CompSci 206 PS2

Xiaoying Li

10/29/2021

1. NLA 6.1

- Prove algebraically

Since P is an orthogonal projector, we have

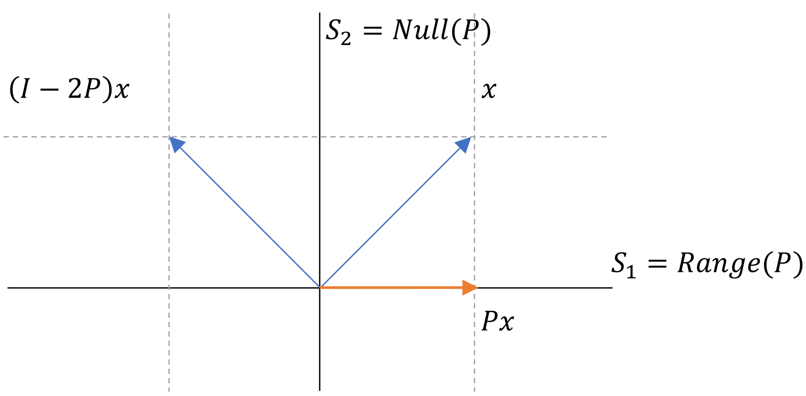
$$P^2 = P = P^*$$

Then,

$$(I - 2P)^*(I - 2P) = (I - 2P)^2 = I^2 - 2P - 2P + 4P^2 = I - 4P - 4P = I$$

Therefore, $I - 2P$ is unitary.

- Geometric interpretation:



(I-2P)x is the mirror image of x. Since the length of (I-2P)x remains the same as x under this transformation, I-2P is unitary.

2. NLA 6.4

(a)

$$range(A) = column\ space\ of\ A$$

$$\rightarrow orthogonal\ basis\ for\ range(A)\ is\ \hat{Q} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & 1 \\ \frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

Then the orthogonal projector P onto range(A) is

$$P = \hat{Q}\hat{Q}^* = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & 1 \\ \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

The image under P of the vector $(1,2,3)^*$ is

$$Im(P) = Px = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = (2,2,2)^*$$

(b)

$$P = B(B^*B)^{-1}B^*$$

We have

$$B^* = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}^* = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$B^*B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

$$(B^*B)^{-1} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}^{-1} = \frac{1}{6} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$$

Then the orthogonal projector P onto range(B) is

$$P = B(B^*B)^{-1}B^* = \frac{1}{6} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 1 & 2 \\ -2 & 2 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{bmatrix}$$

The image under P of the vector $(1,2,3)^*$ is

$$Im(P) = Px = \frac{1}{6}\begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 12 \\ 0 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = (2,0,2)^*$$

3. NLA 7.1

(a)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Perform the Gram-Schmidt Orthogonalization procedure, we obtain,

$$a_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, r_{11} = \|a_1\|_2 = \sqrt{2} \rightarrow q_1 = \frac{a_1}{r_{11}} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$a_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, r_{12} = q_1^* a_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0, r_{22} = \|a_2 - r_{12}q_1\|_2 = \|a_2\|_2 = 1$$

$$\rightarrow q_2 = \frac{a_2 - r_{12}q_1}{r_{22}} = a_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Therefore, reduced QR factorization of A is

$$\rightarrow \hat{Q} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & 1 \\ \frac{\sqrt{2}}{2} & 0 \end{bmatrix}, \hat{R} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix} \rightarrow A = \hat{Q}\hat{R} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & 1 \\ \frac{\sqrt{2}}{2} & 0 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$$

And for full QR factorization of A, let $a_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then,

$$\begin{cases} \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}z = 0 \\ y = 0 \\ x^2 + y^2 + z^2 = 1 \end{cases} \rightarrow \textit{one solution is} \begin{cases} x = \frac{\sqrt{2}}{2} \\ y = 0 \\ z = -\frac{\sqrt{2}}{2} \end{cases} \rightarrow a_3 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

