin m\varphi [A_{mn}\cos(\omega_{mn}t) + B_{mn}\sin(\omega_{mn}t)]$$

18. 解:

(1) 设 $u(r, z) = R(r)Z(z)$

由分离变量法得

$$\begin{cases} R''(r)Z(z) + \frac{1}{r}R'(r)Z(z) + R(r)Z''(z) = 0 \Rightarrow \frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{Z''(z)}{Z(z)} = 0 \\ R(a)Z(z) = 0 \\ R(r)Z(0) = 0, R(r)Z(l) = T_0 \end{cases}$$

设

$$\frac{R''(r)}{R(r)}+\frac{1}{r}\frac{R'(r)}{R(r)}=-\lambda$$

得

$$\begin{cases} r^2R(r)+rR'(r)+\lambda r^2R(r)=0 \\ R(a)=0,R(0)\text{有界} \end{cases}$$

则其固有值为

$$\lambda=\omega_n^2,n=1,2,\cdots$$

其固有函数为

$$R_n(r)=J_0(\omega_nr),n=1,2,\cdots$$

其中 ω_n 为

$$J_0(\omega a)=0$$

的正根

所以

$$\begin{cases} \frac{Z_n''(z)}{Z_n(z)}-\omega_n^2=0 \\ Z(0)=0 \end{cases}$$

则

$$\begin{cases} Z_n(z)=A_ne^{-\omega_nz}+B_ne^{\omega_nz} \\ Z_n(0)=A_n+B_n=0\Rightarrow A_n=-B_n \end{cases}$$

所以

$$Z_n(z)=C_nsh(\omega_nz)$$

所以

$$u(r,z)=\sum_{n=1}^{+\infty}C_nsh(\omega_nz)J_0(\omega_nr)$$

$$u(r,l)=\sum_{n=1}^{+\infty}C_nsh(\omega_nl)J_0(\omega_nr)=T_0$$

$$C_nsh(\omega_nl)=\frac{\langle T_0,J_0(\omega_nr)\rangle}{\langle J_0(\omega_nr),J_0(\omega_nr)\rangle}=\frac{\int_0^aT_0rJ_0(\omega_nr)dr}{\frac{a^2}{2}J_1^2(\omega_na)}=2T_0\frac{\int_0^{\omega_na}tJ_0(t)dt}{a^2\omega_n^2J_1^2(\omega_na)}=2T_0\frac{1}{\omega_naJ_1(\omega_na)}$$

所以

$$u(r,z)=\sum_{n=1}^{+\infty}C_nsh(\omega_nz)J_0(\omega_nr)=2T_0\sum_{n=1}^{+\infty}\frac{1}{\omega_naJ_1(\omega_na)}\frac{sh(\omega_nz)}{sh(\omega_nl)}J_0(\omega_nr)$$

其中 ω_n 为

$$J_0(\omega a)=0$$

的正根

(2) 设 $u(r,t)=R(r)T(t)$

由分离变量法得

$$\begin{cases} T''(t)R(r)+2hT'(t)R(r)=a^2\left(T(t)R''(r)+\frac{1}{r}T(t)R'(r)\right) \\ R(0)T(t)\text{有限},T(t)R'(l)=0 \\ R(r)T(0)=\varphi(r),R(r)T'(0)=0 \end{cases}$$

设

$$\frac{R''(r)}{R(r)}+\frac{1}{r}\frac{R'(r)}{R(r)}=-\lambda$$

得

$$\begin{cases} r^2R(r)+rR'(r)+\lambda r^2R(r)=0 \\ R'(l)=0,R(0)\text{有界} \end{cases}$$

则其固有值为

$$\lambda = \begin{cases} 0 \\ \omega_n^2, n = 1, 2, \dots \end{cases}$$

其固有函数为

$$R_n(r) = \begin{cases} C, C \text{ 为常数} \\ J_0(\omega_n r), n = 1, 2, \dots \end{cases}$$

其中 ω_n 为

$$J_0'(\omega l) = 0, \text{ 即 } J_1(\omega l) = 0$$

的正根, 其中

则当 $\lambda = 0$ 时

$$\begin{cases} \frac{T_0''(t) + 2hT_0'(t)}{a^2 T_0(t)} = 0 \\ T_0'(0) = 0 \end{cases}$$

即

$$\begin{cases} T_0''(t) + 2hT_0'(t) = 0 \\ T_0'(0) = 0 \end{cases}$$

得该方程对应的特征方程为 $\alpha^2 + 2h\alpha = 0$, 特征根为 $\alpha_1 = -2h, \alpha_2 = 0$.

$$\begin{cases} T_0(t) = C_0 + D_0 e^{-h2t} \\ T_0'(0) = 0 \end{cases}$$

则当 $\lambda = \omega_n^2, n = 1, 2, \dots$ 时

$$\begin{cases} D_0 = 0 \\ \frac{T_n''(t) + 2hT_n'(t)}{a^2 T_n(t)} = -\omega_n^2 \\ T_n'(0) = 0 \end{cases}$$

即

$$\begin{cases} T_n''(t) + 2hT_n'(t) + \omega_n^2 a^2 T_n(t) = 0 \\ T_n'(0) = 0 \end{cases}$$

得该方程对应的特征方程为 $\alpha^2 + 2h\alpha + \omega_n^2 a^2 = 0$, 判别式为 $\Delta = 4h^2 - 4\omega_n^2 a^2$.

题中 $h \ll 1$, 结合一阶贝塞尔函数零点及 a 的实际物理意义可知判别式小于 0. (其实就是根据答案凑的说法)

则该特征方程解为 $\alpha_1 = -h + i\sqrt{(\omega_n a)^2 - h^2}, \alpha_2 = -h - i\sqrt{(\omega_n a)^2 - h^2}$

记 $q_n = \sqrt{(\omega_n a)^2 - h^2}$

则该方程解为

$$T_n(t) = e^{-ht}(C_n \cos q_n t + D_n \sin q_n t)$$

$T_n'(0) = 0$ 得

$$-hC_n + q_n D_n = 0 \Rightarrow D_n = \frac{h}{q_n}$$

故原方程解为

$$u(r, t) = C_0 + D_0 e^{-h2t} \text{ (这里本来要乘 } C, \text{ 但将 } C \text{ 规划到两个待定系数显得答案比较美观)}$$

$$+ \sum_{n=1}^{+\infty} e^{-ht}(C_n \cos q_n t + D_n \sin q_n t) J_0(\omega_n r)$$

再考虑 $u(r, 0) = \varphi(r)$

$$u(r, 0) = C_0 + \sum_{n=1}^{+\infty} C_n J_0(\omega_n r) = \sum_{n=0}^{+\infty} C_n J_0(\omega_n r) = \varphi(r)$$

$$C_n = \frac{\langle \varphi(r), J_0(\omega_n r) \rangle}{\langle J_0(\omega_n r), J_0(\omega_n r) \rangle} = \frac{\int_0^l r \varphi(r) J_0(\omega_n r) dr}{\int_0^l r J_0^2(\omega_n r) dr}$$

$$C_0 = \frac{\int_0^l r \varphi(r) dr}{\int_0^l r dr} = \frac{2}{l^2} \int_0^l r \varphi(r) dr$$

$$C_n = \frac{\int_0^l r \varphi(r) J_0(\omega_n r) dr}{N_{01}^2} = \frac{\int_0^l r \varphi(r) J_0(\omega_n r) dr}{\frac{l^2}{2} J_1^2(\omega_n r)} = \frac{2}{l^2} \frac{\int_0^l r \varphi(r) J_0(\omega_n r) dr}{J_1^2(\omega_n r)}, n = 1, 2, \dots$$

所以

$$u(r, t) = C_0 + D_0 e^{-h^2 t} + \sum_{n=1}^{+\infty} e^{-h^2 t} (C_n \cos q_n t + D_n \sin q_n t) J_0(\omega_n r)$$

其中

$$C_0 = \frac{2}{l^2} \int_0^l r \varphi(r) dr$$

$$C_n = \frac{2}{l^2} \frac{\int_0^l r \varphi(r) J_0(\omega_n r) dr}{J_1^2(\omega_n r)}, n = 1, 2, \dots$$

$$D_0 = 0$$

$$D_n = \frac{h}{q_n}$$

19. 解：由分离变量法，设 $u(r, z) = R(r)Z(z)$

$$\begin{cases} R''(r)Z(z) + \frac{1}{r}R'(r)Z(z) + R(r)Z''(z) = 0 \\ R(0)Z(z) = \text{有限}, R'(R)Z(z) + kR(R)Z(z) = 0 \\ R(r)Z(0) = 0, R(r)Z(h) = f(r) \end{cases}$$

即

$$\begin{cases} \frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \frac{Z''(z)}{Z(z)} = 0 \\ R'(R) + kR(R) = 0, R(0) \text{有限} \\ Z(0) = 0, R(r)Z(h) = f(r) \end{cases}$$

设

$$\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} = -\lambda$$

关于 $R(r)$ 的方程为

$$\begin{cases} r^2 R''(r) + rR'(r) + \lambda r^2 R(r) = 0 \\ R'(R) + kR(R) = 0, R(0) \text{有限} \end{cases}$$

其为 0 阶贝塞尔方程，且具有第三类边界条件 1+

故其固有值为

$$\lambda = \omega_n^2, n = 1, 2, \dots$$

固有函数为

$$R_n(r) = J_0(\omega_n r), n = 1, 2, \dots$$

其中 ω_n 为方程 $\omega J_0'(\omega R) + kJ_0(\omega R) = 0$ $[-\omega J_1(\omega R) + kJ_0(\omega R) = 0]$ 的正根。

所以

$$\begin{cases} Z_n''(z) - \omega_n^2 Z_n(z) = 0 \\ Z_n(0) = 0 \end{cases}$$

解得

$$Z_n(z) = A_n \sinh \omega_n z$$

故

$$u(r, z) = \sum_{n=1}^{+\infty} A_n \sinh \omega_n z J_0(\omega_n r)$$

$$u(r, h) = \sum_{n=1}^{+\infty} A_n sh \omega_n h J_0(\omega_n r) = f(r)$$

$$A_n sh \omega_n h = \frac{\langle f(r), J_0(\omega_n r) \rangle}{\langle J_0(\omega_n r), J_0(\omega_n r) \rangle} = \frac{\int_0^R r f(r) J_0(\omega_n r) dr}{N_{03}^2} = \frac{\int_0^R r f(r) J_0(\omega_n r) dr}{\frac{1}{2} \left[R^2 + \left(\frac{R}{\omega_n k} \right)^2 \right] J_0^2(\omega R)} = \frac{2}{R^2} \frac{\int_0^R r f(r) J_0(\omega_n r) dr}{\left(1 + \frac{1}{\omega_n^2 k^2} \right) J_0^2(\omega R)}$$

$$u(r, z) = \frac{2}{R^2} \sum_{n=1}^{+\infty} \frac{\int_0^R r f(r) J_0(\omega_n r) dr}{\left(1 + \frac{1}{\omega_n^2 k^2} \right) J_0^2(\omega R)} \frac{sh \omega_n z}{sh \omega_n h} J_0(\omega_n r)$$

20. 解:

由递推公式

$$(n+1)p_{n+1}(x) - x(2n+1)p_n(x) + np_{n-1}(x) = 0, n \geq 1$$

及

$$p_0(x) = 1$$

$$p_1(x) = x$$

代入 $x = 0$

$$(n+1)p_{n+1}(0) + np_{n-1}(0) = 0, n \geq 1$$

即

$$(n+2)p_{n+2}(0) + (n+1)p_n(0) = 0, n \geq 0$$

当 $n = 2k - 2, k = 1, 2, \dots$ 时

$$2kp_{2k}(0) + (2k-1)p_{2k-2}(0) = 0$$

即

$$\frac{p_{2k}(0)}{p_{2k-2}(0)} = -\frac{(2k-1)}{2k}$$

$$p_{2k}(0) = \frac{p_{2k}(0)}{p_{2k-2}(0)} \frac{p_{2k-2}(0)}{p_{2k-4}(0)} \dots \frac{p_2(0)}{p_0(0)} p_0(0) = \left[-\frac{(2k-1)}{2k} \right] \left[-\frac{(2k-3)}{(2k-2)} \right] \dots \left[-\frac{1}{2} \right] \cdot 1 = (-1)^k \frac{(2k-1)!!}{(2k)!!}$$

k 由来: 观察底下的 $2, \dots, 2k$, 共 k 项

当 $n = 2k - 1, k = 1, 2, \dots$ 时

$$(2k+1)p_{2k+1}(0) + 2kp_{2k-1}(0) = 0$$

$$p_1(x) = x \Rightarrow p_1(0) = 0$$

$$k = 1, 3p_3(0) + 2p_1(0) = 0 \Rightarrow p_3(0) = 0$$

...

所以

$$p_{2k-1}(0) = 0$$

综上

$$p_n(0) = \begin{cases} 1, n = 0 \\ (-1)^k \frac{(2k-1)!!}{(2k)!!}, n = 2k \\ 0, n = 2k-1 \end{cases}, k = 1, 2, \dots$$

由递推公式

$$np_{n-1}(x) - p'_n(x) + xp_{n-1}'(x) = 0$$

代入 $x = 0$

$$np_{n-1}(0) - p'_n(0) = 0$$

$$p'_n(0) = np_{n-1}(0), n \geq 2$$

当 $n = 2k + 1, k = 0, 1, 2, \dots$ 时

$$p'_{2k+1}(0) = (2k+1)p_{2k}(0) = (2k+1)(-1)^k \frac{(2k-1)!!}{(2k)!!} = (-1)^k \frac{(2k+1)!!}{(2k)!!}$$

当 $n = 2k, k = 1, 2, \dots$ 时

$$p'_{2k}(0) = 2kp_{2k-1}(0) = 0$$

$$p_0(0) = 1 \Rightarrow p'_0(0) = 0$$

所以

$$p'_n(0) = \begin{cases} 0, n = 2k \\ (-1)^k \frac{(2k+1)!!}{(2k)!!}, n = 2k+1, k = 0, 1, 2, \dots \end{cases}$$

21. 解:

由

$$p'_{n+1}(x) - p'_{n-1}(x) = (2n+1)p_n(x), n \geq 1$$

得

$$p'_n(x) - p'_{n-2}(x) = (2n-1)p_{n-1}(x), n \geq 2$$

所以

$$p'_{n-2}(x) - p'_{n-4}(x) = (2n-5)p_{n-3}(x), n \geq 3$$

$$p'_{n-4}(x) - p'_{n-6}(x) = (2n-9)p_{n-5}(x), n \geq 3$$

...

