Extension Complexity

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Given a polytope $P \subseteq \mathbb{R}^n$ as a convex hull of finitely many points (called vertices) in $\mathbb{R}^n$, by Minkowski-Weyl theorem, it is equivalent to a bounded polyhedron Q that is described by a system of linear equalities. Let $|Q|$ denote the number of linear inequalities in Q's description. Note that there might be different system of linear inequalities describing the same polytope, hence $|Q|$ depends on the given Q. If for a given polytope P, we can find a description Q such that $|Q| = poly(n)$, hence using Ellipsoid method for linear programming, we can solve any linear optmization problem over P in polynomial time. On the other hand, if the description Q we find for P has exponential size, e.g., $|Q| = 2^n$, then we cannot solve linear optimization problems over P *directly* in polynomial time using the description Q. Now there are two directions to investigate:

(1) Find another description $Q' \subseteq \mathbb{R}^n$ with polynomially many inequalities;

(2) Find a higher dimension polytope (bounded polyhedron) $H \subseteq \mathbb{R}^k$ (where $k > n$) such that it projects to P, and H has a $poly(n)$ description.

In some cases, it can be shown that (1) is not possible by showing that any description in $\mathbb{R}^n$ has to use exponential size, hence we can only turn to (2). Note that any optmization problem over P can be done by optimizing the same objective function over H.

Definition 1. *Let $P \subseteq \mathbb{R}^n$ be a polytope, a polytope $H \subseteq \mathbb{R}^k$ in a higher dimension space is called an* extended formulation *of P if $\pi(H) = P$, where $\pi : \mathbb{R}^k \to \mathbb{R}^n$ is the projection map. The* extension complexity *of P, denoted by $\mathrm{xc}(P)$, is defined to be:*

$$\mathrm{xc}(P) := \min_{Q:\ Q \text{ is an extended formulation of } P} |Q|.$$

We say P has a compact formulation *if $\mathrm{xc}(P) = poly(n)$.*

Given a polytope $P := \{x \in \mathbb{R}^n : Ax \leq b\}$, an extended formulation H might be described by $H := \{(x, y) : Bx + Cy \leq d\}$ for some appropriate B, C and d.

Obviously, linear optimization problems over polytopes with compact formulation can be solved in polytime using polytime algorithm (e.g., ellipsoid method or interior point method) for linear programming. On the other hand, for polytopes with exponential extension complexity, we cannot simply use polytime algorithm for LP to solve linear optimization problems over the polytopes. Note, $\mathrm{xc}(P)$ is exponential with respect to n DOES NOT mean linear optmization problems over P cannot be solved in polytime: it means for each specific linear optimization problem over P, we might have to invent a specific algorithm hopefully in polytime, and perhaps couldn't hope for a general polytime solver for all different problems.

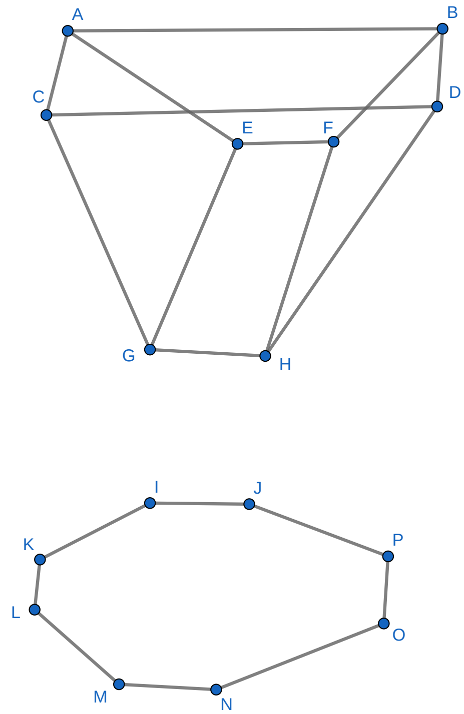

Figure 1: An extended formulation that has less faces

As an example, let P be the matching polytope for K_n (the complete n-node graph), it is recently famously proved by Rothvoss in [Rot17] that $\mathrm{xc}(P) = 2^{\Omega(n)}$. However, Edmonds' *combinatorial* blossom algorithm can solve the maximal matching problem in polytime. On the other hand, for the travelling salesman problem, the underlying polytope Q is the polytope as the convex hull of all Hamiltonion cycles in K_n, Rothvoss's result also implies $\mathrm{xc}(Q) = 2^{\Omega(n)}$, and the travelling salesman problems is **NP**-complete, i.e., it is generally believed that it does not have *any* polytime algorithm.

Next we see two concrete examples that a polytope $P \subseteq \mathbb{R}^n$ has exponential size in $\mathbb{R}^n$ but adopts a compact extended formulation in a higher dimension.