Therefore, full QR factorization of A is

$$A = QR = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(b)

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Perform the Gram-Schmidt Orthogonalization procedure, we obtain,

$$b_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, r_{11} = \|b_1\|_2 = \sqrt{2} \rightarrow q_1 = \frac{b_1}{r_{11}} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$b_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, r_{12} = q_1^* b_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \sqrt{2}$$

$$\rightarrow b_2 - r_{12}q_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \sqrt{2} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \rightarrow r_{22} = \|b_2 - r_{12}q_1\|_2 = \sqrt{3}$$

$$\rightarrow q_2 = \frac{b_2 - r_{12}q_1}{r_{22}} = \frac{\sqrt{3}}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Therefore, reduced QR factorization of B is

$$\rightarrow \hat{Q} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} \end{bmatrix}, \hat{R} = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix} \rightarrow B = \hat{Q}\hat{R} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$$

And for full QR factorization of B

$$\begin{cases} \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}z = 0 \\ \frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}y - \frac{\sqrt{3}}{3}z = 0 \\ x^2 + y^2 + z^2 = 1 \end{cases} \rightarrow one\ solution\ is \begin{cases} x = \frac{\sqrt{6}}{6} \\ y = -\frac{\sqrt{6}}{3} \\ z = -\frac{\sqrt{6}}{6} \end{cases} \rightarrow a_3 = \begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{3} \\ -\frac{\sqrt{6}}{6} \end{bmatrix}$$

Therefore, full QR factorization of B is

$$A = QR = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \\ 0 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$$

4.

$$a_1 = (1, \epsilon, 0,0)^T, a_2 = (1,0, \epsilon, 0)^T, a_3 = (1,0,0, \epsilon)^T$$

- Classical Gram-Schmidt

$$v_1 \leftarrow \begin{bmatrix} 1 \\ \epsilon \\ 0 \\ 0 \end{bmatrix}, r_{11} = \|a_1\|_2 = \sqrt{1+\epsilon^2} \approx 1 \ \rightarrow \ q_1 = \frac{v_1}{r_{11}} = \begin{bmatrix} 1 \\ \epsilon \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 \leftarrow \begin{bmatrix} 1 \\ 0 \\ \epsilon \\ 0 \end{bmatrix}, r_{12} = q_1^* a_2 = \begin{bmatrix} 1 & \epsilon & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \epsilon \\ 0 \end{bmatrix} = 1, v_2 \leftarrow v_2 - \ r_{12} q_1 = \begin{bmatrix} 1 \\ 0 \\ \epsilon \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ \epsilon \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\epsilon \\ \epsilon \\ 0 \end{bmatrix}$$

$$\rightarrow r_{22} = \|v_2\|_2 = \sqrt{\epsilon^2 + \epsilon^2} = \sqrt{2}\epsilon$$

$$\rightarrow \ q_2 = \frac{v_2}{r_{22}} = \frac{1}{\sqrt{2}\epsilon} \begin{bmatrix} 0 \\ -\epsilon \\ \epsilon \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$$

$$v_3 \leftarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ \epsilon \end{bmatrix}, r_{13} = q_1^* a_3 = \begin{bmatrix} 1 & \epsilon & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \epsilon \end{bmatrix} = 1, v_3 \leftarrow v_3 - \ r_{13} q_1 = \begin{bmatrix} 0 \\ -\epsilon \\ 0 \\ \epsilon \end{bmatrix}$$

$$r_{23} = q_2^* a_3 = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \epsilon \end{bmatrix} = 0, v_3 \leftarrow v_3 - r_{23} q_2 = \begin{bmatrix} 0 \\ -\epsilon \\ 0 \\ \epsilon \end{bmatrix}$$

$$\rightarrow r_{33} = \|v_3\|_2 = \sqrt{\epsilon^2 + \epsilon^2} = \sqrt{2}\epsilon$$

$$\rightarrow q_3 = \frac{v_3}{r_{33}} = \frac{1}{\sqrt{2}\epsilon} \begin{bmatrix} 0 \\ -\epsilon \\ 0 \\ \epsilon \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

