CS 206

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Midterm

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1.

(a)

$$A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} \qquad A^T = \begin{bmatrix} -2 & -10 \\ 11 & 5 \end{bmatrix}$$

- Determine U:

$$AA^T = \begin{bmatrix} 125 & 75 \\ 75 & 125 \end{bmatrix} = W$$

Let $W\vec{x} = \lambda\vec{x}$ where $\vec{x} \in C^m$ is a nonzero eigenvector of W, and λ is its corresponding eigenvalue, then,

$$(W - \lambda I)\vec{x} = 0$$

$$\rightarrow W - \lambda I = \begin{bmatrix} 125 - \lambda & 75 \\ 75 & 125 - \lambda \end{bmatrix} = 0$$

$$\rightarrow |W - \lambda I| = (125 - \lambda)^2 - 75^2 = 0$$

$$\rightarrow \lambda = 200 \; or \; \lambda = 50$$

When $\lambda = 200$:

$$(W - \lambda I)\overrightarrow{x_1} = 0, and\ let\ \overrightarrow{x_1} = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 125 - \lambda & 75 \\ 75 & 125 - \lambda \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} -75 & 75 \\ 75 & -75 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = 75 \begin{bmatrix} -x_{11} + x_{12} \\ x_{11} - x_{12} \end{bmatrix} = 0$$

$$\rightarrow x_{11} = x_{12}$$

Since U is unitary, let $x_{11} = -\frac{\sqrt{2}}{2}$, $x_{12} = -\frac{\sqrt{2}}{2}$. So,

$$\overrightarrow{x_1} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

When $\lambda = 50$:

$$(W - \lambda I)\overrightarrow{x_2} = 0, and\ let\ \overrightarrow{x_2} = \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 125-\lambda & 75 \\ 75 & 125-\lambda \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} 75 & 75 \\ 75 & 75 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = 75 \begin{bmatrix} x_{21} + x_{22} \\ x_{21} + x_{12} \end{bmatrix} = 0 = 0$$

$$\rightarrow x_{21} = -x_{22}$$

Since U is unitary, let $x_{21} = -\frac{\sqrt{2}}{2}$, $x_{22} = \frac{\sqrt{2}}{2}$. So,

$$\overrightarrow{x_2} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

Then combine these two eigenvectors we obtain:

$$U = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

- Determine Σ:

$$singular\ values\ \sigma_1 = \sqrt{200} = 10\sqrt{2}, \qquad \sigma_2 = \sqrt{50} = 5\sqrt{2}$$

$$\rightarrow \Sigma = \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix}$$

- Determine V:

$$\sigma_1^{-1} A^T \overrightarrow{x_1} = \frac{1}{10\sqrt{2}} \begin{bmatrix} -2 & -10 \\ 11 & 5 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 0.6 \\ -0.8 \end{bmatrix}$$

$$\sigma_2^{-1} A^T \overrightarrow{x_2} = \frac{1}{5\sqrt{2}} \begin{bmatrix} -2 & -10 \\ 11 & 5 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} -0.8 \\ -0.6 \end{bmatrix}$$

$$\rightarrow V = \begin{bmatrix} 0.6 & -0.8 \\ -0.8 & -0.6 \end{bmatrix}$$

Therefore, $\begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix} \begin{bmatrix} 0.6 & -0.8 \\ -0.8 & -0.6 \end{bmatrix}$.

(b)

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}|, \textit{which is the maximum column sum.}$$

$$= a_{12} + a_{22} = |11| + |5| = 16$$

$$\|A\|_2 = \sqrt{\lambda_{max}(A^*A)} = \sigma_{max}(A), \textit{which is the maximum singular value of } A, = 10\sqrt{2}$$

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{m} |a_{ij}|, \textit{which is the maximum row sum.}$$

$$= a_{11} + a_{12} = |-10| + |5| = 15$$

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}, \textit{which is the Frobenius norm.}$$

$$= \sqrt{a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2} = \sqrt{(-2)^2 + 11^2 + (-10)^2 + 5^2} = 5\sqrt{10}$$

(c)

$$A^{-1} = V\Sigma^{-1}U^T = \begin{bmatrix} 0.6 & -0.8 \\ -0.8 & -0.6 \end{bmatrix} \begin{bmatrix} \frac{1}{10\sqrt{2}} & 0 \\ 0 & \frac{1}{5\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 0.05 & -0.11 \\ 0.1 & -0.02 \end{bmatrix}$$

2.

