CompSci 206 PS4

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CO 2.12

Sets (a), (b), (c), (d), (f), and (g) are convex.

(a)

A slab is an intersection of two halfspaces, therefore, it's convex.

(b)

A rectangle is a finite intersection of halfspaces, therefore, it's convex.

(c)

A wedge is an intersection of two halfspaces, therefore, it's convex.

(d)

$$\{x \mid \|x - x_0\|_2 \le \|x - y\|_2 \; for \; all \; y \in S\} = \bigcap_{y \in S} \{x \mid \|x - x_0\|_2 \le \|x - y\|_2\}$$

Since

$$\begin{aligned}
&\|x - x_0\|_2 \le \|x - y\|_2 \\
&\leftrightarrow (x - x_0)^T (x - x_0) \le (x - y)^T (x - y) \\
&\leftrightarrow x^T x - 2x_0^T x + x_0^T x_0 \le x^T x - 2y^T x + y^T y \\
&\leftrightarrow 2(y - x_0)^T x \le y^T y - x_0^T x_0 \\
&\leftrightarrow \{x \mid \|x - x_0\|_2 \le \|x - y\|_2\} \; is \; a \; halfspace
\end{aligned}$$

So, the set of points closer to a given point than a given set it's an intersection of halfspaces. Therefore, it's convex.

(e)

Let $S = \{-1, 1\}$ and $T = \{0\}$, we obtain,

$$\{x \mid dist(x, S) \le dist(x, T)\} = \{x \in R \mid x \le -\frac{1}{2} \; or \; x \ge \frac{1}{2}\}$$

which obviously not convex.

Therefore, the set of points closer to one set than another is not convex.

(f)

$$x + S_2 \subseteq S_1 \leftrightarrow x + y \in S_1 \; for \; all \; y \in S_2$$

$$\rightarrow \{x \mid x + S_2 \subseteq S_1\} = \bigcap_{y \in S_2} \{x \mid \; x + y \in S_1\} = \bigcap_{y \in S_2} \{S_1 - y\}$$

Since S_1 is convex, the given set is the intersection of convex set $\{S_1 - y\}$. Therefore, it's convex.

(g)

$$\{x \mid \|x-a\|_2 \leq \theta \|x-b\|_2\}$$

$$= \{x \mid \|x-a\|_2^2 \leq \theta^2 \|x-b\|_2^2\}$$

$$= \{x \mid (1-\theta^2)x^T x - 2(a-\theta^2 b)^T x + (a^T a - \theta^2 b^T b) \leq 0\}$$

$$a \neq b, 0 \leq \theta \leq 1$$

When $\theta = 1$, $\{x \mid \|x-a\|_2 \leq \theta \|x-b\|_2\} = \{x \mid -2(a-b)^T x + a^T a - b^T b \leq 0\}$ is a halfspace, which is convex.

When $\theta < 1$, $\{x \mid \|x-a\|_2 \leq \theta \|x-b\|_2\}$ is a ball, which is also convex.

Therefore, the given set is convex.

CO 3.1

(a)

Since $function \; f: R \rightarrow R \; is \; convex, and \; a, b \in dom \; f$, with Jensen's inequality, we obtain

$$f(\theta a + (1-\theta)b) \leq \theta f(a) + (1-\theta) f(b), 0 \leq \theta \leq 1$$

Since $x \in [a, b], a < b$, let $\theta = \frac{b-x}{b-a}$, then

$$f\left(\frac{b-x}{b-a}a + \left(1 - \frac{b-x}{b-a}\right)b\right) \leq \frac{b-x}{b-a}f(a) + \left(1 - \frac{b-x}{b-a}\right)f(b)$$

$$\rightarrow f\left(\frac{b-x}{b-a}a + b - \frac{b-x}{b-a}b\right) \leq \frac{b-x}{b-a}f(a) + \left(\frac{b-a-b+x}{b-a}\right)f(b)$$

$$\rightarrow f\left(\frac{b-x}{b-a}(a-b) + b\right) \leq \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b)$$

$$\rightarrow f(x-b+b) \leq \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b)$$

$$\rightarrow\ f(x) \le \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$$

Therefore, the given inequality is proved.

(b)

From (a) we obtain

$$f(x) \le \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$$

$$\rightarrow f(x) - f(a) \le \frac{b-x}{b-a} f(a) - f(a) + \frac{x-a}{b-a} f(b)$$

$$\rightarrow f(x) - f(a) \le \frac{b-x-b+a}{b-a} f(a) + \frac{x-a}{b-a} f(b)$$

$$\rightarrow f(x) - f(a) \le \frac{a-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$$

Since $x \in [a, b]$, divide both sides of the above inequality by x-a, we obtain

$$\frac{f(x) - f(a)}{x-a} \le \frac{-1}{b-a} f(a) + \frac{1}{b-a} f(b)$$

$$\rightarrow \frac{f(x) - f(a)}{x-a} \le \frac{f(b) - f(a)}{b-a}$$

Thus, the first part of the given equality is proved.