- Modified Gram-Schmidt

$$v_1 \leftarrow \begin{bmatrix} 1 \\ \epsilon \\ 0 \\ 0 \end{bmatrix}, r_{11} = \|a_1\|_2 = \sqrt{1 + \epsilon^2} \approx 1 \ \rightarrow \ q_1 = \frac{v_1}{r_{11}} = \begin{bmatrix} 1 \\ \epsilon \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 \leftarrow \begin{bmatrix} 1 \\ 0 \\ \epsilon \\ 0 \end{bmatrix}, r_{12} = q_1^* v_2 = \begin{bmatrix} 1 & \epsilon & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \epsilon \\ 0 \end{bmatrix} = 1, v_2 \leftarrow v_2 - \ r_{12} q_1 = \begin{bmatrix} 1 \\ 0 \\ \epsilon \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ \epsilon \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\epsilon \\ \epsilon \\ 0 \end{bmatrix}$$

$$\rightarrow r_{22} = \|v_2\|_2 = \sqrt{\epsilon^2 + \epsilon^2} = \sqrt{2}\epsilon$$

$$\rightarrow \ q_2 = \frac{v_2}{r_{22}} = \frac{1}{\sqrt{2}\epsilon} \begin{bmatrix} 0 \\ -\epsilon \\ \epsilon \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$$

$$v_3 \leftarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ \epsilon \end{bmatrix}, r_{13} = q_1^* v_3 = \begin{bmatrix} 1 & \epsilon & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \epsilon \end{bmatrix} = 1, v_3 \leftarrow v_3 - \ r_{13} q_1 = \begin{bmatrix} 0 \\ -\epsilon \\ 0 \\ \epsilon \end{bmatrix}$$

$$r_{23} = q_2^* v_3 = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -\epsilon \\ 0 \\ \epsilon \end{bmatrix} = \frac{\sqrt{2}}{2}\epsilon, v_3 \leftarrow v_3 - r_{23} q_2 = \begin{bmatrix} 0 \\ -\epsilon \\ 0 \\ \epsilon \end{bmatrix} - \begin{bmatrix} 0 \\ -\frac{\epsilon}{2} \\ \frac{\epsilon}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\epsilon}{2} \\ -\frac{\epsilon}{2} \\ \epsilon \end{bmatrix}$$

$$\rightarrow r_{33} = \|v_3\|_2 = \sqrt{(\frac{1}{4} + \frac{1}{4} + 1)\epsilon^2} = \frac{\sqrt{6}}{2}\epsilon$$

$$\rightarrow q_3 = \frac{v_3}{r_{33}} = \frac{2}{\sqrt{6}\epsilon}\begin{bmatrix} 0 \\ -\frac{\epsilon}{2} \\ -\frac{\epsilon}{2} \\ \epsilon \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \end{bmatrix}$$

5. NLA 7.3

- Algebraic proof:

When A is not full rank, obviously,

$$|detA| = det\ A = 0 \leq \prod_{j=1}^{m} \|a_j\|_2$$

When A is full rank, let A=QR be the QR factorization of A, then

$$|detA| = |detQ \cdot detR|$$

Since Q is orthogonal matrix, and all Q's column vectors are unitary, det R=1, then,

$$|detA| = |detQ|$$

Since Q is upper triangular matrix,

$$|detA| = |detQ| = \left|\prod_{j=1}^{m} r_{jj}\right|$$

Per Gram-Schmidt,

$$|r_{jj}| = \left\|a_j - \sum_{i=1}^{j-1} r_{ij} q_i\right\|_2 , and\ r_{ij} = q_i^* a_j (i \neq j)$$

