CompSci 206 PS1

Xiaoying Li

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NLA 2.3

(a)

Since $A \in C^{m\times m}$ is Hermitian, then $A = A^*$, where A^* is the hermitian conjugate of A.

And $Ax = \lambda x$ where $x \in C^m$ is a nonzero eigenvector of A, and λ is its corresponding eigenvalue. Take the hermitian conjugate on both sides:

$$x^*A^* = x^*\lambda^*$$

Since $A = A^*$,

$$x^*A^* = x^*A \;\rightarrow\; x^*A = x^*\lambda^*$$

Then, multiply both sides by x,

$$x^*Ax = \lambda^*x^*x$$

Since $Ax = \lambda x$,

$$x^*Ax = x^*\lambda x \;\rightarrow\; \lambda x^*x = \lambda^*x^*x$$

Since $x \neq 0$, $x^* \neq 0$, then $\lambda = \lambda^*$, so, λ is real.

Therefore, all eigenvalues of A are real is proved.

(b)

Let x and y be eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2, then,

$$Ax = \lambda_1 x$$

$$Ay = \lambda_2 y$$

Take the hermitian conjugate on both sides of the second equation:

$$y^*A^* = \lambda_2 y^*$$

Note: From question (a), we know that λ_1 and λ_2 are both real, so, $\lambda_1^* = \lambda_1$ and $\lambda_2^* = \lambda_2$.

Then, multiply both sides by x,

$$y^*A^*x = \lambda_2 y^*x$$

Since $A = A^*$, $Ax = \lambda_1 x$,

$$y^*A^*x = y^*Ax = y^*\lambda_1 x = \lambda_1 y^*x \rightarrow \lambda_1 y^*x = \lambda_2 y^*x$$

Since $\lambda_1 \neq \lambda_2$, the only way can make $\lambda_1 y^*x = \lambda_2 y^*x$ is $y^*x = 0$, which means x and y are orthogonal.

Therefore, x and y are orthogonal is proved.

NLA 2.4

The eigenvalues of a unitary matrix have norm 1.

Proof:

Let A be a unitary matrix, λ be an eigenvalue of A, and x is its corresponding eigenvector.

Then, we have,

$$A^*A = I, where\ I\ is\ the\ identity\ matrix$$

And,

$$Ax = \lambda x$$

Then,

$$x^*x = x^*(A^*A)x = (Ax)^*Ax = (\lambda x)^*\lambda x = (\lambda^*\lambda)x^*x = \|\lambda\|^2 x^*x$$

$$\rightarrow \|\lambda\|^2 = 1$$

Therefore, $\|\lambda\| = 1$ is proved.

NLA 2.6

- When A is nonsingular, prove its inverse has the form $A^{-1} = I + \alpha uv^*$ for some scalar α:

When $u = 0$ or $v = 0$, $A = I + uv^* = I$. Obviously, $A^{-1} = I = I + \alpha uv^*$ for some scalar α is true.

When $u \neq 0$ and $v \neq 0$, suppose $A^{-1} = [a_1, a_2, a_3, \cdots, a_m]$. Since $A = I + uv^*$ is nonsingular, $AA^{-1} = I$. Then,

$$AA^{-1} = (I + uv^*)[a_1, a_2, a_3, \cdots a_m] = [a_1 + uv^*a_1, a_2 + uv^*a_2, a_3 + uv^*a_3, \cdots, a_m + uv^*a_m] = I$$

Let $I = [e_1, e_2, e_3, \cdots, e_m]$, then,

$$[a_1 + uv^*a_1, a_2 + uv^*a_2, a_3 + uv^*a_3, \cdots, a_m + uv^*a_m] = [e_1, e_2, e_3, \cdots, e_m]$$

$$\rightarrow a_i + uv^*a_i = e_i$$

$$\rightarrow a_i + u(v^*a_i) = e_i$$

Let $v^*a_i = \theta_i$, then,

$$a_i + u\theta_i = e_i$$

$$\rightarrow a_i = e_i - u\theta_i$$

$$\rightarrow A^{-1} = [a_1, a_2, a_3, \cdots, a_m] = [e_1 - u\theta_1, e_2 - u\theta_2, e_3 - u\theta_3, \cdots e_m - u\theta_m] = I - u\theta^*$$

$$\rightarrow \; AA^{-1} = (I + uv^*)(I - u\theta^*) = I^2 - Iu\theta^* + Iuv^* - uv^*u\theta^* = I - u\theta^* + uv^* - uv^*u\theta^* = I$$

$$\rightarrow -\cancel{u}\theta^* + \cancel{u}v^* - \cancel{u}(v^*u)\theta^* = 0$$

$$\rightarrow \theta^* = \frac{v^*}{1 + v^*u}$$

$$\rightarrow A^{-1} = I - u\theta^* = I - \frac{uv^*}{1 + v^*u} = I + (-\frac{1}{1 + v^*u})uv^*$$