等式相加

$$p'_n(x) = (2n-1)p_{n-1}(x) + (2n-5)p_{n-3}(x) + (2n-9)p_{n-5}(x) + \dots, n \geq 2$$

$$p'_1(x) = 1 = (2 \times 1 - 1)p_0(x)$$

综上

$$p'_n(x) = (2n-1)p_{n-1}(x) + (2n-5)p_{n-3}(x) + (2n-9)p_{n-5}(x) + \dots, n \geq 1$$

22. 解: (1)

$m < n$ 时,

$$x^m = \sum_{i=0}^m A_i p_i(x)$$

$$\int_{-1}^1 x^m p_n(x) dx = \int_{-1}^1 \left[\sum_{i=0}^m A_i p_i(x) \right] p_n(x) dx = \int_{-1}^1 \sum_{i=0}^m A_i p_i(x) 1 p_n(x) dx = \sum_{i=0}^m A_i \int_{-1}^1 p_i(x) p_n(x) dx$$

注意到 $\int_{-1}^1 p_i(x) p_n(x) dx = 0, i \neq n$

故

$$\int_{-1}^1 x^m p_n(x) dx = 0$$

$m \geq n$ 时,

由

$$np_n(x) - xp'_n(x) + p'_{n-1}(x) = 0$$

$$n \int_{-1}^1 x^m p_n(x) dx = \int_{-1}^1 x^m [xp'_n(x) - p'_{n-1}(x)] dx = \int_{-1}^1 x^{m+1} p'_n(x) dx - \int_{-1}^1 x^m p'_{n-1}(x) dx$$

$$\int_{-1}^1 x^{m+1} p'_n(x) dx = \int_{-1}^1 x^{m+1} dp_n(x) = x^{m+1} p_n(x) \Big|_{-1}^1 - (m+1) \int_{-1}^1 x^m p_n(x) dx = p_n(1) - (-1)^{m+1} p_n(-1)$$

$$- (m+1) \int_{-1}^1 x^m p_n(x) dx = 1 - (-1)^{m+1+n} - (m+1) \int_{-1}^1 x^m p_n(x) dx$$

$$\int_{-1}^1 x^m p'_{n-1}(x) dx = \int_{-1}^1 x^m dp_{n-1}(x) = x^m p_{n-1}(x) \Big|_{-1}^1 - m \int_{-1}^1 x^{m-1} p_{n-1}(x) dx = 1 - (-1)^{m+n-1} - m \int_{-1}^1 x^{m-1} p_{n-1}(x) dx$$

记

$$\int_{-1}^1 x^m p_n(x) dx = f(m, n)$$

则

$$nf(m, n) = 1 - (-1)^{m+1+n} - (m+1)f(m, n) - [1 - (-1)^{m+n-1} - mf(m-1, n-1)]$$

$$nf(m, n) = -(m+1)f(m, n) + mf(m-1, n-1)$$

$$\begin{aligned} f(m, n) &= \frac{m}{m+n+1} f(m-1, n-1) = \frac{m}{m+n+1} \cdot \frac{m-1}{m+n-1} f(m-2, n-2) \\ &= \frac{m}{m+n+1} \cdot \frac{m-1}{m+n-1} \cdot \frac{m-2}{m+n-3} f(m-3, n-3) \\ &= \cdots \frac{m!}{(m-n)!} \frac{1}{(m+n+1)!!} \int_{-1}^1 x^{m-n} dx = \frac{m!}{(m-n)!} \frac{(m-n+1)!!}{(m+n+1)!!} \frac{1+(-1)^{m-n}}{m-n+1} \\ &= \frac{m!}{(m-n)!} \frac{(m-n-1)!!}{(m+n+1)!!} [1+(-1)^{m-n}] = \frac{m!}{(m-n)!!} \frac{1+(-1)^{m-n}}{(m+n+1)!!} \end{aligned}$$

综上

$$\int_{-1}^1 x^m p_n(x) dx = \begin{cases} 0, m < n \\ \frac{m!}{(m-n)!!} \frac{1+(-1)^{m-n}}{(m+n+1)!!}, m \geq n \end{cases}$$

(2)

$$\int_{-1}^1 x p_m(x) p_n(x) dx$$

依题

$$x p_m(x) = \sum_{i=0}^{m+1} A_i p_i(x)$$

$$\int_{-1}^1 x p_m(x) p_n(x) dx = \int_{-1}^1 \left[\sum_{i=0}^{m+1} A_i p_i(x) \right] p_n(x) dx = \sum_{i=0}^{m+1} A_i \int_{-1}^1 p_i(x) p_n(x) dx$$

所以, 当 $m+1 < n$ 时, 由正交性可知上式等于 0

同理, 对于

$$x p_n(x) = \sum_{i=0}^{n+1} B_i p_i(x)$$

所以, 当 $n+1 < m$ 时, 由正交性可知上式等于 0

故只需讨论, $m-n=0, \pm 1$

$m-n=-1$ 时

$$\text{原式} = \int_{-1}^1 x p_{n-1}(x) p_n(x) dx$$

由递推公式 (1)

$$n p_n - x(2n-1) p_{n-1}(x) + (n-1) p_{n-2}(x) = 0$$

得

$$x p_{n-1}(x) = \frac{n}{2n-1} p_n + \frac{n-1}{2n-1} p_{n-2}(x)$$

代入并由正交性:

$$\text{原式} = \frac{n}{2n-1} \|p_n\|^2 = \frac{n}{2n-1} \frac{2}{2n+1} = \frac{2n}{4n^2-1}$$

$m-n=0$ 时

$$\text{原式} = \int_{-1}^1 x p_n^2(x) dx$$

$$\text{对于 } f(x) = x p_n^2(x), f(-x) = -x p_n^2(-x) = -x[(-1)^2 p_n(x)]^2 = -x p_n^2(x) = -f(x)$$

因此

$$\text{原式} = 0$$

$m - n = 1$ 时

$$\text{原式} = \int_{-1}^1 x p_n(x) p_{n+1}(x) dx$$

由递推公式 (1)

$$(n+1)p_{n+1} - x(2n+1)p_n(x) + np_{n-1}(x) = 0$$

得

$$x p_n(x) = \frac{n+1}{2n+1} p_{n+1} + \frac{n}{2n+1} p_{n-1}(x)$$

代入并由正交性:

$$\text{原式} = \frac{n+1}{2n+1} \|p_{n+1}\|^2 = \frac{n+1}{2n+1} \frac{2}{2n+3} = \frac{2(n+1)}{(2n+1)(2n+3)}$$

综上

$$\text{原式} = \begin{cases} \frac{2n}{4n^2-1}, m-n=-1 \\ 0, m-n \neq \pm 1 \\ \frac{2(n+1)}{(2n+1)(2n+3)}, m-n=1 \end{cases}$$

23. 解:

考虑到

$$\frac{d}{dx} [(1-x^2)p_n'(x)] + n(n+1)p_n(x) = 0$$

$$\begin{aligned} \int_{-1}^1 (1-x^2)[p_n'(x)]^2 dx &= \int_{-1}^1 (1-x^2)p_n'(x) dp_n(x) = (1-x^2)p_n'(x)p_n(x)|_{-1}^1 - \int_{-1}^1 p_n(x) d[(1-x^2)p_n'(x)] dx \\ &= n(n+1) \int_{-1}^1 p_n^2(x) dx = \frac{2n(n+1)}{2n+1} \end{aligned}$$

24. 解: (1)

$$f(x) = x^3, -1 < x < 1$$

$$f(x) = C_3 p_3(x) + C_1 p_1(x)$$

$$p_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x, p_1(x) = x$$

故

$$f(x) = \frac{2}{5}p_3(x) + \frac{3}{5}p_1(x)$$

(2)

$$f(x) = x^4, -1 < x < 1$$

$$f(x) = C_4 p_4(x) + C_2 p_2(x) + C_0 p_0(x)$$

$$p_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}, p_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, p_0(x) = 1$$

则

$$\begin{cases} \frac{35}{8}C_4 = 1 \\ \frac{3}{2}C_2 - \frac{15}{4}C_4 = 0 \\ C_0 - \frac{1}{2}C_2 + \frac{3}{8}C_4 = 0 \end{cases}$$

得

$$\begin{cases} C_0 = \frac{1}{5} \\ C_2 = \frac{4}{7} \\ C_4 = \frac{8}{35} \end{cases}$$

故

$$f(x) = \frac{8}{35}p_4(x) + \frac{4}{7}p_2(x) + \frac{1}{5}p_0(x)$$

(3)

$$f(x) = |x|, -1 < x < 1$$

$$f(x) = \sum_{n=0}^{+\infty} C_n p_n(x)$$

$$C_0 = \frac{1}{2} \int_{-1}^1 f(x) dx = \frac{1}{2}$$

$$C_n = \frac{2n+1}{2} \int_{-1}^1 |x| p_n(x) dx, n \geq 1$$

$$\int_{-1}^1 |x| p_n(x) dx = \int_0^1 x p_n(x) dx - \int_{-1}^0 x p_n(x) dx$$

$$\int_{-1}^0 x p_n(x) dx \xrightarrow{x=-t} - \int_0^1 t p_n(-t) dt = (-1)^{n-1} \int_0^1 x p_n(x) dx$$

$$\int_{-1}^1 |x| p_n(x) dx = [1 + (-1)^n] \int_0^1 x p_n(x) dx = \begin{cases} 2 \int_0^1 x p_{2k}(x) dx & n = 2k, k = 1, 2, \dots \\ 0, & n = 2k - 1 \end{cases}$$

由课本 P280 例 1

$$\int_0^1 x p_{2k}(x) dx = \frac{1}{2k+2} \int_0^1 p_{2k-1}(x) dx$$

由递推公式 (6)

$$\int_0^1 p_{2k-1}(x) dx = \frac{1}{4k-1} [p_{2k-2}(0) - p_{2k}(0)] =$$

注意到

$$w(t, x) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{+\infty} p_n(x) t^n, |t| < 1$$

则

$$w(t, 0) = \frac{1}{\sqrt{1+t^2}} = \sum_{n=0}^{+\infty} p_n(0) t^n, |t| < 1$$

$$\frac{1}{\sqrt{1+t^2}} = (1+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{+\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\cdots\left[-\frac{1}{2}-(n-1)\right]}{n!} (t^2)^n = \sum_{n=0}^{+\infty} (-1)^n \frac{(2n-1)!!}{2^n n!} t^{2n} = \sum_{n=0}^{+\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} t^{2n}$$

所以

$$p_n(0) = \begin{cases} 0, & n = 2k+1 \\ (-1)^k \frac{(2k-1)!!}{(2k)!!}, & n = 2k \end{cases} \quad k = 0, 1, 2, \dots$$

$$p_{2k}(0) = (-1)^k \frac{(2k-1)!!}{(2k)!!}$$

$$p_{2k-2}(0) = (-1)^{k-1} \frac{(2k-3)!!}{(2k-2)!!}$$

$$p_{2k}(0) - p_{2k-2}(0) = (-1)^k \frac{(2k-1)(2k-3)!!}{(2k)!!} - (-1)^{k-1} \frac{2k(2k-3)!!}{2k(2k-2)!!} = (-1)^k \frac{(4k-1)(2k-3)!!}{(2k)!!}$$

所以

$$\begin{aligned}\int_0^1 x p_{2k}(x) dx &= \frac{1}{2k+2} \int_0^1 p_{2k-1}(x) dx = \frac{1}{(2k+2)(4k-1)} [p_{2k-2}(0) - p_{2k}(0)] = (-1)^{k+1} \frac{(2k-3)!!}{(2k+2)!!} \\ C_{2k} &= (4k+1) \int_0^1 x p_{2k}(x) dx = (-1)^k \frac{(2k-3)!! (4k+1)}{(2k+2)!!} = (-1)^{k+1} \frac{(4k+1) (2k-2)!!}{(2k+2)!!} \\ &= (-1)^k \frac{(4k+1)(2k-2)!}{2^{k+1}(k+1)! 2^{k-1}(k-1)!} = (-1)^{k+1} \frac{(4k+1) (2k-2)!}{2^{2k}(k-1)! (k+1)!} \\ f(x) &= \frac{1}{2} + \sum_{k=1}^{+\infty} (-1)^{k+1} \frac{(4k+1) (2k-2)!}{2^{2k}(k-1)! (k+1)!} p_{2k}(x)\end{aligned}$$

25. 解:

根据球坐标系下与方位角 φ 无关的 Laplace 方程通解 (球内形式)

$$\begin{aligned}u(r, \theta) &= \sum_{n=0}^{+\infty} A_n r^n p_n(\cos \theta) \\ u(a, \theta) &= \sum_{n=0}^{+\infty} A_n a^n p_n(\cos \theta) = \cos^2 \theta = \frac{2}{3} p_2(\cos \theta) + \frac{1}{3} p_0(\cos \theta) \\ A_0 &= \frac{1}{3}, A_2 = \frac{2}{3a^2}, A_n = 0 (n \neq 0, 2) \\ u(r, \theta) &= \frac{1}{3} + \frac{2r^2}{3a^2} p_2(\cos \theta) = \frac{1}{3} + \frac{2r^2}{3a^2} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) = \frac{r^2}{a^2} \cos^2 \theta - \frac{r^2}{3a^2} + \frac{1}{3}\end{aligned}$$

26. 解:

根据球坐标系下与方位角 φ 无关的 Laplace 方程通解 (球内形式)

$$\begin{aligned}u(r, \theta) &= \sum_{n=0}^{+\infty} A_n r^n p_n(\cos \theta) \\ u(1, \theta) &= \sum_{n=0}^{+\infty} A_n p_n(\cos \theta) = 3 \cos 2\theta + 1 = 6 \cos^2 \theta - 2 = 4 p_2(\cos \theta) \\ A_2 &= 4, A_n = 0 (n \neq 2) \\ u(r, \theta) &= 4r^2 p_2(\cos \theta) = 4r^2 \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) = 2r^2 (3 \cos^2 \theta - 1)\end{aligned}$$

27. 解:

根据球坐标系下与方位角 φ 无关的 Laplace 方程通解 (球外形式)

$$\begin{aligned}u(r, \theta) &= \sum_{n=0}^{+\infty} B_n r^{-(n+1)} p_n(\cos \theta) \\ u(1, \theta) &= \sum_{n=0}^{+\infty} B_n p_n(\cos \theta) = \cos^2 \theta = \frac{2}{3} p_2(\cos \theta) + \frac{1}{3} p_0(\cos \theta) \\ B_0 &= \frac{1}{3}, B_2 = \frac{2}{3}, B_n (n \neq 0, 2) \\ u(r, \theta) &= \frac{1}{3r} + \frac{2}{3r^3} p_2(\cos \theta) = \frac{1}{3r} + \frac{2}{3r^3} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) = \frac{1}{3} r^{-1} + r^{-3} \cos^2 \theta - \frac{1}{3} r^{-3}\end{aligned}$$

28. 解: 设球的半径为 a

(1) 将半球补成一完整的球, 则温度函数 $u(r, \theta)$ 满足:

$$\begin{cases} \Delta u = 0, (0 \leq r < a, 0 \leq \theta \leq \pi \\ u|_{r=a} = u_0, 0 < \theta \leq \frac{\pi}{2} \\ u|_{\theta=0} = 0 \\ u|_{r=a} = -u_0, -\frac{\pi}{2} \leq \theta < 0 \end{cases}$$

根据球坐标系下与方位角 φ 无关的 Laplace 方程通解 (球内形式)

$$u(r, \theta) = \sum_{n=0}^{+\infty} A_n r^n p_n(\cos \theta)$$

$$u(a, \theta) = \sum_{n=0}^{+\infty} A_n a^n p_n(\cos \theta) = \begin{cases} u_0, 0 \leq \theta < \frac{\pi}{2} \\ 0, \theta = \frac{\pi}{2} \\ -u_0, \frac{\pi}{2} < \theta \leq \pi \end{cases}$$

令 $\cos \theta = x$

则方程转化为求函数 $\begin{cases} u_0, 0 < x \leq 1 \\ 0, x = 0 \\ -u_0, -1 \leq x < 0 \end{cases}$ 的傅里叶-勒让德展开