Then,

$$|r_{jj}|^2 = \left\|a_j - \sum_{i=1}^{j-1} r_{ij} q_i\right\|_2^2 = \left(a_j^* - \sum_{i=1}^{j-1} r_{ij} {q_i}^*\right)\left(a_j - \sum_{i=1}^{j-1} r_{ij} q_i\right) \leq \|a_j\|_2^2$$

Therefore,

$$|detA| = \left|\prod_{j=1}^{m} r_{jj}\right| \leq \prod_{j=1}^{m} \|a_j\|_2\ \ is\ proved.$$

- Geometric interpretation:

|det A| equals the volume of the parallelopiped spanned by the a_i, and the volume of a parallelopiped does not exceed the product of the lengths of its sides, which is $\prod_{j=1}^{m}\left\|a_j\right\|_2$; and it equals this product if and only if its sides are orthogonal.

6. NLA 8.2

```
import numpy

def mgs(A):
  column = A.shape[1]
  Q = A.copy()
  R = numpy.zeros((column, column))
  for i in range(column):
    R[i, i] = numpy.linalg.norm(Q[:, i])
    Q[:, i] = Q[:, i]/numpy.linalg.norm(Q[:, i])
    for j in range(i+1, column):
      R[i, j] = Q[:, j].dot(Q[:, i])
      Q[:,j] = Q[:,j]-Q[:, j].dot(Q[:, i])*Q[:,i]
  return Q, R
```

7.

Apply the [Q, R]=msg(A) function in the previous, we obtain,

$$Q = \begin{bmatrix} 0.70710 & 0.70711 \\ 0.70711 & -0.70710 \end{bmatrix} \; R = \begin{bmatrix} 9.8996e-01 & 1.0000e+00 \\ 0.0000e+00 & 1.0000e-05 \end{bmatrix}$$

And check the orthogonality of Q matrix by calculating,

$$\begin{cases} 0.70710^2 + 0.70711^2 \cong 1 \\ 0.70711^2 + (-0.70710)^2 \cong 1 \\ 0.70710 \cdot 0.70711 + (0.70711 \cdot -0.70710) = 0 \end{cases}$$

Therefore, Q is orthogonal.

The value returned by the python build-in function numpy.linalg.qr is

$$Q = \begin{bmatrix} 0.70710 & 0.70711 \\ 0.70711 & -0.70710 \end{bmatrix} \; R = \begin{bmatrix} 9.8996e-01 & 1.0000e+00 \\ 0.0000e+00 & 1.0000e-05 \end{bmatrix}$$

It is almost same as the result returned by mgs.

8. NLA 10.1

(a)

$$F = I - 2\frac{vv^T}{v^T v}, let\ q = \frac{v}{\|v\|} \rightarrow F = I - 2qq^T$$

Let λ be an eigenvalue of F, and x be its corresponding eigenvector, then we have

$$Fx = \lambda x \rightarrow (I - 2qq^T)x = \lambda x \rightarrow (1-\lambda)x = 2(q^T x)q$$

Therefore, λ=1 is an eigenvalue of F.

When λ≠1, let x=aq with a≠0, then,

$$(1-\lambda)a = 2a \rightarrow \lambda = -1$$

Therefore, the eigenvalues of a Householder reflector are 1 and -1.

(b)

$$\det(A) = \prod_{i=1}^{n} \lambda_i\ , where\ \lambda\ is\ matrix\ A's\ eigenvalue$$

From question(a) we obtain eigenvalues of a Householder reflector are 1 and -1, then,

$$\det(F) = 1 \cdot (-1) = -1$$

Therefore, the determinant of a Householder reflector is -1.

(c)

If $A = A^*$, then the singular values of A are the absolute values of the eigenvalues of A. And for a Householder reflector $F = F^*$, the eigenvalues of a F are 1 and -1, therefore, the singular value of a Householder reflector is 1.

