**Exam 2 Discrete Mathematics 1 Solutions**

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Code snippet

**P 1**) *In this exercise, you prove the Schröder-Bernstein theorem. Suppose that $A$ and $B$ are sets where $|A| \leq |B|$and $|B| \leq |A|$. This means that there are injections $f : A \to B$ and $g : B \to A$. To prove the theorem, we must show that there is a bijection $h : A \to B$, implying that $|A| = |B|$. To build $h : A \to B$, we construct the chain of an element $a \in A$. This chain contains the elements $a, f(a), g(f(a)), f(g(f(a))), g(f(g(f(a)))), \ldots$. It also may contain more elements that precede $a$, extending the chain backwards. So, if there is an element $b \in B$ with $g(b) = a$, then the element $b$ will be the term of the chain just before the element $a$. Because $g : B \to A$ may not be a surjection, there may not be any such $b$, so that $a$ is the first element of the chain. If such an element $b$ exists, because $g : B \to A$ is an injection, it is the unique element of $B$ mapped by $g$ to $a$; hence we denote it by $g^{-1}(a)$. We extend the chain backwards as long as possible in the same way, adding $f^{-1}(g^{-1}(a)), g^{-1}(f^{-1}(g^{-1}(a))), \ldots$ To construct the proof, complete these five parts.*

***(a)*** *Each element in $A \cup B$ belongs to exactly one chain. The functions $f : A \to B$ and $g : B \to A$ define a partition of $A \cup B$ into disjoint chains (sequences or cycles). The relationship is defined by $a \to f(a)$ for $a \in A$ and $b \to g(b)$ for $b \in B$. Since $f$ and $g$ are injective, each element has at most one predecessor and one successor, ensuring the set is partitioned into these unique sequences.*

**(b)** *Chains are categorized by their backward structure (origin):*

- *Type 1 (Loop) / Type 2 ($\mathbf{C}_\infty$): Cyclic or infinite both ways.*
- *Type 3 ($\mathbf{C_A}$): Ends backward at $a_0 \in A$ (no $b \in B$ such that $g(b) = a_0$).*
- *Type 4 ($\mathbf{C_B}$): Ends backward at $b_0 \in B$ (no $a \in A$ such that $f(a) = b_0$).*

*Let $\mathcal{C}_f$ be the union of Type 1, 2, 3 chains, and $\mathcal{C}_{g^{-1}}$ be Type 4 chains.*

**(c)** *Define the bijection candidate $h : A \to B$:*

$$h(a) = \begin{cases} f(a) & \text{if } a \text{ is in } \mathcal{C}_f, \\ g^{-1}(a) & \text{if } a \text{ is in } \mathcal{C}_{g^{-1}}. \end{cases}$$

*For $a \in \mathcal{C}_{g^{-1}} = C_B$, $a$ is necessarily of the form $g(b)$ for some $b \in B$ (since the chain begins at $b_0 \in B$, $a$ cannot be the chain's origin), so $g^{-1}(a)$ is well-defined in $B$.*

**(d)** *To show $h$ is injective, let $h(a_1) = h(a_2)$.*

- *Same Type: If $a_1, a_2 \in \mathcal{C}_f$, $f(a_1) = f(a_2) \implies a_1 = a_2$ (by injectivity of $f$). If $a_1, a_2 \in \mathcal{C}_{g^{-1}}$, $g^{-1}(a_1) = g^{-1}(a_2) \implies a_1 = a_2$ (by injectivity of $g$).*

- *Mixed Types: If* $a_1 \in \mathcal{C}_f$ *and* $a_2 \in \mathcal{C}_{g^{-1}}$, *then* $f(a_1) = g^{-1}(a_2)$. *Applying* $g$ *gives* $g(f(a_1)) = a_2$. *This establishes a connection* $a_1 \to f(a_1) \to a_2$, *meaning* $a_1$ *and* $a_2$ *are in the same chain. This is a contradiction, as connected elements must be in the same type of chain (*$\mathcal{C}_f$ *vs* $\mathcal{C}_{g^{-1}}$*).*

*Thus,* $h$ *is one-to-one.*

**(e)** *To show* $h$ *is onto, take* $b \in B$.

- *If* $b \in \mathcal{C}_f$*: Every* $b \in B$ *in these chains must have a predecessor* $a \in A$ *such that* $f(a) = b$. *For this* $a$, $h(a) = f(a) = b$.

- *If* $b \in \mathcal{C}_{g^{-1}}$*: Let* $a = g(b)$. *Then* $a \in A$ *and* $a$ *is also in* $\mathcal{C}_{g^{-1}}$. *By definition,* $h(a) = h(g(b)) = g^{-1}(g(b)) = b$.

*All* $b \in B$ *are covered, so* $h$ *is onto. Thus,* $h$ *is a bijection, proving* $|A| = |B|$.

**P 2**) *To show* $|(0,1)| = |\mathbb{R}|$, *construct a bijection* $h : (0,1) \to \mathbb{R}$, *defined as:*

$$h(x) = \tan\left(\pi x - \frac{\pi}{2}\right).$$

- *Domain: For* $x \in (0,1)$, $\pi x - \frac{\pi}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2})$, *where* tan *maps to* $\mathbb{R}$.
- *Injectivity: If* $h(x_1) = h(x_2)$, *then* $\tan(\pi x_1 - \frac{\pi}{2}) = \tan(\pi x_2 - \frac{\pi}{2})$. *Since* tan *is injective on* $(-\frac{\pi}{2}, \frac{\pi}{2})$, $\pi x_1 - \frac{\pi}{2} = \pi x_2 - \frac{\pi}{2}$, *so* $x_1 = x_2$.
- *Surjectivity: For* $y \in \mathbb{R}$, *solve* $h(x) = y$: $\tan(\pi x - \frac{\pi}{2}) = y$, *so* $\pi x - \frac{\pi}{2} = \arctan(y)$, *hence* $x = \frac{\arctan(y) + \frac{\pi}{2}}{\pi}$. *Since* $\arctan(y) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $x \in (0,1)$, *and* $h(x) = y$.

*Thus,* $h$ *is a bijection, so* $|(0,1)| = |\mathbb{R}|$.