Assignment 2

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Xiaoying Li

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Problem 1

(a)

A exits the system before any other packets move to server 3 means A leaves Server 1 and moves to Server 3 before B and C move to server 3, and A leaves Server 3 before B and C move to server 3. Let X_1, X_2, X_3 be the service time of Server 1, 2, and 3, then,

$$P(A\ exits\ the\ system\ before\ any\ other\ packets\ move\ to\ server\ 3)$$
$$= P(X_1 < X_2) \cdot P(X_3 < X_2) = \frac{\mu_1}{\mu_1 + \mu_2} \cdot \frac{\mu_3}{\mu_2 + \mu_3} = \frac{1}{1+2} \cdot \frac{3}{2+3} = \frac{1}{5}$$

(b)

C exits the system before any other packets move to server 3 means C leaves Server 2 and moves to Server 3 before A moves to server 3, and C leaves Server 3 before A and B moves to server 3. Let X_1, X_2, X_3 be the service time of Server 1, 2, and 3, then,

$$P(C\ exits\ the\ system\ before\ any\ other\ packets\ move\ to\ server\ 3)$$
$$= P(X_2 < X_1) \cdot P(\min(X_1, X_2, X_3) = X_3) = \frac{\mu_2}{\mu_1 + \mu_2} \cdot \frac{\mu_3}{\mu_1 + \mu_2 + \mu_3} = \frac{2}{1+2} \cdot \frac{3}{1+2+3} = \frac{1}{3}$$

(c)

The expected time T needed by Packet A to exit the system can be divided up into

$$T = T_1 + R$$

where T_1 is the time until the first thing that happens, and R is the rest of the time.

The time until the first thing happens is

$$E(T_1) = \frac{1}{\mu_1 + \mu_2} = \frac{1}{1+2} = \frac{1}{3}$$

Let X_1, X_2, X_3 be the service time of Server 1, 2, and 3.

When the first thing to happen is Packet A leaves Server 1 and moves to Server 3, which occurs with probability

$$P(X_1 < X_2) = \frac{\mu_1}{\mu_1 + \mu_2} = \frac{1}{1+2} = \frac{1}{3}$$

then at that point Packet A's remaining time in the system is the remaining time of Packet A being served in Server 3. Thus,

$$E(R|First\ thing\ is\ A\ leaves\ 1\ and\ moves\ to\ 3) = \frac{1}{\mu_3} \cdot \frac{\mu_1}{\mu_1 + \mu_2} = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

When the first thing to happen is Packet B leaves Server 2 and moves to Server 3, which occurs with probability

$$P(X_2 < X_1) = \frac{\mu_2}{\mu_1 + \mu_2} = \frac{2}{1+2} = \frac{2}{3}$$

The Packet A's remaining time in the system T' can be divided up into

$$T' = T_1' + R'$$

then at that point, the time until the next thing happens is

$$E(T_1') = \frac{1}{\mu_1 + \mu_3} = \frac{1}{1+3} = \frac{1}{4}$$

when the next thing to happen is Packet A leaves Server 1 and moves to Server 3, which occurs with probability

$$P(X_1 < X_3) = \frac{\mu_1}{\mu_1 + \mu_3} = \frac{1}{1+3} = \frac{1}{4}$$

Packet A's remaining time in the system is the remaining time of Packet B being served in Server 3 and the time of Packet A being served in Server 3. Thus,

$$E(R'|Next\ thing\ is\ A\ leaves\ 1\ and\ moves\ to\ 3) = (\frac{1}{\mu_3} + \frac{1}{\mu_3}) \cdot \frac{\mu_1}{\mu_1 + \mu_3} = (\frac{1}{3} + \frac{1}{3}) \cdot \frac{1}{4} = \frac{1}{6}$$

When the next thing to happen is Packet B leaves Server 3 and exits the system, which occurs with probability

$$P(X_3 < X_1) = \frac{\mu_3}{\mu_1 + \mu_3} = \frac{3}{1+3} = \frac{3}{4}$$

Packet A's remaining time in the system is the remaining time of Packet A being served in Server 1 and the time of Packet A being served in Server 3. Thus,

$$E(R'|Next\ thing\ is\ B\ leaves\ 3) = (\frac{1}{\mu_1} + \frac{1}{\mu_3}) \cdot \frac{\mu_3}{\mu_1 + \mu_3} = (\frac{1}{1} + \frac{1}{3}) \cdot \frac{3}{4} = 1$$

