Lecture Notes

Xinyi Li

Mar 29, 2024

We were proving Theorem 5.9 For all $S \in \underset{\sim}{C}$

$$dim(S) = \liminf_{w \to S} dim(w)$$

$\leq$ last time

To see that $dim(S) \geq \liminf_{w \to S} dim(w)$, let s' and s'' be rational numbers such that $s' > s'' > \liminf_{w \in S} dim(w)$. It suffices to show that there exist only $w \sqsubseteq S$ for which $dim(w) \leq s'$. Since $s'' > dim(S)$, there is an algorithmic s''-supergale d such that $S \in S^{\infty}[d]$. Define $d' : \mathcal{T} \to [0, \infty)$ by

$$d'(x) = \begin{cases} d(x) & \text{if } x \in \{0,1\}^* \\ (2^{s'-s''})d(w) & \text{if } x = w\square \in \{0,1\}^*\square \end{cases}$$

Then d' is an algorithmic s'-supergale, so if for each $s \in [0, \infty)$ we define $\tilde{d}^{(s)} : \mathcal{T} \to [0, \infty)$ by $\tilde{d}^{(s)}(x) = d'(x)$ for all $x \in \{0,1\}^*$, $\tilde{d}^{(s)}(x) = 2^{(s-s')}|x|d'(w)$, then the family $\tilde{d} = \{\tilde{d}^{(s)} | s \in [0, \infty)\}$ is an algorithmic termgale. It follows by optimality of $\underset{\sim}{d}_{\square}$ that there is a constant $\alpha > 0$ such that, for all $s \in [0, \infty)$ and all $w \in \{0,1\}^*$,

$$\underset{\sim}{d}_{\square}^{(s)}(w) > \alpha \tilde{d}^{(s)}(w\square)$$

Since $S \in S^{\infty}[d]$, there are infinitely many $w \sqsubseteq S$ such that $\alpha(2^{s'-s''})d(w) > 1$. For all such w, we have

$$\begin{aligned} \underset{\sim}{d}_{\square}^{(s')}(w) &\geq \alpha \tilde{d}^{(s')}(w\square) \\ &= \alpha d'(w\square) \\ &= \alpha(2^{s'-s''})d(w) \\ &> 1 \end{aligned}$$

whence $dim(w) \leq s'$. ■

Theorem (6.1)**.** There is a constant $c \in \mathbb{N}$ such that for all $w \in \{0,1\}^*$,

$$|K(w) - |w|dim(w)| \leq c$$

Proof. Let $\underset{\sim}{m}$ be an optimal algorithmic subprobability measure. [1] The key fact is that for all $w \in \{0,1\}^*$ and $s \in [0, \infty)$,

$$\begin{aligned} d[\underset{\sim}{m}](w\square) > 1 &\Leftrightarrow 2^{s|w\square|}\underset{\sim}{m}(w) > 1 \\ &\Leftrightarrow s > \frac{1}{1-|w|} \log \frac{1}{\underset{\sim}{m}(w)} \end{aligned}$$

so

$$dim_{d[\underset{\sim}{m}]}(w) = \frac{1}{1-|w|} \log \frac{1}{\underset{\sim}{m}(w)}$$

□

[1] $(K(x) \approx \log \frac{1}{\underset{\sim}{m}(x)})$