The first example is the so-called *permutahedron* polytope P_{perm}. Let

$$P_{\text{perm}} := \mathrm{conv}\{x \in \mathbb{R}^n : x = (x_1, x_2, \ldots, x_n) \text{ is a permutation of } (1, 2, \ldots, n)\}.$$

It is known that P_{perm} has exponential size in $\mathbb{R}^n$. Let us now see an extended formulation for P that has polynomial size. Let $Y = (Y_{ij})_{n\times n}$ denote an $n \times n$ matrix, and 1 denote the column vector in $\mathbb{R}^n$ with each entry being 1, and $u = (1, 2, 3, \ldots, n)^T \in \mathbb{R}^n$. Defined a polyhedron Q to be the set of all $(X, Y) \in \mathbb{R}^{n+n^2}$ such that

$$\begin{cases} Y & \geq 0, \\ Y1 & \leq 1, \\ 1^T Y & \leq 1^T, \\ X & = Yu. \end{cases}$$

One can show that $\pi(Q) = P_{\text{perm}}$. This shows $\text{xc}(P_{\text{perm}}) \leq |Q| \leq n^2 + 2n = O(n^2)$. Recently Goemans in [Goe15] determined $\text{xc}(P_{\text{perm}}) = \Theta(n \log n)$.

For the second example, consider the spanning tree polytope P_{span},

$$P_{\text{span}} := \text{conv}\{\chi_T \in \mathbb{R}^{\binom{n}{2}} : T \text{ is a spanning tree of } K_n\}.$$

Edmonds in [Edm71] characterized the spanning tree polytope as all $x \in \mathbb{R}^{\binom{n}{2}}$ satisfying the following system of linear inequalities,

$$\begin{cases} x & \geq 0, \\ \sum_{e \in E} x_e & = n-1, \\ x(\gamma(S)) & \leq |S| - 1, \quad \forall\ S \subseteq V,\ 2 \leq |S| \leq n-1, \end{cases}$$

where $\gamma(S)$ denotes the set of edges lying inside S. And this has exponential size. Now let us give a compact formulation[1] for P_{span}. For each edge $\{v, w\} \in E$, create a variable $x_{\{v,w\}}$. For each ordered triple of nodes (v, w, u), create a variable $z_{v,w,u}$. In total we have $\binom{n}{2} + n(n-1)(n-2)$ variables. Imagine that the vector x is the characteristic vector of a spanning tree. And imagine $z_{v,w,u}$ is interpreted as follows:

$$z_{v,w,u} = \begin{cases} 1, & x_{\{v,w\}} = 1, \text{ and } u \text{ lies in the component of } w \text{ if we remove the edge } \{v,w\} \text{ from the tree}, \\ 0, & \text{otherwise}. \end{cases}$$

Now consider the polyhedron Q defined as the set of $(x, z) \in \mathbb{R}^{\binom{n}{2}+n(n-1)(n-2)}$ satisfying the following set of inequalities,

$$\begin{cases} x & \geq 0, \\ z & \geq 0, \\ x_{\{v,w\}} & = z_{v,w,u} + z_{w,v,u}, \quad \forall\ (v,w,u), \\ x_{\{v,w\}} + \sum_{u \in V - \{v,w\}} z_{v,u,w} & = 1, \quad \forall\ \{v,w\}. \end{cases}$$

One can check that χ_T for any spanning tree T satisfies the above inequalities with z interpreted as above. And it can be shown that $\pi(Q) = P_{\text{span}}$. Hence $\text{xc}(P_{\text{span}}) \leq |Q| = O(n^3)$.

Next we prove a fundamental theorem concerning extension complexity of polytopes due to Yannakakis [Yan91].

Given a polytope $P \subseteq \mathbb{R}^n$, suppose it is described by $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ where $A_{m \times n}$ is a matrix and $b \in \mathbb{R}^m$ is a vector. Let the set of vertices of P be $\{z_1, z_2, \ldots, z_k\}$. The *slack matrix* of P under the above description is defined to be the matrix

$$S_{m \times k} := (b - Az_1, b - Az_2, \ldots, b - Az_k) \in \mathbb{R}^{m \times k}_{\geq 0}.$$

Definition 2. *Given a matrix $S_{m \times k}$, its nonnegative rank, denoted by $\text{rank}_+(S)$, is defined to be the least integer $r \in \mathbb{N}$, such that $S = L_{m \times r} R_{r \times k}$ where L and R are two nonnegative matrices.*

Yannakakis discovered that this notion exactly characterizes the extension complexity of a polytope!

[1]This example is taken from [Kai11], where more examples are discussed.

Theorem 1 (Yannakakis, [Yan91]). *For any polytope $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ whose dimension ≥ 1, let S be its slack matrix, one has*

$$\mathrm{xc}(P) = \mathrm{rank}_+(S).$$

We need a lemma.

