# The factorizations of square of cosecant function

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**Show**

$$\sum_{k=-\infty}^{+\infty} \frac{1}{(k\pi + \frac{w}{2})^2} = \csc^2 \frac{w}{2} \ (w \neq 2n\pi, n \in \mathbb{Z}) \tag{1}$$

*Proof.* Gamma function is defined as

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} \mathrm{d}t, \quad \Re(z) > 0. \tag{2}$$

Digamma function is defined as

$$\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \Gamma(z) = \frac{\Gamma^{'}(z)}{\Gamma(z)}. \tag{3}$$

We begin with Euler's reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad z \notin \mathbb{Z} \tag{4}$$

Then

$$\begin{aligned}
\ln(\Gamma(z)\Gamma(1-z)) &= \ln\left(\frac{\pi}{\sin \pi z}\right) \\
\ln \Gamma(z) + \ln \Gamma(1-z) &= \ln \pi - \ln(\sin(\pi z)) \\
\frac{\Gamma^{'}(z)}{\Gamma(z)} - \frac{\Gamma^{'}(1-z)}{\Gamma(1-z)} &= -\pi \frac{\cos(\pi z)}{\sin(\pi z)} = -\pi \cot \pi z \\
\psi(z) - \psi(1-z) &= -\pi \cot \pi z \\
\psi(1-z) - \psi(z) &= \pi \cot \pi z
\end{aligned}$$

The derivative of above equation is

$$\begin{aligned}
-\psi^{(1)}(1-z) - \psi^{(1)}(z) &= \pi \frac{\mathrm{d}}{\mathrm{d}z} \cot \pi z \\
\psi^{(1)}(z) + \psi^{(1)}(1-z) &= \pi^2 \csc^2 \pi z
\end{aligned}$$

Weierstrass's definition of gamma function is

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}. \tag{5}$$

Thus,

$$
\begin{aligned}
\psi(z) &= \frac{\mathrm{d}}{\mathrm{d}z} \ln \Gamma(z) \\
&= \frac{\mathrm{d}}{\mathrm{d}z} \left( -\gamma z - \ln(z) + \sum_{n=1}^{\infty} \left( -\ln\left(1 + \frac{z}{n}\right) + \frac{z}{n} \right) \right) \\
&= -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+z} \right).
\end{aligned}
$$

The derivative of digamma function is

$$
\psi^{(1)}(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} \frac{1}{(n+z)^2} = \sum_{n=0}^{\infty} \frac{1}{(n+z)^2}. \tag{6}
$$

Thus,

$$
\begin{aligned}
\sum_{k=0}^{\infty} \frac{1}{(z+k)^2} + \sum_{k=0}^{\infty} \frac{1}{(1-z+k)^2} &= \pi^2 \csc^2 \pi z \\
\sum_{k=0}^{\infty} \frac{1}{(z+k)^2} + \sum_{n=-\infty}^{-1} \frac{1}{(n+z)^2} &= \pi^2 \csc^2 \pi z \\
\sum_{k=-\infty}^{\infty} \frac{1}{(z+k)^2} &= \pi^2 \csc^2 \pi z
\end{aligned}
$$

Then set $z = w/(2\pi)$, equation (1) is proofed. $\square$