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CS-225: Discrete Structures in CS
Homework Assignment 5, Part 2
Exercise Set 5.4: Question # 2, 3
Problem provided on Canvas

- **Set 5.4 – Q#2**

For the given statement, to prove it using strong mathematical induction, let the property P(n) be the sentence "b_n *is divisible by* 4 *for all integers* $n \geq 1$".
BASIS STEP – show that P(1) and P(2) are true:
P(1) is the sentence "b_1 *is divisible by* 4". And $b_1 = 4 = 4 \times 1$, thus P(1) is true. P(2) is the sentence "b_2 *is divisible by* 4". And $b_2 = 12 = 4 \times 3$, thus P(2) is true.
INDUCTIVE STEP – show that for any integer $k \geq 2$, if P(i) is true for all integers i with $1 \leq i \leq k$, then P(k+1) is true:
Let $k \geq 2$ be any integer, and suppose b_i is divisible by 4 for all integers i with $1 \leq i \leq k$. [This is the inductive hypothesis.] We must show that b_{k+1} is divisible by 4. We know that $b_{k+1} = b_{k+1-2} + b_{k+1-1} = b_{k-1} + b_k$ by definition of $b_1, b_2, b_3, \cdots$. Moreover, $k - 1 < k$, and $k - 1 \geq 1$, because $k \geq 2$. Thus, by inductive hypothesis, b_{k-1} is divisible by 4. Also, b_k is divisible by 4 because $1 < k \leq k$. So $b_{k-1} = 4r, b_k = 4s$ for some integers r and s. By substitution, $b_{k+1} = b_{k-1} + b_k = 4r + 4s = 4(r + s)$. And r+s is an integer because sum of integers is integer. Hence, by definition of divisibility, b_{k+1} is divisible by 4, the property is true for n=k+1.
This shows that if the inductive hypothesis is true, then P(k+1) must also be true, which complete the inductive argument.
We have completed the basis step and the inductive step, so by strong mathematical induction P(n) is true for all integers $n \geq 1$. This shows that the given statement is true.

- **Set 5.4 – Q#3**

For the given statement, to prove it using strong mathematical induction, let the property P(n) be the sentence "c_n *is even for all integers* $n \geq 0$".
BASIS STEP – show that P(0), P(1) and P(2) are true:
P(0) is the sentence "c_0 *is even*". P(1) is the sentence "c_1 *is even*". P(2) is the sentence "c_2 *is even*". And observe that $c_0 = 2, c_1 = 2, c_2 = 6$ are all even. Thus P(0), P(1) and P(2) are true.
INDUCTIVE STEP – show that for any integer $k \geq 2$, if P(i) is true for all integers i with $0 \leq i \leq k$, then P(k+1) is true:
Let $k \geq 2$ be any integer, and suppose c_i is even for all integers i with $0 \leq i \leq k$. [This is the inductive hypothesis.] We must show that c_{k+1} is even. We know that $c_{k+1} = 3c_{k+1-3} = 3c_{k-2}$ by definition of $c_0, c_1, c_3, \cdots$. Moreover, $k - 2 < k$, and $k - 2 \geq 0$, because $k \geq 2$. Thus, by inductive hypothesis, c_{k-2} is even. By definition of even, $c_{k-2} = 2r$ for some integer r. By substitution, $c_{k+1} = 3c_{k-2} = 3 \cdot 2r = 2 \cdot 3r$. And 3r is

an integer because product of integers is integer. Hence, by definition of even, c_{k+1} is even, the property is true for n=k+1.
This shows that if the inductive hypothesis is true, then P(k+1) must also be true, which complete the inductive argument.
We have completed the basis step and the inductive step, so by strong mathematical induction P(n) is true for all integers $n \geq 0$. This shows that the given statement is true.

- **Problem Provided on Canvas**

(a)
BASIS STEP – show that P(8), P(9) and P(10) are true:
P(8) is the statement "a postage of 8 cents can be formed using just 3-cent and 5-cent stamps". And $8 = 3 \times 1 + 5 \times 1$, thus 8 cents can be formed using just 1 3-cent stamp and 1 5-cent stamp. So P(8) is true.
P(9) is the statement "a postage of 9 cents can be formed using just 3-cent and 5-cent stamps". And $9 = 3 \times 3 + 5 \times 0$, thus 9 cents can be formed using just 3 3-cent stamps and 0 5-cent stamp. So P(9) is true.
P(10) is the statement "a postage of 10 cents can be formed using just 3-cent and 5-cent stamps". And $10 = 3 \times 0 + 5 \times 2$, thus 10 cents can be formed using just 0 3-cent stamp and 2 5-cent stamps. So P(10) is true.
(b)
Inductive hypothesis: Let $k \geq 10$ be any integer, and suppose a postage of i cents can be formed using just 3-cent and 5-cent stamps for all integers i with $8 \leq i \leq k$.
(c)
We must prove that for any integer $k \geq 10$, if P(i) is true for all integers i with $8 \leq i \leq k$, then P(k+1) is true, that is, we must show that a postage of k+1 cents can be formed using just 3-cent and 5-cent stamps.
(d)
INDUCTIVE STEP – show that for any integer $k \geq 10$, if P(i) is true for all integers i with $8 \leq i \leq k$, then P(k+1) is true:
Let $k \geq 10$ be any integer, and suppose a postage of i cents can be formed using just 3-cent and 5-cent stamps for all integers i with $8 \leq i \leq k$. [This is the inductive hypothesis.]
We must show that a postage of k+1 cents can be formed using just 3-cent and 5-cent stamps. We know that $k + 1 = (k - 2) + 3$. Moreover, $8 \leq k - 2 < k$ because $k \geq 10$. Thus, by inductive hypothesis, a postage of k-2 cents can be formed using just 3-cent and 5-cent stamps, which means k-2=3r+5s for some non-negative integers s and r. By substitution, $k + 1 = (k - 2) + 3 = 3r + 5s + 3 = 3(r + 1) + 5s$. And r+1 is an integer because sum of integers is integer. Hence, a postage of k+1 cents can be formed using just r+1 3-cent and s 5-cent stamps, the statement is true for n=k+1.
This shows that if the inductive hypothesis is true, then P(k+1) must also be true, which complete the inductive argument.
We have completed the basis step and the inductive step, so by strong mathematical induction P(n) is true for all integers $n \geq 8$. This shows that the given statement is true.