Practice Exam 1 Discrete Mathematics 1

This is a mandatory homework. Show all your work to get credit.

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Q 1 *) The following sign is at the entrance of a restaurant: "No shoes, no shirt, no service." Write this sentence as a conditional proposition.*

-Lets set p equal to "The user wears shoes", q equal to "The user wears a shirt", and r equal to "Service is provided". Then $\neg p$ is "No shoes", $\neg q$ is "No shirt", and $\neg r$ is "No service". The requirement for service is (shoes $\wedge$ shirt). The conditional proposition is:

$$(\neg p \wedge \neg q) \rightarrow \neg r$$

Q 2 *)Write these system specifications in symbols using the propositions:*
v: "The user enters a valid password,"
a: "Access is granted to the user,"
c: "The user has contacted the network administrator,"
and logical connectives. Then determine if the system specifications are consistent (i.e. all 3 statements can be simultaneously true).
(i) "The user has contacted the network administrator, but does not enter a valid password."
(ii) "Access is granted whenever the user has contacted the network administrator or enters a valid password."
(iii) "Access is denied if the user has not entered a valid password or has not contacted the network administrator."

i) $C \wedge \neg V$

ii) $(C \vee V) \to A$

iii) $(\neg V \vee \neg C) \to \neg A$

1.For (i) to be True, we must have $C = \mathbf{T}$ *and* $V = \mathbf{F}$. *2.Substitute* $C = T$ *and* $V = F$ *into (iii):*

$$(\neg F \vee \neg T) \to \neg A \equiv (T \vee F) \to \neg A \equiv T \to \neg A$$

For (iii) to be True, the conclusion $\neg A$ *must be True, so* $A = \mathbf{F}$. *3.Now, check if the truth assignment* $V = F, A = F, C = T$ *satisfies (ii):*

$$(C \vee V) \to A \equiv (T \vee F) \to F \equiv T \to F$$

The statement $T \to F$ *is False.*

Since the requirement for (i) and (iii) to be true ($V = F, A = F, C = T$) *makes (ii) false, there is no assignment that makes all three true. Therefore, the system specifications are inconsistent.*

Q 3 *)Solve this puzzle: You meet two people, A and B. Each person either always tells the truth or always lies. Person A tells you, "We are not both truthtellers."*
Determine, if possible, which type of person each one is.

Let P_A be the proposition "A is a truthteller" and P_B be the proposition "B is a truthteller". A's statement S is $\neg(P_A \wedge P_B)$.

1.Assume A is a Liar ($\neg P_A$): A's statement S must be False.

$$\neg S \equiv P_A \wedge P_B$$

For $P_A \wedge P_B$ to be True, P_A must be True. This contradicts our assumption that A is a Liar ($\neg P_A$). Thus, this case is impossible.

2.Assume A is a Truthteller (P_A): A's statement S must be True.

$$S \equiv \neg(P_A \wedge P_B)$$

Since P_A is True, $S \equiv \neg(T \wedge P_B) \equiv \neg P_B$. Since S must be True, $\neg P_B$ must be True. This means P_B is False.

Conclusion: A must be a truthteller (Knight) and B must be a liar (Knave).

Q 4 *)Write the following statement and its negation symbolically: "If it is Thursday, then we are going to swim only if it is not raining".*

Let p: "It is Thursday", q: "We are going to swim", r: "It is raining". The phrase "A only if B" means $A \to B$. So, "we are going to swim only if it is not raining" is $q \to \neg r$. The entire statement is in the form "If p, then $q \to \neg r$":

$$\textit{Statement: } p \to (q \to \neg r)$$

To find the negation, we first simplify the statement using logical equivalences:

$$p \to (q \to \neg r) \equiv p \to (\neg q \vee \neg r) \equiv \neg p \vee (\neg q \vee \neg r) \equiv \neg(p \wedge q \wedge r)$$

Now negate the simplified form:

$$\textit{Negation: } \neg(\neg(p \wedge q \wedge r)) \equiv p \wedge q \wedge r$$

In English, the negation is: "It is Thursday, and we are going to swim, and it is raining."

