Learning the Uncertainty Sets for Control Dynamics via Set Membership: A Non-Asymptotic Analysis

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Abstract

Set-membership estimation is commonly used in adaptive/learning-based control algorithms that require robustness over the model uncertainty sets, e.g., online robustly stabilizing control and robust adaptive model predictive control. Despite having broad applications, non-asymptotic estimation error bounds in the stochastic setting are limited. This paper provides such a non-asymptotic bound on the diameter of the uncertainty sets generated by set membership estimation on linear dynamical systems under bounded, i.i.d. disturbances. Further, this result is applied to robust adaptive model predictive control with uncertainty sets updated by set membership. We numerically demonstrate the performance of the robust adaptive controller, which rapidly approaches the performance of the offline optimal model predictive controller, in comparison with the control design based on least square estimation's confidence regions.

1 Introduction

The problem of estimating unknown linear dynamical systems of the form $x_{t+1} = A^*x_t + B^*u_t + w_t$ with unknown parameters (A^*, B^*) has seen considerable progress recently, with many works deriving finite-time estimation bound for the unknown system matrices. Most of this literature focuses on the analysis of the least squares estimator (LSE) and its variants. Recently, modern machine learning techniques have been applied to obtain high-probability guarantees of the LSE in finite time [17, 51, 64, 39, 20, 52], where sharp bounds of the convergence rate for LSE were obtained [55, 21].

As a result, there is now a growing literature on "learning to control" unknown linear systems that leverages least-square *point estimation* (LSE), with extensive results for various control objectives such as stability [30, 25] and regret [39, 13, 29]. Such results pioneered modern machine learning techniques for adaptive and data-driven control design. However, in order to apply such learning-based control methods to safety-critical control applications where robust safety requirements such as robust stability and robust constraint satisfaction are commonly required, it is crucial to quantify the uncertainty of the estimated system. The use of an uncertainty set is central to important classes of control methods such as robust model predictive control (RMPC). Too large of an uncertainty set gives rise to conservative control actions, resulting in poor performance. If the uncertainty set is underestimated, the resulting controller may lead to unsafe behavior. Therefore, works have begun to incorporate the confidence region of the LSE for safe adaptive and learning-based control [57, 8, 33].

Parallel to least-square point estimation-based works, another line of research turns to more direct *uncertainty set estimation* methods in the bounded disturbance setting [42]. Among the algorithms for direct uncertainty set estimation, the set membership (SM) method, which identifies the set of systems models that are consistent with observed data, is one of the most commonly used. The SM method can generate much smaller uncertainty sets than the LSE confidence region, as demonstrated in Figure 1, by leveraging knowledge of the bound on the disturbances. Therefore, numerous robust

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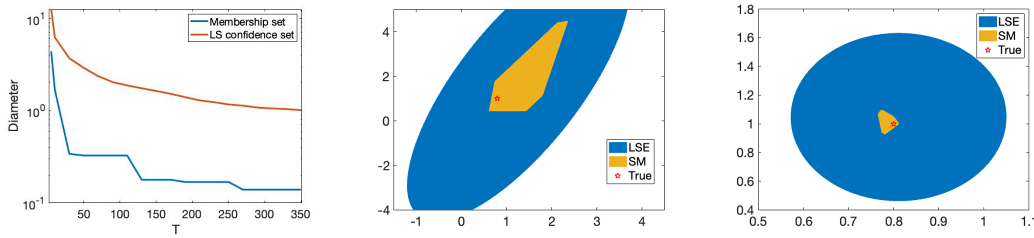


(a) Uncertainty sets' diameters (b) $T = 5$ (c) $T = 250$

Figure 1: Comparison of the diameters of the membership set of SM and the confidence set generated by LSE for $x_{t+1} = A^* x_t + B^* u_t + w_t$, where $A^* = 0.8$ and $B^* = 1$ are unknown. $w_t \sim \mathrm{TrunGauss}(0, \sigma_w, [-w_{\max}, w_{\max}])$ is i.i.d and $u_t \sim \mathrm{TrunGauss}(0, \sigma_u, [-u_{\max}, u_{\max}])$ are also i.i.d generated, where $\sigma_w = \sigma_u = 0.5$, $w_{\max} = u_{\max} = 1$. The confidence sets of LSE is computed via Theorem 1 in [2].

controller designs and empirical algorithm designs leverage SM for uncertainty set estimation. For example, RMPC algorithms such as [42, 3, 22, 56] have long been employing SM for performance improvement, which is sometimes called robust adaptive MPC and has seen a growing literature recently [36, 9, 68, 45, 44, 53]. In addition, SM has also been utilized to stabilize systems despite adversarial disturbances [23, 67, 65, 66].

On the theory side, there is a long history of theoretical analysis of the SM method for both the deterministic setting where w_t is a deterministic sequence and the stochastic setting where w_t is independent or even i.i.d. [5, 31, 4, 27, 6]. However, in the stochastic setting, most papers consider a simpler regression problem: $y_t = \theta^* x_t + w_t$ with a deterministic sequence of x_t, which does not capture the correlation between x_t and history $w_{t-1}, \ldots, w_0$ in the dynamical systems. This issue was largely overlooked in the vast empirical algorithm design research related to SM (for example, see [36, 28], etc.). Not until recently, this unsolved correlation problem on SM resurfaced. Work in [38] provides an initial attempt to establish convergence results on SM for dynamical systems; however, their approach requires a special design of u_t to satisfy persistent excitation requirements deterministically, which can be challenging to satisfy for general control designs in the stochastic settings. Therefore, an open question is:

Can the convergence and convergence rate results of SM on dynamical systems be established in the stochastic disturbance setting for general control designs?

Contributions. In this paper, to the best of our knowledge, we provide the first finite-time convergence rate bound for the set membership estimation error (Theorem 1) for linear dynamical systems with bounded i.i.d. disturbances. Our bound assumes that the control inputs u_t can guarantee the block-martingale small-ball (BMSB) condition in [55], which can be satisfied by adding i.i.d. noises to a general class of controllers as discussed in [33].

Under proper assumptions, we show that the estimation error decays to 0 at a rate of $\tilde{O}(n_x^{1.5}(n_x + n_u)^2/T)$ when the bound on the disturbances w_t is tight. Interestingly, this is better than the LSE's error bound $O(\frac{\sqrt{n_x+n_u}}{\sqrt{T}})$ in terms of the dependence on the number of samples T but worse in terms of the state and control dimensions n_x, n_u. SM's improvement on the rate with respect to T is enabled by taking advantage of the boundedness of w_t. Even when the bound on w_t is not accurately known but overly estimated, we can show that SM's estimation error converges to a neighborhood of 0 with the same decay rate.

Technically, our estimation error bound relies on a novel construction of an event sequence based on designing a sequence of stopping times. This construction, together with the BMSB condition in [55], addresses the challenge caused by the correlation between x_t, u_t, and the history disturbances. For more details, we refer the reader to the proof of Theorem 1.

Moreover, our results lay a foundation for future non-asymptotic analysis of control designs based on SM. To illustrate this, we discuss applications of our results to autonomous system identification and robust-adaptive model predictive control. In each case, we provide numerical comparisons with LSE that demonstrate the promise of SM-based approaches.

2 Problem Formulation and Preliminaries

This paper considers an uncertainty set identification problem for linear dynamical systems with unknown model parameters, i.e.,

$$x_{t+1} = A^* x_t + B^* u_t + w_t \tag{1}$$

where A^*, B^* are unknown, $x_t \in \mathbb{R}^{n_x}, u_t \in \mathbb{R}^{n_u}$. For notational simplicity, we define $\theta^* = (A^*, B^*)$ and $z_t = [x_t^\top, u_t^\top]^\top$, where we write (A, B) as the concatenation of two matrices A, B (the same applies to the vector concatenation). So the system (1) can also be written as $x_{t+1} = \theta^* z_t + w_t$. We define $n_z = n_x + n_u$.

The goal of the uncertainty-set identification problem is to determine a set Θ_T that contains the true parameters (A^*, B^*) based on a sequence of data $\{x_t, u_t, x_{t+1}\}_{t=0}^{T-1}$. Set Θ_T is called an uncertainty set since it captures the remaining uncertainty on the system model after the revelation of the data sequence $\{x_t, u_t, x_{t+1}\}_{t=0}^{T-1}$. Uncertainty sets play an important role in robust control, where one aims to achieve robust stability and/or robust constraint satisfaction for any model in the uncertainty set.[1] Therefore, the diameter of the uncertainty sets heavily influences the conservativeness of robust controllers; thus affecting the control performance. Formally, we define Additionally, we define the diameter as follows.

Definition 1 (Diameter of a set of matrices)**.** Consider a set $\mathbb{S}$ of matrices $\theta \in \mathbb{R}^{n_x \times n_z}$. We define the diameter of $\mathbb{S}$ in Frobenius norm as $\mathrm{diam}(\mathbb{S}) = \sup_{\theta, \theta' \in \mathbb{S}} \|\theta - \theta'\|_F$.

Set Membership. We focus on a widely adopted uncertainty set identification method: set membership, and provide non-asymptotic bounds for estimation error under proper assumptions.

In particular, we consider bounded disturbances, i.e., there exists a bounded set $\mathcal{W}$ such that $w_t \in \mathcal{W}$ for all $t \geq 0$. When $\mathcal{W}$ is known, the set membership method estimates the uncertainty set by

$$\Theta_T = \bigcap_{t=0}^{T-1} \{\hat{\theta} : x_{t+1} - \hat{\theta} z_t \in \mathcal{W}\}. \tag{2}$$

We also refer to (2) as the *membership set*. Notice that $\theta^* \in \Theta_T$ for all $T \geq 0$ for arbitrary sequence of $w_t \in \mathcal{W}$ without assuming any (stochastic) properties on w_t.

Ideally, one hopes Θ_T converges to the singleton of the true model $\{\theta^*\}$ or at least a small neighborhood of θ^*. This usually calls for additional assumptions, such as persistent excitation and additional stochastic properties on w_t. In this paper, we consider the following assumptions to establish convergence rate bounds on the diameter of Θ_T.

Assumption 1 (Bounded i.i.d disturbances)**.** The disturbances are box-constrained, $w_t \in \mathcal{W} = \{w \in \mathbb{R}^{n_x} : \|w\|_\infty \leq w_{\max}\}$ for all $t \geq 0$. Further, w_t is i.i.d.. Let μ_w and $\Sigma_w \succ 0$ denote the mean and covariance of w_t.

In many applications, it is realistic to assume the disturbances are bounded, e.g., wind perturbation for aerial robotics and temperature fluctuations for building control. Therefore, it is advantageous to leverage such prior knowledge about the disturbances for system estimation and control design beyond the sub-Gaussian stochasticity assumption.

Next, we introduce the assumptions on u_t, which relies on the block-martingale small-ball condition proposed in [55]. It can be shown that the block-martingale small-ball condition can guarantee persistent excitation with high probability under proper conditions (see [55, Proposition 2.5] and Lemma 1). In the system identification literature, the persistent excitation (PE) condition is essential for ensuring that the estimation error decays. Notice that PE essentially requires that $(x_t^\top, u_t^\top)$ explores all directions of the state space.

Definition 2 (Persistent excitation)**.** There exists $\alpha > 0$ and $m \in \mathbb{N}+$, such that for any $t_0 \geq 0$,

$$\frac{1}{m} \sum_{t=t_0}^{t_0+m-1} \begin{pmatrix} x_t \\ u_t \end{pmatrix} (x_t^\top, u_t^\top) \succeq \alpha^2 I_{n_x+n_u}.$$

[1]In addition to model uncertainties, robust control may also consider other system uncertainties, e.g., disturbances, measurement noises, etc.

Definition 3 (BMSB [55])**.** Let $\{\mathcal{F}_t\}_{t\geq 1}$ denote a filtration and let $\{Z_t\}_{t\geq 1}$ be an $\{\mathcal{F}_t\}_{t\geq 1}$-adapted random process taking values in $\mathbb{R}^d$. We say that $\{Z_t\}_{t\geq 1}$ satisfies the (k, Γ_{sb}, p)-block martingale small-ball (BMSB) condition for a positive integer k, a positive definite matrix $\Gamma_{sb} \succ 0$, and $0 \leq p \leq 1$, if the following condition holds: for any fixed $\lambda \in \mathbb{R}^d$ such that $\|\lambda\|_2 = 1$, the process $\{Z_t\}_{t\geq 1}$ satisfies $\frac{1}{k}\sum_{i=1}^{k} \mathbb{P}(|\lambda^\top Z_{t+i}| \geq \sqrt{\lambda^\top \Gamma_{sb} \lambda} \mid \mathcal{F}_t) \geq p$ almost surely for any $t \geq 1$.

Our bound depends on the following assumption.

Assumption 2 (BMSB and boundedness)**.** There exists $b_z \geq 0$ such that $\|z_t\|_2 \leq b_z$ almost surely for all $t \geq 0$. Further, considering filtration $\mathcal{F}_t = \mathcal{F}(w_0, \ldots, w_{t-1}, z_0, \ldots, z_t)$, then the $\mathcal{F}_t$-adapted stochastic process $\{z_t\}_{t\geq 0}$ satisfies $(1, \sigma_z^2 I_{n_z}, p_z)$-BMSB.

Assumption 2 can be satisfied in many scenarios. For example, the boundedness of z_t can be achieved by robust stabilizing controllers [59, 67] and robust constraint-satisfying controllers with bounded constraint sets, such as robust model predictive controller, e.g., [49]. Further, the BMSB condition can be achieved by inserting a bounded i.i.d. random noise to any control policies, i.e., $u_t = \pi_t(x_t) + \eta_t$, where η_t and w_t have positive definite covariance matrices (Theorem 1 in [32]).

Finally, we assume that the bound $w_{\max}$ on w_t is tight on all directions.

Assumption 3 (Tight bound on w_t)**.** For any $\epsilon > 0$, there exists $q_w(\epsilon) > 0$, such that for any $1 \leq j \leq n$ and $t \geq 0$, we have

$$\mathbb{P}(w_t^j + w_{\max} \leq \epsilon) \geq q_w(\epsilon) > 0, \quad \mathbb{P}(w_{\max} - w_t^j \leq \epsilon) \geq q_w(\epsilon) > 0.$$

Assumption 3 holds on many different types of distributions, e.g., uniform distributions on $\mathcal{W}$ or truncated Gaussian distributions on $\mathcal{W}$. For example, it can be verified that for uniform distributions on $\mathcal{W}$, $q_w(\epsilon) = \frac{\epsilon}{2w_{\max}}$; and for truncated Gaussian distributions with 0 mean, covariance $\sigma_w^2 I_n$, and truncated region $\mathcal{W}$, $q_w(\epsilon) = \frac{\epsilon}{2w_{\max}\sigma_w} \exp(\frac{-w_{\max}^2}{2\sigma_w^2})$ (see appendix for formal proofs).[2] Notice that another common scenario is that even though w_t is bounded by $\mathcal{W}$, but the system designer only knows a loose upper bound $\hat{w}_{\max} \geq w_{\max}$. In this scenario, it can be shown that the membership set converges to a small region around θ^* with the same convergence rate.