该函数为奇函数, 故只有奇数项系数不为 0

$$C_1 = 3u_0 \int_0^1 x dx = \frac{3}{2}u_0$$

由习题 24. (3)

$n \geq 1$ 时 C_{2n+1}

$$= (4n+3) \int_0^1 u_0 p_{2n+1}(x) dx$$

$$= (4n+3)u_0 \int_0^1 p_{2n+1}(x) dx = (4n+3)u_0 \cdot \frac{1}{4n+3} [p_{2n}(0) - p_{2n+2}(0)] = u_0 (-1)^n \frac{(4n+3)(2n-1)!!}{(2n+2)!!}$$

所以

$$A_1 = \frac{3u_0}{2a}, A_{2n+1} = u_0 (-1)^n \frac{(4n+3)(2n-1)!!}{(2n+2)!!} \frac{1}{a^{2n+1}}, n = 1, 2, \dots$$

因此

$$u(r, \theta) = \frac{3u_0}{2a} \cos \theta + u_0 \sum_{n=1}^{+\infty} (-1)^n \frac{(4n+3)(2n-1)!!}{(2n+2)!!} \left(\frac{r}{a}\right)^{2n+1} p_{2n+1}(\cos \theta)$$

(2) 将半球补成一完整的球, 绝热指底面温度保持不变, 则温度函数 $u(r, \theta)$ 满足:

$$\begin{cases} \Delta u = 0, 0 \leq r \leq a, 0 \leq \theta \leq \pi \\ u|_{r=a} = u_0 \end{cases}$$

根据球坐标系下与方位角 φ 无关的 Laplace 方程通解 (球内形式)

$$u(r, \theta) = \sum_{n=0}^{+\infty} A_n r^n p_n(\cos \theta)$$

$$u(a, \theta) = \sum_{n=0}^{+\infty} A_n a^n p_n(\cos \theta) = u_0 = u_0 p_0(\cos \theta)$$

$$A_0 = u_0, A_n = 0 (n = 1, 2, \dots)$$

所以

$$u(r, \theta) = u_0$$

29. 解: 将半空心球补成全空心球, (注意到 $\theta = \frac{\pi}{2}$ 时, $f\left(\frac{\pi}{2}\right) = \frac{A}{2}$, 说明底面温度没有发生跳变)

则温度函数 $u(r, \theta)$ 满足

$$\begin{cases} \Delta u = 0, \frac{R}{2} \leq r \leq R, 0 \leq \theta \leq \pi \\ u|_{r=R} = u|_{r=\frac{R}{2}} = A \sin^2 \frac{\theta}{2} = \frac{A}{2}(1 - \cos \theta) \\ u|_{\theta=\frac{\pi}{2}} = \frac{A}{2} \end{cases}$$

根据球坐标系下与方位角 φ 无关的 Laplace 方程通解

$$\begin{aligned} u(r, \theta) &= \sum_{n=0}^{+\infty} [A_n r^n + B_n r^{-(n+1)}] p_n(\cos \theta) \\ u\left(\frac{R}{2}, \theta\right) &= \sum_{n=0}^{+\infty} \left[A_n \left(\frac{R}{2}\right)^n + B_n \left(\frac{R}{2}\right)^{-(n+1)} \right] p_n(\cos \theta) = \frac{A}{2}(1 - \cos \theta) \\ u(R, \theta) &= \sum_{n=0}^{+\infty} [A_n R^n + B_n R^{-(n+1)}] p_n(\cos \theta) = \frac{A}{2}(1 - \cos \theta) \end{aligned}$$

比较系数得

$$\begin{cases} A_0 + \frac{2B_0}{R} + \frac{A_1}{2} R \cos \theta + \frac{4B_1}{R^2} \cos \theta = \frac{A}{2}(1 - \cos \theta) \\ A_0 + \frac{B_0}{R} + A_1 R \cos \theta + \frac{B_1}{R^2} \cos \theta = \frac{A}{2}(1 - \cos \theta) \end{cases}$$

$$\begin{cases} A_n = 0, n = 2, 3, \dots \\ B_n = 0, n = 2, 3, \dots \end{cases}$$

得

$$\begin{cases} A_0 + \frac{2B_0}{R} = \frac{A}{2} \\ A_0 + \frac{B_0}{R} = \frac{A}{2} \end{cases} \Rightarrow \begin{cases} A_0 = \frac{A}{2} \\ B_0 = 0 \end{cases}$$

$$\begin{cases} \frac{A_1}{2} R + \frac{4B_1}{R^2} = -\frac{A}{2} \\ A_1 R + \frac{B_1}{R^2} = -\frac{A}{2} \end{cases} \Rightarrow \begin{cases} A_1 = -\frac{3A}{7R} \\ B_1 = -\frac{A}{14} R^2 \end{cases}$$

则

$$u(r, \theta) = \frac{A}{2} - \left(\frac{3r}{7R} + \frac{R^2}{14r} \right) A \cos \theta$$

第四章作业参考解答

刘炜昊

2021 年 4 月 30 日

目录

目录	1
1 课本 $P_{303} T_1$	2
1.1 课本 $P_{303} T_{1(1)}$	2
1.2 课本 $P_{303} T_{1(2)}$	3
1.3 课本 $P_{303} T_{1(3)}$	4
2 课本 $P_{303} T_2$	6
2.1 课本 $P_{303} T_{2(1)}$	6
2.2 课本 $P_{303} T_{2(2)}$	8
2.3 课本 $P_{303} T_{2(3)}$	10
2.4 课本 $P_{304} T_{2(4)}$	12
2.5 课本 $P_{304} T_{2(5)}$	13
2.6 课本 $P_{304} T_{2(6)}$	14
2.7 课本 $P_{304} T_{2(7)}$	15
3 写在最后	18

1 课本 $P_{303} T_1$ 1.1 课本 $P_{303} T_{1(1)}$

用傅里叶变换解下列定解问题:

$$\begin{cases} \Delta_2 u = 0 \quad (-\infty < x < +\infty, y > 0), \\ u(x, 0) = f(x), \\ \text{当 } x^2 + y^2 \rightarrow +\infty \text{ 时, } u(x, y) \rightarrow 0; \end{cases}$$

Sol:

由于 x 的取值范围无界, 可对 x 作 Fourier 变换, 记 $\tilde{u}(\lambda, y) = F[u(x, y)]$, 并记 $\tilde{f}(\lambda) = F[f(x)]$, 因此得到如下 ODE:

$$\begin{cases} \frac{d^2 \tilde{u}}{dy^2} - \lambda^2 \tilde{u} = 0 \\ \tilde{u}(\lambda, 0) = \tilde{f}(\lambda) \end{cases}$$

得到其通解为 $\tilde{u}(\lambda, y) = A(\lambda)e^{\lambda y} + B(\lambda)e^{-\lambda y}$, 考虑边界条件和有界性条件有:

$$\tilde{u}(\lambda, y) = \begin{cases} \tilde{f}(\lambda)e^{-\lambda y}, & \lambda \geq 0 \\ \tilde{f}(\lambda)e^{\lambda y}, & \lambda < 0 \end{cases} = \tilde{f}(\lambda)e^{-|\lambda|y}$$

因此作反变换得到:

$$\begin{aligned} F^{-1}[e^{-|\lambda|y}] &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-|\lambda|y} e^{i\lambda x} d\lambda = \frac{1}{\pi} \int_0^{+\infty} e^{-\lambda y} \cos \lambda x d\lambda \\ &= \frac{1}{\pi} \cdot \left. \begin{vmatrix} \cos \lambda x & e^{-\lambda y} \\ -x \sin \lambda x & -ye^{-\lambda y} \end{vmatrix} \right|_0^{+\infty} = \frac{1}{\pi} \cdot \frac{y}{x^2 + y^2} \end{aligned}$$

$$\begin{aligned} \Rightarrow u(x, y) &= F^{-1}[\tilde{f}(\lambda)] * F^{-1}[e^{-|\lambda|y}] \\ &= f(x) * \left(\frac{y}{\pi} \cdot \frac{1}{x^2 + y^2} \right) \\ &= \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{f(\xi - x) d\xi}{\xi^2 + y^2} \end{aligned}$$

1.2 课本 P_{303} $T_{1(2)}$

用傅里叶变换解下列定解问题:

$$\begin{cases} u_t = a^2 u_{xx} + f(t, x) & (t > 0, -\infty < x < +\infty), \\ u(0, x) = 0; \end{cases}$$

Sol:

由于 x 的取值范围无界, 可对 x 作 Fourier 变换, 记 $\tilde{u}(t, \lambda) = F[u(t, x)]$, 并记 $\tilde{f}(t, \lambda) = F[f(t, x)]$, 因此得到如下 ODE:

$$\begin{cases} \frac{d\tilde{u}}{dt} + a^2 \lambda^2 \tilde{u} = \tilde{f}(t, \lambda) \\ \tilde{u}(0, \lambda) = 0 \end{cases}$$

得到其通解为

$$\tilde{u}(t, \lambda) = e^{-a^2 \lambda^2 t} \int_0^t e^{a^2 \lambda^2 \tau} \tilde{f}(\tau, \lambda) d\tau = \int_0^t e^{a^2 \lambda^2 (\tau-t)} \tilde{f}(\tau, \lambda) d\tau$$

因此作反变换得到:

$$\begin{aligned} F^{-1}[\tilde{u}(t, \lambda)] &= \int_{-\infty}^{+\infty} \left[\int_0^t e^{a^2 \lambda^2 (\tau-t)} \tilde{f}(\tau, \lambda) d\tau \right] e^{i\lambda x} d\lambda \\ &= \int_0^t \left[\int_{-\infty}^{+\infty} e^{a^2 \lambda^2 (\tau-t)} e^{i\lambda x} \tilde{f}(\tau, \lambda) d\lambda \right] d\tau \\ &= \int_0^t F^{-1} \left[e^{a^2 \lambda^2 (\tau-t)} \tilde{f}(\tau, \lambda) \right] d\tau \\ &= \int_0^t F^{-1} \left[e^{-a^2 \lambda^2 (t-\tau)} \right] * F^{-1} [\tilde{f}(\tau, \lambda)] d\tau \\ &= \int_0^t \frac{e^{-\frac{x^2}{4a^2(t-\tau)}}}{2a\sqrt{\pi(t-\tau)}} * f(\tau, x) d\tau \\ &= \frac{1}{2a\sqrt{\pi}} \int_0^t \int_{-\infty}^{+\infty} e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}} \frac{f(\tau, \xi)}{\sqrt{(t-\tau)}} d\xi d\tau \end{aligned}$$

以上已用到公式:

$$\int_{-\infty}^{+\infty} e^{-a^2 \lambda^2 t} e^{i\lambda x} d\lambda = \int_{-\infty}^{+\infty} e^{-a^2 \lambda^2 t} \cos \lambda x d\lambda = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}}$$

1.3 课本 $P_{303} T_{1(3)}$

用傅里叶变换解下列定解问题:

$$\begin{cases} u_t = a^2 u_{xx} (0 < x < +\infty, t > 0), \\ u(t, 0) = \varphi(t), u(0, x) = 0, \quad (\text{用正弦变换}) . \\ u(t, +\infty) = u_x(t, +\infty) = 0 \end{cases}$$

Sol:

由于 x 的取值范围在半直线上, 且, 可对 x 作正弦变换, 记 $\tilde{u}(t, \lambda) = \int_0^{+\infty} u(t, x) \sin \lambda x dx$, 那么则有

$$\begin{aligned} F_s \left[\frac{\partial^2 u}{\partial x^2} \right] &= \int_0^{+\infty} \frac{\partial^2 u}{\partial x^2} \sin \lambda x dx = \frac{\partial u}{\partial x} \sin \lambda x \Big|_0^{+\infty} - \lambda \int_0^{+\infty} \frac{\partial u}{\partial x} \cos \lambda x dx \\ &= -\lambda u \cos \lambda x \Big|_0^{+\infty} - \lambda^2 \int_0^{+\infty} u \sin \lambda x dx \\ &= -\lambda \varphi(t) - \lambda^2 \tilde{u}. \end{aligned}$$

因此得到如下 ODE:

$$\begin{cases} \frac{d\tilde{u}}{dt} + a^2 \lambda^2 \tilde{u} = a^2 \lambda \varphi(t) \\ \tilde{u}(0, \lambda) = 0 \end{cases}$$

得到其通解为

$$\tilde{u}(t, \lambda) = e^{-a^2 \lambda^2 t} \int_0^t e^{a^2 \lambda^2 \tau} a^2 \lambda \varphi(\tau) d\tau = \int_0^t e^{a^2 \lambda^2 (\tau-t)} a^2 \lambda \varphi(\tau) d\tau$$

因此作反变换得到:

$$\begin{aligned}
 u(t, x) &= \frac{2}{\pi} \int_0^{+\infty} e^{-a^2 \lambda^2 t} \int_0^t a^2 \lambda \varphi(\tau) e^{a^2 \lambda^2 \tau} d\tau \sin \lambda x d\lambda \\
 &= \frac{2}{\pi} \int_0^t a^2 \varphi(\tau) d\tau \int_0^{+\infty} \lambda \sin \lambda x e^{-a^2 \lambda^2 (t-\tau)} d\lambda \\
 &= \frac{1}{\pi} \int_0^t \frac{\varphi(\tau)}{t-\tau} d\tau \int_0^{+\infty} \sin \lambda x d e^{-a^2 \lambda^2 (t-\tau)} d\lambda \\
 &= \frac{1}{\pi} \int_0^t \frac{\varphi(\tau)}{t-\tau} d\tau \int_0^{+\infty} x \cos \lambda x e^{-a^2 \lambda^2 (t-\tau)} d\lambda \\
 &= \frac{1}{\pi} \int_0^t \frac{x \varphi(\tau)}{t-\tau} d\tau \cdot \frac{1}{2} \int_{-\infty}^{+\infty} \cos \lambda x e^{-a^2 \lambda^2 (t-\tau)} d\lambda \\
 &= \frac{1}{2\pi} \int_0^t \frac{x \varphi(\tau)}{t-\tau} d\tau \cdot \int_{-\infty}^{+\infty} \operatorname{Re} \{ e^{i\lambda x} \} e^{-a^2 \lambda^2 (t-\tau)} d\lambda \\
 &= \frac{1}{2\pi} \int_0^t \frac{x \varphi(\tau)}{t-\tau} d\tau \cdot \operatorname{Re} \left\{ \int_{-\infty}^{+\infty} e^{-a^2 \lambda^2 (t-\tau) \left(\lambda - \frac{ix}{2a^2(t-\tau)} \right)^2} \cdot e^{-\frac{x^2}{4a^2(t-\tau)}} d\lambda \right\} \\
 &= \frac{1}{2\pi} \int_0^t \frac{x \varphi(\tau)}{t-\tau} \cdot e^{-\frac{x^2}{4a^2(t-\tau)}} \sqrt{\frac{\pi}{a^2(t-\tau)}} d\tau \\
 &= \frac{x}{2a\sqrt{\pi}} \int_0^t (t-\tau)^{-\frac{3}{2}} \varphi(\tau) e^{-\frac{x^2}{4a^2(t-\tau)}} d\tau
 \end{aligned}$$

其中

$$\int_{-\infty}^{+\infty} e^{-a^2(t-\tau) \left(\lambda - \frac{ix}{2a^2(t-\tau)} \right)^2} d\lambda = \int_{-\infty}^{+\infty} e^{-a^2(t-\tau) \lambda^2} d\lambda = \sqrt{\frac{\pi}{a^2(t-\tau)}}.$$

2 课本 $P_{303} T_2$

2.1 课本 $P_{303} T_{2(1)}$

用拉普拉斯变换解下列定解问题:

$$\begin{cases} \frac{\partial^2 u}{\partial x \partial y} = 1 (x > 0, y > 0) \\ u(0, y) = y + 1, u(x, 0) = 1; \end{cases}$$

Sol:

对 x 作 Laplace 变换, 记 $\tilde{u}(p, y) = L[u(x, y)]$, 则有关系:

$$L\left[\frac{\partial u}{\partial x}\right] = p\tilde{u} - u(0, y) = p\tilde{u} - y - 1$$

因此得到如下 ODE:

$$\begin{cases} \frac{\partial}{\partial y}(p\tilde{u} - y - 1) = \frac{1}{p} \\ \tilde{u}(p, 0) = \frac{1}{p} \end{cases}$$

因此得到: $\frac{\partial u}{\partial y} = \frac{1}{p^2} + \frac{1}{p}$, 那么 $\tilde{u}(p, y) = \frac{p+1}{p^2}y + f(p)$, $f \in \mathbb{C}^1$.

