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CS-225: Discrete Structures in CS
Homework Assignment 5, Part 1
Exercise Set 5.2: Question # 9, 13
Exercise Set 5.3: Question # 9, 10, 18

- **Set 5.2 – Q#9**

For the given statement, to prove the formula using mathematical induction, the property P(n) is the equation

$$4^3 + 4^4 + 4^5 + \cdots + 4^n = \frac{4(4^n - 16)}{3}, with\ n \geq 3$$

BASIS STEP - show that P(3) is true:

The left-hand side of P(3) is $4^3 = 64$, and the right-hand side is $\frac{4(4^3-16)}{3} = \frac{4\times 48}{3} = 64$ also. Thus P(3) is true.

INDUCTIVE STEP – show that for all integers $k \geq 3$, if P(k) is true then P(k+1) is true:
Let k be any integer with $k \geq 3$, and suppose P(k) is true. That is, suppose

$$4^3 + 4^4 + 4^5 + \cdots + 4^k = \frac{4\left(4^k - 16\right)}{3}$$

We must show that P(k+1) is true. That is, we must show that

$$4^3 + 4^4 + 4^5 + \cdots + 4^{k+1} = \frac{4\left(4^{k+1} - 16\right)}{3}$$

And the left-hand side of P(k+1) is

$$\begin{aligned}
4^3 + 4^4 + 4^5 + \cdots + 4^{k+1} &= 4^3 + 4^4 + 4^5 + \cdots + 4^k + 4^{k+1} \\
&= \frac{4\left(4^k - 16\right)}{3} + 4^{k+1} \\
&= \frac{4^{k+1} - 4\times 16 + 3\times 4^{k+1}}{3} \\
&= \frac{4\times 4^{k+1} - 4\times 16}{3} \\
&= \frac{4\left(4^{k+1} - 16\right)}{3}
\end{aligned}$$

And this is the right-hand side of P(k+1). Hence the property is true for n=k+1.
This shows that if the inductive hypothesis P(k) is true, then P(k+1) must also be true, which complete the inductive argument.
We have completed the basis step and the inductive step, so by mathematical induction P(n) is true for all integers $n \geq 3$. This shows that the formula is correct.

- **Set 5.2 – Q#13**

For the given statement, to prove the formula using mathematical induction, the property P(n) is the equation

$$\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}, with\ n \geq 2$$

BASIS STEP - show that P(2) is true:

The left-hand side of P(2) is $\sum_{i=1}^{2-1} i(i+1) = \sum_{1}^{1} 1(1+1) = 1 \times 2 = 2$, and the right-hand side is $\frac{2(2-1)(2+1)}{3} = \frac{2\times1\times3}{3} = 2$ also. Thus P(2) is true.

INDUCTIVE STEP – show that for all integers $k \geq 2$, if P(k) is true then P(k+1) is true:

Let k be any integer with $k \geq 2$, and suppose P(k) is true. That is, suppose

$$\sum_{i=1}^{k-1} i(i+1) = \frac{k(k-1)(k+1)}{3}$$

We must show that P(k+1) is true. That is, we must show that

$$\sum_{i=1}^{(k+1)-1} i(i+1) = \frac{(k+1)(k+1-1)(k+1+1)}{3}$$

Or, equivalently,

$$\sum_{i=1}^{k} i(i+1) = \frac{k(k+1)(k+2)}{3}$$

And the left-hand side of P(k+1) is

$$\begin{aligned}\sum_{i=1}^{k} i(i+1) &= \sum_{i=1}^{k-1} i(i+1) + k(k+1)\\ &= \frac{k(k-1)(k+1)}{3} + k(k+1)\\ &= \frac{k(k-1)(k+1) + 3k(k+1)}{3}\\ &= \frac{k(k+1)(k-1+3)}{3}\\ &= \frac{k(k+1)(k+2)}{3}\end{aligned}$$

And this is the right-hand side of P(k+1). Hence the property is true for n=k+1.

This shows that if the inductive hypothesis P(k) is true, then P(k+1) must also be true, which complete the inductive argument.

We have completed the basis step and the inductive step, so by mathematical induction P(n) is true for all integers $n \geq 2$. This shows that the formula is correct.

- **Set 5.3 – Q#9**

For the given statement, to prove it using mathematical induction, the property P(n) is the sentence "$7^n - 1\ is\ divisible\ by\ 6, for\ each\ integer\ n \geq 0$".