Remark 1 (Comparison with least square estimation)**.** On the one hand, compared with the assumptions for least square estimation's convergence rate, set membership requires additional assumptions on bounded $\mathcal{W}$ and tight bound $w_{\max}$ to establish convergence rate. On the other hand, if we do not aim for convergence (rates), SM generates valid uncertainty sets, i.e., $\theta^* \in \Theta_T$, even without any stochastic assumptions, while LSE's uncertainty set is determined based on confidence region, which only holds under proper stochastic properties.

3 Set Membership Convergence Analysis

We now present the main result of this paper, which is a non-asymptotic bound on the estimation errors of SM estimation given bounded i.i.d. stochastic disturbances.

Theorem 1 (Diameter bound on the membership set)**.** *For any $m > 0$ any $\sigma > 0$, when $T > m$, we have*

$$\mathbb{P}(\mathrm{diam}(\Theta_T) > \sigma) \leq \underbrace{\frac{T}{m}\tilde{O}(n_z^{2.5}) a_2^{n_z} \exp(-a_3 m)}_{\text{Term 1}} + \underbrace{\tilde{O}((n_x n_z)^{2.5}) a_4^{n_x n_z} \left(1 - q_w\left(\frac{a_1 \sigma}{4\sqrt{n_x}}\right)\right)^{\lceil T/m \rceil}}_{\text{Term 2}}$$

where $a_1 = \frac{\sigma_z p_z}{4}$, $a_2 = \frac{64 w_{\max}}{\sigma_z^2 p_z^2}$, $a_3 = \frac{p_z^2}{8}$, $a_4 = \max(\frac{4 b_z \sqrt{n_x}}{a_1}, 1)$, and $p_z, \sigma_z, w_{\max}, b_z$ are as defined in Assumption 1-2.

Theorem 1 provides an upper bound on the "failure" probability of the SM method, i.e., the probability that the diameter of the membership set is larger than σ. In this bound, Term 1 decays exponentially with m, so for any small $\epsilon > 0$, m can be chosen such that Term 1 $\leq \epsilon$, which indicates $m \geq$

[2]Throughout the paper, we use TrunGauss$(0, \sigma_w, [-w_{\max}, w_{\max}])$ to refer to the truncated Gaussian distribution generated by Gaussian distribution with Gauss$(0, \sigma_w^2)$ with truncated range $[-w_{\max}, w_{\max}]$.

$O(n_z + \log T + \log(1/\epsilon))$. Term 2 decays exponentially with the number of data points T and involves a distribution-dependent function $q_w(\cdot)$, which characterizes how likely it is for w_t to visit the boundary of $\mathcal{W}$ as defined in Assumption 3. If w_t is more likely to visit the boundary, i.e. $q_w(\cdot)$ is larger, then SM method is less likely to generate an uncertainty set with a diameter bigger than δ.

Estimation error bounds when $q_w(\epsilon) = O(\epsilon)$. To provide intuition for Term 2 and discuss the estimation error bound in Theorem 1 more explicitly, we consider distributions satisfying $q_w(\epsilon) = O(\epsilon)$ for all $\epsilon > 0$. Notice that several common distributions satisfy this additional requirement, such as uniform distribution and truncated Gaussian distribution as discussed after Assumption 3.

Corollary 1 (Estimation error bound when $q_w(\epsilon) = O(\epsilon)$)**.** *For any $\epsilon > 0$, let*

$$m \geq O(n_z + \log T + \log(1/\epsilon)).$$

If w_t is generated i.i.d. by a distribution satisfying $q_w(\epsilon) = O(\epsilon)$ for all $\epsilon > 0$, then with probability at least $1 - 2\epsilon$, for any $\hat{\theta}_T \in \Theta_T$, we have

$$\|\hat{\theta}_T - \theta^*\|_F \leq \text{diam}(\Theta_T) \leq \tilde{O}\left(\frac{n_x^{1.5}(n_x + n_u)^2}{T}\right).$$

Corollary 1 indicates that the estimation error of any point in the membership set Θ_T can be bounded by $\tilde{O}\left(\frac{n_x^{1.5}(n_x+n_u)^2}{T}\right)$ when $q_w(\epsilon) \geq O(\epsilon)$.

It is worth comparing this upper bound with the estimation error bound of LSE in [15], which is $\tilde{O}(\frac{\sqrt{n_z}}{\sqrt{T}})$. Interestingly, the point estimators generated by the SM method achieve better convergence rates in terms of the number of samples T, but are worse with respect to the system dimensions n_x, n_u.

Loose Bound on $w_{\max}$. In many practical scenarios, it is easier to obtain an over-estimation of the range of the disturbances instead of a tight upper bound. Hence, we discuss the scenarios when we only know an over-estimation $\hat{w}_{\max} \geq w_{\max}$. In this case, we can further show that the membership set converges to a small neighborhood around θ^* with the same convergence rate as Theorem 1.

Theorem 2 (Loose bound on $w_{\max}$)**.** *Consider the membership set*

$$\hat{\Theta}_T = \bigcap_{t=0}^{T-1} \{\hat{\theta} : \|x_{t+1} - \hat{\theta} z_t\|_\infty \leq \hat{w}_{\max}\},$$

where $\hat{w}_{\max}$ is a loose upper bound for the disturbances such that $\hat{w}_{\max} \geq w_{\max}$. There exists a constant $\epsilon_0 = O((\hat{w}_{\max} - w_{\max})\sqrt{n_x})$ such that for any $m > 0$, $\sigma > 0$, $T > m$, we have

$$\mathbb{P}(\text{diam}(\hat{\Theta}_T) > \sigma + \epsilon_0) \leq \text{Term 1} + \text{Term 2}$$

where Term1 *and* Term2 *are defined in Theorem 1.*

Similarly to Corollary 1, we can establish $\|\hat{\theta}_T - \theta^*\|_F \leq \epsilon_0 + \tilde{O}(n_x^{1.5} n_z^2/T)$ when $q_w(\epsilon) = O(\epsilon)$ for any $\epsilon > 0$. This establishes a convergence rate towards an ϵ_0-neighborhood around θ^*.

3.1 Proof Sketch of Theorem 1

A formal proof of Theorem 1 is provided in the supplementary material. Here, we provide a sketch to highlight the key technical ideas of the proof.

Specifically, we first define a set Γ_T on the model estimation error $\gamma = \hat{\theta} - \theta^*$ by leveraging the observation that $x_{s+1} - \hat{\theta} z_s = w_s - (\hat{\theta} - \theta^*) z_s$,

$$\Gamma_t = \bigcap_{s=0}^{t-1} \{\gamma : \|w_s - \gamma z_s\|_\infty \leq w_{\max}\}, \quad \forall\, t \geq 0. \tag{3}$$

Notice that $\Theta_t = \theta_* + \Gamma_t$, so $\text{diam}(\Theta_t) = \text{diam}(\Gamma_t)$, and

$$\text{diam}(\Gamma_t) = \sup_{\gamma, \gamma' \in \Gamma_t} \|\gamma - \gamma'\|_F \leq 2 \sup_{\gamma \in \Gamma_t} \|\gamma\|_F.$$

Therefore, we can upper bound our goal event $\{\text{diam}(\Theta_T) > \delta\}$ by the event $\mathcal{E}_1$ defined below, where $\delta = \sigma$.

$$\mathbb{P}(\text{diam}(\Theta_T) > \delta) \leq \mathbb{P}(\mathcal{E}_1), \text{ where } \mathcal{E}_1 := \{\exists\, \gamma \in \Gamma_T, \text{ s.t. } \|\gamma\|_F \geq \frac{\delta}{2}\}. \tag{4}$$

Next, we define an event $\mathcal{E}_2$ below, which is essentially persistent excitation (PE) on every time segments $km \leq t \leq km + m - 1$ for $k \geq 0$, where the choice of m will be specified later.

$$\mathcal{E}_2 = \{\frac{1}{m}\sum_{s=1}^{m} z_{km+s} z_{km+s}^\top \succeq a_1^2 I_{n_z},\ \forall\, 0 \leq k \leq \lceil T/m \rceil - 1\}$$

where $a_1 = \frac{\sigma_z p_z}{4}$. For ease of notation and without loss of generality, we assume T/m is an integer in the following analysis. The proof can now be completed by taking the intersection of $\mathcal{E}_1$ with $\mathcal{E}_2$ and $\mathcal{E}_2^c$ in (4). Specifically, from (4) we have

$$\mathbb{P}(\text{diam}(\Theta_T) > \delta) \leq \mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2^c) + \mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) \leq \mathbb{P}(\mathcal{E}_2^c) + \mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2).$$

The proof concludes by invoking the following bounds on $\mathbb{P}(\mathcal{E}_2^c)$ and $\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2)$.

Lemma 1 (Bound on $\mathbb{P}(\mathcal{E}_2^c)$)**.**

$$\mathbb{P}(\mathcal{E}_2^c) \leq \frac{T}{m} \tilde{O}(n_z^{2.5}) a_2^{n_z} \exp(-a_3 m)$$

where $a_2 = \frac{64 b_z^2}{\sigma_z^2 p_z^2}$ *and* $a_3 = \frac{p_z^2}{8}$.

Lemma 2 (Bound on $\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2)$)**.**

$$\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) \leq \tilde{O}(n_z^{2.5} n_x^{2.5}) a_4^{n_x n_z} (1 - q_w(\frac{a_1 \delta}{4\sqrt{n_x}}))^{T/m}$$

where $a_4 = \max(1, 4 b_z \sqrt{n_x}/a_1)$ *and* $n_z = n_x + n_u$.

Roughly speaking, Lemma 1 indicates that PE holds with high probability. This is proved by leveraging the BMSB assumption and set discretization. The proof of Lemma 2 is more involved and is our major technical contribution. On a high level, the proof relies on the following two technical lemmas.

Lemma 3 (Discretization of $\mathcal{E}_1 \cap \mathcal{E}_2$ (Informal))**.** *Let* $\mathcal{M} = \{\gamma_1, \ldots, \gamma_{v_\gamma}\}$ *denotes an* ϵ_γ*-net of* $\{\gamma : \|\gamma\|_F = 1\}$. *Under a proper choice of* ϵ_γ, *we have* $v_\gamma = \tilde{O}(n_x^{2.5} n_z^{2.5}) a_4^{n_x n_z}$, *and we can construct* $\tilde{\Gamma}_T$,[3] *such that*

$$\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) \leq \mathbb{P}(\{\exists\, 1 \leq i \leq v_\gamma, d \geq 0, \text{ s.t. } d\gamma_i \in \tilde{\Gamma}_T\} \cap \mathcal{E}_2) \leq \sum_{i=1}^{v_\gamma} \mathbb{P}(\mathcal{E}_{1,i} \cap \mathcal{E}_2)$$

where $\mathcal{E}_{1,i} = \{\exists\, d \geq 0, \text{ s.t. } d\gamma_i \in \tilde{\Gamma}_T\}$.

Lemma 3 leverages finite set discretization to bound the existence of a feasible element in an infinite continuous set.

Lemma 4 (Construction of event $G_{i,k}$ via stopping times)**.** *Under the conditions in Lemma 3, we can construct* $G_{i,k}$ *for all* i *and all* $0 \leq k \leq T/m - 1$ *by*

$$G_{i,k} = \{b_{i,km+L_{i,k}} w_{km+L_{i,k}}^{j_{i,km+L_{i,k}}} \geq \frac{a_1 \delta}{4\sqrt{n_x}} - w_{\max}, \text{ and } \frac{1}{m}\sum_{s=1}^{m} z_{km+s} z_{km+s}^\top \succeq a_1^2 I_{n_z}\}.$$

where $1 \leq L_{i,k} \leq m$ *is constructed as a stopping time with respect to* $\{\mathcal{F}_{km+l}\}_{l \geq 0}$ *and* $b_{i,t}, j_{i,t} \in \mathcal{F}_t$.

It can be shown that

$$\mathbb{P}(\mathcal{E}_{1,k} \cap \mathcal{E}_2) \leq \mathbb{P}(\bigcap_{k=0}^{T/m-1} G_{i,k}) = \mathbb{P}(G_{i,0})\mathbb{P}(G_{i,1} \mid G_{i,0}) \cdots \mathbb{P}(G_{i,T/m-1} \mid \bigcap_{k=0}^{T/m-2} G_{i,k})$$

and

$$\mathbb{P}(G_{i,k} \mid \bigcap_{k'=0}^{k-1} G_{i,k'}) \leq 1 - q_w(\frac{a_1 \delta}{4\sqrt{n_x}}).$$

[3] In the Appendix, the accurate version of this lemma will specify the choice of $\tilde{\Gamma}_T$.

The construction of $G_{i,k}$ in Lemma 4 is our major technical contribution. Notice that by constructing a stopping time $L_{i,k}$, conditioning on $L_{i,k} = l$, $w_{km+L_{i,k}} = w_{km+l}$ is independent of $\{L_{i,k} = l\} \in \mathcal{F}_{km+l}$, and $b_{i,km+L_{i,k}}, j_{i,km+L_{i,k}}$ are constant conditioning on $\mathcal{F}_{km+l}$. Hence, we are able to leverage the properties of the probability distribution of w_t in Assumption 3 to bound $\mathbb{P}(G_{i,k} \mid \bigcap_{k'=0}^{k-1} G_{i,k'})$.

By leveraging Lemma 3 and 4, we can now prove Lemma 2. Then, the proof of Theorem 1 is completed by combining Lemma 1 and Lemma 2.

4 Applications and Numerical Experiments

To showcase the results in the previous section, we apply them in the context of two applications. The first application is the estimation problem of an autonomous linear dynamical system with no control inputs. The second is the analysis of an adaptive tube model predictive control (MPC) algorithm. In the following, we provide both analytical discussions and numerical results.

4.1 Estimation of an Autonomous Linear System

In this subsection, we apply Theorem 1 to linear systems with no control inputs: $x_{t+1} = A^* x_t + w_t$. In this setting, the membership set can be defined as

$$\mathbb{A}_T = \bigcap_{t=0}^{T-1} \{\hat{A} : \|x_{t+1} - \hat{A} x_t\|_\infty \leq w_{\max}\}.$$

We will provide a diameter upper bound on $\mathbb{A}_T$ below.

Corollary 2 (System estimation when $B = 0$)**.** *When A^* is (κ, ρ)-stable, i.e., $\|(A^*)^t\|_2 \leq \kappa(1-\rho)^t$ for all t with $\rho < 1$, for any $m > 0$ and any $\sigma > 0$, when $T > m$, we have*

$$\mathbb{P}(\operatorname{diam}(\mathbb{A}_T) > \sigma) \leq \frac{T}{m} \tilde{O}(n_x^{2.5}) a_2^{n_x} \exp(-a_3 m) + \tilde{O}(n_x^5) a_4^{n^2} (1 - q_w(\frac{a_1 \sigma}{4\sqrt{n}_x}))^{\lceil T/m \rceil}$$

where $b_x = \kappa \|x_0\|_2 + \kappa \sqrt{n_x}/\rho$, $p_x = 1/192$, $\sigma_x = \sqrt{\lambda_{\min}(\Sigma_w)/2}$, $a_1 = \frac{\sigma_x p_x}{4}$, $a_2 = \frac{64 w_{\max}}{\sigma_x^2 p_x^2}$, $a_3 = \frac{p_x^2}{8}$, $a_4 = \max(\frac{4 b_x \sqrt{n_x}}{a_1}, 1)$.

