Online switching control with stability and regret guarantees

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Abstract

This paper considers online switching control with a finite candidate controller pool, an unknown dynamical system, and unknown cost functions. The candidate controllers can be unstabilizing policies. We only require at least one candidate controller to satisfy certain stability properties, but we do not know which one is stabilizing. We design an online algorithm that guarantees finite-gain stability throughout the duration of its execution. We also provide a sublinear policy regret guarantee compared with the optimal stabilizing candidate controller. Lastly, we numerically test our algorithm on quadrotor planar flights and compare it with a classical switching control algorithm, falsification-based switching, and a classical multi-armed bandit algorithm, Exp3 with batches.

1. Introduction

This paper considers an online switching control problem with a finite pool of candidate controllers $\{\pi_1, \ldots, \pi_N\}$, an unknown nonlinear system $x_{t+1} = f(x_t, u_t, w_t)$ with process noises w_t, and unknown (time-varying) cost functions $c_t(x_t, u_t)$. Notice that some candidate controllers can be unstabilizing policies. We only require at least one candidate controller to be stabilizing, but we may not know which one(s) are the stabilizing controllers.[1] We consider a single-trajectory setting, where the online switching control algorithm (also called a 'supervisor' in the literature, Hespanha et al., 2003) implements a candidate controller at each stage *without* resetting the system state. Our goal is to design an online algorithm that both stabilizes the system and optimizes the total cost among the candidate controllers.

Online switching control enjoys a long history of research, see e.g., (Hespanha et al., 2003; Stefanovic and Safonov, 2008; Al-Shyoukh and Shamma, 2009; Patil et al., 2021), and wide applications, e.g., power systems (Meng et al., 2016; Dragičević et al., 2013), healthcare (Bin et al., 2021; Marchetti et al., 2008), autonomous vehicles (Aguiar and Hespanha, 2004), Internet of Things (Zolanvari et al., 2019), etc. Online switching control is particularly useful in complex scenarios, such as when the problem has non-continuous uncertainties like unknown system orders (Liu and Yang, 2017) and hybrid systems (Garcia et al., 2013); when multiple control designs are used, for example, comparing model predictive control and PID control (Nikoofard et al., 2014); and when the controller updates are computationally demanding in real-time (Zhou and Doyle, 1998).

In the online switching control literature, most papers focus on system stabilization, and various approaches have been proposed, e.g., estimation-based supervisory control (Hespanha et al., 2003), performance-based falsification (Sajjanshetty and Safonov, 2018; Rosa et al., 2011; Al-Shyoukh and

1. A switching control problem with at least one stabilizing candidate controller is sometimes called a 'feasible' problem in the literature (Sajjanshetty and Safonov, 2018; Stefanovic and Safonov, 2008).

Shamma, 2009; Stefanovic and Safonov, 2008), multi-model adaptive control (Shahab and Miller, 2021; Kuipers and Ioannou, 2010), and others. As for the optimality analysis, most papers either only analyze convergence/asymptotic optimality, e.g., (Shahab and Miller, 2021; Kuipers and Ioannou, 2010), or discuss the optimality with respect to a cost function designed for stability purposes, instead of a cost function from "nature", e.g., (Al-Shyoukh and Shamma, 2009; Sajjanshetty and Safonov, 2018; Stefanovic and Safonov, 2008). Hence, the non-asymptotic optimality on the actual cost $\sum_t c_t(x_t, u_t)$ is largely under-explored for online switching control.

In contrast, there is rich literature in the online learning area that aims to optimize non-asymptotic performance/regret with respect to the actual cost functions (Auer et al., 2002; Arora et al., 2012). Since online switching control is closely related to online learning, especially multi-armed bandit (MAB) with memory (each candidate controller is an arm, and the current cost depends on the controllers used previously), it is tempting to leverage MAB-with-memory algorithms for online switching control (Lin et al., 2022). However, with unstabilizing candidate controllers, our problem does not satisfy the uniform bounded costs in the MAB literature (Auer et al., 2002; Arora et al., 2012; Lin et al., 2022). Further, MAB algorithms may cause *unstable* systems when some candidate controllers are unstabilizing (see, e.g., Figure 1).

Therefore, a natural question arises: *Can we design an online switching control algorithm with both stability and non-asymptotic optimality/regret guarantees on the true cost functions?*

Contributions. We design an online switching control algorithm Exp3-ISS by integrating an MAB algorithm Exp3 with a stability certification rule. Exp3-ISS deactivates controllers that fail the stability certification, then switches to other controllers that have not been deactivated.

Theoretically, our Exp3-ISS guarantees finite-gain stability and a sublinear policy regret when compared with the optimal stabilizing candidate controller. We prove a regret bound that scales as $\tilde{O}(N^{1/3}T^{2/3}) + \exp(O(\mathbb{M}))$, where T is the horizon length, N is the number of candidate controllers, and $\mathbb{M}$ is the number of candidate controllers without the desirable stability properties. Notice that $\tilde{O}(N^{1/3}T^{2/3})$ is the optimal regret for MAB with memory (Dekel et al., 2014), which suggests the optimality for our online switching control problem due to its close relation to MAB with memory. The regret $\exp(O(\mathbb{M}))$ is intuitive if candidate controllers are black boxes, in which case we must try each candidate controller at least once to determine its performance, and trying $\mathbb{M}$ unstabilizing controllers consecutively may result in exponentially large states and regrets.

Numerically, we test Exp3-ISS on quadrotor planar flight simulations and compare it with Exp3 and the falsification-based switching algorithm in (Al-Shyoukh and Shamma, 2009).

