McGill University — COMP/MATH 552 (Winter 2018) HW1

Due date: Feb. 13th, 2018

Instructions(important!): You are allowed to work with your classmates on solving assignments, but you *must* write your *own* solutions independently.

If you read a solution from other resources (e.g. books, articles, ...), you should *cite* each one of the resources. In this case, try to *understand* the solution first, and then write down your solution without looking at the resources.

Present your solution in a *clear* and *rigorous* way. Clarity is as important as correctness. You will lose points for a correct solution that is poorly presented.

If you can not find a solution of a problem, do not answer it, this will earn you 20% of the points. If you have some non-trivial yet incomplete ideas to solve a problem, write them down and indicate the gaps.

Submit your solution in class or to mashbat.suzuki@mail.mcgill.ca. Late submission will not be accepted.

Problems (each worth 20 points, you are free to choose 5 out of 8 problems to answer):

1. (1) In the class we showed the hyperplane separation theorem assuming the convex set in question is bounded. What if the convex set is not bounded?

   (2) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function. If $f(x) = A_{m\times n}x$ where $A_{m\times n}$ is a matrix, show that f transforms any convex sets in $\mathbb{R}^n$ to convex sets in $\mathbb{R}^m$.

   (3) [5 points *extra*, for fun] Can you find another continuous function with the property in (2), i.e., that tranforms any convex sets to other convex sets?

2. Prove that if $P = \{(x, y) \in \mathbb{R} \mid y \leq \sqrt{2}x\}$ then P_I is not a polyhedron, where P_I is the convex hull of integral points in P.

3. Prove the strong duality theorem of LP using Farkas' Lemma.

4. Given an undirected graph $G = (V, E)$ where $|V| = n$, Max Clique problem is to find a set $S \subseteq V$ of maximum cardinality such that if $u, v \in S$ then $(u, v) \in E$.

   (1) Write an integer program to model the Max Clique problem.

   (2) Show that the integrality gap of Max Clique is $\Omega(n)$.

5. Recall given a set $S \subseteq \mathbb{R}^2$, the polar of S, denoted by S^*, is defined as

$$S^* := \{x \in \mathbb{R}^2 : \langle x, y\rangle \leq 1, \forall\, y \in S\}.$$

   In the proof that a polytope is a bounded polyhedron, we assumed without loss of generality the polytope P is of full dimension and $0 \in P$, and showed P^* is also a polytope, and furthermore $P^{**} = P$.

   The following exercise is to understand more details of that proof.

Let P be the square polytope in $\mathbb{R}^2$ with the set S of vertices

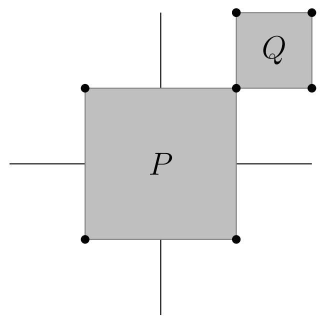
$$S := \{(-1,-1), (-1,1), (1,1), (1,-1)\}.$$

Let Q be the square polytope with the set T of vertices

$$T = \{(1,1), (1,2), (2,2), (2,1)\}$$

See Figure 1. Let $R = [-1, 1]$ be the line segment, viewed as a polytope in $\mathbb{R}^2$ with dimension 1.

Figure 1: Two square polytopes in $\mathbb{R}^2$



(1) Geometrically, what is P^*? Verify that $P^{**} = P$.

(2) What is R^*? Is R^* a polytope? Is $R^{**} = R$?

(3) What is Q^*? Is Q^* a polytope? Is $Q^{**} = Q$?

When you answer this problem, it would be nice to draw the corresponding pictures.

6. (1) Give an example of a matrix $A_{m\times n}$, vectors $b \in \mathbb{R}^m, c \in \mathbb{R}^n$, such that the two polyhedra $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ and $Q := \{y \in \mathbb{R}^m : y \geq 0; y^T A = c^T\}$ are both empty.

   (2) Give an example of a matrix $A_{m\times n} \in \{0, 1, -1\}^{m\times n}$ and an integer vector $b \in \mathbb{Z}^m$ such that the polyhedron $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ is integer; but the matrix A is not totally unimodular.

7. We studied the relation between incidence matrices of a graph and total unimodularity. Let us ask the same question for the adjacency matrix.

   Given a graph G with n vertices, the adjacency matrix A_G is an $n \times n$ matrix defined by $A_G(u, v) = 1$ if uv is an edge in G, and $A_G(u, v) = 0$ otherwise.

   (1) If G is a bipartite graph, is A_G totally unimodular? Give a proof if your answer is yes, and a counter example if your answer is no.

   (2) If A_G is totally unimodular, what can you say about the graph G?

8. We know a polytope is equivalent to a bounded polyhedron. A polytope is described by a set of k points, and a bounded polyhedron is described by a set of r inequalities. Study the following two questions.

   (1) Given P to be a bounded polyhedron, let $k(P)$ denote the number of vertices of P. Let $S(r)$ denote the set of all bounded polyhedron described by r inequalities. Define
   $$k(r) := \sup_{P \in S(r)} k(P).$$
   Is it true that $k(r) = \Omega(2^r)$? Argue your claim.

   (2) Given P to be a polytope, let $r(P)$ denote the minimum number of inequalities needed to describe P. Let $T(k)$ denote the set of all polytopes with k vertices. Define
   $$r(k) := \sup_{P \in T(k)} r(P).$$
   Is it true that $r(k) = \Omega(2^k)$? Argue your claim.

   (3) [5 points *extra*, for fun] What is the point of studying the above two questions, does it have any algorithmic implication?