McGill University — COMP/MATH 552 (Winter 2018) HW2

Due date: Mar. 13th, 2018

Instructions(important!): You are allowed to work with your classmates on solving assignments, but you *must* write your *own* solutions independently.

If you read a solution from other resources (e.g. books, articles, ...), you should *cite* each one of the resources. In this case, try to *understand* the solution first, and then write down your solution without looking at the resources.

Present your solution in a *clear* and *rigorous* way. Clarity is as important as correctness. You will lose points for a correct solution that is poorly presented.

If you can not find a solution of a problem, do not answer it, this will earn you 20% of the points. If you have some non-trivial yet incomplete ideas to solve a problem, write them down and indicate the gaps.

Submit your solution in class or to myCourses. Late submission will not be accepted.

Problems (each worth 20 points, choose at least 3 problems from 1-4, at least 2 problems from 5-8 to solve):

1. Let $D = (V, A)$ be a directed graph. A weight function $x : A \to \mathbb{R}$ is called a *circulation* on D if
$$\sum_{a \in \delta^{\text{in}}(v)} x(a) = \sum_{a \in \delta^{\text{out}}(v)} x(a), \quad \forall\, v \in V.$$
That is, the conservation law holds for *every* vertex of V. Given two weighted functions $f, g : A \to \mathbb{R}$, we say $f \leq g$ if $f(a) \leq g(a)$ for every $a \in A$. Prove the following theorem.

   Theorem 1 (Hoffman's circulation theorem)**.** *Let $l, u : A \to \mathbb{R}$ be such that $l \leq u$. Then there exists a circulation x such that $l \leq x \leq u$ if and only if*
$$\sum_{a \in \delta^{in}(S)} l(a) \leq \sum_{a \in \delta^{out}(S)} u(a), \quad \forall\, S \subseteq V.$$
Hint: use corollary of Farkas' lemma, and the incidence matrix of a directed graph is totally unimodular.

2. Let $A_{m\times n} \in \mathbb{Z}^{m\times n}$ be an integer matrix of rank m. Show: A is unimodular if and only if for every $b \in \mathbb{Z}^m$, the polyhedron
$$P = \{x \in \mathbb{R}^n : x \geq 0; Ax = b\}$$
is integer.

3. Let G be a k-regular graph where $k \geq 1$.
   (1) Using König's theorem to show if G is bipartite then G has a perfect matching.
   (2) For each $k \geq 2$, give an example of a k-regular graph G without a perfect matching.

4. Let $G = (V, E)$ be a tree. Show

   (1) G has at most one perfect matching;

   (2) G has a perfect matching if and only if for every $v \in V$, the subgraph $G - v$ has exactly 1 odd component.

5. Recall we gave two proofs of Tutte-Berge formula. In the 1st proof (Lovász's proof), we showed the following

   Theorem 2. *If G is connected and has no essential vertices, then $2\nu(G) = |V(G)| - 1$.*

   In the 2nd proof (shrinking the odd cycle), we used the following lemma:

   Lemma 1. *Let $G = (V, E)$ be a graph, $u, v \in V$ are two vertices such that $uv \in E$. If u, v are both inessential, then there exists a tight odd cycle containing the edge uv. Moreover, $\{C\}$ is an inessential node in $G \times C$.*

   Using Lemma 1 to show Theorem 2.

6. Given a graph $G = (V, E)$, let B be the set of inessential vertices of G, let C be the set of vertices that are not in B but adjacent to at least one vertex in B, and D denote the set of remaining vertices. The partition $V = B \sqcup C \sqcup D$ is called the *Gallai-Edmonds partition*. Show the set C in the partition is a minimizer in the Tutte-Berge formula.

7. Given a graph G, let **OPT** denote the maximal weight of a perfect matching in G. We want to design simple algorithms giving good approximation to **OPT**. Given an algorithm $\mathcal{A}$ that solves, maybe *approximately*, the maximum-weight perfect matching problem on G, let $\mathbf{ALG}_{\mathcal{A}}$ denote the weight output by $\mathcal{A}$. An algorithm $\mathcal{A}$ is said to have an α-approximation if $\frac{\mathbf{ALG}_{\mathcal{A}}}{\mathbf{OPT}} \geq \alpha$.

   Consider a weighted complete bipartite graph G constructed as follows: given $2n$ distinct points (n red and n black) in $\mathbb{R}^k$ where $k \geq 2$, denote by R and B the set of red and black points, respectively. Let $G = (R \cup B, E)$ be the complete bipartite graph where the edge weights are given by the Euclidean distance in $\mathbb{R}^k$. Show for this graph G:

   (1) Choosing a perfect matching uniformly at random gives a $\frac{1}{3}$- approximation in expectation.

   (2) Choosing edges greedily until we get a perfect matching gives a $\frac{1}{2}$-approximation.

8. Use the algorithm we studied in class to find a maximum matching for the graph G in Fig.1.

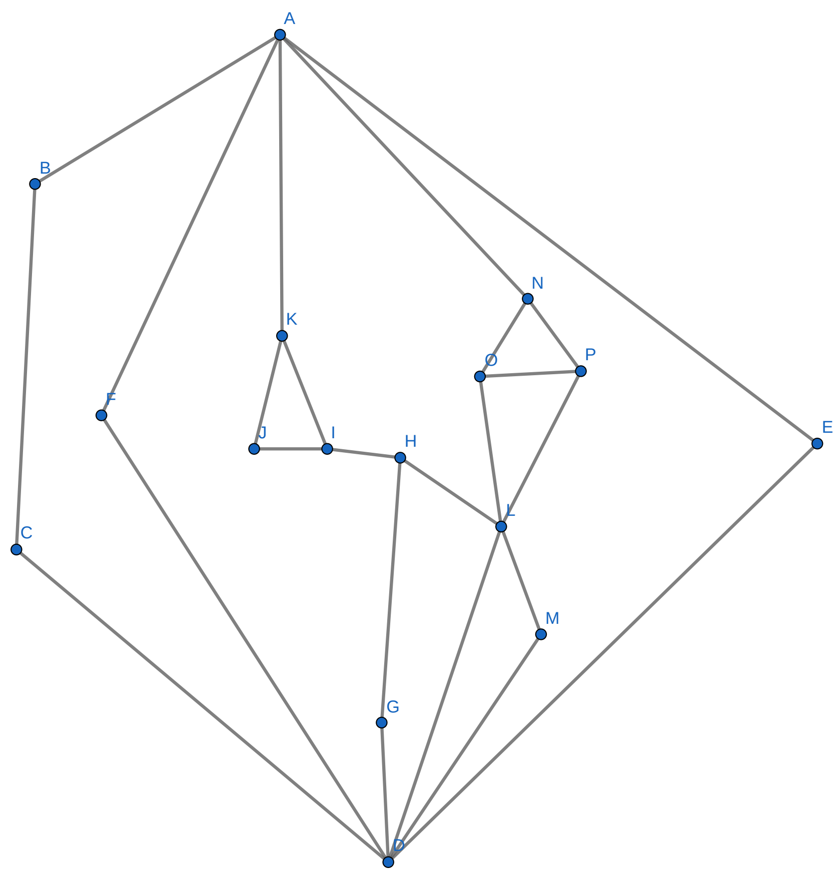

Figure 1: A graph G