The Big-O Problem for Labelled Markov Chains and Weighted Automata[1]

April 9, 2021

[1] Dmitry Chistikov, Stefan Kiefer, Andrzej S Murawski, et al. "The Big-O Problem for Labelled Markov Chains and Weighted Automata". In: *arXiv preprint arXiv:2007.07694* (2020).

Findings

1. The big-O problem for non-negative Weighted Automata (WA) and Labelled Markov Chains (LMCs) turns out to be undecidable in general.
2. For **unambiguous** automata, i.e., where every word has at most one accepting path, the big-O problem becomes decidable and can be solved in polynomial time.
3. In the **unary** case, i.e., if the input alphabet Σ is a singleton, the big-O problem is also decidable and, in fact, **coNP**-complete.
4. In a more general **bounded** case, if the languages of all words w associated with non-zero weight are included in $w_1^* w_2^* \cdots w_m^*$ for some finite words $w_1, w_2, \cdots, w_m \in \Sigma^*$, the big-O problem is decidable *subject to* Schanuel's conjecture.

Overview

1. Background
2. Undecidability Results in General
3. Polynomial-time Deciablity for Unambiguous Cases
4. coNP Upper-bound for Unary Cases
5. Conditional Decidability for Bounded Language

Weighted Automata

$\langle Q, \Sigma, M, F \rangle$: A Weighted Automaton $\mathcal{W}$

Over a semi-ring $(\mathbb{Q}, +, \times)$

Q a finite set of states

Σ a finite alphabet

M $\Sigma \to \mathbb{Q}^{Q \times Q}$, a transition weighting function

F $\subseteq Q$ a set of final states

Non-negative Weighted automata: $\forall a \in \Sigma, \forall q, q' \in Q \to M(a)(q, q') \geq 0$

For an input $w \in \Sigma^*$ and a state $s \in Q$,define a function $f_s : \Sigma' \to \mathbb{R}$

$$f_s(w) = \sum_{t \in F} (M(a_1) \times M(a_2) \times \cdots \times M(a_n))_{s,t} \quad \text{for} \quad a_1 a_2 \ldots a_n \in w$$

stands for the weight of w from state s

Positive Weight Word Set

A set of all w with positive weight from state s

$$\mathcal{L}_s(\mathcal{W}) = \{w | w \in \Sigma^* \wedge f_s(w) > 0\}$$
$$\rightarrow \text{the language of } \mathcal{N}_s(\mathcal{W})$$

$\mathcal{N}_s(\mathcal{W})$: the *non-deterministic finite automaton* (NFA) formed from the same set of states (Q and F) as $\mathcal{W}$, start state s, and transitions $q \xrightarrow{a} q'$ whenever $M(a)(q, q') > 0$

Problem Definition

What is big-O?

Given $s, s' \in Q$:

$$\exists C > 0, \forall w \in \Sigma^*, f_s(w) \leq C \cdot f_{s'}(w)$$

$$\Downarrow$$

$$s \text{ is big-O of } s'$$

Big-O and Big-Θ problems can be reduced to each other:

$$\left.\begin{array}{r} s \text{ is big-O of } s' \\ s' \text{ is big-O of } s \end{array}\right\} \Leftrightarrow s \text{ is big-}\Theta \text{ of } s'$$

Labelled Markov Chain (LMC)

A special class of weighted automata

Definition

A non-negative weighted automaton $\langle Q, \Sigma, M, F \rangle$ that: $\forall q \in Q \backslash F$

- $\sum_{q' \in Q} \sum_{a \in \Sigma} M(a)(q, q') = 1$
- $M(a)(q, q') = 0$

- Final states have no outgoing transitions
- $f_s(E) = \sum_{w \in E} f_s(w) \leq 1$ for a $E \subseteq \Sigma^*$
- The probability (weight) of a transition $q \to q'$ is fixed *independently* of the past sequence of states visited by the machine

Measurements

For two states s and s', define the (asymmetric) ratio variation function as:

$$r(s, s') = \sup_{E \subseteq \Sigma^*} (f_s(E)/f_{s'}(E))$$

The big-O problem $\implies$ whether $r(s, s') < \infty$

The ratio distance (symmetric)

$$rd(s, s') = \max(r(s, s'), r(s', s))$$

A system $\mathcal{M}$ is ϵ-differentially private if

$$\forall s, s' \in Q, \forall E \in \Sigma^*, \quad f_s(E) \leq e^{\epsilon} f_{s'}(e)$$

r captures the level of differential privacy between s and s'