Lemma 1. *Let $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ whose dimension ≥ 1, if $c^T x \leq \delta$ for all $x \in P$, then there exists $y \in \mathbb{R}^m$ such that*

$$\begin{cases} y & \geq 0, \\ y^T A & = c^T, \\ y^T b & = \delta. \end{cases}$$

Proof sketch. Consider the linear programming $\max c^T x$ where $x \in P$. Look at its dual, which is $\min y^T b$ such that

$$\begin{cases} y & \geq 0, \\ y^T A & = c^T. \end{cases}$$

Let Q denote the above polyhedron. The condition implies $\max_{x \in P} c^T x \leq \delta$. By strong duality (assume it applies), one has $\min_{y \in Q} y^T b = \max_{x \in P} c^T x \leq \delta$. Now note that the second set of constraints in Q is an linear equation of matrices. One can solve it, and the result will be some coordinates of y are expressed by some other *free* coordinates. For those free coordinates, we have no furher constraints except $y \geq 0$. Hence we can increase those free coordinates as we like. This shows there must exist some $y \in Q$ such that $y^T b = \delta$. $\square$

Proof of Theorem 1. The $\leq$ direction: Let $r = \mathrm{rank}_+(S)$, and let the set of vertices of P be $\{z_1, z_2, \ldots, z_k\}$. By definition,

$$S_{m \times k} := (b - Az_1, b - Az_2, \ldots, b - Az_k) = L_{m \times r} R_{r \times k}, \tag{1}$$

where L and R are two nonnegative matrices.

We need to find an extended formulation Q of P such that $|Q| \leq r$.

Consider Q to be the set of $(x, y) \in \mathbb{R}^{n+r}$ satisfying the following set of inequalities

$$\begin{cases} y & \geq 0, \\ Ax + Ly & = b. \end{cases} \tag{2}$$

Obviously $|Q| = r$. It suffices to show $\pi(Q) = P$.

Firstly to show $P \subseteq \pi(Q)$. It suffices to show $z_i \in \pi(Q)$, and the inclusion will follow since P is a convex hull of these vertices and $\pi(Q)$ is convex. By (1), one has

$$b - Az_j = LR_j,$$

where $R_j \in \mathbb{R}^r_{\geq 0}$ is the j-th column of the matrix R. Equivalently, one has

$$Az_j + LR_j = b,$$

showing $(z_j, R_j) \in Q$, hence $z_j \in \pi(Q)$.

Next to show $P \supseteq \pi(Q)$. It suffices to show for any $x' \notin P$, there does not exist $y \in \mathbb{R}^r$ such that $(x', y) \in Q$. Indeed, since $x' \notin P$ there exists at least one inequality in $Ax \leq b$ that is violated by this x', so say the i-th inequality is violated by x', then

$$A_i x' > b_i,$$

where A_i denotes the i-th row of A. Note L is nonnegative, hence for any $y \in \mathbb{R}^r_{\geq 0}$, one has $Ly \geq 0$, hence

$$A_i x' + L_i y > b_i,$$

where L_i denotes the i-th row of L. This shows that there does not exists $y \in \mathbb{R}^r_{\geq 0}$ such that $Ax' + Ly = b$.

The $\geq$ direction: given $Q := \{(x, y) : Bx + Cy \leq d\}$ be an extended formulation of P such that $|Q| = r$, we need to construct a decomposition for S with rank not greater than r. Note that the condition $|Q| = r$ means both B and C have r rows, so suppose $B_{r\times n}$, and $C_{r\times l}$, hence $Q \subseteq \mathbb{R}^{n+l}$.

Let A_i be the i-th row of A, consider now the objective functoin $(A_i, 0)\begin{pmatrix} x \\ y \end{pmatrix} = A_i x$ over Q, since $\pi(Q) = P$, and $A_i x \leq b_i$ for all $x \in P$, one has

$$(A_i, 0)\begin{pmatrix} x \\ y \end{pmatrix} \leq b_i$$

for all $(x, y) \in Q$. Note we view x and y as column vectors or row vectors depending on the context. By Lemma 1, this implies there exists a vector $L_i \in \mathbb{R}^r$ such that

$$\begin{cases} L_i & \geq 0, \\ L_i(B, C) & = (A_i, 0), \\ L_i d & = b_i. \end{cases} \tag{3}$$

Since $z_j \in P$ and $\pi(Q) = P$, there exists y_j such that $(z_j, y_j) \in Q$. Define

$$R_j = d - Bz_j - Cy_j.$$

Then $R_j \in \mathbb{R}^r_{\geq 0}$ is nonnegative. One then has

$$L_i R_j = L_i(d - Bz_j - Cy_j) = L_i d - L_i B z_j - 0 = b_i - A_i z_j = S_{ij}.$$

Let

$$L = \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_m \end{pmatrix}, \quad R = \begin{pmatrix} R_1 & R_2 & \cdots & R_k \end{pmatrix}.$$

By our choice of L_i and R_j, we know L and R are both nonnegative matrices. Hence the previous equation showed $S = L_{m\times r} R_{r\times k}$, in particular, it shows $\operatorname{rank}_+(S) \leq r$. $\square$

This beautiful theorem switches the extension complexity to the (maybe more concrete) notion of nonnegative rank of the slack matrix, which is defined solely in terms of P itself. It is a fundamental tool to study extension complexity. Recently, [FFGT12] showed based on Yannakakis' theorem that extension complexity can be characterized as communication complexity of the slack matrix, reveralled a further connection co communication complexity.

References

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