(a)

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Perform the Gram-Schmidt Orthogonalization procedure, we obtain,

$$b_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, r_{11} = \|b_1\|_2 = \sqrt{2} \rightarrow q_1 = \frac{b_1}{r_{11}} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$b_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, r_{12} = q_1^* b_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \sqrt{2}$$

$$\rightarrow b_2 - r_{12} q_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \sqrt{2} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \rightarrow r_{22} = \|b_2 - r_{12} q_1\|_2 = \sqrt{3}$$

$$\rightarrow q_2 = \frac{b_2 - r_{12} q_1}{r_{22}} = \frac{\sqrt{3}}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Therefore, reduced QR factorization of B is

$$\rightarrow \hat{Q} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} \end{bmatrix}, \hat{R} = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix} \rightarrow B = \hat{Q}\hat{R} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$$

(b)

For full QR factorization of B

$$\begin{cases} \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}z = 0 \\ \frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}y - \frac{\sqrt{3}}{3}z = 0 \\ x^2 + y^2 + z^2 = 1 \end{cases} \rightarrow \textit{one solution is} \begin{cases} x = \frac{\sqrt{6}}{6} \\ y = -\frac{\sqrt{6}}{3} \\ z = -\frac{\sqrt{6}}{6} \end{cases} \rightarrow a_3 = \begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{3} \\ -\frac{\sqrt{6}}{6} \end{bmatrix}$$

Therefore, full QR factorization of B is

$$B = QR = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \\ 0 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$$

(c)

$$P = A(A^*A)^{-1}A^*$$

We have

$$A^* = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}^* = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$A^*A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

$$(A^*A)^{-1} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}^{-1} = \frac{1}{6}\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$$

Then the orthogonal projector P onto range(A) is

$$P = A(A^*A)^{-1}A^* = \frac{1}{6}\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 1 & 2 \\ -2 & 2 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{bmatrix}$$

The image under P of the vector $(1,2,3)^T$ is

$$Im(P) = Px = \frac{1}{6}\begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 12 \\ 0 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = (2,0,2)^T$$

3.

(a)

The algorithm reduces A into a matrix form where all entries below the first sub-diagonal are zeros by applying a sequence of unitary matrix multiplications. This form is called Hessenberg form. When A is symmetric, the resulting matrix is tridiagonal.

(b)

The resulting matrix (say H) is related to A through:

$$H = Q_{m-2} \cdots Q_2 Q_1 A Q_1^* Q_2^* \cdots Q_{m-2}^*$$

where $Q_1, \cdots, Q_{m-2}$ are a sequence of unitary matrices generated through Householder reflectors.

Let $A = U\Sigma V^*$. Then $H = (Q_{m-2} \cdots Q_2 Q_1 U)\Sigma(Q_{m-2} \cdots Q_2 Q_1 V)^*$ is a SVD of H.

(c)

The operation count of the algorithm is dominated by the final two lines of the code: the first matrix update involves $4(m - k)(m - k + 1)$ flops, and the second matrix update involves $4m(m - k)$ flops. So overall, the total flops in leading term are:

$$\sim \sum_{k=1}^{m-2} 4(m-k)(m-k+1) + 4m(m-k) \sim \frac{10}{3} m^3$$

4.

(a)

$$F = I - 2\frac{vv^T}{v^T v}, let\ q = \frac{v}{||v||} \rightarrow F = I - 2qq^T$$

Since F is unitary matrix, $F = F^T$, then,

$$FF^T = (I - 2qq^T)(I - 2qq^T) = I - 2qq^T - 2qq^T + 4qq^T qq^T = I - 4qq^T + 4qq^T = I$$

Therefore, F is orthogonal.

(b)

The Householder reflector F has the general form of $F = I - 2qq^*$, where q is a unit vector. If λ is an eignenvalue and x is an associated eignevector, we have

$$Fx = \lambda x \rightarrow (I - 2qq^*)x = \lambda x \rightarrow (1 - \lambda)x = 2(q^T x)q$$

Therefore, λ=1 is an eigenvalue of F, and the space of associated eigenvectors is

$$\perp \text{q} = \{x : q^* x = 0\}$$

and its dimension is n-1. Therefore, the algebraic and geometric multiplicities are both n-1.

If λ≠1, we can conclude x is a scalar multiple of q. Suppose x = μq with μ≠0. Then

$$(1 - \lambda)\mu = 2\mu \rightarrow \lambda = -1$$

So λ = −1 is an eigenvalue and the space of associated eigenvectors is

$$< q >= \{\mu q : \mu \in C\ or\ R\}$$

and its dimension is 1. Therefore, the algebraic and geometric multiplicities are both 1.

Therefore, the eigenvalues of F are 1 and -1, and their algebraic and geometric multiplicities are shown above.

(c)

Yes, the sequence of vectors converges.