Big-O, Threshold and Approximation Problems are Undecidable

- The big-O problem is undecidable, even for LMCs
- Each variation of the problem
  - asymmetric/symmetric (r vs. rd)
  - non-strict/strict ($\leq$ vs. $<$)

  is undecidable, even under the promise of **boundedness**
- All variants of the approximation tasks are unsolvable, even under the promise of boundedness.
  - asymmetric/symmetric (r vs. rd)

  Approximation: Find a x for a given constant γ that

Addictive
$\lvert r(s, s') - x \rvert \leq \gamma$

vs.

Multiplicative
$1 - \gamma \leq \frac{x}{r(s,s')} \leq 1 + \gamma$

The LC condition: A simple necessary (but insufficient) condition of big-O

language containment condition

If s is big-O of s', then

LC condition

If for all words w with $f_s(w) > 0$ we also have $f_{s'}(w) > 0$. Equivalently,

$$\mathcal{L}_s(\mathcal{W}) \subseteq \mathcal{L}_{s'}(\mathcal{W})$$

It can be verfied by constructing NFA $\mathcal{N}_s(\mathcal{W})$ and $\mathcal{N}_{s'}(\mathcal{W})$ that accept $\mathcal{L}_s(\mathcal{W})$ and $\mathcal{L}_{s'}(\mathcal{W})$ respectively and verifying

$$\mathcal{L}(\mathcal{N}_s(\mathcal{W})) \subseteq \mathcal{L}(\mathcal{N}_{s'}(\mathcal{W}))$$

LC condition is the first step in each of verification routines.

Application: Unambiguous Weighted Automata

Definition

A weighted automaton $\mathcal{W}$ is unambiguous from a state s if every word has at most one accepting path in $\mathcal{N}_s(\mathcal{W})$

If a weighted automaton $\mathcal{W}$ is unambiguous from states s and s', the big-O problem is decidable in polynomial time.

Proof.

Construct a product weighted automaton

$$M'(a)((q_1, q_1'), (q_2, q_2')) = \frac{M(a)(q_1, q_2)}{M(a)(q_1', q_2')}$$

Is s big-O of s'?
$\rightarrow$ If there is a cycle on path from (s, s') to (t, t)?
$\rightarrow$ Bellman-Ford algorithm $\square$

Complexity [2]

NFA language containment is

	If the automata are	Note
NL-complete	in fact deterministic	**NL**: problems that can be solved in a **logarithmic** amount of memory space
in **P**	unambiguous	can be solved in a **polynomial** time
coNP-complete	unary	a complement of a **NP**-complete problem
PSPACE-complete	in general	can be solved in a **polynomial** amount space

$\leq$ the complexity level of repsective alogrithm for the big-O problem. (i.e. lower bound)

[2] `https://www.cs.princeton.edu/courses/archive/spring13/cos423/lectures/08IntractabilityII.pdf`

The big-O problem for unary weighted automata is **coNP**-complete

Definition

A unary weighted automata has a singleton alphabet ($|\Sigma| = 1$), its transition can be described as a single matrix $A = M(a)$, so it has $f_s(a^n) = A^n_{s,t}$

There already exists an asymptotic order of growth of

$$A^n_{s,t} + A^{n+1}_{s,t} + \cdots + A^{n+q}_{s,t} \approx \rho^n n^k$$

for some ρ,k and q. This paper accurately gives the correct ρ and k for each word and discovers that

- The big-O problem for unary weighted automata is **coNP**-complete
- It is coNP-hard even for unary labeled Markov chains (LMC)

Upper bound: The unary big-O problem is in **coNP**

Decidability for weighted automata with bounded languages

Assumptions

1. $\mathcal{L}_s(\mathcal{W})$ and $\mathcal{L}_{s'}(\mathcal{W})$ are bounded
2. LC condition is checked: $\mathcal{L}_s(\mathcal{W}) \subseteq \mathcal{L}_{s'}(\mathcal{W})$

Then, the problem is conditionally decidable, subject to Schanuel's conjecture [3].