Therefore, when A is nonsingular, its inverse has the form $A^{-1} = I + \alpha uv^*$ for some scalar α is proved, and the expression for α is $\alpha = -\frac{1}{1+v^*u}(v^*u \neq -1)$.

- When A is singular:

A is singular also means Rank(A)<m, and Ax=0 has some nonzero solution $x \in C\backslash\{0\}$. Then,

$$Ax = x(I + \; uv^*) = x + xuv^* = 0$$

$$\rightarrow x = u(-v^*x), \mathit{which\ is\ a\ scalar\ multiple\ of\ u}$$

Then, let $\beta = -v^*x$, and $x = \beta u$ for some $\beta \in C$. Then,

$$Ax = x + xuv^* = \beta u + u(v^*(\beta u)) = \beta u(1 + v^*u) = 0$$

Since $u \neq 0$ and $v \neq 0$, the only way can let $\beta u(1 + v^*u) = 0$ always has some solution $\beta \in C$ is $1 + v^*u = 0 \rightarrow v^*u = -1$.

Therefore, when $v^*u = -1$, A is singular; and $null(A) = \{\beta u: \beta \in C\}$, since the solution for Ax=0 is $x = \beta u$ for some $\beta \in C$.

NLA 3.2

Let $A \in C^{m\times m}$, λ be an eigenvalue of A, and x is its corresponding eigenvector.

Then, we have,

$$Ax = \lambda x$$

$$\rightarrow \|Ax\| = |\lambda|\|x\|$$

Since,

$$\|Ax\| \leq \|A\|\|x\|$$

$$\rightarrow |\lambda|\|x\| \leq \|A\|\|x\|$$

$$\rightarrow |\lambda| \leq \frac{\|A\|\|x\|}{\|x\|} = \|A\|$$

Since we have, $\rho(A) \leq |\lambda|$, where $\rho(A)$ is the spectral radius of A, so, $\rho(A) \leq \|A\|$.

Therefore, $\rho(A) \leq \|A\|$ is proved.

NLA 4.1

To determine SVD of $A = U\Sigma V^T$, where columns of U are the left singular vectors; Σ has singular values and is diagonal; rows of V^T has are the right singular vectors, we must determine $U,\ \Sigma,\ V^T$.

(a)

$$A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \qquad A^T = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

- Determine U:

$$AA^T = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} = W$$

Let $W\vec{x} = \lambda\vec{x}$ where $\vec{x} \in C^m$ is a nonzero eigenvector of W, and λ is its corresponding eigenvalue, then,

$$(W - \lambda I)\vec{x} = 0$$

$$\rightarrow W - \lambda I = \begin{bmatrix} 9-\lambda & 0 \\ 0 & 4-\lambda \end{bmatrix} = 0$$

$$\rightarrow |W - \lambda I| = (9-\lambda)(\ 4-\lambda) = 0$$

$$\rightarrow \lambda = 9\ or\ \lambda = 4$$

When $\lambda = 9$:

$$(W - \lambda I)\overrightarrow{x_1} = 0, and\ let\ \overrightarrow{x_1} = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 9-\lambda & 0 \\ 0 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ -5x_{12} \end{bmatrix} = 0$$

$$\rightarrow -5x_{12} = 0$$

Since U is unitary, let $x_{11} = 1$, $x_{12} = 0$. So,

$$\overrightarrow{x_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

When $\lambda = 4$:

$$(W - \lambda I)\overrightarrow{x_2} = 0, and\ let\ \overrightarrow{x_2} = \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 9-\lambda & 0 \\ 0 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 5x_{21} \\ 0 \end{bmatrix} = 0$$

$$\rightarrow 5x_{21} = 0$$

Since U is unitary, let $x_{21} = 0$, $x_{22} = 1$. So,

$$\overrightarrow{x_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then combine these two eigenvectors we obtain:

