Lecture Notes

Xinyi Li

Feburary 16, 2024

Definition. Given a TM M, let $\hat{M}$ be a TM that does the following:

input $\pi \in \Sigma^*$, dovetail computation of $M(\sigma)$ for all $\sigma \in \Sigma^*$ (in a fixed dovetailing order that does not depend on π) with $M(\sigma)\downarrow$ for some σ that $\sigma \sqsubseteq \pi$ or $\pi \sqsubseteq \sigma$. If $\sigma = \pi$ then halt, else run forever.

Observation. This construction has the following two properties:

1. $\hat{M}$ is a prefix TM.
2. If M is a prefix TM, then for all $\pi \in \Sigma^*$, $\hat{M}(\pi) \cong M(\pi)$

Hence the sequence of TMs $\hat{M}_0, \hat{M}_1, \ldots$ is an enumeration of all prefix TMs.
We call this the standard enumeration of prefix TMs.

Definition. A universal prefix TM is a prefix TM $\hat{U}$ such that for all $n \in \mathbb{N}$ and for all $\pi \in \Sigma^*$, $\hat{U}(0^n|\pi) \cong \hat{M}_n(\pi)$.

Definition. Let M be a TM,

1. for each $x \in \Sigma^*$, the set of programs for n on M is
$$PROG_M(x) = \{\pi \in \Sigma^* | M(\pi) = x\}$$
2. The set of valid programs on M is
$$PROG_M = \{\pi \in \Sigma^* | M(\pi)\downarrow\}$$

Observation. For every TM M, $PROG_M = \cup_{x\in\Sigma^*} PROG_M(x)$ and this is a union of disjoint sets $PROG_M(x)$.

A prefix TM is a TM M for which $PROG_M$ is a prefix set.

Definition. 1. The Kolmogorov complexity of a string $x \in \Sigma^*$ is
$$K_M(x) = \min\{|\pi| | \pi \in PROG_M(x)\}$$
where $\min \emptyset = \infty$.

2. The Kolmogorov complexity of a string $x \in \Sigma^*$ is
$$K(x) = K_{\hat{U}}(x)$$

Notation. $PROG(x) = PROG_{\hat{U}}(x)$ and $PROG = PROG_{\hat{U}}$ Note, then, that $K(x) = \min\{|\pi| | \pi \in PROG(x)\}$

Theorem 0.1. *(Optimality Theorem): For each prefix TM M, there is a optimality constant c_M such that for all $x \in \Sigma^*$, $K(x) \le K_M(x) + c_M$.*