Maths Notes

Li WeiHua

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Part I

Real Analysis

Chapter 1

sequences and series

theorem vs proposition vs lemma

The main results are theorems, smaller results are called propositions. A Lemma is a technical intermediate step which has no standing as an independent result. Lemmas are only used to chop big proofs into handy pieces.

1.1 sequence

Theorem 1.1.1. *A sequence $(a_n)_{n=1}^{\infty}$ of real numbers is convergent if and only if (iff) it is a Cauchy sequence.*

Proposition 1.1.2 (Monotone bounded sequences converge)**.** *Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers which has some finite upper bound $M \in \mathbb{R}$, and which is also increasing (i.e., $a_{n+1} \geq a_n$ for all $n \geq m$). Then $(a_n)_{n=m}^{\infty}$ is convergent, and in fact*

$$\lim_{n\to\infty} a_n = sup(a_n)_{n=m}^{\infty} \leq M$$

Part II

Functional Analysis

Chapter 2

Metric Space

Definition 2.0.1 (Metric space)**.** A Metric space is a pair (X, d), where X is a set and d is a distance function or metric on X such that $d : X \times X \to [0, +\infty)$. Furthermore, the metric must satisfy the following four axioms:

(a) $d(x, y) = 0$ if and only if $x = y$

(b) (Positivity) for any distinct $x, y \in X, d(x, y) > 0$

(c) (Symmetry) $d(x, y) = d(y, x)$

(d) (Triangle inequality) $d(x, y) <= d(x, z) + d(z, y)$

Chapter 3

Calculus of Variations

3.1 Euler-Lagrange Equation

Lemma 3.1.1. *If $f(x) \in C[a,b]$, and if*

$$\int_a^b f(x)g(x)\,dx = 0$$

for $\forall g(x) \in L(a,b)$ such that $g(a) = g(b) = 0$, then $f(x) = 0, \forall x \in [a,b]$.

Proof. Suppose $f(x) > 0$ in some $[x_1, x_2]$, if we set $g(x) = (x - x_1)(x_2 - x)$, then

$$\int_{x_1}^{x_2} f(x)(x - x_1)(x_2 - x)\,dx > 0\,,$$

since the integrand is positive. This contradiction proves the lemma. □

Remark 3.1.2. The Lemma 3.1.1 still holds if we replace linear space $L(a,b)$ by normed linear space $N_n(a,b)$. To proof this, just set

$$g(x) = [(x - x_1)(x_2 - x)]^n$$

Problem: find the extremum of equation (3.1).

$$J[y] = \int_a^b F[x, y, y']\,dx \tag{3.1}$$

Solution 1:

$$\begin{aligned}
0 &= \delta J[y] \\
&= \int_a^b \left[F(x, y+\delta y, y'+\delta y') - F(x, y, y')\right]\, dx \\
&= \int_a^b \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right]\, dx \\
&= \int_a^b \left[\frac{\partial F}{\partial y}\delta y - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y\right]\, dx + \frac{\partial F}{\partial y'}\delta y\bigg|_a^b \\
&= \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y\, dx
\end{aligned}$$

So,

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \tag{3.2}$$

which is called Euler-Lagrange equation.

Part III

Ordinary Differential Equations

Chapter 4

Brachistochrone Problem

Chapter 5

First Order Equations

5.1 Homogeneous Equations

$$\frac{dy}{dx} = f(x, y)$$

Part IV

Numerical Methods

Chapter 6

Coursera: Numerical Methods for Engineers

6.1 Logistic Map

Mathematically, the logistic map is written

$$x_{n+1} = rx_n(1 - x_n), \tag{6.1}$$

where r (sometimes also denoted μ) is a positive constant.

6.1.1 Fixed Point

The fixed point means that some point satisfies $x = f(x)$. By solving

$$x = rx(1 - x), \quad x = 0 \quad \text{or} \quad x = 1 - \frac{1}{r}$$

6.1.2 Period-2 Point

We say that x_1 and x_2 are a period-2 cycle of a one-dimensional map $f(x)$ if

$$x_2 = f(x_1) \quad \text{and} \quad x_1 = f(x_2) \quad \text{and} \quad x_1 \neq x_2.$$

Determine the period-2 cycle for the logistic map by solving the equation $x = f(f(x))$, with $f(x) = rx(1 - x)$.