For the second part of the given equality, subtract f(b) form both sides of the equality in (a), we obtain,

$$f(x) - f(b) \le \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) - f(b)$$

$$\rightarrow f(x) - f(b) \le \frac{b-x}{b-a} f(a) + \frac{x-a-b+a}{b-a} f(b)$$

$$\rightarrow f(x) - f(b) \le \frac{b-x}{b-a} f(a) + \frac{x-b}{b-a} f(b)$$

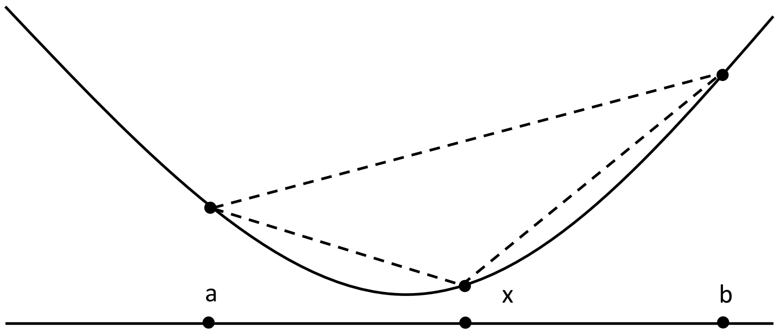
Then divide both sides of the above inequality by b-x, we obtain

$$\frac{f(x) - f(b)}{b-x} \le \frac{1}{b-a} f(a) + \frac{-1}{b-a} f(b)$$

$$\rightarrow \frac{f(x) - f(b)}{b-x} \le \frac{f(a) - f(b)}{b-a}$$

$$\rightarrow \frac{f(b) - f(a)}{b-a} \le \frac{f(b) - f(x)}{b-x}$$

Therefore, the inequality $\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a} \leq \frac{f(b)-f(x)}{b-x}$ is proved.



The sketch that illustrates the inequality is shown above.

(c)

From (b) we obtain

$$\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a} \leq \frac{f(b)-f(x)}{b-x}$$

Taking limit for $x \to a$ on both sides of the first part of the above inequality, we obtain

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} \leq \lim_{x\to a} \frac{f(b)-f(a)}{b-a}$$

$$\to f'(a) \ \leq \frac{f(b)-f(a)}{b-a}$$

Thus, the first part of the given equality is proved.

Taking limit for $x \to b$ on both sides of the second part of the above inequality, we obtain

$$\lim_{x\to b} \frac{f(b)-f(a)}{b-a} \leq \lim_{x\to b} \frac{f(b)-f(x)}{b-x}$$

$$\to \frac{f(b)-f(a)}{b-a} \leq f'(b)$$

Therefore, the inequality $f'(a) \ \leq \frac{f(b)-f(a)}{b-a} \leq f'(b)$ is proved using the result in (b).

(d)

From (c) we obtain

$$f'(a) \leq f'(b)$$

$$\to \frac{f'(b)-f'(a)}{b-a} \geq 0, with\ a < b$$

Taking limit for $b \to a$ we obtain

$$\lim_{b\to a} \frac{f'(b)-f'(a)}{b-a} \geq 0$$

$$\rightarrow f''(a) \geq 0$$

Taking limit for $a \rightarrow b$ we obtain

$$\lim_{a \rightarrow b} \frac{f'(b) - f'(a)}{b - a} \geq 0$$

$$\rightarrow f''(b) \geq 0$$

Therefore, $f''(a) \geq 0$ and $f''(b) \geq 0$ is proved using the result in (c).

CO 3.16

(a)

$$f(x) = e^x - 1 \rightarrow f'(x) = e^x \rightarrow f''(x) = e^x > 0$$

Therefore, $f(x) = e^x - 1 \; on \; R$ is convex and quasiconvex. $f(x) = e^x - 1 \; on \; R$ is also quasiconcave, but not concave.

(b)

$$f(x_1, x_2) = x_1 x_2 \rightarrow \frac{\partial f}{\partial x_1} = x_2, \frac{\partial f}{\partial x_2} = x_1 \rightarrow \frac{\partial^2 f}{\partial x_1^2} = 0, \frac{\partial^2 f}{\partial x_2^2} = 0, \frac{\partial^2 f}{\partial x_1 \partial x_2} = 1, \frac{\partial^2 f}{\partial x_2 \partial x_1} = 1$$

$$\rightarrow \nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Let $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

$$|H - \lambda I| = \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$$

So, H's eigenvalues are 1 and -1. Since H has one positive and one negative eigenvalue, H is neither positive semidefinite nor negative semidefinite. Therefore, $f(x_1, x_2) = x_1 x_2$ is neither convex nor concave. However, since its superlevel sets $\{(x_1, x_2) \in R_{++}^2 \mid x_1 x_2 \geq \alpha\}$ are convex sets for all α, $f(x_1, x_2) = x_1 x_2$ is quasiconcave.