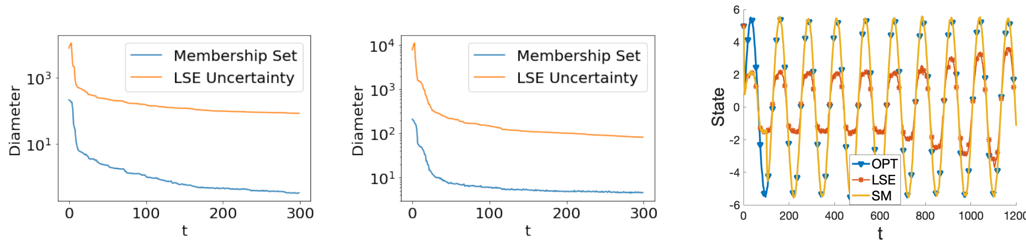
Consequently, when the distribution of w_t satisfies $q_w(\epsilon) = O(\epsilon)$, e.g. uniform or truncated Gaussian, we have $\|\hat{\theta} - \theta_*\| \leq \tilde{O}(n_x^{3.5}/T)$.

Corollary 2 can be proved by verifying the assumptions of Theorem 1: we can show $\|x_t\|_2 \leq b_x$ for stable A^* and prove the $(1, p_x, \sigma_x)$-BMSB condition for $\{x_t\}_{t \geq 0}$ based on [15].

Interestingly, though [55] shows that $\Omega(\sqrt{n}_x/\sqrt{T})$ is the best possible estimation rate for (unbounded) Gaussian distribution, when the random w_t are *bounded*, Corollary 2 shows that SM can achieve a better rate $\tilde{O}(n_x^3/T)$ when $T >> n_x$ by taking advantage of the additional information of the bounded support. The fundamental lower bound for bounded random w_t are left for future work.

Experimental Results. To illustrate the quantitative behavior of Corollary 2, we provide simulation results in Figure 2. Our experiments illustrate the membership set diameter as a function of time for an autonomous dynamical system $x_{t+1} = A^* x_t + w_t$, where w_t is generated by $\text{Unif}([-w_{\max}, w_{\max}]^{n_x})$. We consider two scenarios. In the first, $w_{\max}$ is accurately known (Figure 2a), and in the second, we only have a loose upper bound on $w_{\max}$ (Figure 2b). In particular, we consider a system with $n_x = 6$, with $n_x^2 = 36$ total unknown parameters. We consider $w_{\max} = 2$, and an inaccurate estimation of $w_{\max}$ as $\hat{w}_{\max} = 2 w_{\max}$.

The quantitative behavior of the SM method is consistent with that predicted by Corollary 2, outperforming the 90% confidence region of the LSE in both magnitude and rate of convergence. Moreover, even when the bound is loose, i.e., Assumption 3 fails to hold, the SM method is still orders of magnitude better than the LSE in terms of the size of the uncertainty set.



(a) Uniform Disturbance

(b) Loose Bound

(c) RA-MPC Trajectory Tracking

Figure 2: Figure (a)-(b) compares the diameters of LSE confidence sets and membership sets with accurate $w_{\max}$ and inaccurate estimation of $w_{\max}$ respectively. Figure (c) demonstrates the tracking performance of an adaptive robust model predictive controller based on LSE confidence sets and membership sets.

4.2 Adaptive Tube Model Predictive Control

The set membership method is extensively used in the robust adaptive model predictive control (RA-MPC) literature, e.g., [57, 37]. In RA-MPC, the model uncertainty sets can be updated online based on set membership estimation as more data is collected. Since the model uncertainty sets heavily influence the conservativeness of RA-MPC controllers, SM updates are able to quickly improve control performance by reducing the size of the uncertainty/membership sets.

In the following, we provide a brief recap of RA-MPC and show the implication of our SM result. At every time step k, RA-MPC solves the following robust optimization with horizon T:

$$\begin{array}{ll} J_T^* = & \min_\pi \sum_{t=1}^T c_t(x_t, \pi(x_t)) \\ \text{s.t.} & x_{t+1} = \left(\hat{A} + \Delta_A\right) x_t + \left(\hat{B} + \Delta_B\right) u_t + w_t \\ & u_t = \pi_t\left(x_{0:t}\right) \\ & x_0 = x_k \\ & x_t \in \mathcal{X},\ u_t \in \mathcal{U},\ x_T \in \mathcal{X}_T,\ t = 0, 1, \cdots, T-1 \\ & \forall \left(\Delta_A, \Delta_B\right) \in \mathcal{P}_t,\ \forall w_t \in \mathcal{W},\ t = 0, 1, \cdots, T-1 \end{array} \tag{5}$$

with respect to some cost objective c_t. It is evident in (5) that smaller uncertainty set $\mathcal{P}_t$ leads to better performance of the MPC algorithm. We refer readers to the supplementary material for more details on RMPC and RA-MPC.

There is a vast literature of algorithms that approximate the solutions to (5) for fixed uncertainty sets, e.g., [40, 47, 11]. We will leverage such methods to solve (5) for the adaptive control task. For concreteness, we consider the basic adaptive tube MPC algorithm [49, 40] for the FTCOP problem here. We parameterize the control policy π as $u_k = Kx_k + v_k + \eta_k$, where K is a given robustly stabilizing feedback gain for the entire uncertainty set $\mathcal{P}_0$ (and thus for all subsequent $\mathcal{P}_k$), v_k is the FTCOP optimization variable, and η_k is a bounded exploration injection with $\|\eta_k\|_\infty \leq \eta_{\max}$ with $\mathbb{E}[\eta_k \eta_k^\top] = \sigma_\eta I$. Note that η_k is left out for (5).

Applying Theorem 1, we obtain a finite-time convergence guarantee for this RA-MPC algorithm that integrates SM with tube MPC.

Corollary 3. *Assume that* (5) *is feasible for $\mathcal{P}_0$ and Assumption 3 holds, the membership set under the adaptive tube MPC controller has the following uncertainty set convergence guarantee,* $\|\hat{\theta}_T - \theta^*\|_F \leq \tilde{O}(\frac{n_x^{3.5} n_u^2}{T})$ *with high probability.*

This result is immediate by verifying Assumption 2 using [32] and verifying the boundedness assumption by noting that $\mathcal{X}, \mathcal{U}$ are bounded. For completeness, we include the constants of Corollary 3 in the supplementary material.

Experimental Results. To illustrate the quantitative impact of using SM for adaptive tube MPC, we provide simulation results in Figure 2. Our experiments study tube MPC for a single-input-single-output system with nominal system $\hat{A} = 1.1$, $\hat{B} = 1$ and true uncertainty parameter $\Delta_A = 0.1$, $\Delta_B = -0.1$. The initial uncertainty set $\mathcal{P}_0$ is set to be $[-0.2, 0.2]^2$. We use the basic tube MPC method [49, 40] and parameterize the control policy as $u_k = Kx_k + v_k + \eta_k$, where $K = -1$ and η_k is a bounded exploration injection with $\eta_k \sim \text{Unif}([-0.01, 0.01])$. Note that η_k is left out when solving for (5). The disturbance w_k has a known bound of $w_{\max} = 0.1$ and is generated to be i.i.d. $\text{Unif}([-0.1, 0.1])$. The horizon of (5) is set to be 5. The state and input constraints are such that $x_k \in [-10, 10]$ and $u_k \in [-10, 10]$ for all $k \geq 0$. We consider the task of constrained LQ tracking problem with a time-varying cost function $c_t := (x_t - g_t)^\top Q(x_t - g_t) + u_t^\top R u_t$ where the target trajectory is generated as $g_t = 8\sin(t/20)$.

We compare the performance of an adaptive tube MPC controller that uses the SM method for uncertainty set estimation (SM) against one that uses the LSE confidence region (LSE). We also plot the offline optimal RMPC controller, i.e., the controller that has knowledge of the true underlying system parameters (OPT). The resulting state trajectories are shown in Figure 2c. Due to the fast convergence of the membership set, the tracking performance of SM quickly coincides with OPT, while LSE converges more slowly. Since the controller has to robustly satisfy constraints against the worst-case model in the uncertainty set, smaller uncertainty set for (5) means more optimal trajectories are available to the RA-MPC controller. Therefore, Corollary 3 verifies that the SM method is indeed a good candidate for the uncertainty set estimation problem arising in RA-MPC problems as have been extensively observed empirically in existing literature [36, 38, 28].

5 Conclusion, Limitations, and Future Work

Conclusion. This work provides an explicit bound on the convergence rate of set membership estimation methods for linear control dynamical systems under the assumption of i.i.d. stochastic disturbances that are bounded by box constraints. When the box constraints are accurately known, we establish the rate of convergence to the true model parameters. When the box constraints on w_t are over-estimated, we establish the rate of convergence to a neighborhood of the true model parameters. We demonstrate the main result of this paper (Theorem 1) in both the linear system identification problem and the adaptive robust model predictive control setting. Numerical experiments on these two settings are also provided in comparison with least square estimation and its confidence region.

Limitations and future work. One limitation of our work is that our bound heavily relies on the assumption of the box constraints on the disturbances, which may be overly conservative in practice. Therefore, an important future direction is to analyze set membership estimation for general polyhedron constraints on the disturbances. Moreover, our results have focused exclusively on the *diameter* of the membership set. Other metrics such as the volume are also of importance in the control literature and therefore needs more investigation. For example, if the uncertainty set has a shape similar to a narrow ellipse, it may have a large diameter yet small volume, and whether this has high impact or low impact on control performance is also worth discussing. Another limitation of this work is that we only consider linear systems, and it would be interesting to explore whether some of the results in this paper can be extended to some nonlinear systems, such as bi-linear systems, or nonlinear systems with linear parametrization, etc.

Interesting directions for future work also include obtaining a lower bound for set membership estimation. Further, we note that Corollary 3 also lays a foundation for non-asymptotic optimality analysis for RA-MPC based on set membership, which has seen a significant amount of empirical research but lacks formal optimality analysis. Finally, it is also worth applying the proof techniques of this paper to analyze the set membership performance on other types of uncertainties, such as measurement errors, etc.

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Appendix

Roadmap

- Appendix A introduces additional notation used throughout the Appendix.
- Appendix B provides more literature review on LSE and SM, and a more detailed discussion on the **technical novelty** of this paper.
- Appendix C provides more discussions on examples that satisfy Assumptions 2 and 3.
- Appendix D presents the proof of Theorem 1. In particular, we provide helper lemmas in Appendix D.1 and prove Lemma 1, Lemma 2 in Appendix D.2 and Appendix D.3 respectively.
- Appendix E presents a proof of Corollary 1.
- Appendix F presents a proof of Theorem 2.
- Appendix G presents a proof of Corollary 2.
- Appendix H reviews robust (adaptive) model predictive control and proves Corollary 3.
- Appendix I provides more details on the simulations in Section 4.

A Additional notations

Let $\mathbb{S}_n(0,1)$ denote the unit sphere in $\mathbb{R}^n$ in l_2 norm, i.e., $\mathbb{S}_n(0,1) = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$. Let $\mathbb{S}_{n\times m}(0,1)$ denote the unit sphere in $\mathbb{R}^{n\times m}$ with respect to the Frobenius norm, i.e., $\mathbb{S}_{n\times m}(0,1) = \{M \in \mathbb{R}^{n\times m} : \|M\|_F = 1\}$. Let $\bar{B}_n(0,1)$ denote the closed unit ball in $\mathbb{R}^n$ in l_2 norm, i.e., $\bar{B}_n(0,1) = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$. Let $\bar{B}_{n\times m}(0,1)$ denote the closed unit ball in $\mathbb{R}^{n\times m}$ in Frobenius norm, i.e., $\bar{B}_{n\times m}(0,1) = \{M \in \mathbb{R}^{n\times m} : \|M\|_F \leq 1\}$. For a matrix $M \in \mathbb{R}^{n\times m}$, vec$(M)$ is the vectorization of M. Moreover, we define the inverse mapping of vec$(\cdot)$ as mat$(\cdot)$, i.e., for a vector $d \in \mathbb{R}^{nm}$, mat$(d) \in \mathbb{R}^{n\times m}$. Consider a σ-algebra $\mathcal{F}$ and a random variable X, we write $X \in \mathcal{F}$ if X is measurable with respect to $\mathcal{F}$, i.e., for all Borel measurable sets $B \subseteq \mathbb{R}$, we have $X^{-1}(B) \in \mathcal{F}$. We can similarly define $\mathcal{F}$-measurable random matrices and random vectors. Further, consider a polyhedral $\mathbb{D} = \{x : Ax \leq b\}$, we write $\mathbb{D} \in \mathcal{F}$ if matrix A and vector b are measurable with respect to $\mathcal{F}$. Consider two symmetric matrices $A, B \in \mathbb{R}^{n\times n}$, we write $A \succeq B$ if $A - B$ is a positive definite matrix. We define $\min \emptyset = +\infty$. For a set $\mathcal{E}$, let $\mathbb{1}_{\mathcal{E}}$ denote the indicator function on $\mathcal{E}$. For a vector $x \in \mathbb{R}^n$, we use x^j to denote the jth coordinate of x.

B More discussions on least square and set membership

System identification studies the problem of estimating the parameters of an unknown dynamical systems from trajectory data. There are two main classes of estimation methods: point estimator such as least square estimation (LSE), and set estimator such as set membership (SM). In the following, we provide more discussions and literature review on LSE and SM. We will also discuss the major technical novelties of this work.

B.1 Least square estimation

For linear dynamical systems $x_{t+1} = A^* x_t + B^* u_t + w_t = \theta^* z_t + w_t$, given a trajectory of data $\{x_t, u_t\}_{t\geq 0}$, least square estimation generates a point estimator that minimizes the following quadratic error [60, 35]:

$$\hat{\theta}_{\text{LSE}} = \min_{\hat{\theta}} \sum_{t=0}^{T-1} \|x_{t+1} - \hat{\theta} z_t\|_2^2.$$

Least-square estimation is widely used and its convergence (rate) guarantees have been investigated for a long time. In particular, non-asymptotic convergence rate guarantees of LSE has become increasingly important as these guarantees are the foundations for non-asymptotic performance analysis of learning-based/adaptive control algorithms. Earlier non-asymptotic analysis of LSE focused on the simpler regression model $y_t = \theta^* x_t + w_t$, where x_t and y_t are independent [10, 63, 24].