由 $\tilde{u}(p, 0) = f(p) = \frac{1}{p}$, 得到 $\tilde{u}(p, y) = \frac{p+1}{p^2}y + \frac{1}{p}$.

因此作反变换得到:

$$u(x, y) = L^{-1}[\tilde{u}(p, y)] = L^{-1}\left[\frac{p+1}{p^2}y + \frac{1}{p}\right] = xy + y + 1.$$

PS: 也可对 y 作 Laplace 变换, 记 $\tilde{u}(x, p) = L[u(x, y)]$, 则有关系:

$$L\left[\frac{\partial u}{\partial y}\right] = p\tilde{u} - u(x, 0) = p\tilde{u} - 1$$

因此得到如下 ODE:

$$\begin{cases} \frac{\partial}{\partial x}(p\tilde{u} - 1) = \frac{1}{p} \\ \tilde{u}(0, p) = \frac{1}{p} + \frac{1}{p^2} \end{cases}$$

因此得到: $\frac{\partial u}{\partial x} = \frac{1}{p^2}$, 那么 $\tilde{u}(p, y) = \frac{x}{p^2}y + f(p)$, $f \in \mathbb{C}^1$.

由 $\tilde{u}(0, p) = f(p) = \frac{1}{p} + \frac{1}{p^2}$, 得到 $\tilde{u}(x, p) = \frac{x+1}{p^2} + \frac{1}{p}$.

因此作反变换得到:

$$u(x, y) = L^{-1}[\tilde{u}(x, p)] = L^{-1}\left[\frac{x+1}{p^2} + \frac{1}{p}\right] = y + xy + 1.$$

PS: 也可以不用 Laplace 变换解题, 直接用第一章的知识解题也可.

2.2 课本 $P_{303} T_{2(2)}$

用拉普拉斯变换解下列定解问题:

$$\begin{cases} u_t = a^2 u_{xx} (t > 0, 0 < x < l) \\ u_x(t, 0) = 0, u(t, l) = u_0 (\text{常数}) \\ u(0, x) = u_1 (\text{常数}) ; \end{cases}$$

Sol:

对 t 作 Laplace 变换, 记 $\tilde{u}(p, x) = L[u(t, x)]$, 得到如下 ODE:

$$\begin{cases} p\tilde{u} - u_1 = a^2 \tilde{u}_{xx} \\ \tilde{u}_x(p, 0) = 0, \tilde{u}(p, l) = \frac{u_0}{p} \end{cases}$$

其中以上的方程 $p\tilde{u} - u_1 = a^2 \tilde{u}_{xx}$ 等价于以下方程

$$\tilde{u} - \frac{u_1}{p} = \frac{a^2}{p} \frac{\partial^2}{\partial x^2} \left(\tilde{u} - \frac{u_1}{p} \right)$$

得到通解: $\tilde{u}(p, x) = \frac{u_1}{p} + A \operatorname{sh} \frac{\sqrt{p}}{a} x + B \operatorname{ch} \frac{\sqrt{p}}{a} x$

代入边界条件有:

$$\begin{cases} \tilde{u}_x(p, 0) = A \cdot \frac{\sqrt{p}}{a} = 0 \Rightarrow A = 0 \\ \tilde{u}(p, l) = \frac{u_1}{p} + B \operatorname{ch} \frac{\sqrt{p}}{a} l = \frac{u_0}{p} \Rightarrow B = \frac{u_0 - u_1}{p \operatorname{ch} \frac{\sqrt{p}}{a} l} \end{cases}$$

$$\Rightarrow \tilde{u}(p, x) = \frac{u_1}{p} + \frac{u_0 - u_1}{p} \cdot \frac{\operatorname{ch} \frac{\sqrt{p}}{a} x}{\operatorname{ch} \frac{\sqrt{p}}{a} l}$$

对应的奇点 (令分母为 0) 的 p 值为:

$$p = 0, p_k = \left[\frac{(2k+1)\pi a l}{2l} \right]^2, k \in \mathbb{Z}_+$$

因此作 Laplace 反变换得到:

$$\begin{aligned}
 u(t, x) &= L^{-1}[\tilde{u}(p, x)] = u_1 + \sum_{k=0}^{+\infty} \text{Res} \left[\frac{u_0 - u_1}{p} \cdot \frac{\text{ch} \frac{\sqrt{p}}{a} x}{\text{ch} \frac{\sqrt{p}}{a} l}, p_k \right] \\
 &= u_1 + (u_0 - u_1) + (u_0 - u_1) \sum_{k=0}^{+\infty} \left[\frac{p - p_k}{p} \cdot \frac{\text{ch} \frac{\sqrt{p}}{a} x}{\text{ch} \frac{\sqrt{p}}{a} l}, p_k \right] \Big|_{p \rightarrow p_k} \\
 &= u_0 - (u_0 - u_1) \sum_{k=0}^{+\infty} \left\{ \left[\frac{2l}{(2k+1)\pi a} \right]^2 \cdot \cos \frac{(2k+1)\pi x}{2l} \cdot e^{-\left[\frac{(2k+1)\pi a}{2l} \right]^2 t} \cdot \lim_{p \rightarrow p_k} \frac{p - p_k}{\text{ch} \frac{\sqrt{p}}{a} l} \right\}
 \end{aligned}$$

其中

$$\lim_{p \rightarrow p_k} \frac{p - p_k}{\text{ch} \frac{\sqrt{p}}{a} l} = \lim_{p \rightarrow p_k} \frac{1}{\frac{l}{2a\sqrt{p}} \text{sh} \frac{\sqrt{p}}{a} l} = \frac{a^2(2k+1)\pi}{l^2 \sin(k\pi + \frac{\pi}{2})} = (-1)^k \cdot \frac{a^2(2k+1)\pi}{l^2}$$

因此得到

$$u(t, x) = u_0 + \frac{4}{\pi}(u_0 - u_1) \sum_{k=0}^{+\infty} \frac{(-1)^{k+1}}{2k+1} \cdot \cos \frac{(2k+1)\pi x}{2l} \cdot e^{-\left[\frac{(2k+1)\pi a}{2l} \right]^2 t}$$

2.3 课本 P_{303} $T_{2(3)}$

用拉普拉斯变换解下列定解问题:

$$\begin{cases} u_t = a^2 u_{xx} - hu (x > 0, t > 0, h > 0, h \text{ 为常数}) \\ u(0, x) = b \text{ (常数)}, u(t, 0) = 0 \\ \lim_{x \rightarrow +\infty} u_x = 0; \end{cases}$$

Sol:

对 t 作 Laplace 变换, 记 $\tilde{u}(p, x) = L[u(t, x)]$, 得到如下 ODE:

$$\begin{cases} p\tilde{u} - b = a^2 \tilde{u}_{xx} - h\tilde{u} \\ \tilde{u}_x(p, 0) = 0, \lim_{x \rightarrow +\infty} \tilde{u}(p, x) = 0 \end{cases}$$

其中以上的方程 $p\tilde{u} - b = a^2 \tilde{u}_{xx} - h\tilde{u}$ 等价于以下方程

$$\tilde{u} - \frac{b}{p+h} = \frac{a^2}{p+h} \frac{\partial^2}{\partial x^2} \left(\tilde{u} - \frac{b}{p+h} \right)$$

$$\text{得到通解: } \tilde{u}(p, x) = \frac{b}{p+h} + Ae^{\frac{\sqrt{p+h}}{a}x} + Be^{-\frac{\sqrt{p+h}}{a}x}$$

代入边界条件有:

$$\begin{cases} \lim_{x \rightarrow +\infty} \tilde{u}_x = 0 \Rightarrow A = 0 \\ \tilde{u}(p, 0) = \frac{b}{p+h} + B = 0 \Rightarrow B = -\frac{b}{p+h} \end{cases} \\ \Rightarrow \tilde{u}(p, x) = \frac{b}{p+h} \cdot \left(1 - e^{-\frac{\sqrt{p+h}}{a}x} \right)$$

因此作 Laplace 反变换得到:

$$u(t, x) = L^{-1}[\tilde{u}(p, x)] = be^{-ht} \left[1 - \operatorname{erfc} \left(\frac{x}{2a\sqrt{t}} \right) \right]$$

其中余误差函数 $\operatorname{erfc}(x)$ 与 $\operatorname{erf}(x)$ 的关系为:

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$$

因此有

$$u(t, x) = be^{-ht} \operatorname{erf}\left(\frac{x}{2a\sqrt{t}}\right) = \frac{2b}{\sqrt{\pi}} e^{-ht} \int_0^{\frac{x}{2a\sqrt{t}}} e^{-y^2} dy$$

2.4 课本 P_{304} $T_{2(4)}$

用拉普拉斯变换解下列定解问题:

$$\begin{cases} u_{tt} = a^2 u_{xx} (x > 0, t > 0) \\ u(0, x) = 0, u_x(0, x) = b \\ u(t, 0) = 0, \lim_{x \rightarrow +\infty} u_x = 0; \end{cases}$$

Sol:

对 t 作 Laplace 变换, 记 $\tilde{u}(p, x) = L[u(t, x)]$, 得到如下 ODE:

$$\begin{cases} p^2 \tilde{u} - b = a^2 \tilde{u}_{xx} \\ \tilde{u}(p, 0) = 0, \lim_{x \rightarrow +\infty} \tilde{u}_x = 0 \end{cases}$$

得到通解为: $u(\tilde{p}, x) = Ae^{\frac{p}{a}x} + Be^{-\frac{p}{a}x} + \frac{b}{p^2}$

代入边界条件有:

$$\begin{cases} \tilde{u}(p, 0) = A + B + \frac{b}{p^2} = 0 \\ \lim_{x \rightarrow +\infty} \tilde{u}_x = \lim_{x \rightarrow +\infty} (Ae^{\frac{p}{a}x} + Be^{-\frac{p}{a}x}) = 0 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = -\frac{b}{p^2} \end{cases} \Rightarrow \tilde{u}(p, x) = \frac{b}{p^2}(1 - e^{-\frac{x}{a}p})$$

因此作 Laplace 反变换得到:

$$u(t, x) = L^{-1}[\tilde{u}(p, x)] = bt - b\left(t - \frac{x}{a}\right)h\left(t - \frac{x}{a}\right)$$

其中 $h(t)$ 为单位函数,

$$h(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0 \end{cases}$$

2.5 课本 $P_{304} T_{2(5)}$

用拉普拉斯变换解下列定解问题:

$$\begin{cases} u_t = u_{xx} (t > 0, 0 < x < 4) \\ u(t, 0) = u(t, 4) = 0 \\ u(0, x) = 6 \sin \frac{\pi x}{2} + 3 \sin \pi x; \end{cases}$$

Sol:

对 t 作 Laplace 变换, 记 $\tilde{u}(p, x) = L[u(t, x)]$, 得到如下 ODE:

$$\begin{cases} p\tilde{u} - 6 \sin \frac{\pi x}{2} - 3 \sin \pi x = \tilde{u}_{xx} \\ \tilde{u}(p, 0) = 0, \tilde{u}(p, 4) = 0 \end{cases}$$

得到齐通解: $\tilde{u}(p, x) = Ae^{\sqrt{p}x} + Be^{-\sqrt{p}x}$ 和特解 $\tilde{u}^*(p, x) = C \sin \frac{\pi x}{2} + D \sin \pi x$

其中特解代入原方程得:

$$p(C \sin \frac{\pi x}{2} + D \sin \pi x) - 6 \sin \frac{\pi x}{2} - 3 \sin \pi x = -\frac{\pi^2}{4}C \sin \frac{\pi x}{2} - \pi^2 D \sin \pi x$$

由于 $\sin \frac{\pi x}{2}$ 和 $\sin \pi x$ 相互正交, 因此可得

$$\begin{cases} pC - 6 = -\frac{\pi^2}{4}C \\ pD - 3 = -\pi^2 D \end{cases} \Rightarrow \begin{cases} C = \frac{6}{p + \frac{\pi^2}{4}} \\ D = \frac{3}{p + \pi^2} \end{cases}$$

因此通解为 $\tilde{u}(p, x) = Ae^{\sqrt{p}x} + Be^{-\sqrt{p}x} + \frac{6}{p + \frac{\pi^2}{4}} \sin \frac{\pi x}{2} + \frac{3}{p + \pi^2} \sin \pi x$

代入边界条件有:

$$\begin{cases} \tilde{u}(p, 0) = A + B = 0 \\ \tilde{u}(p, 4) = Ae^{4\sqrt{p}} + Be^{-4\sqrt{p}} = 0 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = 0 \end{cases} \Rightarrow \tilde{u}(p, x) = \frac{6}{p + \frac{\pi^2}{4}} \sin \frac{\pi x}{2} + \frac{3}{p + \pi^2} \sin \pi x$$

因此作 Laplace 反变换得到:

$$u(t, x) = L^{-1}[\tilde{u}(p, x)] = 6e^{-\frac{\pi^2}{4}t} \sin \frac{\pi x}{2} + 3e^{-\pi^2 t} \sin \pi x$$

2.6 课本 P_{304} $T_{2(6)}$

用拉普拉斯变换解下列定解问题:

$$\begin{cases} u_t = u_{xx} (0 < x < 2) \\ u_x(t, 0) = u_x(t, 2) = 0 \\ u(0, x) = 4 \cos \pi x - 2 \cos 3\pi x; \end{cases}$$

Sol:

对 t 作 Laplace 变换, 记 $\tilde{u}(p, x) = L[u(t, x)]$, 得到如下 ODE:

$$\begin{cases} p\tilde{u} - 4 \cos \pi x + 2 \cos 3\pi x = \tilde{u}_{xx} \\ \tilde{u}_x(p, 0) = 0, \tilde{u}_x(p, 2) = 0 \end{cases}$$

得到齐通解: $\tilde{u}(p, x) = Ae^{\sqrt{p}x} + Be^{-\sqrt{p}x}$ 和特解 $\tilde{u}^*(p, x) = C \cos \pi x + D \cos 3\pi x$

其中特解代入原方程得:

$$p(C \cos \pi x + D \cos 3\pi x) - 4 \cos \pi x + 2 \cos 3\pi x = -\pi^2 C \cos \pi x - 9\pi^2 D \cos 3\pi x$$