Therefore, $f(x_1, x_2) = x_1 x_2 \; on \; R_{++}^2$ is quasiconcave, but not convex, concave, or quasiconvex.

(c)

$$f(x_1, x_2) = \frac{1}{x_1 x_2} \rightarrow \frac{\partial f}{\partial x_1} = -\frac{1}{x_1^2 x_2}, \frac{\partial f}{\partial x_2} = -\frac{1}{x_1 x_2^2}$$

$$\rightarrow \frac{\partial^2 f}{\partial x_1^2} = \frac{2}{x_1^3 x_2}, \frac{\partial^2 f}{\partial x_2^2} = \frac{2}{x_1 x_2^3}, \frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{1}{x_1^2 x_2^2}, \frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{1}{x_1^2 x_2^2}$$

$$\rightarrow \nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \frac{1}{x_1 x_2} \begin{bmatrix} \frac{2}{x_1^2} & \frac{1}{x_1 x_2} \\ \frac{1}{x_1 x_2} & \frac{2}{x_2^2} \end{bmatrix} \geq 0$$

Therefore, $f(x_1, x_2) = \frac{1}{x_1 x_2}\ on\ R_{++}^2$ is convex and quasiconvex, not concave or quasiconcave.

(d)

$$f(x_1, x_2) = \frac{x_1}{x_2} \rightarrow \frac{\partial f}{\partial x_1} = \frac{1}{x_2}, \frac{\partial f}{\partial x_2} = -\frac{x_1}{x_2^2} \rightarrow \frac{\partial^2 f}{\partial x_1^2} = 0, \frac{\partial^2 f}{\partial x_2^2} = \frac{2x_1}{x_2^3}, \frac{\partial^2 f}{\partial x_1 \partial x_2} = -\frac{1}{x_2^2}, \frac{\partial^2 f}{\partial x_2 \partial x_1} = -\frac{1}{x_2^2}$$

$$\rightarrow \nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} \end{bmatrix}$$

Obviously, the Hessian matrix of $f(x_1, x_2) = \frac{x_1}{x_2}$ is not positive semidefinite nor negative semidefinite. Therefore, $f(x_1, x_2) = \frac{x_1}{x_2}$ is neither convex nor concave.

And since the sublevel and superlevel sets of $f(x_1, x_2) = \frac{x_1}{x_2}$ is halfspaces, it's quasiconvex and quasiconcave.

Therefore, $f(x_1, x_2) = \frac{x_1}{x_2}\ on\ R_{++}^2$ is quasiconvex and quasiconcave, but not convex or concave.

(e)

$$f(x_1, x_2) = \frac{x_1^2}{x_2} \rightarrow \frac{\partial f}{\partial x_1} = \frac{2x_1}{x_2}, \frac{\partial f}{\partial x_2} = -\frac{x_1^2}{x_2^2} \rightarrow \frac{\partial^2 f}{\partial x_1^2} = \frac{2}{x_2}, \frac{\partial^2 f}{\partial x_2^2} = \frac{2x_1^2}{x_2^3}, \frac{\partial^2 f}{\partial x_1 \partial x_2} = -\frac{2x_1}{x_2^2}, \frac{\partial^2 f}{\partial x_2 \partial x_1} = -\frac{2x_1}{x_2^2}$$

$$\rightarrow \nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} \frac{2}{x_2} & -\frac{2x_1}{x_2^2} \\ -\frac{2x_1}{x_2^2} & \frac{2x_1^2}{x_2^3} \end{bmatrix} = \frac{2}{x_2} \begin{bmatrix} 1 \\ -\frac{2x_1}{x_2} \end{bmatrix} \begin{bmatrix} 1 & -\frac{2x_1}{x_2} \end{bmatrix} \geq 0$$

Therefore, $f(x_1, x_2) = \frac{x_1^2}{x_2}\ on\ R \times R_{++}^2$ is convex and quasiconvex, not concave or quasiconcave.