Recently, there is one major breakthrough in [55] that provides LSE's convergence rate analysis for linear dynamical system $x_{t+1} = \theta^* z_t + w_t$, where x_{t+1} and $z_t = [x_t^\top, u_t^\top]^\top$ are correlated. More specifically, [55] establishes a fundamental property, *block-martingale small-ball (BMSB)*, to analyze LSE under correlated data. BMSB enables a long list of subsequent literature on LSE's non-asymptotic analysis for different types of dynamical systems, e.g., [43, 14, 70, 48, 17, 64, 58, 69, 32].

Though LSE is a point estimator, one can establish confidence region of LSE based on proper statistical assumptions on w_t. The pioneer works on the confidence region of LSE for linear dynamical systems are [1, 2], which construct ellipsoid confidence regions for LSE. Moreover, the non-asymptotic bounds on estimation errors established in [55, 15] can also be viewed as confidence bounds. Further, the estimation error $\tilde{O}(\frac{\sqrt{n_x+n_z}}{\sqrt{T}})$ has been shown to match the fundamental lower bound for any estimation methods for unbounded disturbances in [55]. However, these confidence bounds all rely on statistical inequalities, which may result in loose constant factors despite an optimal convergence rate. When applying these confidence bounds to robust control, where the controller is required to satisfy certain stability and constraint satisfaction properties for every possible system in the confidence region, a loose constant factor will result in a larger confidence region and a more conservative control design. Finally, in robust control and many practical applications, the disturbances are usually bounded, and it will be interesting to see how the knowledge of the boundedness will improve the uncertainty set estimation.

B.2 Set membership

Set membership is commonly used in robust control for uncertainty set estimation [42, 3, 56, 9, 68, 45, 44, 53]. There is a long history of research on SM for both deterministic disturbances, such as [6, 19, 26, 41, 31], and stochastic disturbances, such as [5, 6, 27, 4, 38]. For the stochastic disturbances, both convergence and convergence rate analysis have been investigated under the persistent excitation (PE) condition. However, the existing convergence rates are only established for simpler regression problems, $y_t = \theta^* x_t + w_t$, where y_t and x_t are independent [4, 6, 5, 27].

Recently, [38] provided an initial attempt to establish the convergence guarantee of SM for linear dynamical systems $x_{t+1} = \theta^* z_t + w_t$ for correlated data x_{t+1} and z_t. However, [38] assumes that PE holds deterministically, and designs a special control design based on constrained optimization to satisfy PE deterministically. Therefore, the convergence for general control design and the convergence rate analysis remain open questions for correlated data arising from dynamical systems.

In this paper, we establish the convergence rate guarantees of SM on linear dynamical systems under the BMSB conditions in [55]. Compared with [38], BMSB condition can be satisfied by adding an i.i.d. random noise to a general class of control designs [32].

Technically, one major challenge of SM analysis compared with the LSE analysis is that the diameter of the membership set does not have an explicit formula, which is in stark contrast with LSE, where the point estimator is the solution to a quadratic program and has explicit form. A common trick to address this issue in the analysis of SM is to connect the diameter bound with the values of disturbances subsequences $\{w_{s_k}\}_{k\geq 0}$: it can be generally shown that a large diameter indicates that a long subsequence of disturbances are far away from the boundary of $\mathcal{W}$. However, existing construction methods of $\{w_{s_k}\}_{k\geq 0}$ will cause the time indices $\{s_k\}_{k\geq 0}$ to *correlate* with the realization of the sequences $\{x_t, u_t, w_t\}_{t\geq 0}$ [4, 38, 6].[4] Consequently, in the correlated-data scenario and when PE does not hold deterministically, under the existing construction methods in [4, 38, 6], the probability of $\{w_{s_k}\}_{k\geq 0}$ with correlated time indices *cannot* be bounded by the probability of the *independent* sequence $\{w_t\}_{t\geq 0}$. One major **technical contribution** of this paper is to provide a novel construction of $\{w_{s_k}\}_{k\geq 0}$ based on a sequence of stopping times and establish conditional independence properties despite correlated data and stochastic PE condition (BMSB). More details can be found in Lemma 4 and the proof or Lemma 2.

Though we only consider box constraints for w_t, it is worth mentioning that SM can be applied to much more general forms of disturbances. For example, a common alternative is the ellipsoidal-

[4] In [38], the correlation between $\{s_k\}_{k\geq 0}$ and $\{x_t, u_t, w_t\}_{t\geq 0}$ is via the PE condition, but [38] assume deterministic PE to avoid this correlation issue.

bounded disturbance where $\mathcal{W} := \{w \in \mathcal{R}^{n_x} : w^\top P w \leq 1\}$ with positive definite $P \in \mathcal{R}^{n_x \times n_x}$ [6, 61, 16, 34] and polytopic-bounded disturbance $\mathcal{W} := \{w \in \mathcal{R}^{n_x} : Gw \leq h\}$ for positive definite $G \in \mathcal{R}^{n_x \times n_x}$ and $h \in \mathcal{R}^{n_x}$ [19, 38, 37]. There are also SM literature assuming bounded energy of the disturbance sequences [6]. It is an interesting future direction to extend the analysis in this paper to more general disturbance constraints.

A separate but important challenge is that the knowledge of $\mathcal{W}$ is not always available a priori. There is literature discussing the estimation of $\mathcal{W}$ [31, 5]. We leave for future work how to simultaneously estimate $\mathcal{W}$ and perform non-asymptotic analysis on the size of the membership set.

C More discussions on Assumptions 2 and 3

C.1 More discussions on Assumption 2

The BMSB condition has been widely used in learning-based control. It has been shown that BMSB can be satisfied in many scenarios. For example, [55, 59] showed that linear systems with i.i.d. perturbed linear control policies, i.e., $x_{t+1} = Ax_t + B(Kx_t + \eta_t) + w_t$,[5] satisfy BMSB if the disturbances w_t and η_t are i.i.d. and follow Gaussian distributions with positive definite covariance matrices. Later, [15] showed that $x_{t+1} = Ax_t + B(Kx_t + \eta_t) + w_t$ can still satisfy BMSB even for non-Gaussian distributions of w_t, η_t, as long as w_t and η_t have independent coordinates and finite fourth moments. Recently, [32] extended the results to linear systems with nonlinear policies, i.e., $x_{t+1} = Ax_t + B(\pi_t(x_t) + \eta_t) + w_t$, and showed that BMSB still holds as long as the nonlinear policies π_t generate bounded trajectories of states and control inputs, and w_t, η_t are bounded and follow distributions with certain anti-concentrated properties (a special case is positive definite covariance matrix).

C.2 More discussions on Assumption 3

In this subsection, we provide two example distributions, truncated Gaussian and uniform distributions, and discuss their corresponding $q_w(\epsilon)$ functions. It will be shown that for both distributions below, $q_w(\epsilon) = O(\epsilon)$.

Lemma 5 (Example of uniform distribution)**.** *Consider w_t that follows a uniform distribution on $[-w_{\max}, w_{\max}]^{n_x}$. Then, $q_w(\epsilon) = \frac{\epsilon}{2w_{\max}}$.*

Proof. Since Unif($\mathcal{W}$) is symmetric, we only need to consider one direction $j = 1$.

$$\begin{aligned}
\mathbb{P}(w^j + w_{\max} \leq \epsilon) &= \int_{w^1 + w_{\max} \leq \epsilon} \int_{w^2, \ldots, w^{n_x} \in [-w_{\max}, w_{\max}]} \frac{1}{(2w_{\max})^{n_x}} \mathbb{1}_{(w \in \mathcal{W})} \, \mathrm{d}w \\
&= \int_{w^1 \leq \epsilon - w_{\max}} \frac{1}{2w_{\max}} \mathbb{1}_{(w \in \mathcal{W})} \mathrm{d}w^1 = \frac{\epsilon}{2w_{\max}}
\end{aligned}$$

Similarly, $\mathbb{P}(w_{\max} - w^1 \leq \epsilon) = \int_{w^1 \geq w_{\max} - \epsilon} \frac{1}{2w_{\max}} \mathbb{1}_{(w \in \mathcal{W})} dw^1 = \frac{\epsilon}{2w_{\max}}$. □

Lemma 6 (Example of truncated Gaussian distribution)**.** *Consider w_t follows a truncated Gaussian distribution on $[-w_{\max}, w_{\max}]^{n_x}$ generated by a Gaussian distribution with zero mean and $\sigma_w I_{n_x}$ covariance matrix. Then, $q_w(\epsilon) = \frac{1}{\min(\sqrt{2\pi}\sigma_w, 2w_{\max})} \exp(\frac{-w_{\max}^2}{2\sigma_w^2})\epsilon$.*

Proof. Since this distribution is symmetric and each coordinate is independent, we only need to consider one direction j. Let X denote a Gaussian distribution with zero mean and σ_w^2 variance. By the definition of truncated Gaussian distributions, we have

$$\mathbb{P}(w^j + w_{\max} \leq \epsilon) = \frac{\mathbb{P}(-w_{\max} \leq X \leq -w_{\max} + \epsilon)}{\mathbb{P}(-w_{\max} \leq X \leq w_{\max})}$$

[5] Though we only describe a static linear policy $u_t = Kx_t$ here, the results in [55, 59, 15] hold for dynamic linear policies.

Notice that X/σ_w follows the standard Gaussian distribution, so we can obtain the following bounds.

$$\begin{aligned}\mathbb{P}(-w_{\max} \leq X \leq -w_{\max}+\epsilon) &= \int_{-w_{\max}/\sigma_w}^{(-w_{\max}+\epsilon)/\sigma_w} \frac{1}{\sqrt{2\pi}} \exp(-\frac{z^2}{2})\,\mathrm{d}z \\ &\geq \frac{1}{\sqrt{2\pi}} \exp(-w_{\max}^2/(2\sigma_w^2))\frac{\epsilon}{\sigma_w}\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(-w_{\max} \leq X \leq w_{\max}) &= \int_{-w_{\max}/\sigma_w}^{w_{\max}/\sigma_w} \frac{1}{\sqrt{2\pi}} \exp(-\frac{z^2}{2})\,\mathrm{d}z \\ &\leq \min(1, \frac{1}{\sqrt{2\pi}}\frac{2w_{\max}}{\sigma_w})\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\mathbb{P}(w^j + w_{\max} \leq \epsilon) &= \frac{\mathbb{P}(-w_{\max} \leq X \leq -w_{\max}+\epsilon)}{\mathbb{P}(-w_{\max} \leq X \leq w_{\max})} \\ &\geq \max(\frac{1}{\sqrt{2\pi}} \exp(-w_{\max}^2/\sigma_w^2)\frac{\epsilon}{\sigma_w}, \frac{\epsilon}{2w_{\max}} \exp(\frac{-w_{\max}^2}{2\sigma_w^2})) \\ &= \frac{1}{\min(\sqrt{2\pi}\sigma_w, 2w_{\max})} \exp(\frac{-w_{\max}^2}{2\sigma_w^2})\epsilon\end{aligned}$$

Finally, $\mathbb{P}(w_{\max} - w^1 \leq \epsilon)$ can be bounded similarly. $\square$

D Proof of Theorem 1

The section provides more details for the proof of Theorem 1. In particular, we first provide technical lemmas for set discretization, then prove Lemma 1 and Lemma 2 respectively. The proof of Theorem 1 follows naturally by combining Lemma 1 and Lemma 2.

D.1 Technical lemmas: set discretization

In this subsection, we provide technical results of finite-ball covering. The results are classical and are based on [50] [62].

Theorem 3 (Ball covering [50] [62])**.** *Consider a closed ball $\bar{\mathbb{B}}_n(0,1) = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$ in l_2 norm. Considering covering this ball $\bar{\mathbb{B}}_n(0,1)$ with smaller closed balls $\bar{\mathbb{B}}_n(z,\epsilon)$ for $z \in \mathbb{R}^n$. Let $v_{\epsilon,n}$ denote the minimal number of smaller balls needed to cover $\bar{\mathbb{B}}_n(0,1)$. Then, we have*

$$v_{\epsilon,n} \leq \begin{cases} \tilde{O}(n)\left(\frac{1}{\epsilon}\right)^n, & \text{if } \epsilon \leq \frac{1}{n}, \\ \tilde{O}(n^{5/2})\left(\frac{1}{\epsilon}\right)^n, & \text{if } \epsilon \leq 1. \end{cases}$$

Corollary 4. *There exists a finite set $\mathcal{M}' = \{\lambda_1, \ldots, \lambda_{v_\lambda}\} \subseteq \mathbb{S}_{n_z}(0,1)$ such that for any $\lambda \in \mathbb{R}^{n_z}$ with $\|\lambda\|_2 = 1$, there exists $\lambda_i \in \mathcal{M}'$ such that $\|\lambda - \lambda_i\|_2 \leq 2\epsilon_\lambda$. Further, $v_\lambda \leq \tilde{O}(n_z^{2.5})\left(\frac{1}{\epsilon_\lambda}\right)^{n_z}$ if $\epsilon_\lambda \leq 1$.*

In the following, we consider $\epsilon_\lambda = \min(\sigma_z^2 p_z^2/(64 b_z^2), 1)$. Hence,

$$v_\lambda \leq \tilde{O}(n_z^{2.5}) \max(1, \left(\frac{64 b_z^2}{\sigma_z^2 p_z^2}\right)^{n_z}). \tag{6}$$

Next, we apply Theorem 3 to a space of matrices.

Lemma 7. *There exists a finite set $\mathcal{M} = \{\gamma_1, \ldots, \gamma_{v_\gamma}\} \subseteq \mathbb{S}_{n_x \times n_z}(0,1)$ such that for any $\gamma \in \mathbb{R}^{n_x \times n_z}$ and $\|\gamma\|_F = 1$, there exists $\gamma_i \in \mathcal{M}$ such that $\|\gamma - \gamma_i\|_F \leq 2\epsilon_\gamma$. Further, $v_\gamma \leq \tilde{O}(n_x^{2.5} n_z^{2.5})\left(\frac{1}{\epsilon_\gamma}\right)^{n_z n_x}$ if $\epsilon_\gamma \leq 1$.*

In the following, we consider $\epsilon_\gamma = \min(\frac{a_1}{4 b_z \sqrt{n_x}}, 1)$.

Proof. The proof is basically by mapping the matrices to vectors based on matrix vectorization, then mapping the vectors back to matrices. These two mappings are isomorphism.