Q 5 *)Prove that* $p \to (q \vee r) \equiv (p \wedge \neg q) \to r$ *by using a series of logical equivalences.*

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\begin{aligned}
(p \wedge \neg q) \to r &\equiv \neg(p \wedge \neg q) \vee r && (\textit{Implication Law: } A \to B \equiv \neg A \vee B) \\
&\equiv (\neg p \vee \neg(\neg q)) \vee r && (\textit{De Morgan's Law}) \\
&\equiv (\neg p \vee q) \vee r && (\textit{Double Negation}) \\
&\equiv \neg p \vee (q \vee r) && (\textit{Associative Law}) \\
&\equiv p \to (q \vee r) && (\textit{Implication Law})
\end{aligned}
$$

The two expressions are equivalent.

Q 6 *)Consider this sentence, which is Section 2 of Article I of the U. S. Constitution: "No person shall be a Representative who shall not have attained the age of twenty-five years, and been seven years a citizen of the United States, and who shall not, when elected, be an inhabitant of that state in which he shall be chosen."*
(a) Rewrite the sentence in English in the form "If . . . , then . . . ".
(b) Using the predicates A(x): "x is at least twenty-five years old," C(x): "x has been a citizen of the United States for at least seven years," I(x): "x, when elected, is an inhabitant of the state in which he is chosen," and R(x): "x can be a Representative," where the universe for x in all four predicates consists of all people, rewrite the sentence using quantifiers and these predicates. [Note: At the time at which the U. S. Constitution was ratified, the universe for x consisted of landowning males.]

a) If a person is a Representative, then that person is at least 25 years old, has been a citizen of the United States for at least 7 years, and is an inhabitant of the state from which they are chosen when elected.

b) The sentence establishes the necessary conditions for being a Representative. The form is $R(x) \to (requirements)$.

$$\forall x(R(x) \to (A(x) \wedge C(x) \wedge I(x)))$$

Q 7 *)Suppose that the universe for x and y is* $\{1, 2, 3\}$*. Also, assume that P(x,y) is a predicate that is true in the following cases, and false otherwise: P(1,3),P(2,1),P(2,2),P(3,1),P(3,2),P(3,3). Determine whether each of the following is true or false:*
(a) $\forall y \exists x (x \neq y \wedge P(x, y))$.
(b) $\forall x \exists y (x \neq y \wedge \neg P(x, y))$.
(c) $\forall y \exists x (x \neq y \wedge \neg P(x, y))$.

The false cases $\neg P(x, y)$ *are:* $(1, 1), (1, 2), (2, 3)$.
a) $\forall y \exists x (x \neq y \wedge P(x, y))$. *(True)*

- $y = 1$: $\exists x \neq 1 \mid P(x, 1)$? *Yes,* $P(2, 1)$ *and* $P(3, 1)$ *are True.*
- $y = 2$: $\exists x \neq 2 \mid P(x, 2)$? *Yes,* $P(3, 2)$ *is True.*
- $y = 3$: $\exists x \neq 3 \mid P(x, 3)$? *Yes,* $P(1, 3)$ *is True.*

b) $\forall x \exists y (x \neq y \wedge \neg P(x, y))$. *(False)*

- $x = 1$: $\exists y \neq 1 \mid \neg P(1, y)$? *Yes,* $\neg P(1, 2)$ *is True.*
- $x = 2$: $\exists y \neq 2 \mid \neg P(2, y)$? *Yes,* $\neg P(2, 3)$ *is True.*
- $x = 3$: $\exists y \neq 3 \mid \neg P(3, y)$? *No.* $P(3, 1), P(3, 2), P(3, 3)$ *are all True, so* $\neg P(3, y)$ *is never True.*

Since it fails for $x = 3$*, the statement is False.*
c) $\forall y \exists x (x \neq y \wedge \neg P(x, y))$. *(False)*