(f)

$$f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha} \rightarrow \frac{\partial f}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{1-\alpha}, \frac{\partial f}{\partial x_2} = (1-\alpha) x_1^{\alpha} x_2^{-\alpha}$$

$$\rightarrow \frac{\partial^2 f}{\partial x_1^2} = \alpha(\alpha-1)x_1^{\alpha-2}x_2^{1-\alpha}, \frac{\partial^2 f}{\partial x_2^2} = -\alpha(1-\alpha)x_1^{\alpha}x_2^{-\alpha-1},$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha}, \frac{\partial^2 f}{\partial x_2 \partial x_1} = \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha}$$

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} \alpha(\alpha-1)x_1^{\alpha-2}x_2^{1-\alpha} & \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} \\ \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} & -\alpha(1-\alpha)x_1^{\alpha}x_2^{-\alpha-1} \end{bmatrix}$$

$$= \alpha(\alpha-1)x_1^{\alpha}x_2^{1-\alpha} \begin{bmatrix} \frac{1}{x_1^2} & -\frac{1}{x_1x_2} \\ -\frac{1}{x_1x_2} & \frac{1}{x_2^2} \end{bmatrix} = \alpha(\alpha-1)x_1^{\alpha}x_2^{1-\alpha} \begin{bmatrix} \frac{1}{x_1} \\ -\frac{1}{x_2} \end{bmatrix} \begin{bmatrix} \frac{1}{x_1} & -\frac{1}{x_2} \end{bmatrix}$$

Since $0 \le \alpha \le 1$, $\nabla^2 f(x)\ \alpha(\alpha-1)x_1^{\alpha}x_2^{1-\alpha} \begin{bmatrix} \frac{1}{x_1} \\ -\frac{1}{x_2} \end{bmatrix} \begin{bmatrix} \frac{1}{x_1} & -\frac{1}{x_2} \end{bmatrix} \le 0$.

Therefore, $f(x_1, x_2) = x_1^{\alpha}x_2^{1-\alpha}\ on\ R_{++}^2$ is concave and quasiconcave, not convex or quasiconvex.

CO 3.17

$$f(x) = \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \rightarrow \frac{\partial f(x)}{\partial x_i} = \frac{1}{p}\left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}-1} \cdot p x_i^{p-1} = \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1-p}{p}} x_i^{p-1} = \left(\frac{f(x)}{x_i}\right)^{1-p}$$

$$\rightarrow \frac{\partial^2 f(x)}{\partial x_i \partial x_j} = \left(\frac{1}{x_i}\right)^{1-p} \cdot (1-p)\big(f(x)\big)^{-p} \cdot \left(\frac{f(x)}{x_j}\right)^{1-p} = \frac{1-p}{f(x)}\left(\frac{f(x)^2}{x_i x_j}\right)^{1-p}, i \ne j$$

$$\frac{\partial^2 f(x)}{\partial x_i^2} = \frac{1-p}{f(x)}\left(\frac{f(x)^2}{x_i^2}\right)^{1-p} - \frac{1-p}{x_i}\left(\frac{f(x)}{x_i}\right)^{1-p}$$

In order to show the given function $f(x)$ is concave with $dom\ f = R_{++}^n$, we need to show the Hessian matrix of $f(x)$ is negative semidefinite, i.e.

$$y^T \nabla^2 f(x) y = \frac{1-p}{f(x)}\left(\left(\sum_{i=1}^{n} \frac{y_i f(x)^{1-p}}{x_i^{1-p}}\right)^2 - \sum_{i=1}^{n} \frac{y_i^2 f(x)^{2-p}}{x_i^{2-p}}\right) \le 0$$

From the Cauchy-Schwarz inequality, we obtain $a^T b \le \|a\|_2 \|b\|_2$.

$$Let\ a_i = \left(\frac{f(x)}{x_i}\right)^{-\frac{p}{2}}, b_i = y_i\left(\frac{f(x)}{x_i}\right)^{1-\frac{p}{2}}$$

Then, we obtain

$$\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} (\frac{f(x)}{x_i})^{-p} = \sum_{i=1}^{n} \frac{((\sum_{i=1}^{n} x_i^p)^{\frac{1}{p}})^{-p}}{x_i^{-p}} = \sum_{i=1}^{n} \frac{\left(\sum_{i=1}^{n} x_i^p\right)^{-1}}{x_i^{-p}} = 1$$

$$\rightarrow y^T \nabla^2 f(x) y = \frac{1-p}{f(x)} \left(\left(\sum_{i=1}^{n} y_i (\frac{f(x)}{x_i})^{1-\frac{p}{2}} (\frac{f(x)}{x_i})^{-\frac{p}{2}} \right)^2 - \sum_{i=1}^{n} (y_i (\frac{f(x)}{x_i})^{1-\frac{p}{2}})^2 \right)$$

$$= \frac{1-p}{f(x)} \left(\left(\sum_{i=1}^{n} a_i b_i \right)^2 - \sum_{i=1}^{n} b_i^2 \right)$$

$$\leq \frac{1-p}{f(x)} \left(\sum_{i=1}^{n} b_i^2 - \sum_{i=1}^{n} b_i^2 \right) = 0$$

Thus, the Hessian matrix of $f(x)$ is negative semidefinite. Therefore, $f(x)$ is concave with $dom\ f = R_{++}^n$ is proved.