Submodular functions: basic examples

Yaqiao Li

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In this lecture we give definitions and basic examples.

Definition 1. *Given a finite set V, a function $f : 2^V \to \mathbb{R}$ is called submodular if for any $A \subseteq B \subseteq X$ and $x \notin B$,*

$$f(B \cup \{x\}) - f(B) \leq f(A \cup \{x\}) - f(A). \tag{1}$$

or equivalently, for any two subsets $X, Y \subseteq V$,

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y). \tag{2}$$

The condition in (1) is sometimes called the *diminishing return*: you get less return if you start from a larger set. This diminishing return property immediately suggests that many functions in economics and algorithmic game theory are submodular.

We showed (1) and (2) are equivalent.

Proof. (2) implies (1): trivial.

In the other direction, we showed that if a function is submodular, then (1) is still true if one replaces the element x with any subset S such that $S \cap B = \emptyset$, and this can be used to show (2). □

Recall in analysis we learned that a function g is concave if and only if $g'' \leq 0$. By checking the definition of g'' and comparing it with the definition of submodular functions, we found that submodular functions look like concave functions. This might suggest maximizing a submodular function is easy. However, later we will see the opposite is true: minimizing submodular functions are easier than maximizing, and submodular functions behaves like convex functions.

A few basic examples.

- The weight function: let $w : X \to \mathbb{R}$ be a weight on X, define $f(S) = \sum_{x \in S} w(x)$. Obviously f is submodular. Many combinatorial optimization problems are equivalent to find a set S maximizing $f(S)$, perhaps under some constraints.
- The cut capacity functions. See this example in [Iwa10]. The maximal flow problem is then equivalent to a problem of minimizing some submodular function.
- set cover function. See this example in [Iwa10]. Here one has a maximizing submodular function example.
- Entropy. See this example in [Iwa10].

One more important example is the matroid rank. It turns out that the matroid rank functions are submodular, and a submodular function (with some conditions) defines a matroid.

Theorem 1. *The following are true:*

(1). Given V to be a finite set, and M be a matroid defined on V. Then the rank function $r_M : 2^V \to \mathbb{N}$ is submodular.

(2). Given a submodular function $f : 2^V \to \mathbb{N}$, and suppose futher $f(S) \leq f(T)$ whenever $S \subseteq T \subseteq V$, and $f(S) \leq |S|$. Let

$$M := \{S \subseteq V : f(S) = |S|\}.$$

Then M is a matroid, and $r_M = f$.

Proof. Consider (1) first. Let us verify the condition (2). Pick $A \in M$ to be a maximal independent set lying in $X \cap Y$. Let $B \in M$ be an independent set such that B is maximal in $X \cup Y$ and $A \subseteq B$. Then $B \cap X$ is an independent set in X, and $B \cap Y$ is an independent set in Y. Hence

$$r_M(X) + r_M(Y) \geq |B \cap X| + |B \cap Y| \geq |B| + |A| = r_M(X \cup Y) + r_M(X \cap Y).$$

Consider (2). Firstly, by assumption,

$$0 \leq f(\emptyset) \leq |\emptyset| = 0 \Longrightarrow f(\emptyset) = |\emptyset| = 0.$$

That is $\emptyset \in M$.

Secondly, let $A \subseteq B$ and $B \in M$. Then

$$|B - A| + f(A) \geq f(B - A) + f(A) \geq f(B) + f(\emptyset) = |B| \Longrightarrow f(A) \geq |A|.$$

By assumption we also have $f(A) \leq |A|$. Hence $f(A) = |A|$, i.e., $A \in M$.

Lastly, to show for any $S \subseteq V$, and $A, B \in M$ are both maximal subsets lying inside S, then $|A| = |B|$. It suffices to show if $A \in M$ is maximal inside S, then $|A| = f(S)$. Let us use induction on the size of S. If $|S| = |A| + 1$, suppose $S = A \cup \{x\}$, then by the fact that $A \in M$ is a maximal subset inside S, one has $S \notin M$. By definition of M and our assumption on f, this implies

$$|A| = f(A) \leq f(S) < |S| = |A| + 1 \Longrightarrow f(S) = |A|.$$

Now the induction hypothesis says what we want to show is true for any S of size k, let us look at S of size $k+1$. Choose $x \in S \backslash A$, and let $X = S \backslash \{x\}$ and $Y = A \cup \{x\}$, then

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) = f(S) + f(A).$$

Now that since A is maximal in S, it must also be maximal in X and Y. By induction hypothesis, one has $f(X) = f(Y) = |A|$. Hence the above inequality implies $f(S) \leq |A|$, implying $f(S) = |A|$, becasue $f(S) \geq f(A) = |A|$ by assumption. □

References

[Iwa10] Satoru Iwata, *Submodular functions: Optimization and approximation*, Proceedings of the International Congress of Mathematicians 2010 (ICM 2010) (In 4 Volumes) Vol. I: Plenary Lectures and Ceremonies Vols. II–IV: Invited Lectures, World Scientific, 2010, pp. 2943–2963.