Specifically, consider a closed unit ball in $\mathbb{R}^{n_x n_z}$. There exist $v_{\epsilon,n_x n_z}$ smaller closed balls to cover it, denoted by $\mathbb{B}_1,\ldots,\mathbb{B}_{v_{\epsilon,n_x n_z}}$. Consider the non-empty sets from $\mathbb{B}_1\cap\mathbb{S}_{n_x n_z}(0,1),\ldots,\mathbb{B}_{v_{\epsilon,n_x n_z}}\cap\mathbb{S}_{n_x n_z}(0,1)$. For any $1\le i\le v_{\epsilon,n_x n_z}$, if $\mathbb{B}_i\cap\mathbb{S}_{n_x n_z}(0,1)\neq\emptyset$, select a point $\mathrm{vec}(\gamma)\in\mathbb{B}_i\cap\mathbb{S}_{n_x n_z}(0,1)$. Notice that $\|\mathrm{vec}(\gamma)\|_2=1$. In this way, we construct a finite sequence $\{\mathrm{vec}(\gamma_1),\ldots,\mathrm{vec}(\gamma_{v_\gamma})\}$ where $v_\gamma\le v_{\epsilon_\gamma,n_x n_z}$.[6]

For any $\gamma\in\mathbb{R}^{n_x\times n_z}$, we have $\mathrm{vec}(\gamma)\in\mathbb{R}^{n_x n_z}$ and $\|\mathrm{vec}(\gamma)\|_2=1$. Hence, there exists $1\le i\le v_\gamma$ such that $\mathrm{vec}(\gamma)\in\mathbb{B}_i\cap\mathbb{S}_{n_x n_z}(0,1)$. Hence, $\|\mathrm{vec}(\gamma)-\mathrm{vec}(\gamma_i)\|_2\le 2\epsilon_\gamma$. Moreover, $\|\gamma_i\|_F=\|\mathrm{vec}(\gamma_i)\|_2=1$. Therefore, $\|\gamma_i-\gamma\|_F\le 2\epsilon_\gamma$. So the set $\mathcal{M}=\{\gamma_1,\ldots,\gamma_{v_\gamma}\}$ satisfies our requirement. ☐

D.2 Proof of Lemma 1

The proof leverages the following results related to Proposition 2.5 in [55].

Definition 4 (BMSB for scalar sequence)**.** Let $\{Z_t\}_{t\ge1}$ be an $\{\mathcal{F}_t\}_{t\ge1}$-adapted random process taking values in $\mathbb{R}$. We say $\{Z_t\}_{t\ge1}$ satisfies the (k,v,p)-block martingale small-ball (BMSB) condition if, for any $j\ge0$, one has $\frac{1}{k}\sum_{i=1}^k\mathbb{P}(|Z_{j+i}|\ge v\mid\mathcal{F}_t)\ge p$ almost surely.

Theorem 4 (Proposition 2.5 in [55] when $k=1$)**.** *Let $\{Z_t\}_{t\ge0}$ be a scalar process that is $(1,v,p)$-BMSB and $Z_t\in\mathcal{F}_t$ for $t\ge0$, where Z_0 is given. Then*

$$\mathbb{P}(\sum_{t=1}^T Z_t^2\le v^2p^2T/8)\le\exp(-Tp^2/8)$$

Corollary 5 (Implications of Proposition 2.5 in [55] conditioning on $\mathcal{F}_t$)**.** *Let $\{Z_t\}_{t\ge0}$ be a scalar process that is $(1,v,p)$-BMSB and $Z_t\in\mathcal{F}_t$ for $t\ge0$, where Z_0 is given. Then*

$$\mathbb{P}(\sum_{t=s+1}^T Z_t^2\le v^2p^2(T-s)/8\mid\mathcal{F}_s)\le\exp(-(T-s)p^2/8),\quad\forall\, s\ge0$$

Proof. Consider $\{Y_t\}_{t\ge0}$ where $Y_t=Z_{t+s}$ for $t\ge0$. Conditioning on $\mathcal{F}_s$, Y_0 is given. Further, conditioning on $\mathcal{F}_s$, $\{Y_t\}$ is BMSB because

$$\mathbb{P}(|Y_{t+1}|\ge v\mid\mathcal{F}_t^y\cap\mathcal{F}_s)=\mathbb{P}(|Z_{t+1+s}|\ge v\mid\mathcal{F}_{t+s})\ge p.$$

Hence, we can apply Theorem 4 on $\{Y_t\}$ for $t=1,\ldots,T-s$. ☐

Lemma 8. *For any $\lambda\in\mathbb{R}^{n_z}$ such that $\|\lambda\|_2=1$, for any $k\ge0$, $m\ge1$, we have*

$$\mathbb{P}(\sum_{i=1}^m\lambda^\top z_{km+i}z_{km+i}^\top\lambda\le\sigma_z^2p_z^2m/8\mid\mathcal{F}_{km})\le\exp(-mp_z^2/8)$$

Proof. Firstly, by Assumption 2, $\{z_t\}$ is $\mathcal{F}_t$ adapted and $(1,\sigma_z^2I_{n_z},p_z)$-BMSB. Therefore, for any λ with $\|\lambda\|_2=1$, $\lambda^\top z_t$ is $(1,\sigma_z,p_z)$-BMSB. Therefore, by Corollary 5, we have

$$\mathbb{P}(\sum_{t=km+1}^{km+m}\lambda^\top z_tz_t^\top\lambda\le\sigma_z^2p_z^2m/8\mid\mathcal{F}_{km})\le\exp(-mp_z^2/8),\quad\forall\, k\ge0,m\ge1$$

This completes the proof. ☐

Lemma 9 (Definition of $H_{i,k}$)**.** *Consider $\mathcal{M}'=\{\lambda_1,\ldots,\lambda_{v_\lambda}\}$ as defined in Corollary 4. For any $m\ge1$, define events*

$$H_{i,k}=\left\{\sum_{t=km+1}^{km+m}\lambda_i^\top z_tz_t^\top\lambda_i>\sigma_z^2p_z^2m/8\right\},\quad\forall\, k\ge0,1\le i\le v_\lambda$$

Then $\mathbb{P}(H_{i,k}^c\mid\mathcal{F}_{km})\le\exp(-mp_z^2/8)),\quad\forall\, k\ge0,1\le i\le v_\lambda.$

[6] Here, without loss of generality, we consider $\mathbb{B}_1\cap\mathbb{S}_{n_xn_z}(0,1),\ldots,\mathbb{B}_{v_\gamma}\cap\mathbb{S}_{n_xn_z}(0,1)$ are not empty.

Proof. For any $\lambda_i \in \mathcal{M}'$, $\|\lambda_i\|_2 = 1$, so by Lemma 8, we have $\mathbb{P}(H_{i,k}^c \mid \mathcal{F}_{km}) \leq \exp(-mp_z^2/8))$. $\square$

Lemma 10 (Probability of intersections of $H_{i,k}$)**.** $\mathbb{P}(\bigcap_{i=1}^{v_\lambda} H_{i,k} \mid F_{km}) \geq 1 - v_\lambda \exp(-mp_z^2/8))$.

Proof. We can establish the following:

$$\mathbb{P}(\bigcap_{i=1}^{v_\lambda} H_{i,k} \mid F_{km}) = 1 - \mathbb{P}(\bigcup_{i=1}^{v_\lambda} H_{i,k}^c \mid F_{km}) \geq 1 - \sum_{i=1}^{v_\lambda} \mathbb{P}(H_{i,k}^c \mid F_{km}) \geq 1 - v_\lambda \exp(-mp_z^2/8))$$

where the last inequality is by Lemma 9. $\square$

Lemma 11 (Probability of PE in one segment)**.** *For any $m \geq 1$, for any $k \geq 0$, we have*

$$\mathbb{P}(\sum_{t=km+1}^{km+m} z_t z_t^\top \succ (\sigma_z^2 p_z^2 m/16) I_{n_z} \mid \mathcal{F}_{km}) \geq 1 - v_\lambda \exp(-mp_z^2/8))$$

Proof. By Corollary 4, for any λ such that $\|\lambda\|_2 = 1$, there exists $\lambda_i \in \mathcal{M}'$ such that $\|\lambda - \lambda'\|_2 \leq 2\epsilon_\lambda$. Suppose for this λ_i, we have $\sum_{t=km+1}^{km+m} \lambda_i^\top z_t z_t^\top \lambda_i > \sigma_z^2 p_z^2 m/8$. Then

$$\begin{aligned}
\sum_{t=km+1}^{km+m} \lambda^\top z_t z_t^\top \lambda &= \sum_{t=km+1}^{km+m} \lambda_i^\top z_t z_t^\top \lambda_i + \sum_{t=km+1}^{km+m} (\lambda + \lambda_i)^\top z_t z_t^\top (\lambda - \lambda_i) \\
&> \sigma_z^2 p_z^2 m/8 - \sum_{t=km+1}^{km+m} (\lambda + \lambda_i)^\top z_t z_t^\top (\lambda_i - \lambda) \\
&\geq \sigma_z^2 p_z^2 m/8 - \sum_{t=km+1}^{km+m} \|\lambda + \lambda_i\|_2 \|z_t\|_2^2 \|\lambda_i - \lambda\|_2 \\
&\overset{(a)}{\geq} \sigma_z^2 p_z^2 m/8 - \sum_{t=km+1}^{km+m} 4 b_z^2 \epsilon_\lambda \\
&= \sigma_z^2 p_z^2 m/8 - 4 b_z^2 \epsilon_\lambda m \overset{(b)}{\geq} \sigma_z^2 p_z^2 m/16
\end{aligned}$$

where (a) is by Assumption 2 and $\|\lambda - \lambda'\|_2 \leq 2\epsilon_\lambda$, and (b) is by $\epsilon_\lambda \leq \sigma_z^2 p_z^2/(64 b_z^2)$.

Therefore, together with Lemma 10, we complete the proof.

$$\begin{aligned}
\mathbb{P}(\bigcap_{i=1}^{v_\lambda} H_{i,k} \mid F_{km}) &\leq \mathbb{P}(\forall \|\lambda\|_2 = 1, \sum_{t=km+1}^{km+m} \lambda^\top z_t z_t^\top \lambda > \sigma_z^2 p_z^2 m/16 \mid F_{km}) \\
&= \mathbb{P}(\sum_{t=km+1}^{km+m} z_t z_t^\top \succ (\sigma_z^2 p_z^2 m/16) I_{n_z} \mid \mathcal{F}_{km})
\end{aligned}$$

$\square$

Proof of Lemma 1. Recall that $\mathcal{E}_2 = \{\frac{1}{m} \sum_{s=1}^m z_{km+s} z_{km+s}^\top \succeq a_1^2 I_{n_z}, \forall\, 0 \leq k \leq \lceil (T-1)/m \rceil - 1\}$. Hence

$$\mathcal{E}_2 = \bigcap_{k=0}^{(T-1)/m-1} \{\sum_{t=km+1}^{km+m} z_t z_t^\top \succ (\sigma_z^2 p_z^2 m/16) I_{n_z}\}$$

where $a_1 = \sigma_z p_z/4$.

For notational simplicity, we define $\bar{H}_k = \{\sum_{t=km+1}^{km+m} z_t z_t^\top \succ (\sigma_z^2 p_z^2 m/16) I_{n_z}\}$. So

$$\begin{aligned}\mathbb{P}(\mathcal{E}_2) &= \mathbb{P}(\bigcap_{k=0}^{(T-1)/m-1} \bar{H}_k) = 1 - \mathbb{P}(\bigcup_{k=0}^{(T-1)/m-1} \bar{H}_k^c) \\ &\geq 1 - \sum_{k=0}^{(T-1)/m-1} \mathbb{P}(\bar{H}_k^c) = 1 - \sum_{k=0}^{(T-1)/m-1} (1 - \mathbb{P}(\bar{H}_k))\end{aligned}$$

By Lemma 11, we have $\mathbb{P}(\bar{H}_k \mid \mathcal{F}_{km}) \geq 1 - v_\lambda \exp(-mp_z^2/8))$. Therefore, $1 - \mathbb{P}(\bar{H}_k \mid \mathcal{F}_{km}) \leq v_\lambda \exp(-mp_z^2/8))$. Therefore,

$$\begin{aligned}\mathbb{P}(\mathcal{E}_2) &\geq 1 - ((T-1)/m) v_\lambda \exp(-mp_z^2/8)) \\ &\geq 1 - ((T-1)/m) \tilde{O}(n_z^{2.5}) \max(1, \left(\frac{64 b_z^2}{\sigma_z^2 p_z^2}\right)^{n_z}) \exp(-mp_z^2/8)) \\ &\geq 1 - \frac{T}{m} \tilde{O}(n_z^{2.5}) \max(1, \left(\frac{64 b_z^2}{\sigma_z^2 p_z^2}\right)^{n_z}) \exp(-mp_z^2/8)),\end{aligned}$$

where in the second inequality we used (6). Finally

$$\mathbb{P}(\mathcal{E}_2^c) = 1 - \mathbb{P}(\mathcal{E}_2) \leq \frac{T}{m} \tilde{O}(n_z^{2.5}) \max(1, \left(\frac{64 b_z^2}{\sigma_z^2 p_z^2}\right)^{n_z}) \exp(-mp_z^2/8)).$$

□

D.3 Proof of Lemma 2

This proof takes four major steps:

(i) Define $b_{i,t}, j_{i,t}, L_{i,k}$.

(ii) Provide a formal definition of $\mathcal{E}_{1,k}$ based on $b_{i,t}, j_{i,t}, L_{i,k}$ and prove a formal version of Lemma 3.

(iii) Prove Lemma 4.

(iv) Prove Lemma 2 by the formal version of Lemma 3 and Lemma 4.

It is worth mentioning that the formal definition of $\mathcal{E}_{1,k}$ is slightly different from the definition in Lemma 3, but we still have $\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) \leq \sum_{i=1}^{v_\gamma} \mathbb{P}(\mathcal{E}_{1,k} \cap \mathcal{E}_2)$, which is the key property that will be used in the proof of Lemma 2.

D.3.1 Step (i): Definitions of $b_{i,t}, j_{i,t}, L_{i,k}$.

Recall the discretization of $\mathbb{S}_{n_x \times n_z}(0,1)$ in Lemma 7, which generates the set $\mathcal{M} = \{\gamma_1, \ldots, \gamma_{v_\gamma}\}$. We are going to define $b_{i,t}, j_{i,t}, L_{i,k}$ for $\gamma_i \in \mathcal{M}$ for each $1 \leq i \leq v_\gamma$. Notice that $\mathcal{M}$ is a deterministic set of matrices.

Lemma 12 (Definition of $b_{i,t}, j_{i,t}$)**.** *For any $\gamma_i \in \mathcal{M}$, any $0 \leq t \leq T$, there exist $b_{i,t} \in \{-1, 1\}$ and $1 \leq j_{i,t} \leq n_x$ such that $b_{i,t}, j_{i,t} \in \mathcal{F}(z_t) \subseteq \mathcal{F}_t$ and*

$$\|\gamma_i z_t\|_\infty = b_{i,t} (\gamma_i z_t)^{j_{i,t}}.$$

Note that one way to determine $b_{i,t}, j_{i,t}$ from z_t is by the following: first pick the smallest j such that $|(\gamma_i z_t)^j| = \|\gamma_i z_t\|_\infty$, then let $b_{i,t} = \mathrm{sgn}((\gamma_i z_t)^j)$, where $\mathrm{sgn}(\cdot)$ denotes the sign of a scalar argument.