Chapter 7

Linear Algebra

Part V

Others To Be Orgnized

Chapter 8

Vector Calculus

8.1 Kronecker-Delta and Levi-Civita symbol

$$\begin{aligned}
\epsilon_{ijk}\epsilon_{lmn} &= \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \\
&= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) \\
&\quad + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})
\end{aligned} \tag{8.1}$$

We use **Einstein sum convention**, then we can prove the following simpler and useful identities:

(a) $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$

(b) $\epsilon_{ijk}\epsilon_{ijn} = 2\delta_{kn}$

Proof. (a)

$$\begin{aligned}
\epsilon_{ijk}\epsilon_{imn} &= \epsilon_{ijk}\epsilon_{lmn}\delta_{il} \\
&= \delta_{il}\delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{il}\delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) \\
&\quad + \delta_{il}\delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) \\
&= \delta_{ii}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{ji}\delta_{kn} - \delta_{jn}\delta_{ki}) \\
&\quad + \delta_{in}(\delta_{ij}\delta_{km} - \delta_{jm}\delta_{ki}) \\
&= 3(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) + (\delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn}) \\
&= \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}
\end{aligned}$$

(b)

$$\begin{aligned}\epsilon_{ijk}\epsilon_{ijn} &= \epsilon_{ijk}\epsilon_{imn}\delta_{jm} \\ &= \delta_{jm}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) \\ &= \delta_{jj}\delta_{kn} - \delta_{jn}\delta_{kj} \\ &= 2\delta_{kn}\end{aligned}$$

□

8.2 Scalar and vector product

Definition 8.2.1. The **scalar triple product** is defined by $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ and the **vector triple product** is defined by $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ where $\mathbf{A},\mathbf{B},\mathbf{C}\in\mathbf{R}^3$.

Property 8.2.2. The scalar triple product has identities

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) \tag{8.2}$$

The vector triple product can be expanded as

$$\begin{aligned}\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} \\ \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) &= (\mathbf{B}\cdot\mathbf{A})\mathbf{C} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A} \\ \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) &= (\mathbf{C}\cdot\mathbf{B})\mathbf{A} - (\mathbf{C}\cdot\mathbf{A})\mathbf{B}\end{aligned} \tag{8.3}$$

These identities can be proved by using Kronecker-Delta and Levi-Civita symbol.

Proof. Equation (8.2) can be expanded as following

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= A_i[\mathbf{B}\times\mathbf{C}]_i \\ &= A_i\epsilon_{ijk}B_jC_k \\ &\overset{i\leftrightarrow j}{=\!=} B_i\epsilon_{jik}A_jC_k \\ &= B_i\epsilon_{ikj}C_kA_j \\ &= B_i[\mathbf{C}\times\mathbf{A}]_i \\ &= \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})\end{aligned}$$

The i-th component of equation (8.3) can be expanded as following

$$\begin{aligned}[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})]_i &= \epsilon_{ijk}A_j[\mathbf{B}\times\mathbf{C}]_k \\ &= \epsilon_{ijk}A_j\epsilon_{klm}B_lC_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})A_jB_lC_m \\ &= A_jB_iC_j - A_jB_jC_i \\ &= [(\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}]_i\end{aligned}$$

□

Furthermore, we can get following identities,

1. Jacobi identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0. \tag{8.4}$$

2. scalar quadruple product

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \tag{8.5}$$

3. Lagrange's identity

$$|\mathbf{A} \times \mathbf{B}|^2 = |\mathbf{A}|^2 |\mathbf{B}|^2 - (\mathbf{A} \cdot \mathbf{B})^2 \tag{8.6}$$

8.3 Partial differentiation with non-independent variables

Example 8.3.1. Let $w = x^2 + y^2 + z^2$ where $z = x^2 + y^2$. Calculate $\partial w/\partial x$. There are two possible:

(a) x, y are independent

$$\begin{aligned} \frac{\partial w}{\partial x} &= \frac{\partial}{\partial x}(x^2 + y^2 + (x^2 + y^2)^2) \\ &= 2x + 2(x^2 + y^2)^2 \\ &= 4x^2 + 4y^2 x + 2x. \end{aligned}$$

(b) x, z are independent

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(z + z^2) = 0.$$

To sum up, the value of $\partial w/\partial x$ depends on which variables are independent. To avoid the ambiguity, we should using the following notation:

Case (a) x, y are the independent variables: $\left(\frac{\partial w}{\partial x}\right)_y$

Case (b) x, z are the independent variables: $\left(\frac{\partial w}{\partial x}\right)_z$.

These are read, "the partial of w with respect to x, with y (resp. z) held constant".

Example 8.3.2.

$$w = f(x, y, z, t), \quad \text{where} \quad xy = zt,$$

then only three of the variables x, y, z, t can be independent. Thus we would write expressions like

(a) x, y are the independent variables: $\left(\frac{\partial w}{\partial x}\right)_y$

(b) x, z are the independent variables: $\left(\frac{\partial w}{\partial x}\right)_z$.

Appendix A

LaTeX Notes

A.1 LaTeX list: enumitem

List structures in LaTeX have three types:

- itemize for a bullet list
- enumerate for an enumerated list
- description for a descriptive list.

The description environment can have optional label:

```
\item[label text] Text of your description goes here...
```

A.1.1 Customizing lists

A.2 How to produce png file?

Fist use *standalone* package with option **preview**:

```
\documentclass[preview, convert={outext=.png}]{standalone}
```

Then compile tex source code:

```
latexmk -shell-escape textfile
```