- $y = 1$: $\exists x \neq 1 \mid \neg P(x, 1)$? *No.* $P(1, 1)$ *is False (so* $\neg P(1, 1)$ *is True, but* $x \neq 1$ *is required).* $P(2, 1)$ *and* $P(3, 1)$ *are True, so* $\neg P(2, 1)$ *and* $\neg P(3, 1)$ *are False.*

Since it fails for $y = 1$*, the statement is False.*

Q 8 *)Determine whether this argument is valid:*
Lynn works part time or full time. ($P \vee F$)
If Lynn does not play on the team, then she does not work part time. ($\neg T \to \neg P$)
If Lynn plays on the team, she is busy. ($T \to B$)
Lynn does not work full time. ($\neg F$)
Therefore, Lynn is busy. ($\therefore B$)

The argument is valid.

1.	$P \vee F$	(*Premise 1*)
2.	$\neg T \to \neg P$	(*Premise 2*)
3.	$T \to B$	(*Premise 3*)
4.	$\neg F$	(*Premise 4*)
5.	P	(*Disjunctive Syllogism from 1 and 4*)
6.	$P \to T$	(*Contrapositive from 2*)
7.	T	(*Modus Ponens from 5 and 6*)
8.	B	(*Modus Ponens from 7 and 3*)

The conclusion B logically follows from the premises.

Q 9 *)Give a proof by contradiction of: "If n is an even integer, then $3n + 7$ is odd."*

We want to prove $p \to q$, where p: "n is an even integer" and q: "$3n + 7$ is odd". Assume the negation of the conclusion, $\neg q$, holds: $3n + 7$ is even. Since n is an even integer (premise p), we can write $n = 2k$ for some integer k. Substituting $n = 2k$ into the assumption:

$$3n + 7 = 3(2k) + 7 = 6k + 7$$

By our assumption, $6k + 7$ is even, so $6k + 7 = 2m$ for some integer m.

$$6k + 7 = 2m$$

$$6k - 2m = -7$$

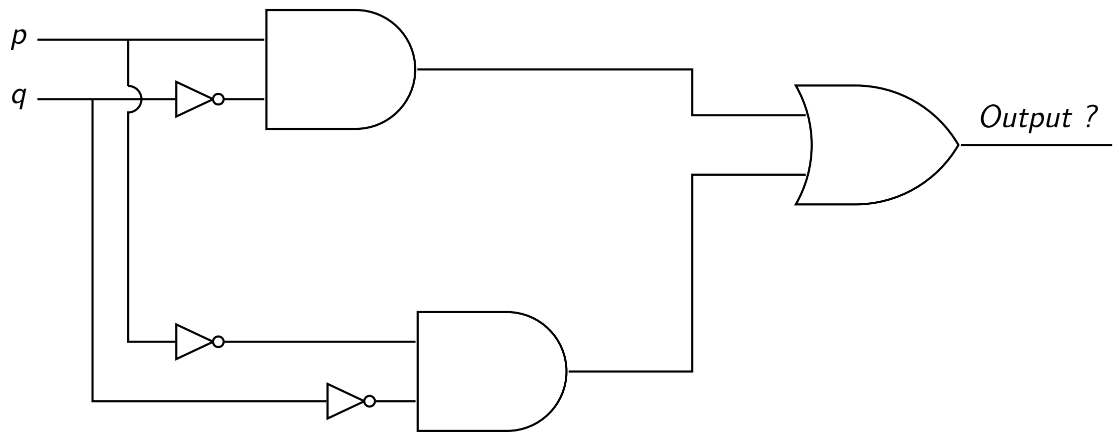
$$2(3k - m) = -7$$

Let $j = 3k - m$. Since k and m are integers, j is an integer.

$$2j = -7$$

This equation states that an even integer $2j$ is equal to an odd integer -7. This is a contradiction. Therefore, our assumption ($\neg q$) must be false, and the original conclusion (q) that $3n + 7$ is odd must be true.