Proof. For any $\gamma_i \in \mathcal{M}$, any $0 \leq t \leq T$, we have

$$\|\gamma_i z_t\|_\infty = \max_{1 \leq j \leq n_x} \max_{b \in \{-1,1\}} b (\gamma_i z_t)^j$$

Hence, there exist $b_{i,t}, j_{i,t}$ such that $\|\gamma_i z_t\|_\infty = b_{i,t} (\gamma_i z_t)^{j_{i,t}}$. Further, $b_{i,t}, j_{i,t}$ only depend on γ_i and z_t, so they are $\mathcal{F}(z_t)$-measurable, and $\mathcal{F}(z_t) \subseteq \mathcal{F}_t$. □

Lemma 13 (Definition of stopping times $L_{i,k}$). *Let $\eta = \frac{a_1}{\sqrt{n_x}}$. For any $\gamma_i \in \mathcal{M}$, any $0 \le k \le (T-1)/m - 1$, we can define a random time index $1 \le L_{i,k} \le m+1$ by*

$$L_{i,k} = \min(m+1, \min\{l \ge 1 : \|\gamma_i z_{km+l}\|_\infty \ge \eta\}).$$

Then, we have $1 \le L_{i,k} \le m+1$. Further, for any $1 \le l \le m$, $\{L_{i,k} = l\} \in \mathcal{F}_{km+l}$, and $\{L_{i,k} = m+1\} \in \mathcal{F}_{km+m} \subseteq \mathcal{F}_{km+m+1}$. In other words, $L_{i,k}$ is a stopping time with respect to filtration $\{F_{km+l}\}_{l\ge 1}$.

Proof. For any i and any k, it is straightforward to see that $L_{i,k}$ is well-defined and $1 \le L_{i,k} \le m+1$.

When $L_{i,k} = l \le m$, this is equivalent with $\|\gamma_i z_{km+l}\|_\infty \ge \eta$ but $\|\gamma_i z_{km+s}\| < \eta$ for $1 \le s < l$. Notice that this event is only determined by $z_{km+l}, \ldots, z_{km+1}$, so $\{L_{i,k} = l\} \in \mathcal{F}_{km+l}$.

When $L_{i,k} = m+1$, this is equivalent with $\|\gamma_i z_{km+s}\| < \eta$ for $1 \le s \le m$. Notice that this event is only determined by $z_{km+m}, \ldots, z_{km+1}$, so $\{L_{i,k} = m+1\} \in \mathcal{F}_{km+m}$.

Therefore, by definition, $L_{i,k}$ is a stopping time with respect to filtration $\{\mathcal{F}_{km+l}\}_{l\ge 1}$. $\square$

D.3.2 Step (ii): a formal version of Lemma 3 and its proof

Lemma 14 (Discretization of $\mathcal{E}_1 \cap \mathcal{E}_2$ (Formal version of Lemma 3)). *Let $\mathcal{M} = \{\gamma_1, \ldots, \gamma_{v_\gamma}\}$ be an ϵ_γ-net of $\{\gamma : \|\gamma\|_F = 1\}$ as defined in Lemma 7, where $\epsilon_\gamma = \min(\frac{a_1}{4b_z\sqrt{n_x}}, 1)$, $v_\gamma = \tilde{O}(n_x^{2.5} n_z^{2.5}) a_4^{n_x n_z}$, and $a_4 = \max(\frac{4b_z\sqrt{n_x}}{a_1}, 1)$. Define*

$$\mathcal{E}_{1,i} = \{\exists\, \gamma \in \Gamma_T, \text{ s.t. } b_{i,km+L_{i,k}}(\gamma z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \ge \frac{a_1\delta}{4\sqrt{n_x}},\ \forall\, k \ge 0\}.$$

Then, we have

$$\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) \le \sum_{i=1}^{v_\gamma} \mathbb{P}(\mathcal{E}_{1,i} \cap \mathcal{E}_2).$$

The rest of this subsubsection is dedicated to the proof of Lemma 14. As an overview: firstly, we will discuss the implications of $\mathcal{E}_2$ on $\gamma_i \in \mathcal{M}$. Then, we discuss the implications of $\mathcal{E}_2$ on any γ. Lastly, we prove Lemma 14 by combining the implications of $\mathcal{E}_2$ on any γ and $\|\gamma\|_F \ge \delta/2$.

Lemma 15 (The implication of $\mathcal{E}_2$ on γ_i). *If $\mathcal{E}_2$ happens, then for any $\gamma_i \in \mathcal{M}$, any $0 \le k \le (T-1)/m - 1$, we have*

$$\max_{1\le s\le m} \|\gamma_i z_{km+s}\|_\infty \ge \frac{a_1}{\sqrt{n_x}}.$$

Therefore, almost surely, we have $1 \le L_{i,k} \le m$ and

$$b_{i,km+L_{i,k}}(\gamma_i z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \ge \frac{a_1}{\sqrt{n_x}}.$$

Proof. If $\mathcal{E}_2$ happens, then by definition, we have

$$\frac{1}{m}\sum_{s=1}^m z_{km+s} z_{km+s}^\top \succeq a_1^2 I_{n_z},$$

for all $0 \le k \le (T-1)/m - 1$.

Now, for any $\gamma_i \in \mathcal{M}$, we have that

$$\frac{1}{m}\sum_{s=1}^m \gamma_i z_{km+s} z_{km+s}^\top \gamma_i^\top \succeq a_1^2 \gamma_i \gamma_i^\top. \tag{7}$$

Therefore, by taking trace at each side of (7), we obtain

$$\frac{1}{m}\sum_{s=1}^m \mathrm{tr}(\gamma_i z_{km+s} z_{km+s}^\top \gamma_i^\top) \ge a_1^2 \mathrm{tr}(\gamma_i \gamma_i^\top) \tag{8}$$

Since $\gamma_i \in \mathbb{S}_{n_x \times n_z}(0,1)$, we have $\|\gamma_i\|_F = 1$, so $\mathrm{tr}(\gamma_i\gamma_i^\top) = \mathrm{tr}(\gamma_i^\top\gamma_i) = \|\gamma_i\|_F^2 = 1$. Further, we have

$$\mathrm{tr}(\gamma_i z_{km+s} z_{km+s}^\top \gamma_i^\top) = \mathrm{tr}(z_{km+s}^\top \gamma_i^\top \gamma_i z_{km+s}) = z_{km+s}^\top \gamma_i^\top \gamma_i z_{km+s} = \|\gamma_i z_{km+s}\|_2^2.$$

Consequently, we have

$$\frac{1}{m}\sum_{s=1}^{m} \|\gamma_i z_{km+s}\|_2^2 \geq a_1^2$$

for all k.

By the pigeonhole principle, we have that

$$\max_{1\leq s\leq m} \|\gamma_i z_{km+s}\|_2^2 \geq a_1^2.$$

This is equivalent with $\max_{1\leq s\leq m} \|\gamma_i z_{km+s}\|_2 \geq a_1$.

Notice that $\|\gamma_i z_{km+s}\|_2 \leq \sqrt{n_x}\|\gamma_i z_{km+s}\|_\infty$, so $\max_{1\leq s\leq m} \sqrt{n_x}\|\gamma_i z_{km+s}\|_\infty \geq a_1$, which completes the proof of the first inequality in the lemma statement.

Next, we prove the second inequality in the lemma statement. Notice that by the definition of $L_{i,k}$ in Lemma 13 and by $\eta = \frac{a_1}{\sqrt{n_x}}$, we have $1 \leq L_{i,k} \leq m$ and $\|\gamma_i z_{km+L_{i,k}}\|_\infty \geq \frac{a_1}{\sqrt{n_x}}$ for all k. Further, by Lemma 12, we have $\|\gamma_i z_{km+L_{i,k}}\|_\infty = b_{i,km+L_{i,k}}(\gamma_i z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}}$ almost surely. Hence, we have $b_{i,km+L_{i,k}}(\gamma_i z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \geq \frac{a_1}{\sqrt{n_x}}$, which completes the proof.

□

Lemma 16 (The implication of $\mathcal{E}_2$ on γz_t)**.** *If $\mathcal{E}_2$ happens, then for any $\gamma \in \mathbb{R}^{n_x\times n_z}$, there exists $1 \leq i \leq v_\gamma$, such that*

$$b_{i,km+L_{i,k}}(\gamma z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \geq \frac{a_1}{2\sqrt{n_x}}\|\gamma\|_F,$$

for all $0 \leq k \leq (T-1)/m - 1$.

Proof. Firstly, when $\gamma = 0$, the inequality holds because both sides are 0.

Next, when $\gamma \neq 0$, it suffices to prove $b_{i,km+L_{i,k}}(\frac{\gamma}{\|\gamma\|_F} z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \geq \frac{a_1}{2\sqrt{n_x}}$. Therefore, we will only consider $\gamma \in \mathbb{S}_{n_x\times n_z}(0,1)$. By Lemma 7, there exists $\gamma_i \in \mathcal{M}$ such that $\|\gamma - \gamma_i\|_F \leq 2\epsilon_\gamma = \min(\frac{a_1}{2b_z\sqrt{n_x}}, 2)$. Notice that by Lemma 15, if $\mathcal{E}_2$ happens, for all k, we have

$$b_{i,km+L_{i,k}}(\gamma_i z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \geq \frac{a_1}{\sqrt{n_x}}.$$

Therefore,

$$\begin{aligned}
b_{i,km+L_{i,k}}(\gamma z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} &= b_{i,km+L_{i,k}}(\gamma_i z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \\
&\quad - b_{i,km+L_{i,k}}((\gamma_i - \gamma) z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \\
&\geq \frac{a_1}{\sqrt{n_x}} - |b_{i,km+L_{i,k}}((\gamma_i - \gamma) z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}}| \\
&\geq \frac{a_1}{\sqrt{n_x}} - \|(\gamma_i - \gamma) z_{km+L_{i,k}}\|_2 \\
&\geq \frac{a_1}{\sqrt{n_x}} - \|\gamma_i - \gamma\|_2 \|z_{km+L_{i,k}}\|_2 \\
&\geq \frac{a_1}{\sqrt{n_x}} - 2\epsilon_\gamma b_z \geq \frac{a_1}{2\sqrt{n_x}}
\end{aligned}$$

□

Proof of Lemma 14. By Lemma 16, under $\mathcal{E}_2$, for any $\gamma \in \mathbb{R}^{n_x\times n_z}$, there exists $1 \leq i \leq v_\gamma$, such that

$$b_{i,km+L_{i,k}}(\gamma z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \geq \frac{a_1}{2\sqrt{n_x}}\|\gamma\|_F,$$

for all $0 \leq k \leq (T-1)/m - 1$. Therefore, if $\mathcal{E}_1 \cap \mathcal{E}_2$ happens, there exists $\gamma \in \Gamma_T$ and a corresponding i, such that

$$b_{i,km+L_{i,k}}(\gamma z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \geq \frac{a_1}{2\sqrt{n_x}}\|\gamma\|_F \geq \frac{a_1\delta}{4\sqrt{n_x}}.$$

Therefore,

$$\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) \leq \mathbb{P}(\bigcup_{i=1}^{v_\gamma} \mathcal{E}_{1,i} \cap \mathcal{E}_2) \leq \sum_{i=1}^{v_\gamma} \mathbb{P}(\mathcal{E}_{1,i} \cap \mathcal{E}_2),$$

which completes the proof. □

D.3.3 Proof of Lemma 4

Notice that Lemma 4 states two inequalities: in the following, we will first prove the first inequality $\mathbb{P}(\mathcal{E}_{1,i} \cap \mathcal{E}_2) \leq \mathbb{P}(\cap_{k=0}^{(T-1)/m-1} G_{i,k})$, then prove the second inequality on $\mathbb{P}(G_{i,k} \mid \cap_{k'=0}^{k-1} G_{i,k'})$.

Lemma 17 (Bound $\mathcal{E}_{1,i} \cap \mathcal{E}_2$ by $G_{i,k}$)**.** *Under the conditions in Lemma 4, for any $1 \leq i \leq v_\gamma$, we have*

$$\mathbb{P}(\mathcal{E}_{1,i} \cap \mathcal{E}_2) \leq \mathbb{P}(\bigcap_{k=0}^{(T-1)/m-1} G_{i,k}).$$

Proof. Firstly, for any $\gamma \in \Gamma_T$, we have $\|w_t - \gamma z_t\|_\infty \leq w_{\max}$ for all $t \geq 0$. This suggests that, for any $1 \leq j \leq n_x$, we have

$$-w_{\max} \leq w_t^j - (\gamma z_t)^j \leq w_{\max}.$$

Hence, we have $b(\gamma z_t)^j \leq b w_t^j + w_{\max}$ for any $b \in \{-1, 1\}$, $1 \leq j \leq n_x$, and $t \geq 0$.

Next, by $\mathcal{E}_{1,i}$, there exists $\gamma \in \Gamma_T$ such that $b_{i,km+L_{i,k}}(\gamma z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \geq \frac{a_1\delta}{4\sqrt{n_x}}$ for all $k \geq 0$. Therefore, $b_{i,km+L_{i,k}} w_{km+L_{i,k}}^{j_{i,km+L_{i,k}}} + w_{\max} \geq \frac{a_1\delta}{4\sqrt{n_x}}$ for all k.

Finally, $\mathcal{E}_{1,i} \cap \mathcal{E}_2$ implies that $b_{i,km+L_{i,k}} w_{km+L_{i,k}}^{j_{i,km+L_{i,k}}} + w_{\max} \geq \frac{a_1\delta}{4\sqrt{n_x}}$ and $\frac{1}{m}\sum_{s=1}^m z_{km+s} z_{km+s}^\top \succeq a_1^2 I_{n_z}$ for all k, which is $\bigcap_k G_{i,k}$ by the definition of $G_{i,k}$.