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Determine Σ:

$$singular\ values\ \lambda_1 = \sqrt{9} = 3, \qquad \lambda_2 = \sqrt{4} = 2$$

$$\rightarrow \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

- Determine V^T:

$$A = U\Sigma V^T \rightarrow V^T = A(U\Sigma)^{-1} = A\Sigma^{-1}, since\ U = I$$

Then,

$$\det(\Sigma) = 3 \times 2 - 0 = 6$$

$$\rightarrow \Sigma^{-1} = \frac{adj(\Sigma)}{\det(\Sigma)} = \frac{1}{6}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\rightarrow V^T = A\Sigma^{-1} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Therefore, $\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(b)

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

- Determine U:

$$AA^T = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} = W$$

Let $W\vec{x} = \lambda\vec{x}$ where $\vec{x} \in C^m$ is a nonzero eigenvector of W, and λ is its corresponding eigenvalue, then,

$$(W - \lambda I)\vec{x} = 0$$

$$\rightarrow W - \lambda I = \begin{bmatrix} 4-\lambda & 0 \\ 0 & 9-\lambda \end{bmatrix} = 0$$

$$\rightarrow |W - \lambda I| = (4-\lambda)(9-\lambda) = 0$$

$$\rightarrow \lambda = 4 \; or \; \lambda = 9$$

When $\lambda = 4$:

$$(W - \lambda I)\overrightarrow{x_1} = 0, and\ let\ \overrightarrow{x_1} = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 4-\lambda & 0 \\ 0 & 9-\lambda \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 5x_{12} \end{bmatrix} = 0$$

$$\rightarrow 5x_{12} = 0$$

Since U is unitary, let $x_{11} = 1$, $x_{12} = 0$. So,

$$\overrightarrow{x_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

When $\lambda = 9$:

$$(W - \lambda I)\overrightarrow{x_2} = 0, and\ let\ \overrightarrow{x_2} = \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 4-\lambda & 0 \\ 0 & 9-\lambda \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} -5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} -5x_{21} \\ 0 \end{bmatrix} = 0$$

$$\rightarrow -5x_{21} = 0$$

Since U is unitary, let $x_{21} = 0$, $x_{22} = 1$. So,

$$\overrightarrow{x_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then combine these two eigenvectors we obtain:

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Determine Σ:

$$singular\ values\ \lambda_1 = \sqrt{4} = 2, \qquad \lambda_2 = \sqrt{9} = 3$$

$$\rightarrow \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Since diagonal entries of Σ are in nonincreasing order, $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. Thus, $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

- Determine V^T:

$$A = U\Sigma V^T \rightarrow V^T = A(U\Sigma)^{-1} = AU\Sigma^{-1}$$

Then,

$$\det(\Sigma) = 3 \times 2 - 0 = 6$$

$$\rightarrow \Sigma^{-1} = \frac{adj(\Sigma)}{\det(\Sigma)} = \frac{1}{6}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\rightarrow V^T = AU\Sigma^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Therefore, $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(c)

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

- Determine U:

$$AA^T = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = W$$

Let $W\vec{x} = \lambda\vec{x}$ where $\vec{x} \in C^m$ is a nonzero eigenvector of W, and λ is its corresponding eigenvalue, then,

$$(W - \lambda I)\vec{x} = 0$$

$$\rightarrow W - \lambda I = \begin{bmatrix} 4-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = 0$$

$$\rightarrow |W - \lambda I| = (4-\lambda)[(-\lambda)(-\lambda) - 0] = 0$$

$$\rightarrow (4-\lambda)\lambda^2 = 0$$

$$\rightarrow \lambda_1 = 4, \lambda_2 = \lambda_3 = 0$$

When $\lambda = \lambda_1 = 4$,

$$(W - \lambda I)\overrightarrow{x_1} = 0, and\ let\ \overrightarrow{x_1} = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 4-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ -4x_{12} \\ -4x_{13} \end{bmatrix} = 0$$

$$\rightarrow -4x_{12} = 0, \qquad -4x_{13} = 0$$

Since U is unitary, let $x_{11} = 1$, $x_{12} = 0, x_{13} = 0$. So,

$$\overrightarrow{x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

When $\lambda = \lambda_2 = 0$,

$$(W - \lambda I)\overrightarrow{x_2} = 0, and\ let\ \overrightarrow{x_2} = \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 4-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} = \begin{bmatrix} 4x_{21} \\ 0 \\ 0 \end{bmatrix} = 0$$

$$\rightarrow 4x_{21} = 0,$$

Since U is unitary, let $x_{21} = 0$, $x_{22} = 1, x_{23} = 0$. So,

$$\overrightarrow{x_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

When $\lambda = \lambda_3 = 0$, same as when $\lambda = \lambda_2 = 0$,

$$\overrightarrow{x_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Then combine these three eigenvectors we obtain:

