Logical Operators: NAND and NOR

In this document, we address a series of questions to demonstrate the properties and equivalences of logical operators, including the newly introduced NAND ($|$), defined as true when at least one of p or q is false, and NOR ($\downarrow$), defined as true only when both p and q are false. We use truth tables to verify equivalences and construct compound propositions.

Q1: Conjunction, Disjunction, and Conditional using $\neg$ and $\vee$

To express conjunction ($p \wedge q$), disjunction ($p \vee q$), and conditional ($p \rightarrow q$) using only $\neg$ and $\vee$, we use the following equivalences:

- Disjunction: $p \vee q$ (already given).
- Conjunction: $p \wedge q \equiv \neg(\neg p \vee \neg q)$
- Conditional: $p \rightarrow q \equiv \neg p \vee q$.

The truth table verifies these equivalences:

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$\neg(\neg p \vee \neg q)$
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

The column for $\neg(\neg p \vee \neg q)$ matches $p \wedge q$. For the conditional:

p	q	$\neg p$	$\neg p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

The column for $\neg p \vee q$ matches $p \rightarrow q$. Thus, $\neg$ and $\vee$ can express all three.

Q2: Conjunction, Disjunction, and Conditional using $\neg$ and $\wedge$

To express conjunction ($p \wedge q$), disjunction ($p \vee q$), and conditional ($p \rightarrow q$) using only $\neg$ and $\wedge$, we use:

- Conjunction: $p \wedge q$ (already given).
- Disjunction: $p \vee q \equiv \neg(\neg p \wedge \neg q)$
- Conditional: $p \rightarrow q \equiv \neg(p \wedge \neg q)$.

The truth table verifies these:

p	q	$\neg p$	$\neg q$	$\neg p \land \neg q$	$\neg(\neg p \land \neg q)$	$\neg(p \land \neg q)$
T	T	F	F	F	T	T
T	F	F	T	F	T	F
F	T	T	F	F	T	T
F	F	T	T	T	F	T

The column for $\neg(\neg p \land \neg q)$ matches $p \lor q$, and $\neg(p \land \neg q)$ matches $p \to q$. Thus, $\neg$ and $\land$ can express all three.

Q3: Truth Table for NAND ($p|q$)

The NAND operator ($p|q$) is true when at least one of p or q is false, and false when both are true:

p	q	$p\|q$
T	T	F
T	F	T
F	T	T
F	F	T

Q4: Showing $p|q \equiv \neg(p \land q)$

We compare $p|q$ with $\neg(p \land q)$ using a truth table:

p	q	$p \land q$	$\neg(p \land q)$	$p\|q$
T	T	T	F	F
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

The columns for $\neg(p \land q)$ and $p|q$ are identical, confirming $p|q \equiv \neg(p \land q)$.

Q5: Truth Table for NOR ($p \downarrow q$)

The NOR operator ($p \downarrow q$) is true only when both p and q are false:

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

Q6: Showing $p \downarrow q \equiv \neg(p \vee q)$

We compare $p \downarrow q$ with $\neg(p \vee q)$ using a truth table:

p	q	$p \vee q$	$\neg(p \vee q)$	$p \downarrow q$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

The columns for $\neg(p \vee q)$ and $p \downarrow q$ are identical, confirming $p \downarrow q \equiv \neg(p \vee q)$.

Q7: Showing $\{\downarrow\}$ is Functionally Complete

A set of operators is functionally complete if it can express negation, conjunction, and disjunction (or equivalently, $\neg$ and $\vee$ or $\neg$ and $\wedge$). We show this for $\{\downarrow\}$ in two parts.

Q7a: Showing $p \downarrow p \equiv \neg p$

We construct a truth table for $p \downarrow p$:

p	$p \downarrow p$	$\neg p$
T	F	F
F	T	T

Since $p \downarrow p$ matches $\neg p$, we have $p \downarrow p \equiv \neg p$.

Q7b: Showing $(p \downarrow q) \downarrow (p \downarrow q) \equiv p \vee q$

We construct a truth table:

p	q	$p \downarrow q$	$(p \downarrow q) \downarrow (p \downarrow q)$	$p \vee q$
T	T	F	T	T
T	F	F	T	T
F	T	F	T	T
F	F	T	F	F

The column for $(p \downarrow q) \downarrow (p \downarrow q)$ matches $p \vee q$. Since we can express $\neg p$ (from Q7a) and $p \vee q$, and Q1 shows that $\{\neg, \vee\}$ is functionally complete, $\{\downarrow\}$ is functionally complete.

Q8: Expressing $p \to q$ using only $\downarrow$

We proved $p \to q \equiv \neg p \vee q$ in class so using this and the work from Q7:

- $\neg p \equiv p \downarrow p$.
- $p \vee q \equiv (p \downarrow q) \downarrow (p \downarrow q)$.

Thus, $p \to q \equiv \neg p \vee q \equiv (p \downarrow p) \vee q \equiv ((p \downarrow p) \downarrow q) \downarrow ((p \downarrow p) \downarrow q)$. The truth table verifies:

p	q	$p \downarrow p$	$(p \downarrow p) \downarrow q$	$((p \downarrow p) \downarrow q) \downarrow ((p \downarrow p) \downarrow q)$	$p \to q$
T	T	F	T	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The columns match, so $p \to q \equiv ((p \downarrow p) \downarrow q) \downarrow ((p \downarrow p) \downarrow q)$.

Q9: Showing $p|q \equiv q|p$

We compare $p|q$ and $q|p$ using a truth table:

p	q	$p\|q$	$q\|p$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

The columns for $p|q$ and $q|p$ are identical, confirming $p|q \equiv q|p$.