9. NLA 10.2

```
import numpy

# Function [W,R]=house(A) computes an implicit representation of a full QR factorization A=QR of an m*n matrix A with m⩾n using Householder reflections.
# Output: a lower-triangular matrix W ∈ C^{m×n} whose columns are the vectors v_k defining the successive Householder reflections, and a triangular matrix R ∈ C^{n,n}.
def house(A):
```

```
    m, n = A.shape
    W = numpy.zeros((m, n))

    for k in range(n):
        x = A[k:m, k]
        if x[0] >= 0:
            sign = 1
        else:
            sign = -1
        v = sign*(numpy.linalg.norm(x)*numpy.eye(m-k, 1)).squeeze()+x
        v = v/numpy.linalg.norm(v)
        A[k:m, k:n] = A[k:m, k:n]-2*numpy.outer(v, numpy.dot(v.transpose(), A[k:m, k:n]))
        W[k:m, k] = v
    R = A[0:n, :]

    return W, R


# Function Q=formQ(W) that takes the matrix W produced by house as input and generates a
corresponding m*M orthogonal matrix Q.
def formQx(W, x):
    m, n = W.shape

    for k in reversed(range(n)):
        x[k:m] = x[k:m]-2*W[k:m, k]*numpy.dot(W[k:m, k].transpose(), x[k:m])

    return x


def formQ(W):
    m, n = W.shape
    Q = numpy.eye(m, m)

    for k in range(m):
        Q[:, k] = formQx(W, Q[:, k])

    return Q
```

10. NLA 10.3

Reduced QR factorization of Z by the Gram-Schmidt routine mgs of exercise 8.2 is

$$Q = \begin{bmatrix} 0.10102 & 0.31617 & 0.54100 \\ 0.40406 & 0.35337 & 0.51619 \\ 0.70711 & 0.39057 & -0.52479 \\ 0.40406 & -0.55795 & 0.38714 \\ 0.40406 & -0.55795 & -0.12044 \end{bmatrix} \quad R = \begin{bmatrix} 9.89949 & 9.49543 & 9.69746 \\ 0 & 3.29192 & 3.01294 \\ 0 & 0 & 1.97012 \end{bmatrix}$$

Reduced QR factorization of Z by the Householder routines house and formQ of exercise 10.2 is

$$Q = \begin{bmatrix} -0.10102 & 0.31617 & 0.54200 & -0.68421 & -0.35767 \\ -0.40406 & -0.35337 & 0.51619 & 0.32801 & 0.58123 \\ -0.70711 & -0.39057 & -0.52479 & 0.00940 & -0.26826 \\ -0.40406 & 0.55795 & 0.38714 & 0.36560 & -0.49182 \\ -0.40406 & 0.55795 & -0.12044 & -0.53900 & 0.46947 \end{bmatrix}$$

$$R = \begin{bmatrix} -9.89949 & -9.49543 & -9.69746 \\ 0 & -3.29192 & -3.01294 \\ 0 & 0 & 1.97012 \end{bmatrix}$$

Reduced QR factorization of Z by the python build-in function numpy.linalg.qr is

$$Q = \begin{bmatrix} 0.10102 & 0.31617 & 0.54100 \\ 0.40406 & 0.35337 & 0.51619 \\ 0.70711 & 0.39057 & -0.52479 \\ 0.40406 & -0.55795 & 0.38714 \\ 0.40406 & -0.55795 & -0.12044 \end{bmatrix} \quad R = \begin{bmatrix} 9.89949 & 9.49543 & 9.69746 \\ 0 & 3.29192 & 3.01294 \\ 0 & 0 & 1.97012 \end{bmatrix}$$

The reduced QR factorization of Z by the Gram-Schmidt routine mgs of exercise 8.2 and by the python build-in function numpy.linalg.qr are almost the same, but the reduced QR factorization of Z by the Householder routines house and formQ of exercise 10.2 is the negative Q and R of the other two. All results are valid QR factorization of Z.