由于 $\cos \pi x$ 和 $\cos 3\pi x$ 相互正交, 因此可得

$$\begin{cases} pC - 4 = -\pi^2 C \\ pD + 2 = -9\pi^2 D \end{cases} \Rightarrow \begin{cases} C = \frac{4}{p + \pi^2} \\ D = -\frac{2}{p + 9\pi^2} \end{cases}$$

因此通解为 $\tilde{u}(p, x) = Ae^{\sqrt{p}x} + Be^{-\sqrt{p}x} + \frac{4}{p + \pi^2} \cos \pi x - \frac{2}{p + 9\pi^2} \cos 3\pi x$

代入边界条件有:

$$\begin{cases} \tilde{u}_x(p, 0) = \sqrt{p}(A - B) = 0 \\ \tilde{u}_x(p, 2) = \sqrt{p}(Ae^{2\sqrt{p}} + Be^{-2\sqrt{p}}) = 0 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = 0 \end{cases} \Rightarrow \tilde{u}(p, x) = \frac{4 \cos \pi x}{p + \pi^2} - \frac{2 \cos 3\pi x}{p + 9\pi^2}$$

因此作 Laplace 反变换得到:

$$u(t, x) = L^{-1}[\tilde{u}(p, x)] = 4e^{-\pi^2 t} \cos \pi x - 2e^{-9\pi^2 t} \cos 3\pi x$$

2.7 课本 $P_{304} T_{2(7)}$

用拉普拉斯变换解下列定解问题:

$$\begin{cases} u_{tt} = a^2 \Delta_3 u (t > 0, r > 0) \\ u|_{r=0} \text{有界}, r = \sqrt{x^2 + y^2 + z^2} \\ u|_{t=0} = 0, u_t|_{t=0} = (1 + r^2)^{-2}; \end{cases}$$

[提示: 利用在复变函数部分第 5 章中已求得的积分

$$\int_0^{+\infty} \frac{\cos \omega x}{1 + x^2} = \frac{\pi}{2} e^{-\omega}.]$$

Sol:

由于初始条件只与 r 有关, 故 $u = u(t, r)$, 那么得到: $a^2 \Delta_3 u = a^2(u_{rr} + \frac{2}{r}u_r)$, 因此令 $v(t, r) = ru$, 化简得到如下定解问题:

$$\begin{cases} v_{tt} = a^2 v_{rr} (t > 0, r > 0) \\ v|_{r=0} \text{有界}, r = \sqrt{x^2 + y^2 + z^2} \\ v|_{t=0} = 0, v_t|_{t=0} = \frac{r}{(1 + r^2)^2}; \end{cases}$$

对 $v(t, r)$ 做 Laplace 变换, 设 $\tilde{v}(p, r) = L[v(t, r)]$, 因此可得到:

$$\begin{cases} p^2 \tilde{v}(p, r) - \frac{r}{(1 + r^2)^2} = a^2 \tilde{v}(p, r) \\ |\tilde{v}(p, 0)| < +\infty \end{cases}$$

对 $\tilde{v}(p, r)$ 作正弦变换, 记 $\tilde{V}(p, \omega) = F_s[\tilde{v}(p, r)] = \int_0^{+\infty} \tilde{v}(p, r) \sin \omega r dr$, 则有

$$\begin{aligned} F_s[\tilde{v}_{rr}] &= \int_0^{+\infty} \tilde{v}_{rr} \sin \omega r dr \\ &= \tilde{v}_r \sin \omega r \Big|_0^{+\infty} - \omega \int_0^{+\infty} \tilde{v}_r \cos \omega r dr \\ &= -\omega \int_0^{+\infty} \cos \omega r d\tilde{v} \\ &= -\omega \left[\tilde{v} \cos \omega r \Big|_0^{+\infty} + \omega \int_0^{+\infty} \tilde{v} \sin \omega r dr \right] \\ &= -\omega^2 \tilde{V} \end{aligned}$$

另外还有

$$\begin{aligned}
 F_s \left[\frac{r}{(1+r^2)^2} \right] &= \int_0^{+\infty} \frac{r}{(1+r^2)^2} \sin \omega r dr \\
 &= -\frac{1}{2} \int_0^{+\infty} \sin \omega r d \left(\frac{1}{1+r^2} \right) \\
 &= -\frac{1}{2} \left[\frac{\sin \omega r}{1+r^2} \Big|_0^{+\infty} - \omega \int_0^{+\infty} \frac{\cos \omega r}{1+r^2} dr \right] \\
 &= \frac{\omega}{2} \int_0^{+\infty} \frac{\cos \omega r}{1+r^2} dr \\
 &= \frac{\pi}{4} \omega e^{-\omega}
 \end{aligned}$$

因此可以得到: $p^2 \tilde{V} - \frac{\pi}{4} \omega e^{-\omega} = -a^2 \omega^2 \tilde{V}$, 那么

$$\tilde{V}(p, \omega) = \frac{1}{p^2 + a^2 \omega^2} \cdot \frac{\pi}{4} \omega e^{-\omega} = \left(\frac{1}{p - a\omega i} - \frac{1}{p + a\omega i} \right) \cdot \frac{\pi}{8ai} e^{-\omega}$$

因此

$$\begin{aligned}
 v(t, r) &= L^{-1}[F_s^{-1}[\tilde{V}(p, \omega)]] = F_s^{-1}[L^{-1}[\tilde{V}(p, \omega)]] \\
 &= F_s^{-1} \left[\frac{e^{a\omega i t} - e^{-a\omega i t}}{8ai} \pi e^{-\omega} \right] \\
 &= F_s^{-1} \left[\frac{\pi}{4a} e^{-\omega} \sin a\omega t \right] \\
 &= \frac{2}{\pi} \int_0^{+\infty} \frac{\pi}{4a} e^{-\omega} \sin a\omega t \sin \omega r d\omega \\
 &= \frac{1}{4a} \int_0^{+\infty} e^{-\omega} [\cos \omega(r - at) - \cos \omega(r + at)] d\omega \\
 &= \frac{1}{4a} \left[\frac{1}{1 + (r - at)^2} - \frac{1}{1 + (r + at)^2} \right] \\
 &= \frac{t}{[1 + (r + at)^2][1 + (r - at)^2]}
 \end{aligned}$$

Sol* (未用到 Laplace 变换, 直接使用 d'Alembert 公式):

由于初始条件只与 r 有关, 故 $u = u(t, r)$, 那么得到: $a^2 \Delta_3 u = a^2(u_{rr} + \frac{2}{r}u_r)$, 因此令 $v(t, r) = ru$, 化

简得到如下定解问题:

$$\begin{cases} v_{tt} = a^2 v_{rr} (t > 0, r > 0) \\ v|_{r=0} \text{有界}, r = \sqrt{x^2 + y^2 + z^2} \\ v|_{t=0} = 0, v_t|_{t=0} = \frac{r}{(1+r^2)^2}; \end{cases}$$

由于 v 满足的问题是半直线问题, 可考虑使用奇延拓得到的全直线问题:

$$\begin{cases} V_{tt} = a^2 V_{xx} (t > 0, x > 0) \\ V|_{x=0} \text{有界} \\ V|_{t=0} = 0, V_t|_{t=0} = \frac{x}{(1+x^2)^2}; \end{cases}$$

可利用 d'Alembert 公式得到:

$$\begin{aligned} V(t, x) &= \frac{1}{2a} \int_{x-at}^{x+at} \frac{\xi}{(1+\xi^2)^2} d\xi = -\frac{1}{4a} \cdot \frac{1}{1+\xi^2} \Big|_{x-at}^{x+at} \\ &= \frac{1}{4a} \cdot \left[\frac{1}{1+(x+at)^2} - \frac{1}{1-(x+at)^2} \right] \\ &= \frac{xt}{[1+(x+at)^2][1+(x-at)^2]} \end{aligned}$$

此时把解 $V(t, x)$ 中限制在 $x > 0$ (对应 $r > 0$), 可得到原半无界弦振动的定解问题的解为:

$$v(t, r) = \frac{rt}{[1+(r+at)^2][1+(r-at)^2]}$$

因此原三位波动方程定解问题的解为:

$$u(t, r) = \frac{v(t, r)}{r} = \frac{t}{[1+(r+at)^2][1+(r-at)^2]}$$

3 写在最后

本答案仅基于本人对数理方程的粗略理解，为方便同学们复习及纠正自己的答案而编写，很多题目也仅用了一种方法，仅提供参考，具体到对每个同学的意见，在批改作业的过程中也已经写了批注。

此外，考虑到本参考答案可能会存在一定的错误，后期可能会有修正版，可以[点击此处](#)查看最新版，对这些可能存在的错误，还请同学们海涵，同时也很感谢朱健同学在我编写答案的过程中提供的帮助。

2020-2021 春季学期数理方程 B 助教
本科 18 级 地球和空间科学学院 刘炜昊
2021 年 4 月 于合肥

数理方程第五章参考答案

1. 解:

(1) 由 $\delta(x)$ 的性质, 有

$$\int_{-\infty}^{+\infty} x\delta(x)dx = x|_{x=0} = 0$$

又当 $x \neq 0$ 时, $\delta(x) = 0, x\delta(x) = 0$

所以当 $x = 0$ 时, $x\delta(x) = 0$

所以

$$x\delta(x) \equiv 0$$

(2) 对于任意函数 $g(x) \in K$, 有

$$\begin{aligned}\int_{-\infty}^{+\infty} g(x)f(x)\delta(x-a)dx &= f(a)g(a) \\ &= f(a)\int_{-\infty}^{+\infty} g(x)\delta(x-a)dx = \int_{-\infty}^{+\infty} g(x)f(a)\delta(x-a)dx\end{aligned}$$

所以

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$

(3) 对于任意函数 $g(x) \in K$, 有

$a > 0$ 时

$$\begin{aligned}\int_{-\infty}^{+\infty} g(x)\delta(ax)dx &\stackrel{t=ax}{\Longrightarrow} \int_{-\infty}^{+\infty} g\left(\frac{t}{a}\right)\frac{1}{a}\delta(t)dt = \frac{1}{a}g(0) \\ &= \frac{1}{a}\int_{-\infty}^{+\infty} g(x)\delta(x)dx = \int_{-\infty}^{+\infty} g(x)\frac{1}{a}\delta(x)dx\end{aligned}$$

所以

$$\delta(ax) = \frac{1}{a}\delta(x)$$

$a < 0$ 时

$$\begin{aligned}\int_{-\infty}^{+\infty} g(x)\delta(ax)dx &\stackrel{t=ax}{\Longrightarrow} \int_{-\infty}^{+\infty} g\left(\frac{t}{a}\right)\frac{1}{-a}\delta(t)dt = \frac{1}{-a}g(0) = \\ &= \frac{1}{-a}\int_{-\infty}^{+\infty} g(x)\delta(x)dx = \int_{-\infty}^{+\infty} g(x)\frac{1}{-a}\delta(x)dx\end{aligned}$$

所以

$$\delta(ax) = \frac{1}{-a}\delta(x)$$

综上

$$\delta(ax) = \frac{1}{|a|}\delta(x), a \neq 0$$

(4) 对于任意函数 $f(x) \in K$, 有

$$\begin{aligned}\int_{-\infty}^{+\infty} f(x)\delta'(-x)dx &\stackrel{-x=t}{\Longrightarrow} \int_{-\infty}^{+\infty} f(-t)\delta'(t)dt = -[f(-t)]'|_{t=0} = f'(-t)|_{t=0} = f'(0) \\ \int_{-\infty}^{+\infty} f(x)[- \delta'(x)]dx &= -\int_{-\infty}^{+\infty} f(x)\delta'(x)dx = f'(0)\end{aligned}$$

所以

$$\delta'(-x) = -\delta'(x)$$

(5) 对于任意函数 $f(x) \in K$, 有

$$\int_{-\infty}^{+\infty} f(x)x\delta'(x)dx = \int_{-\infty}^{+\infty} \textcolor{red}{xf(x)}\delta'(x)dx = -[xf(x)]'|_{x=0} = -f(0)$$

$$\int_{-\infty}^{+\infty} f(x)[- \delta(x)]dx = - \int_{-\infty}^{+\infty} f(x)\delta(x)dx = -f(0)$$

所以

$$x\delta'(x) = -\delta(x)$$

2. 解:

对于任意函数 $f(x, y) \in K$, 有

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)\delta(x - x_0, y - y_0)dxdy$$

做变量替换

$$\begin{cases} x = x(\xi, \eta) \\ y = y(\xi, \eta) \end{cases}$$

所以

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)\delta(x - x_0, y - y_0)dxdy &= \\ \iint_D f[x(\xi, \eta), y(\xi, \eta)] \delta[x(\xi, \eta) - x_0, y(\xi, \eta) - y_0] \frac{1}{|J|} d\xi d\eta &= \\ \iint_D f[x(\xi, \eta), y(\xi, \eta)] \delta[x(\xi, \eta) - x_0, y(\xi, \eta) - y_0] \frac{1}{|J|} d\xi d\eta &= \\ = \iint_D f[x(\xi_0, \eta_0), y(\xi_0, \eta_0)] \delta[x(\xi, \eta), y(\xi, \eta)] \frac{1}{|J|} d\xi d\eta &= \\ \iint_D f[x(\xi_0, \eta_0), y(\xi_0, \eta_0)] \delta[x(\xi, \eta), y(\xi, \eta)] \frac{1}{|J|} d\xi d\eta = \iint_D f(\xi_0, \eta_0) \delta(\xi, \eta) \frac{1}{|J|} d\xi d\eta &= \\ = \iint_D f(\xi, \eta) \delta(\xi - \xi_0, \eta - \eta_0) \frac{1}{|J|} d\xi d\eta \end{aligned}$$

因此

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)\delta(x - x_0, y - y_0)dxdy = \iint_D f(\xi, \eta) \delta(\xi - \xi_0, \eta - \eta_0) \frac{1}{|J|} d\xi d\eta$$

故

$$\delta(x - x_0, y - y_0) = \frac{1}{|J|} \delta(\xi - \xi_0, \eta - \eta_0)$$

对于变量替换

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}, r \geq 0, 0 \leq \theta < 2\pi$$

$$|J| = r$$

$$\delta(x - x_0, y - y_0) = \frac{1}{r} \delta(r - r_0, \theta - \theta_0)$$

3. 解:

(1) 由分离变量法, 设 $u(t, x) = T(t)X(x)$

完成分量变量手续得关于 x, t 的方程

$$\begin{cases} X''(x) + \lambda = 0 \\ X(0) = X(l) = 0 \\ T'(t) + \lambda a^2 T(t) = 0 \end{cases}$$

由关于 x 的方程得固有值 $\lambda_n = \left(\frac{n\pi}{l}\right)^2$, 固有函数 $X_n(x) = \sin \frac{n\pi x}{l}, n = 1, 2, \dots$

所以 $T_n(t) = \exp\left\{-\left(\frac{n\pi a}{l}\right)^2 t\right\}$

$$u(t, x) = \sum_{n=1}^{+\infty} A_n \exp\left\{-\left(\frac{n\pi a}{l}\right)^2 t\right\} \sin \frac{n\pi x}{l}$$

$$u(0, x) = \sum_{n=1}^{+\infty} A_n \sin \frac{n\pi x}{l} = \delta(x - \xi)$$

$$A_n = \frac{2}{l} \int_0^l \delta(x - \xi) \sin \frac{n\pi x}{l} dx = \frac{2}{l} \sin \frac{n\pi \xi}{l}$$

$$u(t, x) = \frac{2}{l} \sum_{n=1}^{+\infty} \exp\left\{-\left(\frac{n\pi a}{l}\right)^2 t\right\} \sin \frac{n\pi \xi}{l} \sin \frac{n\pi x}{l}$$