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Determine Σ:

$$singular\ values\ \lambda_1 = \sqrt{4} = 2, \qquad \lambda_2 = \sqrt{0} = 0, \qquad \lambda_3 = \sqrt{0} = 0$$

And since Σ has the same shape as A,

$$\rightarrow \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- Determine V^T:

$$A^T A = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} = W$$

Let $W\vec{x} = \lambda\vec{x}$ where $\vec{x} \in C^m$ is a nonzero eigenvector of W, and λ is its corresponding eigenvalue, then,

$$(W - \lambda I)\vec{x} = 0$$

$$\rightarrow W - \lambda I = \begin{bmatrix} -\lambda & 0 \\ 0 & 4 - \lambda \end{bmatrix} = 0$$

$$\rightarrow |W - \lambda I| = (-\lambda)(4 - \lambda) = 0$$

$$\rightarrow \lambda = 4\ or\ 0$$

When $\lambda = 4$:

$$(W - \lambda I)\overrightarrow{x_1} = 0, and\ let\ \overrightarrow{x_1} = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} -\lambda & 0 \\ 0 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} -4x_{11} \\ 0 \end{bmatrix} = 0$$

$$\rightarrow x_{11} = 0$$

Since U is unitary, let $x_{11} = 0$, $x_{12} = 1$. So,

$$\overrightarrow{x_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

When $\lambda = 0$:

$$(W - \lambda I)\overrightarrow{x_2} = 0, and\ let\ \overrightarrow{x_2} = \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} -\lambda & 0 \\ 0 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 4x_{22} \end{bmatrix} = 0$$

$$\rightarrow x_{22} = 0$$

Since U is unitary, let $x_{21} = 1$, $x_{22} = 0$. So,

$$\overrightarrow{x_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Then combine these two eigenvectors we obtain:

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow V^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Therefore, $\begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(d)

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

- Determine U:

$$AA^T = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = W$$

Let $W\vec{x} = \lambda\vec{x}$ where $\vec{x} \in C^m$ is a nonzero eigenvector of W, and λ is its corresponding eigenvalue, then,

$$(W - \lambda I)\vec{x} = 0$$

$$\rightarrow W - \lambda I = \begin{bmatrix} 2-\lambda & 0 \\ 0 & -\lambda \end{bmatrix} = 0$$

$$\rightarrow |W - \lambda I| = (2-\lambda)(-\lambda) = 0$$

$$\rightarrow \lambda = 2 \ or \ \lambda = 0$$

When $\lambda = 2$:

$$(W - \lambda I)\overrightarrow{x_1} = 0, and\ let\ \overrightarrow{x_1} = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2-\lambda & 0 \\ 0 & -\lambda \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ -2x_{12} \end{bmatrix} = 0$$

$$\rightarrow -2x_{12} = 0$$

Since U is unitary, let $x_{11} = 1$, $x_{12} = 0$. So

$$\overrightarrow{x_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

When $\lambda = 0$:

$$(W - \lambda I)\overrightarrow{x_2} = 0, and\ let\ \overrightarrow{x_2} = \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2-\lambda & 0 \\ 0 & -\lambda \end{bmatrix}\begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 2x_{21} \\ 0 \end{bmatrix} = 0$$

$$\rightarrow 2x_{21} = 0,$$

Since U is unitary, let $x_{21} = 0$, $x_{22} = 1$. So,

$$\overrightarrow{x_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then combine these two eigenvectors we obtain:

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Determine Σ:

$$singular\ values\ \lambda_1 = \sqrt{2}, \qquad \lambda_2 = 0$$

$$\rightarrow \Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$$

- Determine V^T:

$$A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = W$$

Let $W\vec{x} = \lambda\vec{x}$ where $\vec{x} \in C^m$ is a nonzero eigenvector of W, and λ is its corresponding eigenvalue, then,

$$(W - \lambda I)\vec{x} = 0$$

$$\rightarrow W - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = 0$$

$$\rightarrow |W - \lambda I| = (1-\lambda)^2 - 1 = 0$$

$$\rightarrow (1-\lambda)^2 = 1$$

$$\rightarrow \lambda = 0 \; or \; 2$$

When $\lambda = 2$:

$$(W - \lambda I)\overrightarrow{x_1} = 0, and \; let \; \overrightarrow{x_1} = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} -x_{11} + x_{12} \\ x_{11} - x_{12} \end{bmatrix} = 0$$

$$\rightarrow x_{11} - x_{12} = 0$$

So, $x_{11} = x_{12}$, let $x_{11} = x_{12} = 1$. Then,

$$\overrightarrow{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Since V^T is unitary, divide $\overrightarrow{x_1}$ by its length.

$$L_{x_1} = \sqrt{1+1} = \sqrt{2}$$

$$\rightarrow \overrightarrow{x_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

When $\lambda = 0$:

$$(W - \lambda I)\overrightarrow{x_2} = 0, and \; let \; \overrightarrow{x_2} = \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} x_{21} + x_{22} \\ x_{21} + x_{22} \end{bmatrix} = 0$$

$$\rightarrow x_{21} + x_{22} = 0$$

So, $x_{21} = -x_{22}$, let $x_{21} = 1$, $x_{22} = -1$. Then,

$$\overrightarrow{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Since V^T is unitary, divide $\overrightarrow{x_2}$ by its length.

$$L_{x_2} = \sqrt{1+1} = \sqrt{2}$$

$$\rightarrow \overrightarrow{x_2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

Then combine these two eigenvectors we obtain:

$$V = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \rightarrow V^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

Therefore, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$.

(e)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

- Determine U:

$$AA^T = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = W$$

Let $W\vec{x} = \lambda\vec{x}$ where $\vec{x} \in C^m$ is a nonzero eigenvector of W, and λ is its corresponding eigenvalue, then,

$$(W - \lambda I)\vec{x} = 0$$

$$\rightarrow W - \lambda I = \begin{bmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} = 0$$

$$\rightarrow |W - \lambda I| = (2-\lambda)(2-\lambda) - 4 = 0$$

$$\rightarrow (2-\lambda)^2 = 4$$

$$\rightarrow \lambda = 4 \; or \; \lambda = 0$$

When $\lambda = 4$:

$$(W - \lambda I)\overrightarrow{x_1} = 0, and\ let\ \overrightarrow{x_1} = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} -2x_{11} + 2x_{12} \\ 2x_{11} - 2x_{12} \end{bmatrix} = 0$$

$$\rightarrow x_{11} - x_{12} = 0$$

So, $x_{11} = x_{12}$, let $x_{11} = x_{12} = 1$. Then,

$$\overrightarrow{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Since U is unitary, divide $\overrightarrow{x_1}$ by its length.

$$L_{x_1} = \sqrt{1+1} = \sqrt{2}$$

$$\rightarrow \overrightarrow{x_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

When $\lambda = 0$:

$$(W - \lambda I)\overrightarrow{x_2} = 0, and\ let\ \overrightarrow{x_2} = \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 2x_{11} + 2x_{12} \\ 2x_{11} + 2x_{12} \end{bmatrix} = 0$$

$$\rightarrow x_{21} + x_{22} = 0$$

So, $x_{21} = -x_{22}$, let $x_{21} = 1,\ x_{22} = -1$. Then,

$$\overrightarrow{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Since U is unitary, divide $\overrightarrow{x_2}$ by its length.

$$L_{x_2} = \sqrt{1+1} = \sqrt{2}$$

$$\rightarrow \overrightarrow{x_2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

Then combine these two eigenvectors we obtain:

$$U = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

- Determine Σ:

$$singular\ values\ \lambda_1 = \sqrt{4} = 2, \qquad \lambda_2 = \sqrt{0} = 0$$

$$\rightarrow \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

- Determine V^T:

$$A^T A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = AA^T$$

So, the eigenvalues and eigenvectors of $A^T A$ should be same as $AA^T's$. Obviously, V=U. Then,

$$V^T = U^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

Therefore, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$.