□

Lemma 18 (Bound on $\mathbb{P}(G_{i,k} \mid \cap_{k'=0}^{k-1} G_{i,k'})$)**.** *Under the conditions in Lemma 4, for any $1 \leq i \leq v_\gamma$ and any $k \geq 0$, we have*

$$\mathbb{P}(G_{i,k} \mid \bigcap_{k'=0}^{k-1} G_{i,k'}) \leq 1 - q_w(\frac{a_1\delta}{4\sqrt{n_x}}).$$

Proof. Firstly, notice that when $\frac{1}{m}\sum_{s=1}^m z_{km+s} z_{km+s}^\top \succeq a_1^2 I_{n_z}$, we have $1 \leq L_{i,k} \leq m$ by the proof of Lemma 15. Therefore, we have

$$\begin{aligned}
&\mathbb{P}(G_{i,k} \mid \bigcap_{k'=0}^{k-1} G_{i,k'}) \leq \mathbb{P}(b_{i,km+L_{i,k}} w_{km+L_{i,k}}^{j_{i,km+L_{i,k}}} + w_{\max} \geq \frac{a_1\delta}{4\sqrt{n_x}}, 1 \leq L_{i,k} \leq m \mid \bigcap_{k'=0}^{k-1} G_{i,k'}) \\
&\leq \sum_{l=1}^m \mathbb{P}(b_{i,km+l} w_{km+l}^{j_{i,km+l}} + w_{\max} \geq \frac{a_1\delta}{4\sqrt{n_x}}, L_{i,k} = l \mid \bigcap_{k'=0}^{k-1} G_{i,k'}) \\
&\leq \sum_{l=1}^m \mathbb{P}(b_{i,km+l} w_{km+l}^{j_{i,km+l}} + w_{\max} \geq \frac{a_1\delta}{4\sqrt{n_x}} \mid L_{i,k} = l, \bigcap_{k'=0}^{k-1} G_{i,k'}) \mathbb{P}(L_{i,k} = l \mid \bigcap_{k'=0}^{k-1} G_{i,k'}) \\
&\overset{(a)}{\leq} (1 - q_w(\frac{a_1\delta}{4\sqrt{n_x}})) \sum_{l=1}^m \mathbb{P}(L_{i,k} = l \mid \bigcap_{k'=0}^{k-1} G_{i,k'}) \\
&\leq 1 - q_w(\frac{a_1\delta}{4\sqrt{n_x}})
\end{aligned}$$

The inequality (c) is proved in the following:

$$\mathbb{P}(b_{i,km+l}w_{km+l}^{j_{i,km+l}} + w_{\max} \geq \frac{a_1\delta}{4\sqrt{n_x}} \mid L_{i,k} = l, \bigcap_{k'=0}^{k-1} G_{i,k'})$$

$$= \int_{v_{0:km+l}} \mathbb{P}(b_{i,km+l}w_{km+l}^{j_{i,km+l}} + w_{\max} \geq \frac{a_1\delta}{4\sqrt{n_x}}, w_{0:km+l} = v_{0:km+l} \mid L_{i,k} = l, \bigcap_{k'=0}^{k-1} G_{i,k'})\mathrm{d}v_{0:km+l}$$

$$= \int_{v_{0:km+l}\in S_{km+l}} \mathbb{P}(b_{i,km+l}w_{km+l}^{j_{i,km+l}} + w_{\max} \geq \frac{a_1\delta}{4\sqrt{n_x}} \mid w_{0:km+l} = v_{0:km+l})$$

$$\times \mathbb{P}(w_{0:km+l} = v_{0:km+l} \mid L_{i,k} = l, \bigcap_{k'=0}^{k-1} G_{i,k'})\mathrm{d}v_{0:km+l}$$

$$\overset{(b)}{\leq} (1 - q_w(\frac{a_1\delta}{4\sqrt{n}_x})) \int_{v_{0:km+l}\in S_{km+l}} \mathbb{P}(w_{0:km+l} = v_{0:km+l} \mid L_{i,k} = l, \bigcap_{k'=0}^{k-1} G_{i,k'})\mathrm{d}v_{0:km+l}$$

$$= 1 - q_w(\frac{a_1\delta}{4\sqrt{n}_x}),$$

where we define a shorthand notation $w_{0:km+l} = (w_0, \ldots, w_{km+l-1})$, and we use $v_{0:km+l}$ to denote a realization of $w_{0:km+l}$, then we define the set of values of $w_{0:km+l}$ as S_{km+l} such that $L_{i,k} = l, \bigcap_{k'=0}^{k-1} G_{i,k'}$ holds. Notice that $L_{i,k} = l$ can be determined by a set of values of $w_{0:km+l}$ because $L_{i,k}$ is a stopping time of $\{F_{km+l}\}_{l\geq 1}$ and thus $\{L_{i,k} = l\} \in \mathcal{F}_{km+l}$. The inequality (b) above is because of the following: firstly, notice that $b_{i,km+l}, j_{i,km+l} \in \mathcal{F}_{km+l}$, so $b_{i,km+l}, j_{i,km+l}$ are deterministic values when $w_{0:km+l} = v_{0:km+l}$. Further, since w_{km+l} is independent of $w_{0:km+l}$, we have $\mathbb{P}(w_{max} + bw_{km+l}^{j} \geq \epsilon \mid w_{0:km+l} = v_{0:km+l}) \leq 1 - q_w(\epsilon)$ for any deterministic b, j and any $\epsilon > 0$ by Assumption 3. Hence, we have $\mathbb{P}(b_{i,km+l}w_{km+l}^{j_{i,km+l}} + w_{\max} \geq \frac{a_1\delta}{4\sqrt{n_x}} \mid w_{0:km+l} = v_{0:km+l}) \leq 1 - q_w(\frac{a_1\delta}{4\sqrt{n}_x})$. □

D.3.4 Proof of Lemma 2

The proof is by leveraging Lemma 14 and Lemma 4.

$$\begin{aligned}
\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) &\leq \sum_{i=1}^{v_\gamma} \mathbb{P}(\mathcal{E}_{1,i} \cap \mathcal{E}_2) \\
&\leq \sum_{i=1}^{v_\gamma} \mathbb{P}(\bigcap_{k=0}^{(T-1)/m-1} G_{i,k}) \\
&= \sum_{i=1}^{v_\gamma} \mathbb{P}(G_{i,0})\mathbb{P}(G_{i,1} \mid G_{i,0}) \cdots \mathbb{P}(G_{i,(T-1)/m-1} \mid \bigcap_{k=0}^{(T-1)/m-2} G_{i,k}) \\
&\leq \sum_{i=1}^{v_\gamma} (1 - q_w(\frac{a_1\delta}{4\sqrt{n}_x}))^{(T-1)/m} \\
&\leq \tilde{O}(n_x^{2.5}n_z^{2.5})a_4^{n_x n_z}(1 - q_w(\frac{a_1\delta}{4\sqrt{n}_x}))^{(T-1)/m}.
\end{aligned}$$

E Proof of Corollary 1

The proof involves two parts. Firstly, we will show that Term $1 \leq \epsilon$ under our choice of m. Secondly, we will let Term $2 = \epsilon$, then we will show $\delta \leq \tilde{O}(n_x^{1.5}n_z^2/T)$, which completes the proof.

Step 1: show Term 1 $\leq \epsilon$**.** Notice that when $m \geq \frac{1}{a_3}O(\log T + 2.5\log n_z + n_z \log a_2 + \log(1/\epsilon)) = O(n_z + \log T + \log(1/\epsilon))$, we have $T\tilde{O}(n_z^{2.5})a_2^{n_z}\exp(-a_3 m) \leq \epsilon$. Since $m \geq 1$, we obtain Term $1 \leq \epsilon$.

Step 2: let Term 2 $=\epsilon$ and show $\delta \leq \tilde{O}(n_x^{1.5}n_z^2/T)$. Let Term 2 $=\epsilon$, then we have $(1-q_w(\frac{a_1\delta}{4\sqrt{n}_x}))^{T/m} = \frac{\epsilon}{\tilde{O}(n_x^{2.5}n_z^{2.5})a_4^{n_xn_z}}$. Then, we obtain $(1-q_w(\frac{a_1\delta}{4\sqrt{n}_x})) = \left(\frac{\epsilon}{\tilde{O}(n_x^{2.5}n_z^{2.5})a_4^{n_xn_z}}\right)^{m/T}$, which is equivalent with

$$q_w(\frac{a_1\delta}{4\sqrt{n}_x}) = 1-\left(\frac{\epsilon}{\tilde{O}(n_x^{2.5}n_z^{2.5})a_4^{n_xn_z}}\right)^{m/T}.$$

When $q_w(\frac{a_1\delta}{4\sqrt{n}_x}) = O(\frac{a_1\delta}{4\sqrt{n}_x})$, we obtain

$$\begin{aligned}
\delta &= O(\frac{4\sqrt{n}_x}{a_1})\left(1-\left(\frac{\epsilon}{\tilde{O}(n_x^{2.5}n_z^{2.5})a_4^{n_xn_z}}\right)^{m/T}\right)\\
&\leq O(\frac{-4\sqrt{n}_x}{a_1})\log\left(\left(\frac{\epsilon}{\tilde{O}(n_x^{2.5}n_z^{2.5})a_4^{n_xn_z}}\right)^{m/T}\right)\\
&= O(\frac{4\sqrt{n}_x m}{a_1T})(\log(1/\epsilon)+n_xn_z+\log(n_xn_z))\\
&= \tilde{O}\left(\frac{n_x^{1.5}n_z^2}{T}\right).
\end{aligned}$$

Step 3: prove Corollary 1. By leveraging the bounds above and Theorem 1, we have $\mathbb{P}(\mathrm{diam}(\Theta_T) \leq \tilde{O}\left(\frac{n_x^{1.5}n_z^2}{T}\right)) \geq \mathbb{P}(\mathrm{diam}(\Theta_T)\leq\delta) \geq 1-2\epsilon$.

Since $\theta^*\in\Theta_T$ by definition, for any $\hat{\theta}_T\in\Theta_T$, we have $\|\hat{\theta}_T-\theta^*\|_F \leq \mathrm{diam}(\Theta_T) \leq \tilde{O}\left(\frac{n_x^{1.5}n_z^2}{T}\right)$ with probability at least $1-2\epsilon$.

F Proof of Theorem 2

Specifically, we define $\epsilon_0 = \frac{4\sqrt{n}_x}{a_1}(\hat{w}_{\max}-w_{\max})$.

The proof is similar to the proof of Theorem 1. Firstly, we define $\hat{\Gamma}_T$ as a translation of the set $\hat{\Theta}_T$:

$$\hat{\Gamma}_t = \bigcap_{s=0}^{t-1}\{\gamma : \|w_s-\gamma z_s\|_\infty \leq \hat{w}_{\max}\}, \quad \forall\, t\geq 0. \tag{9}$$

Notice that

$$\hat{\Theta}_T = \theta^* + \hat{\Gamma}_T$$

by considering $\gamma = \hat{\theta}-\theta^*$. Therefore, we can upper bound our goal event $\{\mathrm{diam}(\hat{\Theta}_T) > \delta+\epsilon_0\}$ by the event $\mathcal{E}_3$ defined below.

$$\mathbb{P}(\mathrm{diam}(\hat{\Theta}_T) > \delta+\epsilon_0) \leq \mathbb{P}(\mathcal{E}_3),\ \text{where } \mathcal{E}_3 := \{\exists\,\gamma\in\hat{\Gamma}_T,\ \text{s.t. } \|\gamma\|_F \geq \frac{\delta+\epsilon_0}{2}\}. \tag{10}$$

Next, notice that

$$\mathbb{P}(\mathrm{diam}(\hat{\Theta}_T) > \delta+\epsilon_0) \leq \mathbb{P}(\mathcal{E}_3) \leq \mathbb{P}(\mathcal{E}_3\cap\mathcal{E}_2) + \mathbb{P}(\mathcal{E}_2^c)$$

By Lemma 1, we have already shown $\mathbb{P}(\mathcal{E}_2^c) \leq$ Term 1. So we only need to discuss $\mathbb{P}(\mathcal{E}_3\cap\mathcal{E}_2)$.

Lemma 19.

$$\mathbb{P}(\mathcal{E}_3\cap\mathcal{E}_2) \leq \text{Term 2}$$

Proof. Firstly, define

$$\mathcal{E}_{3,i} = \{\exists\,\gamma\in\hat{\Gamma}_T,\ \text{s.t. } b_{i,km+L_{i,k}}(\gamma z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \geq \frac{a_1(\delta+\epsilon_0)}{4\sqrt{n}_x},\ \forall\, k\geq 0\}.$$

We have $\mathbb{P}(\mathcal{E}_3\cap\mathcal{E}_2) \leq \sum_{i=1}^{v_\gamma}\mathbb{P}(\mathcal{E}_{3,i}\cap\mathcal{E}_2)$ based on the same proof ideas of Lemma 14.

Next, we will show that

$$\mathbb{P}(\mathcal{E}_{3,k} \cap \mathcal{E}_2) \leq \mathbb{P}(\bigcap_{k=0}^{T/m-1} G_{i,k}) \tag{11}$$

This is because for any $\gamma \in \hat{\Gamma}_T$, we have $b(\gamma z_t)^j \leq b w_t^j + \hat{w}_{\max}$ for any $b \in \{-1, 1\}$, $1 \leq j \leq n_x$, and $t \geq 0$. By $\mathcal{E}_{3,i}$, there exists $\gamma \in \hat{\Gamma}_T$ such that $b_{i,km+L_{i,k}}(\gamma z_{km+L_{i,k}})^{j_{i,km+L_{i,k}}} \geq \frac{a_1(\delta+\epsilon_0)}{4\sqrt{n_x}}$ for all $k \geq 0$. Thus, $b_{i,km+L_{i,k}} w_{km+L_{i,k}}^{j_{i,km+L_{i,k}}} + \hat{w}_{\max} \geq \frac{a_1(\delta+\epsilon_0)}{4\sqrt{n_x}}$ for all k. Notice that this is equivalent with $b_{i,km+L_{i,k}} w_{km+L_{i,k}}^{j_{i,km+L_{i,k}}} + w_{\max} \geq \frac{a_1\delta}{4\sqrt{n_x}}$ for all k because $\epsilon_0 = \frac{4\sqrt{n_x}}{a_1}(\hat{w}_{\max} - w_{\max})$. In this way, we can prove (11).

Finally, we can complete the proof by the following.

$$\begin{aligned}
\mathbb{P}(\mathcal{E}_3 \cap \mathcal{E}_2) &\leq \sum_{i=1}^{v_\gamma} \mathbb{P}(\mathcal{E}_{3,i} \cap \mathcal{E}_2) \leq \sum_{i=1}^{v_\gamma} \mathbb{P}(\bigcap_{k=0}^{(T-1)/m-1} G_{i,k}) \\
&= \sum_{i=1}^{v_\gamma} \mathbb{P}(G_{i,0})\mathbb{P}(G_{i,1} \mid G_{i,0}) \cdots \mathbb{P}(G_{i,(T-1)/m-1} \mid \bigcap_{k=0}^{(T-1)/m-2} G_{i,k}) \\
&\leq \sum_{i=1}^{v_\gamma} (1 - q_w(\frac{a_1\delta}{4\sqrt{n_x}}))^{(T-1)/m} \leq \text{Term 1}
\end{aligned}$$

where the second last inequality is by Lemma 18 and the last inequality uses the definition of v_γ in Lemma 7. $\square$

G Proof of Corollary 2

The proof of Corollary 2 is exactly the same as the proofs of Theorem 1 and Corollary 1. When A^* is stable, we can show that $\|x_t\|_2 \leq b_x$ for all t. Further, by [15], the sequence $\{x_t\}_{t\geq 0}$ satisfies the $(1, \sigma_x, p_x)$-BMSB condition. Therefore, we complete the proof.

H Robust adaptive model predictive control

In this section, we provide an overview of robust model predictive control (RMPC) and its adaptive variants.

H.1 RMPC

Consider the following uncertain linear dynamics,

$$x_{k+1} = \underbrace{\left(\hat{A} + \Delta_A\right)}_{A^*} x_k + \underbrace{\left(\hat{B} + \Delta_B\right)}_{B^*} u_k + w_k, \tag{12}$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$, and $\|w_k\|_\infty \leq w_{max}$ is bounded disturbances with unknown distributions. The true system parameters A^*, B^* has two components, a nominal part where $\hat{A}$ and $\hat{B}$ are known nominal parameters and an uncertain part Δ_A and Δ_B, which denotes the unknown additive uncertainty parameter. In particular, there is a known uncertainty set $\mathcal{P}_0$ such that $(\Delta_A, \Delta_B) \in \mathcal{P}_0$.