(2) 由分离变量法, 设 $u(t, x) = T(t)X(x)$

完成分量变量手续得关于 x, t 的方程

$$\begin{cases} X''(x) + \lambda = 0 \\ X'(0) = X'(l) = 0 \\ T''(t) + \lambda a^2 T(t) = 0 \end{cases}$$

由关于 x 的方程得固有值 $\lambda_n = 0, \left(\frac{n\pi}{l}\right)^2$, 固有函数 $X_0(x) = 1$,

$$X_n(x) = \cos \frac{n\pi x}{l}, n = 1, 2, \dots$$

所以 $T_0(t) = A_0 + B_0 t$, $T_n(t) = A_n \cos \frac{n\pi a}{l} t + B_n \sin \frac{n\pi a}{l} t$

$$u(t, x) = A_0 + B_0 t + \sum_{n=1}^{+\infty} A_n \cos \frac{n\pi a}{l} t \cos \frac{n\pi x}{l} + B_n \sin \frac{n\pi a}{l} t \cos \frac{n\pi x}{l}$$

$$u(0, x) = A_0 + \sum_{n=1}^{+\infty} A_n \cos \frac{n\pi x}{l} = 0 \Rightarrow A_0 = A_n = 0$$

$$u_t(0, x) = B_0 + \sum_{n=1}^{+\infty} \frac{n\pi a}{l} B_n \cos \frac{n\pi x}{l} = \delta(x - \xi)$$

相当于对 $\delta(x - \xi)$ 进行余弦级数展开, 第一项为 $\frac{1}{l} \int_0^l \delta(x - \xi) dl = \frac{1}{l}$

$$B_0 = \frac{1}{l}$$

$$\frac{n\pi a}{l} B_n = \frac{2}{l} \int_0^l \delta(x - \xi) \cos \frac{n\pi x}{l} dx = \frac{2}{l} \cos \frac{n\pi \xi}{l}$$

$$B_n = \frac{2}{n\pi a} \cos \frac{n\pi \xi}{l}$$

$$u(t, x) = \frac{t}{l} + \sum_{n=1}^{+\infty} \frac{2}{n\pi a} \cos \frac{n\pi \xi}{l} \sin \frac{n\pi a t}{l} \cos \frac{n\pi x}{l}$$

4. (1)

解:

设

$$\begin{cases} \bar{x} = x \\ \bar{y} = \frac{1}{\beta} y \end{cases}$$

$$u_{xx} = u_{\bar{x}\bar{x}}$$

$$u_{yy} = \frac{1}{\beta^2} u_{\bar{y}\bar{y}}$$

所以

$$u_{xx} + \beta^2 u_{yy} = u_{\bar{x}\bar{x}} + u_{\bar{y}\bar{y}} = 0$$

由二维拉普拉斯方程基本解

$$\tilde{u}_{xx} + \beta^2 \tilde{u}_{yy} = \tilde{u}_{\bar{x}\bar{x}} + \tilde{u}_{\bar{y}\bar{y}} = \delta(x, y) = \frac{1}{\beta} \delta(\bar{x}, \bar{y})$$

$$u(\bar{x}, \bar{y}) = \frac{1}{2\pi\beta} \ln \bar{r}, \bar{r} = \sqrt{\bar{x}^2 + \bar{y}^2} = \sqrt{x^2 + \left(\frac{y}{\beta}\right)^2}$$

所以所得基本解为

$$u(x, y) = \frac{1}{4\pi\beta} \ln \left[x^2 + \left(\frac{y}{\beta}\right)^2 \right]$$

书上答案给的是

$$u(x, y) = \frac{1}{4\pi\beta} \ln[\beta^2 x^2 + y^2]$$

其实是

$$u(x, y) = \frac{1}{4\pi\beta} \ln \left[x^2 + \left(\frac{y}{\beta}\right)^2 \right] = \frac{1}{4\pi\beta} \ln(\beta^2 x^2 + y^2) - \frac{1}{4\pi\beta} \ln \beta^2$$

其中

$$-\frac{1}{4\pi\beta} \ln \beta^2$$

为常数项，不能反应基本解在原点处无界的性质，故可以舍去

(2)

解:

$$\Delta_2 \Delta_2 u = 0$$

设

$$v(x, y) = \Delta_2 u$$

则有

$$\Delta_2 v = 0$$

则有基本解方程

$$\Delta_2 v = \delta(x, y)$$

$$v(x, y) = \frac{1}{2\pi} \ln r$$

$$\Delta_2 u = \frac{1}{2\pi} \ln r$$

该方程右边只有 r ，所以设 $u = u(r)$

则有

$$\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} = \frac{1}{2\pi} \ln r$$

该方程为一欧拉方程，做变量替换

$$r = e^t$$

则

$$\frac{d^2 u}{dt^2} = \frac{te^{2t}}{2\pi}$$

$$u = \frac{\frac{1}{4}te^{2t} - \frac{1}{4}e^{2t}}{2\pi} + Dt + C$$

$$u(r) = \frac{1}{8\pi} r^2 \ln r - \frac{1}{8\pi} r^2 + D \ln r + C$$

$$\Delta_2 C = 0$$

$$\Delta_2 D \ln r = 2\pi D \delta(x, y)$$

$$\Delta_2 - \frac{1}{8\pi} r^2 = -\frac{1}{2\pi}$$

其中

$$\Delta_2 C = 0, \Delta_2 - \frac{1}{8\pi} r^2 = -\frac{1}{2\pi}$$

为常数，无法体现基本解在原点出无界的性质。

$$\Delta_2 D \ln r = 2\pi D \delta(x, y)$$

$$\Delta_2 \Delta_2 D \ln r = \Delta_2 2\pi D \delta(x, y) = 2\pi D \Delta_2 \delta(x, y) \neq 0$$

无法体现基本解满足齐次方程。

故基本解为

$$u = \frac{1}{8\pi} r^2 \ln r$$

(3)

解：

$$\Delta_3 \Delta_3 u = 0$$

设

$$v(x, y, z) = \Delta_3 u$$

则有

$$\Delta_3 v = 0$$

则有基本解方程

$$\Delta_3 v = \delta(x, y, z)$$

$$v = -\frac{1}{4\pi r}$$

$$\Delta_3 u = -\frac{1}{4\pi r}$$

该方程右边只有 r ，所以设 $u = u(r)$

则有

$$\frac{d^2 u}{dr^2} + \frac{2}{r} \frac{du}{dr} = -\frac{1}{4\pi r}$$

该方程为一欧拉方程，做变量替换

$$r = e^t$$

则

$$\frac{d^2 u}{dt^2} + \frac{du}{dt} = -\frac{e^t}{4\pi}$$

对应的非齐次方程特解为（特解主要体现非齐次项影响，对应原方程体现基本解的影响，故该特解符合条件，不需要再进行 Δ_3 之后的分析）

$$u = -\frac{1}{8\pi} e^t$$

所以所得方程基本解为

$$u = -\frac{1}{8\pi} r$$

5. 解：

依题

$$\Delta_3 u + k^2 u = \delta(x, y, z)$$

两边做三维傅里叶变换

设

$$\tilde{u} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u(x, y, z) e^{i\lambda x + i\mu y + i\nu z} dx dy dz$$

所以

$$(-\lambda^2 - \mu^2 - \nu^2) \tilde{u} + k^2 \tilde{u} = 1$$

$$\tilde{u} = \frac{1}{k^2 - (\lambda^2 + \mu^2 + \nu^2)}$$

$$u(x, y, z) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{k^2 - (\lambda^2 + \mu^2 + \nu^2)} e^{-(i\lambda x + i\mu y + i\nu z)} d\lambda d\mu d\nu$$

记

$$\vec{\rho} = (\lambda, \mu, \nu), \vec{r} = (x, y, z)$$

则

$$\rho = \sqrt{\lambda^2 + \mu^2 + \nu^2}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

设

θ 为 $\vec{\rho}, \vec{r}$ 的夹角，将 ν 轴作为 \vec{r} 的方向，做球坐标代换

$$\begin{cases} \lambda = \rho \sin\theta \cos\varphi \\ \mu = \rho \sin\theta \sin\varphi \\ \nu = \rho \cos\theta \end{cases}$$

$$\begin{aligned}
 u &= \frac{1}{(2\pi)^3} \int_0^{2\pi} d\varphi \int_0^\pi \sin\theta d\theta \int_0^{+\infty} \frac{\rho^2}{k^2 - \rho^2} e^{-i\rho r \cos\theta} d\rho \\
 u &= \frac{1}{4\pi^2} \int_0^{+\infty} \frac{\rho^2}{k^2 - \rho^2} d\rho \int_0^\pi \sin\theta e^{-i\rho r \cos\theta} d\theta \\
 \int_0^\pi \sin\theta e^{-i\rho r \cos\theta} d\theta &= - \int_0^\pi e^{-i\rho r \cos\theta} d\cos\theta \stackrel{t=\cos\theta}{\Longrightarrow} \int_{-1}^1 e^{-i\rho r t} dt = 2 \frac{\sin\rho r}{\rho r} \\
 u &= \frac{1}{2\pi^2 r} \int_0^{+\infty} \frac{\rho \sin\rho r}{k^2 - \rho^2} d\rho
 \end{aligned}$$

计算

$$\int_0^{+\infty} \frac{\rho \sin\rho r}{k^2 - \rho^2} d\rho$$

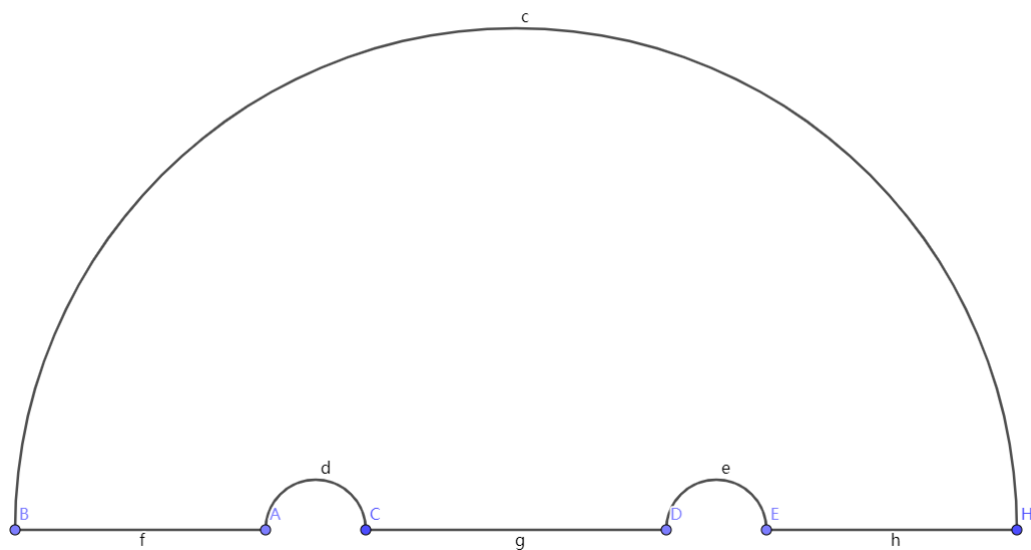
利用复变函数的方法计算该积分，首先

$$\int_0^{+\infty} \frac{\rho \sin\rho r}{k^2 - \rho^2} d\rho = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\rho \sin\rho r}{k^2 - \rho^2} d\rho$$

设

$$f(z) = \frac{ze^{irz}}{k^2 - z^2}$$

取围道如图：其中设大半圆的半径为 R ，小半圆的半径为 a 。围道方向为逆时针方向。



$$\int_c f(z)dz + \int_f f(z)dz + \int_d f(z)dz + \int_g f(z)dz + \int_e f(z)dz + \int_h f(z)dz = 0$$

令 $R \rightarrow +\infty, a \rightarrow 0$

则由约当引理

$$\int_c f(z)dz = 0$$

由小圆弧引理

$$\int_d f(z)dz = i(0 - \pi) \lim_{z \rightarrow -k} (z + k)f(z) = i\pi \frac{e^{-ikr}}{2}$$

$$\int_e f(z)dz = i(0 - \pi) \lim_{z \rightarrow k} (z - k)f(z) = i\pi \frac{e^{ikr}}{2}$$

X 轴部分

$$\int_f f(z)dz = \int_{-\infty}^{-k} f(x)dx$$

$$\int_g f(z)dz = \int_{-k}^k f(x)dx$$

$$\int_h f(z)dz = \int_k^{+\infty} f(x)dx$$

所以

$$\int_{-\infty}^{+\infty} f(x)dx + i\pi \cos kr = 0$$

$$\int_{-\infty}^{+\infty} \frac{x e^{irx}}{k^2 - x^2} dx = \int_{-\infty}^{+\infty} \frac{x}{k^2 - x^2} (\cos rx + i \sin rx) dx = i \int_{-\infty}^{+\infty} \frac{x \sin rx}{k^2 - x^2} dx = -i\pi \cos kr$$

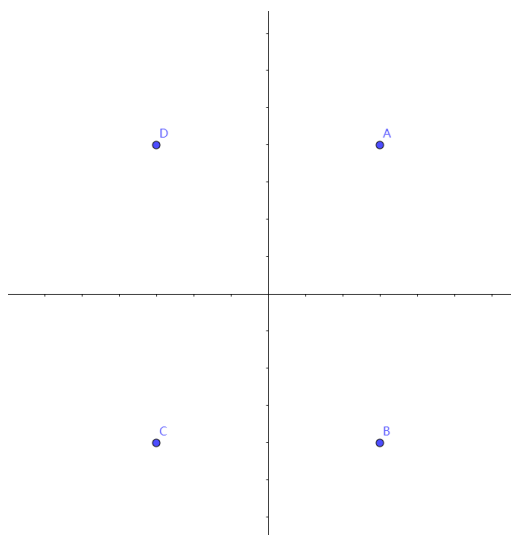
故

$$\int_0^{+\infty} \frac{\rho \sin pr}{k^2 - \rho^2} d\rho = -\frac{\pi \cos kr}{2}$$

$$u = \frac{1}{2\pi^2 r} \int_0^{+\infty} \frac{\rho \sin pr}{k^2 - \rho^2} d\rho = \frac{1}{2\pi^2 r} \cdot \left(-\frac{\pi \cos kr}{2} \right) = -\frac{\cos kr}{4\pi r}$$

6. 解:

(1) 如图所示 xoy 截面, 在 A 点 (ξ, η, ζ) 处放置一大小为 ε 的电荷, 为了使 $x = 0, y = 0$ 处电势均为 0, 需要在 A 点关于平面 $x=0$ 对称点 B $(\xi, -\eta, \zeta)$ 处放置一大小为 $-\varepsilon$ 的电荷, 以此类推, B 点关于平面 $y=0$ 对称点 C $(-\xi, -\eta, \zeta)$ 处放置一大小为 ε 的电荷, C 点关于平面 $x=0$ 对称点 D $(-\xi, \eta, \zeta)$ 处放置一大小为 $-\varepsilon$ 的电荷。



