McGill University — COMP/MATH 552 (Winter 2018) HW3

Due date: Apr. 10th, 2018

Instructions(important!): You are allowed to work with your classmates on solving assignments, but you *must* write your *own* solutions independently.

If you read a solution from other resources (e.g. books, articles, ...), you should *cite* each one of the resources. In this case, try to *understand* the solution first, and then write down your solution without looking at the resources.

Present your solution in a *clear* and *rigorous* way. Clarity is as important as correctness. You will lose points for a correct solution that is poorly presented.

If you can not find a solution of a problem, do not answer it, this will earn you 20% of the points. If you have some non-trivial yet incomplete ideas to solve a problem, write them down and indicate the gaps.

Submit your solution in class or to myCourses. Late submission will not be accepted.

Problems (each worth 25 points):

1. (Extension complexity) Given $P = \{x \in \mathbb{R}^n : A_{m\times n}x \leq b\} \subseteq \mathbb{R}^n$ to be a polytope with vertices $\{z_1, z_2, \ldots, z_k\}$, we defined its associated slack matrix $S_{m\times k}$ in class. This problem connects extension complexity to communication complexity through Yannakakis' theorem $\mathrm{xc}(P) = \mathrm{rank}_+(S)$.

   Given a matrix $S_{m\times k}$ known to two players Alice and Bob, suppose Alice is given a row index $i \in [m]$ and Bob is given a column index $j \in [k]$, and their task is to compute $S(i,j)$. Obviously they need to communicate to each other to solve this problem. A communication protocol is a binary tree, in which each node is associated with a player's name and a strategy with which the player uses to send a bit to the other player. At the end of the protocol both Alice and Bob should know the value $S(i,j)$ correctly. Let $\mathrm{CC}(S_{m\times k})$ denote the minimal depth of such communication protocol trees that compute $S(i,j)$ correctly for all possible $(i,j) \in [m] \times [k]$. Show

   $$\mathrm{CC}(S_{m\times k}) \geq \log \mathrm{rank}_+(S_{m\times k}).$$

   Hence $\mathrm{CC}(S_{m\times k}) \geq \log \mathrm{xc}(P)$. This result is also due to Yannakakis.

   Hint: consult necessary material on definition of communication complexity if the above explanation is insufficient.

2. (Matroid) Let X be a finite set and M_1, M_2 be two matroids on X. In class we studied the Edmonds' algorithm for finding max-cardinality independent set in the intersection of M_1 and M_2. In the algorithm suppose we are now having an intermediate set $Y \in M_1 \cap M_2$, the algorithm goes by constructing sets $X_1 := \{x \in X : Y + \{x\} \in M_1\}$ and $X_2 := \{x \in X : Y + \{x\} \in M_2\}$, and then checks if there are directed paths from X_1 to X_2 in the directed graph $H(M_1, M_2, Y)$. In the case when there is none, let $U \subseteq X$ be the set of vertices that can not be reached from X_1, we showed $|Y \cap U| = \mathrm{rank}_{M_1}(U)$. Show

   $$|Y - U| = \mathrm{rank}_{M_2}(X - U).$$

This concludes $|Y| = \mathrm{rank}_{M_1}(U) + \mathrm{rank}_{M_2}(X - U)$, as well as the duality characterization

$$\max_{Y \in M_1 \cap M_2} |Y| = \min_{U \subseteq X} \Big(\mathrm{rank}_{M_1}(U) + \mathrm{rank}_{M_2}(X - U)\Big).$$

Hint: consult references of Lecture 13.

3. (Matroid) Let X be a finite set with $|X| = n$, let M_1, M_2 be two matroids on X. The *matroid intersection polytope* is defined to be $\mathrm{conv}(T)$ where

$$T := \{\chi_A \in \{0,1\}^n : A \in M_1 \cap M_2\}.$$

Let $P \subseteq \mathbb{R}^n$ be a polyhedron defined by the following system of constraints:

$$\begin{cases} x & \geq 0; \\ x(Y) & \leq \mathrm{rank}_{M_1}(Y), \quad \forall\, Y \subseteq X; \\ x(Y) & \leq \mathrm{rank}_{M_2}(Y), \quad \forall\, Y \subseteq X. \end{cases}$$

Show that $\mathrm{conv}(T) = P$.

4. (Submodular functions) Let $f : 2^V \to \mathbb{R}^+$ be a submodular function and suppose it is monotone: for any $A \subseteq B \subseteq V$, one has $f(A) \leq f(B)$, suppose further $f(\emptyset) = 0$. Given $A \subseteq V$ and $j \in V - A$, the *marginal value* of j with respect to A is defined to be $f_A(j) := f(A \cup \{j\}) - f(A)$. Define the *total curvature* of f, denoted by c, to be

$$c = 1 - \min_{\substack{A \subseteq V \\ j \in V - A}} \frac{f_A(j)}{f(j)}.$$

Show:

(1). $c \in [0,1]$. Moreover, $c = 0$ if and only if f is additive.

(2). For any $j \in V$, and $A, B \subseteq V - \{j\}$, one has

$$f_B(j) \geq (1 - c) f_A(j).$$

(3). Suppose $|V| = n$. Let f^L and f^M denote the Lovász extension and multilinear extension, respectively. Then for all $x \in [0,1]^n$, one has

$$f^L(x) \leq f^M(x) \leq \frac{1}{1-c} f^L(x).$$