Let $\lambda^{(k)} = (v^{(k)})^T A v^{(k)}$, expand initial $v^{(0)}$ in orthonormal eigenvectors q_i and apply A^k.

$$x^{(0)} = a_1 q_1 + a_2 q_2 + \cdots + a_m q_m$$

$$x^{(k)} = c_k A^k x^{(0)}$$

$$= c_k \big(a_1 \lambda_1^k q_1 + a_2 \lambda_2^k q_2 + \cdots + a_m \lambda_m^k q_m\big)$$

$$= c_k \lambda_1^k \left(a_1 q_1 + a_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k q_2 + \cdots + a_m \left(\frac{\lambda_m}{\lambda_1}\right)^k q_m \right)$$

If $|\lambda_1| > |\lambda_2| \geq \cdots |\lambda_m| \geq 0$ and $q_1^T x^{(0)} \neq 0$, this gives,

$$\left\|x^{(k)} - (\pm q_1)\right\| = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right) \approx \left\|x^{(k-1)} - q_1\right\| \left|\frac{\lambda_2}{\lambda_1}\right| < 1$$

$$\left|\lambda^{(k)} - \lambda_1\right| = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^{2k}\right)$$

Finds the largest eigenvalue, unless eigenvector orthogonal to $x^{(0)}$.

Therefore, the sequence of vectors converges in the pattern of linear convergence, factor $\approx \frac{\lambda_2}{\lambda_1}$ at each iteration.

5.

(a)

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

(b)

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

(c)

$$L = D - A = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

(d)

Let $e_i \in \{0,1\}^n$ be the standard basis vectors (1 in the i-th coordinate, 0's elsewhere). Then the Laplacian L,

$$L = \sum_{(i,j)\in E} (e_i - e_j)(e_i - e_j)^T$$

Each term $(e_i - e_j)(e_i - e_j)^T$ is an $|V| \times |V|$ matrix that has +1 in the (i, i) and (j, j) coordinate, -1 in the in the (i, j) and (j, i) coordinate and the rest of the entries are all zero. Then,

$$x^T L x = x^T \left(\sum_{(i,j)\in E} (e_i - e_j)(e_i - e_j)^T \right) x$$

$$= \sum_{(i,j)\in E} x^T (e_i - e_j)(e_i - e_j)^T x$$

$$= \sum_{(i,j)\in E} \big(x(i) - x(j)\big)\big(x(i) - x(j)\big)$$

$$= \sum_{(i,j)\in E} (x(i) - x(j))^2 \geq 0$$

Therefore, the graph Laplacian L is always positive semi-definite.

(e)

From (d) we know that all eigenvalues of L are real and non-negative, and L is always positive semi-definite, then the smallest eigenvalue of L is 0, otherwise, L is positive definite, not positive semi-definite. Therefore, the smallest eigenvalue of L is 0.

The corresponding eigenvector is $\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$.

(f)

$$L^{sym} = D^{-\frac{1}{2}} L D^{-\frac{1}{2}}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{4}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{4}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{6}}{6} & 0 & 0 \\ -\frac{\sqrt{2}}{4} & 1 & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{6} & 1 & -\frac{1}{3} & 0 \\ 0 & -\frac{\sqrt{3}}{6} & -\frac{1}{3} & 1 & -\frac{\sqrt{6}}{6} \\ 0 & -\frac{\sqrt{2}}{4} & 0 & -\frac{\sqrt{6}}{6} & 1 \end{bmatrix}$$

(g)

Let A be an adjacency matrix, $NA = D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$ be the normalized adjacency matrix, then,

$$L^{sym} = I - NA = D^{-\frac{1}{2}}(D - A)D^{-\frac{1}{2}} = D^{-\frac{1}{2}} L D^{-\frac{1}{2}}$$

Then,

$$x^T L^{sym} x = x^T (I - NA) x$$

$$= \sum_{i=1}^{n} x_i^2 - \sum_{\sigma=ij} \frac{2x_i x_j}{\sqrt{d_i}\sqrt{d_j}}$$

$$= \sum_{i=1}^{n} \sum_{j=ij\epsilon E} \frac{x_i^2}{d_i} - \sum_{\sigma=ij} \frac{2x_i x_j}{\sqrt{d_i}\sqrt{d_j}}$$

$$= \sum_{\sigma=ij} \left(\frac{x_i^2}{d_i} + \frac{x_j^2}{d_j}\right) - \sum_{\sigma=ij} \frac{2x_i x_j}{\sqrt{d_i}\sqrt{d_j}}$$

$$= \sum_{\sigma=ij} \left(\frac{x_i}{\sqrt{d_i}} - \frac{x_j}{\sqrt{d_j}}\right)^2 \geq 0$$

Therefore, the normalized graph Laplacian L^{sym} is positive semi-definite.

(h)

```
import numpy


def power_iteration(A, v0):
    v0 = numpy.random.rand(n,)
    L = graph_laplacian(G)
    while (1):
        lam, x = power_iteration(L, v0)


    print(lam)
    print(x)
```