Q 10 *)Find the output of the following logic gate:*



$$(p \wedge \neg q) \vee (\neg p \wedge \neg q)$$

Q 11 *)Prove, using whatever method you want, that at least one of the real numbers* $a_1, a_2, ..., a_n$ *is greater than or equal to the average of these numbers.*

1. Let $\bar{a}$ *be the average of the* n *real numbers* $a_1, a_2, \ldots, a_n$*. 2. By the definition of the average, we have:*

$$\bar{a} = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

3. Multiplying by n *gives:*

$$n\bar{a} = a_1 + a_2 + \cdots + a_n$$

4. Rearranging the terms shows that the sum of the differences between each number and the average is zero:

$$0 = (a_1 - \bar{a}) + (a_2 - \bar{a}) + \cdots + (a_n - \bar{a})$$

For a sum of n *real numbers to equal zero, it is impossible for all* n *numbers to be strictly negative. If* $(a_i - \bar{a}) < 0$ *for all* i*, the sum would be strictly less than zero.*

Therefore, at least one of the difference terms, say $(a_k - \bar{a})$*, must be greater than or equal to zero:*

$$a_k - \bar{a} \geq 0$$

Adding $\bar{a}$ *to both sides yields the desired result:*

$$a_k \geq \bar{a}$$

We conclude that at least one of the numbers is greater than or equal to the average.

Q 12 *)Prove by contraposition:* $x \in \mathbb{R}, x^2 - 6x + 5 > 0 \to x \geq 5 \ \vee \ x \leq 1$

Let P *be* $x^2 - 6x + 5 > 0$ *and* Q *be* $x \geq 5 \vee x \leq 1$*. We prove the contrapositive* $\neg Q \to \neg P$*.*

- *Hypothesis* ($\neg Q$)*: The negation of* $x \geq 5 \vee x \leq 1$ *is* $x < 5 \wedge x > 1$*, or* $1 < x < 5$*.*
- *Conclusion* ($\neg P$)*: The negation of* $x^2 - 6x + 5 > 0$ *is* $x^2 - 6x + 5 \leq 0$*.*

Proof: Assume $1 < x < 5$*. Factor the quadratic:* $x^2 - 6x + 5 = (x - 5)(x - 1)$*. Since* $x < 5$*, the factor* $(x - 5)$ *is negative. Since* $x > 1$*, the factor* $(x - 1)$ *is positive. The product of a negative number and a positive number is negative:*

$$(x - 5)(x - 1) < 0$$

Thus, $x^2 - 6x + 5 \leq 0$ *is true. Since the contrapositive is true, the original statement is also true.*

Q 13 *)Prove by contradiction $\sqrt{3}$ is irrational*

Assume, for the sake of contradiction, that $\sqrt{3}$ is rational. If $\sqrt{3}$ is rational, it can be written as a fraction $\frac{a}{b}$, where a and b are integers, $b \neq 0$, and $\frac{a}{b}$ is in simplest form (a and b have no common factors other than 1).

$$\sqrt{3} = \frac{a}{b}$$

Squaring both sides gives:

$$3 = \frac{a^2}{b^2}$$

$$3b^2 = a^2 \quad (*)$$

Since a^2 equals 3 times an integer (b^2), a^2 must be divisible by 3. By the fundamental theorem of arithmetic, if 3 divides a^2, then 3 must divide a. Therefore, we can write $a = 3k$ for some integer k. Substitute $a = 3k$ back into equation $(*)$:

$$3b^2 = (3k)^2$$

$$3b^2 = 9k^2$$

Divide both sides by 3:

$$b^2 = 3k^2$$

Since b^2 equals 3 times an integer (k^2), b^2 must be divisible by 3. Again, by the fundamental theorem of arithmetic, if 3 divides b^2, then 3 must divide b. We have established that 3 divides a (since $a = 3k$) and 3 divides b. This means that a and b have a common factor of 3. This contradicts our initial assumption that $\frac{a}{b}$ was in simplest form (i.e., a and b have no common factors other than 1). Therefore, the initial assumption that $\sqrt{3}$ is rational must be false. Hence, $\sqrt{3}$ is irrational.