So, when the first thing to happen is Packet B leaves Server 2 and moves to Server 3, Packet A's remaining time in the system is

$$E(R|First\ thing\ is\ B\ leaves\ 2\ and\ moves\ to\ 3) = \frac{2}{3}\cdot\left(\frac{1}{4}+\frac{1}{6}+1\right) = \frac{17}{18}$$

Therefore, the expected time T needed by Packet A to exit the system is

$$T = \frac{1}{3}+\frac{1}{9}+\frac{17}{18} = \frac{25}{18}$$

Problem 2

(a)

$$\lambda = 3, t = 2, k = 1$$

$$P(N(2) = 1) = \frac{(\lambda t)^k}{k!}e^{-\lambda t} = \frac{(3\cdot 2)^1}{1!}e^{-3\cdot 2} = 6e^{-6}$$

(b)

$$\lambda = 3, t = 2$$

$$E(\lambda t) = 3\cdot 2 = 6$$

(c)

$$P(N(5) - N(3) = 4) = P(N(2) = 4) \rightarrow \lambda = 3, t = 2, k = 4$$

$$P(N(2) = 4) = \frac{(\lambda t)^k}{k!}e^{-\lambda t} = \frac{(3\cdot 2)^4}{4!}e^{-3\cdot 2} = 54e^{-6}$$

Therefore, the probability that between the second 5 and the second 3 there are exactly 4 packets in the router is $54e^{-6}$.

Problem 3

(a)

Let $X\sim Exponential(\lambda)$ be the time in between two packets sent out, $Y\sim Poisson(\lambda)$ be the number of packets leaving the router per second. Then,

$$\lambda = \frac{60}{3} = 20\ packets/second$$

$$P(x < 4) = 1 - e^{-\lambda t} = 1 - e^{-20\cdot 4} = 1 - e^{-80}$$

Therefore, the probability that a packet will be sent out in less than 4 seconds is $1 - e^{-80}$.

(b)

$$P(y=1)=\frac{\lambda^y}{y!}e^{-y}=\frac{20^1}{1!}e^{-1}=20e^{-1}$$

Therefore, the probability that exactly 1 packet will leave in the next second is $20e^{-1}$.

(c)

$$P(y<2)=P(y=0)+P(y=1)=20e^{-1}+\frac{20^1}{1!}e^{-1}=21e^{-1}$$

Therefore, the probability that less than 2 packet will leave in the next second is $21e^{-1}$.

(d)

$$P(N(3)=2)=\frac{(\lambda t)^k}{k!}e^{-\lambda t}=\frac{(20\cdot 3)^2}{2!}e^{-20\cdot 3}=1800e^{-60}$$

Therefore, the probability that exactly 2 packets will leave in the next 3 second is $1800e^{-60}$.

Problem 4

(a)

Let the time between the generation of two consecutive packets at each client be X_1 and X_2.

$$P(X_2<X_1)=P(\min(X_1,X_2)=X_2)=\frac{\lambda_2}{\lambda_1+\lambda_2}=\frac{2}{1+2}=\frac{2}{3}$$

Therefore, the probability that the next packet will come from node 2 is $\frac{2}{3}$.

(b)

$$P(N(2)=3)=\frac{(\lambda t)^k}{k!}e^{-\lambda t}=\frac{((\lambda_1+\lambda_2)t)^k}{k!}e^{-(\lambda_1+\lambda_2)t}=\frac{((1+2)\cdot 2)^3}{3!}e^{-(1+2)\cdot 2}=36e^{-6}$$

Therefore, the probability that the router will receive exactly 3 packets in the next 2 second is $36e^{-6}$.

(c)

$$P(N(4-2)\geq 1)=P(N(2)\geq 1)=1-P(N(2)=0)=1-e^{-(1+2)\cdot 2}=1-e^{-6}$$

Therefore, the probability that at least one packet will arrive by t=4 is $1-e^{-6}$.