RMPC algorithms aim to guarantee the robust satisfaction of state and input constraints, i.e., $x_k \in \mathcal{X}$ and $u_k \in \mathcal{U}$ for all $k \geq 0$ and for all possible uncertainty parameters in $\mathcal{P}_t$, as well as for all admissible disturbance realizations. To do so, RMPC algorithms solve a finite-time constrained optimal control problem (FTCOP) at each time step k based on the latest state observation and apply the first optimal control input u_0^* from the solution of the FTCOP. In particular the FTCOP with horizon T at time step k is defined to be:

$$\begin{array}{ll}\min_{u_1,\ldots,u_T} & J_T^{nom}(u_0,\ldots,u_T\,;x_k) \\ \text{s.t.} & x_{t+1} = \left(\hat{A}+\widehat{\Delta}_A\right)x_t + \left(\hat{B}+\widehat{\Delta}_B\right)u_t + w_t \\ & x_0 = x_k \\ & x_t \in \mathcal{X}, u_t \in \mathcal{U}, x_{T+1} \in \mathcal{X}_T \quad \forall \left(\widehat{\Delta}_A,\, \widehat{\Delta}_B\right) \in \mathcal{P}_t, \forall w_t \in \mathcal{W} \\ & \forall t = 0,1,\cdots,T\end{array} \tag{13}$$

where $\mathcal{X}$, $\mathcal{U}$, $\mathcal{X}_T$ are the state, input and terminal state constraints respectively, while $J_T^{nom}(u_0,\ldots,u_T\,;x_k)$ is commonly chosen to be the cost incurred by the control actions with respect to the nominal model without perturbation:

$$J_T^{nom}(u_0,\ldots,u_T\,;x_k) := \min \sum_{t=0}^{T} c_t(\hat{x}_t,u_t) + c_f(\hat{x}_{T+1}) \tag{14a}$$

$$\text{s.t. } \hat{x}_{t+1} = \hat{A}\hat{x}_t + \hat{B}u_t \quad \forall t = 0,1,\cdots,T \tag{14b}$$

$$\hat{x}_0 = x(k), \tag{14c}$$

where the stage cost c_t is commonly defined as a quadratic penalty $c_t(\hat{x}_t,u_t) := \hat{x}_t^\top Q\hat{x}_t + u_t^\top Ru_t$ for $Q, R \succ 0$ and $c_f(\hat{x}_{T+1})$ is the terminal cost on the terminal state.

The optimal control problem (13) is challenging to solve because the state and input constraints must be satisfied robustly under the presence of system parameter mismatch. Therefore (13) has to be reformulated to be numerically tractable. There is a vast and growing literature that gives efficient algorithms for (13), e.g., [18, 7, 12, 47, 11].

H.2 Adaptive RMPC

Adaptive RMPC (RA-MPC) algorithms, e.g., [57, 9, 38, 37] can be thought of as the adaptive variant of RMPC algorithms. Instead of solving the fix FTCOP (13) every time step k, RA-MPC shrinks the initial uncertainty set $\mathcal{P}_0$ to $\mathcal{P}_k$ online as more data becomes available and solves (13) based on the latest uncertainty set estimation $\mathcal{P}_k$. This procedure is summarized in Algorithm 1 below where we inject exploration noise η_k together with the control action. We choose η_k to be a bounded i.i.d. random vector with $\|\eta_k\| \le \eta_{\max}$ and $\mathbb{E}[\eta_k\eta_k^\top] = \sigma_\eta I$. It is straightforward to see that the faster the uncertainty set shrinks, the better the performance of the overall RA-MPC algorithm performs.

Algorithm 1: Robust adaptive model predictive control (RA-MPC)

Input: initial uncertainty set $\mathcal{P}_0$, bounds $w_{\max}$, $\eta_{\max}$, constraint sets $\mathcal{X}, \mathcal{U}$, nominal model $(A_0,\, B_0)$, robustly stabilizing ontroller K

for $t = 1, 2, \ldots$ **do**
- Observe x_t
- Construct membership set for uncertainty $\mathcal{P}_t := \bigcap_{k=0}^{t-1}\left\{(\hat{\Delta}_A,\hat{\Delta}_B) \in \mathcal{P}_0 : \left\|x_{k+1} - \left(\hat{A}+\hat{\Delta}_A\right)x_k - \left(\hat{B}+\hat{\Delta}_B\right)u_k\right\|_\infty \le w_{\max}\right\}$
- $u^*_{0:T} \leftarrow$FTCOP (13) with $\mathcal{P}_t$ and x_t
- $u_t \leftarrow u^*_0 + \eta_t$ where $\|\eta_k\| \le \eta_{\max}$ and $\mathbb{E}[\eta_k\eta_k^\top] = \sigma_\eta I$

end

The set membership method is extensively used in RA-MPC literature, e.g., [36, 57, 37, 42, 3, 22, 56]. Due to its direct quantification of all possible models that are consistent with the online observations, the membership set can be updated online during deployment and reduce uncertainty about the system in order to improve performance. Thanks to Theorem 1, we can obtain finite-time convergence guarantee for any set membership based RA-MPC. This result is summarized in Corollary 3 and we present its proof and explicit constants in Appendix H.3. Corollary 3 corroborates existing works that choose SM method for RA-MPC. Compared to LSE, which does not leverage the knowledge of $w_{\max}$, SM provides faster convergence rate, thus reducing the conservativeness of the controller rapidly.

H.3 Proof of Corollary 3

We provide the explicit constants and proof of Corollary 3 in this section.

Corollary 6 (Robust adaptive MPC)**.** *Assume that* (5) *is recursively feasible in Algorithm 1 and Assumption 1 and 3 hold. Under any adaptive robust MPC algorithms in the form of Algorithm 1 with bounded state and input constraint sets* $\mathcal{X}$, $\mathcal{U}$*, the membership set* $\mathcal{P}_T$ *at time* T *generated as* (2) *by the closed loop of* (1) *has the following convergence guarantee: For any* $m > 0$ *any* $\sigma > 0$*, when* $T > m$*, we have*

$$\mathbb{P}(\mathrm{diam}(\mathcal{P}_T) > \sigma) \leq \frac{T}{m}\tilde{O}(n_z^{2.5})a_2^{n_z}\exp(-a_3 m) + \tilde{O}((n_x n_z)^{2.5})a_4^{n_x n_z}\left(1 - q_w\left(\frac{a_1\sigma}{4\sqrt{n_x}}\right)\right)^{\lceil T/m \rceil}$$

where $n_z = n_x + n_u$, $a_1 = \frac{\sigma_z p_z}{4}$, $a_2 = \frac{64 w_{\max}}{\sigma_z^2 p_z^2}$, $a_3 = \frac{p_z^2}{8}$, $a_4 = \max(\frac{4 b_z\sqrt{n_x}}{a_1}, 1)$*, and*

$$p_z := \min\left(\frac{\sigma_w^2}{4w_{\max}^2}, \frac{\sigma_\eta^2}{4\eta_{\max}^2}\right)$$
$$\sigma_z := \min\left(\frac{\sigma_w^2}{16 w_{\max}}, \frac{\sqrt{3}\sigma_\eta^2}{8\eta_{\max}}, \frac{\sigma_w^2\sigma_\eta^2}{64\eta_{\max}w_{\max}b_z}\right)$$
$$b_z := \sqrt{n_x x_{\max}^2 + n_u u_{\max}^2},$$

where $x_{\max} := \max_{x\in\mathcal{X}} \|x\|_2$ *and* $u_{\max} := \max_{u\in\mathcal{U}} \|u\|_2$

Proof. It is sufficient to verify Assumption 2 in order to apply Theorem 1 to the membership set generated by the robust adaptive MPC controller. To see why Assumption 2 holds, we first note that the boundedness in the state and control actions are enforced by the state and input constraints $\mathcal{X}$ and $\mathcal{U}$ in Algorithm 1. We invoke [32, lemma 9] and conclude that the states and inputs of the closed loop under Algorithm 1 satisfy the BMSB condition (Definition 3) with p_z and σ_z listed above. The diameter bounds of $\mathcal{P}_T$ follows naturally by using the same arguments in the proof of Corollary 1.

□

I Simulation details for Section 4

I.1 Numerical settings for system identification

Figure 2a-2b illustrates the relative size between the LSE uncertainty set and the membership set. The system investigated has $n_x = 6$ with

$$A^* = \begin{bmatrix} -0.3 & 0.1 & -0.1 & -0. & 0.6 & -0.4 \\ -0.3 & -0.6 & 0.4 & -0.2 & -0.1 & 0.5 \\ -0.4 & 0.1 & -0. & -0.1 & 0.4 & -0.2 \\ -0.3 & 0.6 & 0. & -0.2 & -0.3 & -0.3 \\ -0.4 & -0.3 & 0.3 & 0.2 & 0.4 & 0. \\ 0.2 & -0.2 & -0.5 & -0.5 & 0.6 & 0.1 \end{bmatrix}$$

The largest disturbance is bounded by $W_{max} = 2$ and for the loose case, $W_{max} = 4$ when the true disturbance is bounded by 2. The Gaussian disturbances were set to be zero-mean with variance $\sigma = 0.5$. We use [2, Theorem 1] to compute the uncertainty set for LSE, with the confidence level set to be 90% and regularization weight $\lambda = 0.01$, as well as sub-Gaussian parameter $L = 16$.

I.2 RA-MPC

For Figure 2c, we empirically verify Theorem 1 with an instantiation of set membership based adaptive RMPC algorithms described in Appendix H. In particular, we adapt a popular RMPC algorithm, tube model predictive control (tube MPC) in our setting. For simplicity, we consider here the basic tube MPC formulation [40, 49] and note that there has been significant progress on tube MPC algorithms, see for example,[36, 54, 28, 46].

To generate Figure 2c, we parameterize the control policy in (13) as $u_k = Kx_k + v_k + \eta_k$. Here K is a given robustly stabilizing feedback gain for the entire uncertainty set $\mathcal{P}_0$, v_k is the new FTCOP optimization variable, and η_k is a bounded exploration injection with $\|\eta_k\| \le \eta_{\max}$ and $\mathbb{E}[\eta_k \eta_k^\top] = \sigma_\eta I$. Moreover, in order to accommodate the parametric uncertainty and the additive disturbances for robust constraint satisfaction, tube MPC generates a nominal trajectory governed by (14b) as well as a tube around the nominal trajectory such that the true trajectory generated by (12) always lies within such tube. Tube MPC guarantees the robust satisfaction of the constraints by ensuring that all trajectories within the tube satisfy the input and state constraints. Specifically, we plug in this parameterized controller and rewrite (12) as the lumped disturbance model,

$$x_{t+1} = A_0 x_t + B_0 u_t + \underbrace{\left(w_t + [\Delta_A \; \Delta_B] \begin{bmatrix} x_t \\ u_t + \eta_t \end{bmatrix} \right)}_{\text{lumped disturbances}}$$

and directly apply the tube MPC formulation from [40]. In particular, at every time step k, we solve the following FTCOP:

$$\begin{aligned} \Psi\left(\mathcal{P}_k, x_k\right) := \min_{v_{0:T}} & \sum_{t=0}^{T} c_t(x_t, u_t) + c_f(x_{T+1}) \\ \text{s.t. } & \hat{x}_{t+1} = \hat{A}\hat{x}_t + \hat{B} u_t \quad \forall t = 0, 1, \cdots, T \\ & \hat{x}_0 = x(k) \\ & u_t = K x_t + v_t \\ & x_t \in \mathcal{X} \ominus \mathcal{S}_\infty, \, u_t \in \mathcal{U} \ominus K\mathcal{S}_\infty, \quad \forall t = 0, 1, \cdots, T \\ & x_{T+1} \in \mathcal{X}_F \subset \mathcal{X} \ominus \mathcal{S}_\infty \end{aligned} \tag{15}$$

where $\ominus$ denotes Minkowski set subtraction, and $\mathcal{S}_\infty$ denotes the minimal (lumped) disturbance invariant set which is defined to be

$$\mathcal{S}_\infty := \sum_{i=0}^{\infty} (A_0 + B_0 K)^i \mathcal{S}$$

with

$$\mathcal{S} := \left\{ w + [\Delta_A \; \Delta_B] \begin{bmatrix} x \\ u + \eta \end{bmatrix} : \|w\|_\infty \le w_{\max}, \, \|\eta\|_\infty \le \eta_{\max}, \, x \in \mathcal{X}, \, u \in \mathcal{U}, \, (\Delta_A, \Delta_B) \in \mathcal{P}_k \right\}.$$

We denote the optimal solution sequence $v_0^*, \ldots, v_T^*$ to (15) at time step k as $\Psi\left(\mathcal{P}_k, x_k\right)$ to emphasize the dependency on the current state and uncertainty set. The RA-MPC algorithm applies $u_k = Kx_k + v_0^* + \eta_k$, observes the next state x_{k+1}, update the uncertainty set based on this new observation, compute $\Psi\left(\mathcal{P}_{k+1}, x_{k+1}\right)$ and repeat this process.

To generate Figure 2c, we consider a single-input-single-output system with nominal system $\hat{A} = 1.1$, $\hat{B} = 1$ and true uncertainty parameter $\Delta_A = 0.1, \Delta_B = -0.1$. The initial uncertainty set $\mathcal{P}_0$ is set to be $[-0.2, 0.2]^2$. We use the basic tube MPC method [49, 40] and parameterize the control policy as $u_k = Kx_k + v_k + \eta_k$, where $K = -1$ and η_k is a bounded exploration injection with $\eta_k \sim \text{Unif}([-0.01, 0.01])$. The disturbance w_k has a known bound of $w_{\max} = 0.1$ and is generated to be i.i.d. $\text{Unif}([-0.1, 0.1])$. The horizon of (13) is set to be 5. The state and input constraints are such that $x_k \in [-10, 10]$ and $u_k \in [-10, 10]$ for all $k \ge 0$. We consider the task of constrained LQ tracking problem with the stage cost at every time step k being $c_t(x_t, u_t) := (x_t - g_{k+t})^\top Q (x_t - g_{k+t}) + u_t^\top R u_t$ where the target trajectory is generated as $g_t = 8\sin(t/20)$, and the terminal cost $c_f(x_{T+1}) := (x_{T+1} - g_{k+T+1})^\top Q (x_{T+1} - g_{k+T+1})$.

For set membership based tube MPC, we used (2) to generate $\mathcal{P}_k$. For the LSE uncertainty set, we used the uncertainty set description in [2, Theorem 1] with regularization parameter $\lambda = 0.1$, sub-Gaussian parameter $L = \sqrt{n_x} w_{\max} = 0.1$ for 90% confidence level.