由三维形式点电荷的 Green 函数

$$G = \frac{1}{4\pi} \left(\frac{1}{r_A} + \frac{1}{r_C} - \frac{1}{r_B} - \frac{1}{r_D} \right)$$

其中

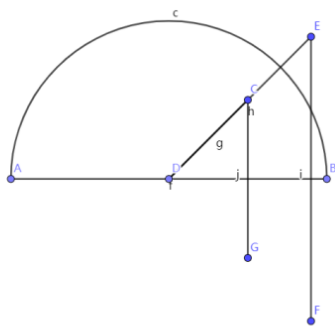
$$r_A = \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}$$

$$r_B = \sqrt{(x - \xi)^2 + (y + \eta)^2 + (z - \zeta)^2}$$

$$r_C = \sqrt{(x + \xi)^2 + (y + \eta)^2 + (z - \zeta)^2}$$

$$r_D = \sqrt{(x + \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}$$

- (2) 如图所示 xoz 截面，在 C 点 (ξ, η, ζ) 处放置一大小为 ε 的电荷，为了使 $x = 0$ 和球面处电势均为 0，需要在 C 点关于平面 $z=0$ 对称点 G 处放置一大小为 $-\varepsilon$ 的电荷， C 点关于球面对称点 E 处放置一大小为 $-\frac{a}{\rho_0} \varepsilon$ 的电荷， E 点关于平面 $z=0$ 对称点 F 处放置一大小为 $\frac{a}{\rho_0} \varepsilon$ 的电荷，其中 $\rho_0 = |DC| = \sqrt{\xi^2 + \eta^2 + \zeta^2}$ 。



由三维形式点电荷的 Green 函数

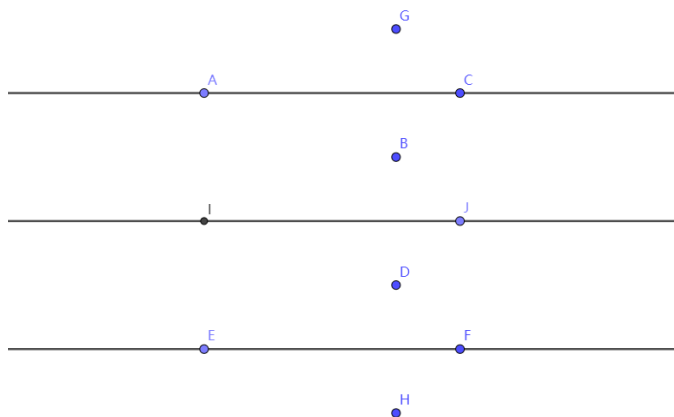
$$G = \frac{1}{4\pi} \left[\frac{1}{r(M, C)} + \frac{a}{\rho_0} \frac{1}{r(M, F)} - \frac{1}{r(M, G)} - \frac{a}{\rho_0} \frac{1}{r(M, E)} \right]$$

其中 $M(x, y, z)$.

- (3) 如图所示， xoz 截面，在 $0 < z < H$ 间 B 点 (ξ, η, ζ) 放置一大小为 ε 的电荷，先只考虑 B 点处电荷，为了使得 $z = 0$ 处电势为 0，需要在 D 点 $(\xi, \eta, -\zeta)$ (B 点关于 $z=0$ 的对称点) 放置一大小为 $-\varepsilon$ 的电荷，为了使 $z = H$ 需要在 G 点 $(\xi, \eta, \zeta + 2H)$ (B 点关于 $z=H$ 的对称点) 放置一大小为 $-\varepsilon$ 的电荷。

加入新电荷之后，原来的平衡体系进一步被打破，需要加入新的电荷平衡。

故在 $(\xi, \eta, 2nH + \zeta)$ 处放置一大小为 ε 的电荷，故在 $(\xi, \eta, 2nH - \zeta)$ 处放置一大小为 $-\varepsilon$ 的电荷。 $n = \dots, -2, -1, 0, 1, 2, \dots$



由三维形式点电荷的格林函数

$$G = \frac{1}{4\pi} \sum_{n=-\infty}^{n=+\infty} \left(\frac{1}{r_n} - \frac{1}{r'_n} \right)$$

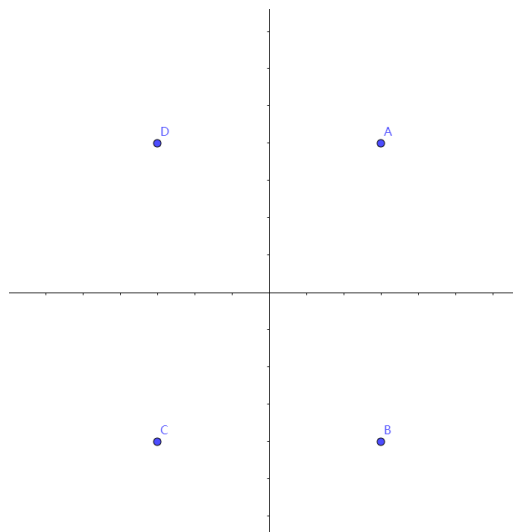
其中

$$r_n = \sqrt{(x - \xi)^2 + (y - \eta)^2 + [z - (2nH + \zeta)]^2}$$

$$r'_n = \sqrt{(x - \xi)^2 + (y - \eta)^2 + [z - (2nH - \zeta)]^2}$$

7. 解:

- (1) 如图在 A 点 (ξ, η) 处放置一大小为 ε 的电荷, 为了使 $x = 0, y = 0$ 处电势均为 0, 需要在 A 点关于 $x=0$ 对称点 B $(\xi, -\eta)$ 处放置一大小为 $-\varepsilon$ 的电荷, 以此类推, B 点关于 $y=0$ 对称点 C $(-\xi, -\eta)$ 处放置一大小为 ε 的电荷, C 点关于 $x=0$ 对称点 D $(-\xi, \eta)$ 处放置一大小为 $-\varepsilon$ 的电荷



由二维形式点电荷的 Green 函数

$$G = \frac{1}{2\pi} \left(\ln \frac{1}{r_A} + \ln \frac{1}{r_C} - \ln \frac{1}{r_B} - \ln \frac{1}{r_D} \right)$$

其中

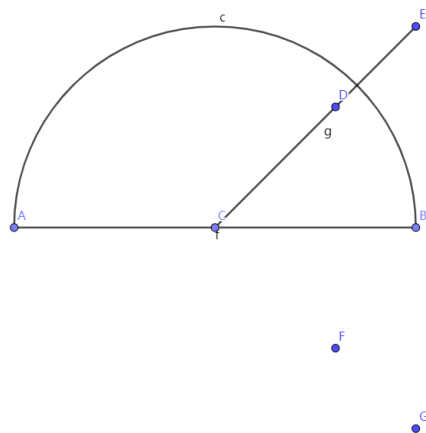
$$r_A = \sqrt{(x - \xi)^2 + (y - \eta)^2}$$

$$r_B = \sqrt{(x - \xi)^2 + (y + \eta)^2}$$

$$r_C = \sqrt{(x + \xi)^2 + (y + \eta)^2}$$

$$r_D = \sqrt{(x + \xi)^2 + (y - \eta)^2}$$

- (2) 如图所示在 D 点 (ξ, η) 处放置一大小为 ε 的电荷, 为了使 $x = 0$ 和圆边处电势均为 0, 需要在 D 点关于 $x=0$ 对称点 F 处放置一大小为 $-\varepsilon$ 的电荷, D 点关于球面对称点 E 处放置一大小为 $-\frac{1}{\rho_0}\varepsilon$ 的电荷, E 点关于 $x=0$ 对称点 G 处放置一大小为 $\frac{1}{\rho_0}\varepsilon$ 的电荷, 其中 $\rho_0 = |DC| = \sqrt{\xi^2 + \eta^2}$ 。



由二维形式点电荷的 Green 函数

$$G = \frac{1}{2\pi} \left(\ln \frac{1}{r(M, D)} + \frac{1}{\rho_0} \ln \frac{1}{r(M, G)} - \ln \frac{1}{r(M, F)} - \frac{1}{\rho_0} \ln \frac{1}{r(M, E)} \right)$$

其中 $M(x, y)$

8. 解: 依题基本解 $U(t, x)$ 满足

$$\begin{cases} U_t = a^2 U_{xx} + bU & (t > 0, -\infty < x < +\infty) \\ U|_{t=0} = \delta(x) \end{cases}$$

由傅里叶变换, 设 $\tilde{U}(t, \lambda) = \int_{-\infty}^{+\infty} U(t, x) e^{i\lambda x} dx$

所以

$$\begin{cases} \frac{d\tilde{U}}{dt} = (b - \lambda^2 a^2) \tilde{U} \\ \tilde{U}|_{t=0} = 1 \end{cases}$$

故

$$\tilde{U}(t, \lambda) = \exp\{(b - \lambda^2 a^2)t\} = e^{bt} e^{-\lambda^2 a^2 t}$$

所以

$$U(t, x) = e^{bt} F^{-1}[e^{-\lambda^2 a^2 t}]$$

$$\begin{aligned}
F^{-1}[e^{-\lambda^2 a^2 t}] &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\lambda x - \lambda^2 a^2 t} d\lambda \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left\{-a^2 t \left(\lambda + \frac{ix}{2a^2 t}\right)^2 - \frac{x^2}{4a^2 t}\right\} d\lambda \\
&= \frac{\exp\left\{-\frac{x^2}{4a^2 t}\right\}}{2\pi a \sqrt{t}} \int_{-\infty}^{+\infty} \exp\left\{-\left[a\sqrt{t}\left(\lambda + \frac{ix}{2a^2 t}\right)\right]^2\right\} da \sqrt{t} \left(\lambda + \frac{ix}{2a^2 t}\right) \\
&= \frac{\exp\left\{-\frac{x^2}{4a^2 t}\right\}}{2\pi a \sqrt{t}} \sqrt{\pi} = \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}} \\
U(t, x) &= \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t} + bt}
\end{aligned}$$

9. 解: (1)

该问题基本解满足

$$\begin{cases} U_t = -aU_x, (t > 0, -\infty < x < +\infty) \\ U|_{t=0} = \delta(x) \end{cases}$$

设

$$\begin{aligned}
\tilde{U}(t, \lambda) &= \int_{-\infty}^{+\infty} U(t, x) e^{i\lambda x} dx \\
\begin{cases} \frac{d\tilde{U}}{dt} = ia\lambda \tilde{U} \\ \tilde{U}|_{t=0} = 1 \end{cases} \\
\tilde{U} &= e^{-ia\lambda t} \\
U(t, x) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\lambda at} e^{-i\lambda x} d\lambda = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(-x+at)\lambda} d\lambda = \delta(-x+at) = \delta(x-at) \\
u(t, x) &= U(t, x) * \varphi(x) + \int_0^t U(t-\tau, x) * f(\tau, x) d\tau \\
&= \delta(x-at) * \varphi(x) + \int_0^t \delta[x-a(t-\tau)] * f(\tau, x) d\tau \\
&= \varphi(x-at) + \int_0^t f[\tau, x-a(t-\tau)] d\tau
\end{aligned}$$

(2) 方程基本解满足

$$\begin{cases} \frac{\partial^2 U}{\partial t^2} + 2 \frac{\partial U}{\partial t} = a^2 \frac{\partial^2 U}{\partial x^2} - 2U \\ U|_{t=0} = 0, U_t|_{t=0} = \delta(x) \end{cases}$$

设

$$\tilde{U}(t, \lambda) = \int_{-\infty}^{+\infty} U(t, x) e^{i\lambda x} dx$$

则有

$$\begin{aligned}
\begin{cases} \frac{d^2 \tilde{U}}{dt^2} + 2 \frac{d\tilde{U}}{dt} = -(\lambda^2 a^2 + 2) \tilde{U} \\ \tilde{U}|_{t=0} = 0, \tilde{U}_t|_{t=0} = 1 \end{cases} \\
\tilde{U} = \frac{1}{\sqrt{\lambda^2 a^2 + 1}} e^{-t} \sin \sqrt{\lambda^2 a^2 + 1} t
\end{aligned}$$

$$\begin{aligned}
U(t, x) &= F^{-1} \left[\frac{1}{\sqrt{\lambda^2 a^2 + 1}} e^{-t} \sin \sqrt{\lambda^2 a^2 + 1} t \right] = e^{-t} F^{-1} \left[\frac{\sin at \sqrt{\lambda^2 + \frac{1}{a^2}}}{a \sqrt{\lambda^2 + \frac{1}{a^2}}} \right] \\
&= \frac{e^{-t}}{2a} J_0 \left(\frac{1}{a} \sqrt{a^2 t^2 - x^2} \right) h(at - |x|) \\
&= \frac{e^{-t}}{2a} J_0 \left(\frac{1}{a} \sqrt{a^2 t^2 - x^2} \right) (-at < x < at)
\end{aligned}$$

注：课本给的公式有误，应该是

$$\begin{aligned}
F^{-1} \left[\frac{\sin a \sqrt{\lambda^2 + b^2}}{\sqrt{\lambda^2 + b^2}} \right] &= \frac{1}{2} J_0 \left(b \sqrt{a^2 - x^2} \right) h(a - |x|) \\
u(t, x) &= U(t, x) * \psi(x) = \int_{-at}^{at} \frac{e^{-t}}{2a} J_0 \left(\frac{1}{a} \sqrt{a^2 t^2 - \xi^2} \right) \psi(x - \xi) d\xi
\end{aligned}$$

10. 解：

由二维波动方程的基本解

$$U(t, x, y) = \begin{cases} \frac{1}{2\pi a} \iint_{D_{at}} \frac{\delta(\xi, \eta)}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}} d\xi d\eta, & x^2 + y^2 \leq a^2 t^2 \\ 0, & x^2 + y^2 > a^2 t^2 \end{cases}$$

其中

$$D_{at}: (\xi - x)^2 + (\eta - y)^2 < (at)^2$$

$$\begin{aligned}
U(t, x, y) &= \frac{1}{2\pi a} \iint_{D_{at}} \frac{\delta(\xi, \eta)}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}} d\xi d\eta \\
&= \begin{cases} \frac{1}{2\pi a} \frac{1}{\sqrt{a^2 t^2 - x^2 - y^2}}, & x^2 + y^2 \leq a^2 t^2 \\ 0, & x^2 + y^2 > a^2 t^2 \end{cases}
\end{aligned}$$

其中

$$D_{at}: (\xi - x)^2 + (\eta - y)^2 < (at)^2$$

$$\text{记 } \mathcal{D}_{at}: (\xi - x)^2 + (\eta - y)^2 < [a(t - \tau)]^2$$

$$\mathcal{D}_{at}: x^2 + y^2 \leq [a(t - \tau)]^2$$

$$\begin{aligned}
&\iint_{\mathcal{D}_{at}} \frac{\delta(\xi, \eta)}{\sqrt{a^2(t - \tau)^2 - (\xi - x)^2 - (\eta - y)^2}} d\xi d\eta * f(\tau, x, y) \\
&= \iint_{\mathcal{D}_{at}} \iint_{\mathcal{D}_{at}} \frac{\delta(\xi, \eta)}{\sqrt{a^2(t - \tau)^2 - (\xi - x)^2 - (\eta - y)^2}} d\xi d\eta f(\tau, x - x, y - y) dx dy \\
&= \iint_{\mathcal{D}_{at}} \iint_{\mathcal{D}_{at}} \frac{\delta(\xi, \eta)}{\sqrt{a^2(t - \tau)^2 - (\xi - x)^2 - (\eta - y)^2}} f(\tau, x - x, y - y) dx dy d\xi d\eta \\
&= \iint_{\mathcal{D}_{at}} \frac{f(\tau, \xi, \eta)}{\sqrt{a^2(t - \tau)^2 - (\xi - x)^2 - (\eta - y)^2}} d\xi d\eta
\end{aligned}$$