BASIS STEP - show that P(0) is true:

P(0) is the sentence "$7^0 - 1\ is\ divisible\ by\ 6$". And $7^0 - 1 = 1 - 1 = 0$, and 0 is

divisible by 6 because $0 = 6 \times 0$. Thus P(0) is true.
INDUCTIVE STEP – show that for all integers $k \geq 0$, if P(k) is true then P(k+1) is true:
Let k be any integer with $k \geq 0$, and suppose P(k) is true. That is, suppose $7^k - 1\ is\ divisible\ by\ 6$. [This is the inductive hypothesis.] We must show that P(k+1) is true. That is, we must show that $7^{k+1} - 1\ is\ divisible\ by\ 6$. Now

$$7^{k+1} - 1 = 7^k \times 7 - 1 = 7^k \times (6+1) - 1 = 7^k \times 6 + (7^k - 1)$$

By the inductive hypothesis $7^k - 1$ is divisible by 6, so $7^k - 1 = 6r$ for some integer r. By substitution,

$$7^{k+1} - 1 = 7^k \times 6 + 6r = 6(7^k + r)$$

And $7^k + r$ is an integer because power and sum of integers are integers. Hence, by definition of divisibility, $7^{k+1} - 1$ is divisible by 6, the property is true for n=k+1.
This shows that if the inductive hypothesis P(k) is true, then P(k+1) must also be true, which complete the inductive argument.
We have completed the basis step and the inductive step, so by mathematical induction P(n) is true for all integers $n \geq 0$. This shows that the given statement is true.

- **Set 5.3 – Q#10**

For the given statement, to prove it using mathematical induction, the property P(n) is the sentence $"n^3 - 7n + 3\ is\ divisible\ by\ 3, for\ each\ integer\ n \geq 0"$.
BASIS STEP - show that P(0) is true:
P(0) is the sentence $"0^3 - 7 \times 0 + 3\ is\ divisible\ by\ 3"$. And $0^3 - 7 \times 0 + 3 = 0 - 0 + 3 = 3$, and 3 is divisible by 3 because $3 = 3 \times 1$. Thus P(0) is true.
INDUCTIVE STEP – show that for all integers $k \geq 0$, if P(k) is true then P(k+1) is true:
Let k be any integer with $k \geq 0$, and suppose P(k) is true. That is, suppose $n^3 - 7n + 3\ is\ divisible\ by\ 3$. [This is the inductive hypothesis.] We must show that P(k+1) is true. That is, we must show that $(n+1)^3 - 7(n+1) + 3\ is\ divisible\ by\ 3$. Now

$$\begin{aligned}(n+1)^3 - 7(n+1) + 3 &= (n+1)(n+1)^2 - 7n - 7 + 3 \\ &= (n+1)(n^2 + 2n + 1) - 7n - 4 \\ &= n^3 + 2n^2 + n + n^2 + 2n + 1 - 7n - 4 \\ &= n^3 + 3n^2 - 4n - 3 \\ &= n^3 - 7n + 3 + 3n^2 + 3n - 6 \\ &= (n^3 - 7n + 3) + 3(n^2 + n - 2)\end{aligned}$$

By the inductive hypothesis $n^3 - 7n + 3\ is\ divisible\ by\ 3$, so $n^3 - 7n + 3 = 3r$ for some integer r. By substitution,

$$(n+1)^3 - 7(n+1) + 3 = 3r + 3(n^2 + n - 2) = 3(r + n^2 + n - 2)$$

And $r + n^2 + n - 2$ is an integer because power, sum and difference of integers are integers. Hence, by definition of divisibility, $(n+1)^3 - 7(n+1) + 3$ is divisible by 3, the property is true for n=k+1.
This shows that if the inductive hypothesis P(k) is true, then P(k+1) must also be true, which complete the inductive argument.
We have completed the basis step and the inductive step, so by mathematical induction P(n) is true for all integers $n \geq 0$. This shows that the given statement is true.

- **Set 5.3 – Q#18**

For the given statement, to prove it using mathematical induction, the property P(n) is the inequality "$5^n + 9 < 6^n, for\ all\ integers\ n \geq 2$".
BASIS STEP - show that P(2) is true:
P(2) is the inequality $5^2 + 9 < 6^2$. The left-hand side of P(2) is $5^2 + 9 = 25 + 9 = 34$, and the right-hand side is $6^2 = 36$. Because 34<36, P(2) is true.
INDUCTIVE STEP – show that for all integers $k \geq 2$, if P(k) is true then P(k+1) is true:
Let k be any integer with $k \geq 2$, and suppose P(k) is true. That is, suppose $5^k + 9 < 6^k$. [This is the inductive hypothesis.] We must show that P(k+1) is true. That is, we must show that $5^{k+1} + 9 < 6^{k+1}$. And the left-hand side of P(k+1) is

$$5^{k+1} + 9 = 5^k \times 5 + 9 \times 5 - 9 \times 4 = 5\left(5^k + 9\right) - 36$$

By the inductive hypothesis $5^k + 9 < 6^k$,

$$5^{k+1} + 9 = 5\left(5^k + 9\right) - 36 < 5 \times 6^k - 36 < 6 \times 6^k - 36 < 6^{k+1}$$

And this is the right-hand side of P(k+1). Hence the property is true for n=k+1.
This shows that if the inductive hypothesis P(k) is true, then P(k+1) must also be true, which complete the inductive argument.
We have completed the basis step and the inductive step, so by mathematical induction P(n) is true for all integers $n \geq 2$. This shows that the given statement is true.