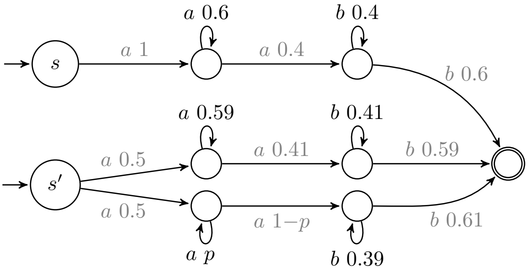
Theorem

Given a weighted automaton $\mathcal{W} = \langle Q, \Sigma, M, F \rangle$, $s, s' \in Q$ with $\mathcal{L}_s(\mathcal{W})$ and $\mathcal{L}_{s'}(\mathcal{W})$ are bounded, it is decidable whether s is big-O of s', subject to Schanuel's conjecture.

[3] Refined definition, explanation and example available: `https://ncatlab.org/nlab/show/Schanuel%27s+conjecture`

Boundedness Matters: Relative order is not sufficient

Different from the *unary* case: even if Relative orderings are the same, the boundness questions are different. For example,



$p \leq 0.62$

$$\begin{cases} f_s(a^m b^n) & = \Theta(0.6^m 0.4^n) \\ f_{s'}(a^m b^n) & = \Theta(p^m 0.39^n + 0.59^m 0.41^n) \end{cases}$$

Big-O problem instance

When $n, m \to \infty$, does $f_s(a^m b^n)/f_{s'}(a^m b^n) \to \infty$ always hold?

Language Boundedness

Let $L \subseteq \Sigma'$:

bounded

L is bounded if $L \subseteq w_1^* w_2^* \cdots w_m^*$ for some $w_1, w_2, \cdots, w_m \in \Sigma^*$

- Constructed by repeating some patterns

letter-bounded

L is letter-bounded if $L \subseteq a_1^* a_2^* \cdots a_m^*$ for some $a_1, a_2, \cdots, a_m \in \Sigma$

- Constructed by repeating some letters

plus-letter-bounded

L is plus-letter-bounded if $L \subseteq a_1^+ a_2^+ \cdots a_m^+$ for some $a_1, a_2, \cdots, a_m \in \Sigma$

- Constructed by repeating certain letters ≥ 1

Language Boundedness

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plus-letter-bounded **Conditional Decidable!**

L is plus-letter-bounded if $L \subseteq a_1^+ a_2^+ \cdots a_m^+$ for some $a_1, a_2, \cdots, a_m \in \Sigma$

- Constructed by repeating certain letters ≥ 1

Language Boundedness

Let $L \subseteq \Sigma'$:

bounded

L is bounded if $L \subseteq w_1^* w_2^* \cdots w_m^*$ for some $w_1, w_2, \cdots, w_m \in \Sigma^*$

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letter-bounded Conditional Decidable!

L is letter-bounded if $L \subseteq a_1^* a_2^* \cdots a_m^*$ for some $a_1, a_2, \cdots, a_m \in \Sigma$

- Constructed by repeating some letters

$\Downarrow$ Reduce

plus-letter-bounded Conditional Decidable!

L is plus-letter-bounded if $L \subseteq a_1^+ a_2^+ \cdots a_m^+$ for some $a_1, a_2, \cdots, a_m \in \Sigma$

- Constructed by repeating certain letters ≥ 1

Language Boundedness

Let $L \subseteq \Sigma'$:

bounded Conditional Decidable!

L is bounded if $L \subseteq w_1^* w_2^* \cdots w_m^*$ for some $w_1, w_2, \cdots, w_m \in \Sigma^*$

- Constructed by repeating some patterns

$\Downarrow$ Reduce

letter-bounded Conditional Decidable!

L is letter-bounded if $L \subseteq a_1^* a_2^* \cdots a_m^*$ for some $a_1, a_2, \cdots, a_m \in \Sigma$

- Constructed by repeating some letters

$\Downarrow$ Reduce

plus-letter-bounded Conditional Decidable!

L is plus-letter-bounded if $L \subseteq a_1^+ a_2^+ \cdots a_m^+$ for some $a_1, a_2, \cdots, a_m \in \Sigma$

- Constructed by repeating certain letters ≥ 1

Conclusions

This paper

- keeps undecidability results of weighted automata Big-O problem in general
- for bounded languages,
  - the result depends on a conjecture from number theory
  - leaves open the exact borderline between decidability and undecidability

Natural directions in future work

- analogous problem for infinite words
- further analysis on ambiguity
  - e.g., is the big-O problem decidable for k-ambiguous weighted automata?
- extension to negative edge weights