这一步是 $\delta(\xi, \eta)$ 将 \mathcal{D}_{at} 区域中 $f(\tau, x-x, y-y)$ 中的 $f(\tau, \xi, \eta)$ 筛选出来了。

因此

$$u(t, x, y) = \frac{1}{2\pi a} \int_0^t d\tau \iint_{\mathcal{D}_{at}} \frac{f(\tau, \xi, \eta)}{\sqrt{a^2(t-\tau)^2 - (\xi-x)^2 - (\eta-y)^2}} d\xi d\eta$$

$$\mathcal{D}_{at}: (\xi-x)^2 + (\eta-y)^2 < [a(t-\tau)]^2$$

11. 解:

考虑一维波动方程:

$$\begin{cases} u_{tt} = a^2 u_{xx}, (-\infty < x < +\infty, t > 0) \\ u|_{t=0} = \varphi(x), u_t|_{t=0} = \psi(x) \end{cases}$$

此问题可以看成是二维波动方程

$$\begin{cases} u_{tt} = a^2 \Delta_2 u, (-\infty < x, y < +\infty, t > 0) \\ u|_{t=0} = \varphi(x, y), u_t|_{t=0} = \psi(x, y) \end{cases}$$

自变量限制在 $y=0$ 时的特殊情形。

由二维情形解的表达式:

$$u(t, x, y) = \frac{1}{2\pi a} \iint_{\mathcal{D}_{at}} \frac{\psi(\xi, \eta) d\xi d\eta}{\sqrt{(at)^2 - (\xi-x)^2 - (\eta-y)^2}}$$

$$+ \frac{1}{2\pi a} \frac{\partial}{\partial t} \left[\iint_{\mathcal{D}_{at}} \frac{\varphi(\xi, \eta) d\xi d\eta}{\sqrt{(at)^2 - (\xi-x)^2 - (\eta-y)^2}} \right]$$

$$\mathcal{D}_{at}: (\xi-x)^2 + (\eta-y)^2 < a^2 t^2$$

则一维情形:

$$u(t, x) = \frac{1}{2\pi a} \iint_D \frac{\psi(\xi) d\xi d\eta}{\sqrt{(at)^2 - (\xi-x)^2 - \eta^2}} + \frac{1}{2\pi a} \frac{\partial}{\partial t} \left[\iint_D \frac{\varphi(\xi) d\xi d\eta}{\sqrt{(at)^2 - (\xi-x)^2 - \eta^2}} \right]$$

$$D: (\xi-x)^2 + \eta^2 < a^2 t^2$$

$$\iint_D \frac{\psi(\xi) d\xi d\eta}{\sqrt{(at)^2 - (\xi-x)^2 - \eta^2}} = \int_{x-at}^{x+at} \psi(\xi) d\xi \int_{-\sqrt{(at)^2 - (\xi-x)^2}}^{+\sqrt{(at)^2 - (\xi-x)^2}} \frac{1}{\sqrt{(at)^2 - (\xi-x)^2 - \eta^2}} d\eta$$

对于

$$\int_{-\sqrt{(at)^2 - (\xi-x)^2}}^{+\sqrt{(at)^2 - (\xi-x)^2}} \frac{1}{\sqrt{(at)^2 - (\xi-x)^2 - \eta^2}} d\eta = \arcsin \frac{\eta}{\sqrt{(at)^2 - (\xi-x)^2}} \Big|_{-\sqrt{(at)^2 - (\xi-x)^2}}^{+\sqrt{(at)^2 - (\xi-x)^2}}$$

$$= \pi$$

所以

$$\iint_D \frac{\psi(\xi) d\xi d\eta}{\sqrt{(at)^2 - (\xi-x)^2 - \eta^2}} = \pi \int_{x-at}^{x+at} \psi(\xi) d\xi$$

$$\frac{\partial}{\partial t} \left[\iint_D \frac{\varphi(\xi) d\xi d\eta}{\sqrt{(at)^2 - (\xi-x)^2 - \eta^2}} \right] = \pi \frac{\partial}{\partial t} \left[\int_{x-at}^{x+at} \varphi(\xi) d\xi \right] = \pi a [\varphi(x+at) + \varphi(x-at)]$$

故

$$u(t, x) = \frac{1}{2a} \int_{x-at}^{x+at} \psi(\xi) d\xi + \frac{[\varphi(x+at) + \varphi(x-at)]}{2}$$

12. 解:

(1) 依题

已知一维热传导方程的基本解为

$$U(t, x) = \frac{1}{2a\sqrt{\pi t}} \exp\left\{-\frac{x^2}{4a^2 t}\right\}$$

则

$$\begin{aligned} u(t, x) &= U(t, x) * e^{-x^2} = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} \exp\left\{-\frac{\xi^2}{4a^2 t}\right\} e^{-(x-\xi)^2} d\xi \\ &= \int_{-\infty}^{+\infty} \exp\left\{-\frac{\xi^2}{4a^2 t}\right\} e^{-(x-\xi)^2} d\xi \\ &= \int_{-\infty}^{+\infty} \exp\left\{-\frac{\xi^2(4a^2 t + 1) - 8a^2 t x \xi + 4a^2 t x^2}{4a^2 t}\right\} d\xi \\ &= e^{-x^2} \int_{-\infty}^{+\infty} \exp\left\{-\frac{\xi^2(4a^2 t + 1) - 8a^2 t x \xi}{4a^2 t}\right\} d\xi \\ &= e^{-x^2} e^{\frac{4a^2 t x}{4a^2 t + 1} x^2} \int_{-\infty}^{+\infty} \exp\left\{-\frac{4a^2 t + 1}{4a^2 t} \left(\xi - \frac{4a^2 t x}{4a^2 t + 1}\right)^2\right\} d\xi \\ &= e^{-\frac{x^2}{4a^2 t + 1}} \frac{2a}{\sqrt{4a^2 t + 1}} \sqrt{\pi t} \\ u(t, x) &= \frac{e^{-\frac{x^2}{4a^2 t + 1}}}{\sqrt{4a^2 t + 1}} \end{aligned}$$

(2) 依题

$$\begin{aligned} u(t, x, y) &= \frac{1}{2\pi a} \frac{\partial}{\partial t} \left[\iint_{D_{at}} \frac{\xi^2(\xi + \eta) d\xi d\eta}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}} \right] \\ D_{at}: (\xi - x)^2 + (\eta - y)^2 &< a^2 t^2 \\ &\iint_{D_{at}} \frac{\xi^3 d\xi d\eta}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}} \\ &\iint_{D_{at}} \frac{\xi^3 d\xi d\eta}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}} \\ &= \int_{x-at}^{x+at} \xi^3 d\xi \int_{y-\sqrt{(at)^2 - (\xi-x)^2}}^{y+\sqrt{(at)^2 - (\xi-x)^2}} \frac{1}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}} d\eta \\ &\int_{y-\sqrt{(at)^2 - (\xi-x)^2}}^{y+\sqrt{(at)^2 - (\xi-x)^2}} \frac{1}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}} d\eta \\ &\xrightarrow{\eta-y=\rho} \int_{-\sqrt{(at)^2 - (\xi-x)^2}}^{\sqrt{(at)^2 - (\xi-x)^2}} \frac{1}{\sqrt{(at)^2 - (\xi - x)^2 - \rho^2}} d\rho = \pi \end{aligned}$$

$$\begin{aligned}
& \iint_{D_{at}} \frac{\xi^3 d\xi d\eta}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}} = \pi 2xat[x^2 + a^2t^2] \\
& \iint_{D_{at}} \frac{\xi^2 \eta d\xi d\eta}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}} \\
& = \int_{x-at}^{x+at} \xi^2 d\xi \int_{-\sqrt{(at)^2 - (\xi - x)^2}}^{+\sqrt{(at)^2 - (\xi - x)^2}} \frac{\eta}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}} d\eta \\
& \int_{y-\sqrt{(at)^2 - (\xi - x)^2}}^{y+\sqrt{(at)^2 - (\xi - x)^2}} \frac{\eta}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}} d\eta \\
& \xrightarrow{\eta-y=\rho} \int_{-\sqrt{(at)^2 - (\xi - x)^2}}^{+\sqrt{(at)^2 - (\xi - x)^2}} \frac{\rho + y}{\sqrt{(at)^2 - (\xi - x)^2 - \rho^2}} d\rho \\
& \int_{-\sqrt{(at)^2 - (\xi - x)^2}}^{+\sqrt{(at)^2 - (\xi - x)^2}} \frac{\rho}{\sqrt{(at)^2 - (\xi - x)^2 - \rho^2}} d\rho = 0 \\
& \int_{-\sqrt{(at)^2 - (\xi - x)^2}}^{+\sqrt{(at)^2 - (\xi - x)^2}} \frac{y}{\sqrt{(at)^2 - (\xi - x)^2 - \rho^2}} d\rho = \pi y \\
& \iint_{D_{at}} \frac{\xi^2 \eta d\xi d\eta}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}} = \frac{2\pi ayt(3x^2 + a^2t^2)}{3}
\end{aligned}$$

所以

$$u(t, x, y) = \frac{\partial}{\partial t} \left[xt(x^2 + a^2t^2) + \frac{yt(3x^2 + a^2t^2)}{3} \right] = x^3 + 3a^2xt + x^2y + a^2yt^2$$

(3) 依题

$$\begin{aligned}
u(t, x, y) &= \frac{1}{2\pi a} \iint_{D_{at}} \frac{\xi + \eta d\xi d\eta}{\sqrt{(at)^2 - (\xi - x)^2 - (\eta - y)^2}} + \int_0^t U(t - \tau, x, y) * (x + y) d\tau \\
& D_{at}: (\xi - x)^2 + (\eta - y)^2 < a^2t^2
\end{aligned}$$

$$U(t - \tau, x, y) = \frac{1}{2\pi a} \iint_{D_{at}} \frac{\delta(\xi, \eta)}{\sqrt{a^2(t - \tau)^2 - (\xi - x)^2 - (\eta - y)^2}} d\xi d\eta$$

其中 $D_{at}: (\xi - x)^2 + (\eta - y)^2 < [a(t - \tau)]^2$

$$\begin{aligned}
& \int_0^t U(t - \tau, x, y) * (x + y) d\tau \\
& = \frac{1}{2\pi a} \int_0^t \iint_{D_{at}} \frac{\delta(\xi, \eta)}{\sqrt{a^2(t - \tau)^2 - (\xi - x)^2 - (\eta - y)^2}} d\xi d\eta \\
& \quad * (x + y) d\tau
\end{aligned}$$

$$\begin{aligned}
& \iint_{\mathcal{D}_{at}} \frac{\delta(\xi, \eta)}{\sqrt{a^2(t-\tau)^2 - (\xi-x)^2 - (\eta-y)^2}} d\xi d\eta * (x+y) \\
&= \iint_D \iint_{\mathcal{D}_{at}} \frac{\delta(\xi, \eta)}{\sqrt{a^2(t-\tau)^2 - (\xi-x)^2 - (\eta-y)^2}} (x-x_0+y-y_0) d\xi d\eta dx_0 dy_0 \\
&= \iint_D \iint_{\mathcal{D}_{at}} \frac{\delta(\xi, \eta)}{\sqrt{a^2(t-\tau)^2 - (\xi-x)^2 - (\eta-y)^2}} (x-x_0+y-y_0) d\xi d\eta dx_0 dy_0 \\
&= \iint_{\mathcal{D}_{at}} \frac{\xi + \eta}{\sqrt{a^2(t-\tau)^2 - (\xi-x)^2 - (\eta-y)^2}} d\xi d\eta \\
&= \iint_{\mathcal{D}_{at}} \frac{\xi d\xi d\eta}{\sqrt{a^2(t-\tau)^2 - (\xi-x)^2 - (\eta-y)^2}} = 2\pi x a(t-\tau) \\
&= \iint_{\mathcal{D}_{at}} \frac{\eta d\xi d\eta}{\sqrt{a^2(t-\tau)^2 - (\xi-x)^2 - (\eta-y)^2}} = 2\pi y a(t-\tau) \\
&= \iint_{\mathcal{D}_{at}} \frac{\xi d\xi d\eta}{\sqrt{(at)^2 - (\xi-x)^2 - (\eta-y)^2}} \\
&= \int_{x-at}^{x+at} \xi d\xi \int_{-\sqrt{(at)^2 - (\xi-x)^2}}^{+\sqrt{(at)^2 - (\xi-x)^2}} \frac{1}{\sqrt{(at)^2 - (\xi-x)^2 - (\eta-y)^2}} d\eta \\
&= 2\pi x at \\
&= \iint_{\mathcal{D}_{at}} \frac{\eta d\xi d\eta}{\sqrt{(at)^2 - (\xi-x)^2 - (\eta-y)^2}} \\
&= \int_{x-at}^{x+at} d\xi \int_{-\sqrt{(at)^2 - (\xi-x)^2}}^{+\sqrt{(at)^2 - (\xi-x)^2}} \frac{\eta}{\sqrt{(at)^2 - (\xi-x)^2 - (\eta-y)^2}} d\eta \\
&= 2\pi y at \\
&= \frac{1}{2\pi a} \iint_{\mathcal{D}_{at}} \frac{\xi + \eta d\xi d\eta}{\sqrt{(at)^2 - (\xi-x)^2 - (\eta-y)^2}} + \int_0^t U(t-\tau, x, y) * (x+y) d\tau \\
&= \frac{1}{2\pi a} [2\pi x at + 2\pi y at] \\
&+ (x+y) \int_0^t (t-\tau) d\tau = (x+y) \left(t + \frac{t^2}{2} \right)
\end{aligned}$$

故

$$u(t, x) = (x+y) \left(t + \frac{t^2}{2} \right)$$

(4) 此题既可以效仿 (3) 的做法, 直接由三维波动方程的公式直接做, 也可以用特

解法求解。设 $u = v + u^*$

观察方程的非齐次项，为 $x + y + z$

且该非齐次项与初始位置，初始速度均一致。

且非齐次项的 Δ_3 之后的结果为 0。

因此我们可以设所求方程的特解为：

$$u^*(t, x, y, z) = f(t)(x + y + z)$$

则由题目已知得到三个关于 $f(t)$ 的条件

$$\begin{cases} f''(t) = 0 \\ f(0) = 1 \\ f'(0) = 1 \end{cases}$$

则可以解得

$$f(t) = \frac{1}{2}t^2 + t + 1$$

$$u^*(t, x, y, z) = \left(\frac{1}{2}t^2 + t + 1\right)(x + y + z)$$

之后所得方程为

$$\begin{cases} v_{tt} = a^2 \Delta_3 v \\ v|_{t=0} = 0 \\ v_t|_{t=0} = 0 \end{cases}$$

由三维波动方程解的公式可知

$$v \equiv 0$$

所以原方程解为

$$u = \left(\frac{1}{2}t^2 + t + 1\right)(x + y + z)$$

与 (3) 过程做对比，说明恰当的特解选取对于解决问题而言效果十分显著。