Related work. *Online switching control* has been studied under different names, e.g., supervisory control (Hespanha et al., 2003), logic-based switching control (Aguiar and Hespanha, 2007), and multi-model adaptive control (Kuipers and Ioannou, 2010). There are two major types of switching rules: model-estimation-based rules (Hespanha et al., 2003) and performance-based rules that do not estimate models (Al-Shyoukh and Shamma, 2009). This paper belongs to the second type.

Our stability certification is inspired by Rosa et al. (2011) and Al-Shyoukh and Shamma (2009) but is slightly different because our certification is checked at every stage, while the certification in Rosa et al. (2011); Al-Shyoukh and Shamma (2009) is only checked every Δ_T stages, where Δ_T is determined by their algorithm. The combination of a stability certification and a performance-optimization algorithm was also discussed in Rosa et al. (2011), but without optimality guarantees.

There are other stability certificates, e.g., control Lyapunov functions (Brunke et al., 2022).

Online control and online learning. Online control and its connection with online learning (with memory) have attracted a lot of attention recently (Wang and Boyd, 2009; Lin et al., 2022; Li et al., 2021a; Kakade et al., 2020; Boffi et al., 2021). Most papers consider linear systems, but there is a growing interest in nonlinear systems (Kakade et al., 2020; Boffi et al., 2021; Lin et al., 2022). This work is mostly related to (Lin et al., 2022; Arora et al., 2012; Dekel et al., 2014). However, these papers all assume uniform bounded cost functions, which corresponds to all candidate controllers being stabilizing in our case. One major contribution of this paper is to guarantee stability via a novel online control design despite unstabilizing candidate controllers.

Many online control and learning-based control papers assume to know a stabilizing policy beforehand (Lin et al., 2022; Agarwal et al., 2019; Fazel et al., 2018; Li et al., 2021a), which can be restrictive in certain applications. There is a growing interest on online (learning-based) control without prior knowledge of a stabilizing policy. This paper contributes to this area since we do not know which candidate controller is stabilizing. Besides, our result is related with Chen and Hazan (2021), which consider online linear control and provide a regret bound of $\tilde{O}(\text{poly}(d)T^{2/3}) + \exp(\text{poly}(d))$, where d is the system dimension. Notice that Chen and Hazan (2021) only consider linear policies so their regret can depend on the system dimension, while our problem considers black-box controllers without restrictions or knowledge of controller structures for nonlinear systems, so our regret bound depends on the number of unstabilizing candidate controllers. It is an interesting future direction to study how to leverage controller structures in online nonlinear control to generate regret bounds that also depend on the system dimension instead of the number of controllers.

Reinforcement learning. This work is also related to model-free reinforcement learning, especially zeroth-order policy gradient for control, which also updates policies based on observed cost performance (Fazel et al., 2018; Malik et al., 2019; Li et al., 2021b). The major difference is that we consider a finite policy pool while policy gradient considers a continuous policy pool. Further, under proper conditions, policy gradient can guarantee every selected controller updates with small enough gradient steps to be stabilizing, while our problem allows quick updates of controllers at a cost of potential encounters with unstabilizing policies.

Notations. $\|\cdot\|$ refers to the Euclidean norm.

2. Problem formulation

This paper focuses on an online supervisory/switching control problem. We consider an unknown nonlinear dynamical system $x_{t+1} = f(x_t, u_t, w_t)$ and unknown time-varying cost functions $c_t(x_t, u_t)$, with state $x_t \in \mathbb{R}^n$, action $u_t \in \mathbb{R}^m$, and process noise $w_t \in \mathbb{R}^n$. We consider a *bandit* setting, i.e., we can only observe the value of $c_t(x_t, u_t)$ after observing x_t and implementing u_t at stage t. The process noise w_t is bounded by a known set $\mathcal{W} = \{w : \|w\|_2 \leq w_{\max}\}$ and can be obliviously adversarial, i.e., w_t does not depend on the history states and actions. We consider a finite pool of candidate controllers

$$\{i \in \mathcal{P}_0 = \{1, \ldots, N\} : u_t = \pi_i(x_t)\}. \tag{1}$$

Some candidate controllers may not stabilize the system, and we do not know which controllers stabilize the system. Further, we treat the candidate controllers as black boxes in this paper and do not assume knowledge of their explicit forms, which is convenient for complex controllers, e.g., when the controllers are represented by neural networks. It is left as future work to consider candidate controllers with known structures.

Our goal is to design an online algorithm $\mathcal{A}$ that selects a candidate controller $I_t \in \mathcal{P}_0$ at each stage t in order to both *stabilize* the system and *optimize* the total cost $J_T(\mathcal{A})$ defined below.

$$J_T(\mathcal{A}) = \sum_{t=0}^{T} c_t(x_t(\mathcal{A}), u_t(\mathcal{A})), \quad \text{where } u_t(\mathcal{A}) = \pi_{I_t}(x_t(\mathcal{A})).$$

In the supervisory control literature, this online algorithm is often called a "supervisor" (Hespanha, 2001; Hespanha et al., 2003; Tsao and Safonov, 2001; Al-Shyoukh and Shamma, 2009). We now formally introduce our assumptions and our performance metric, policy regret.

1) Assumptions on the candidate controllers. In our problem, we do not need all the candidate controllers to be stabilizing controllers. In fact, we only require at least one of them to satisfy desirable stability properties, which are formally introduced below.

Firstly, we consider input-to-state stability (ISS), which is commonly used in nonlinear systems with process noises w_t (Sontag, 2008). Further, for the purpose of non-asymptotic analysis, we consider exponential-ISS (E-ISS) below (see e.g., Shi et al. (2021); Kolathaya et al. (2018)).

Definition 1 (E-ISS) *A controller π is called exponential-ISS (E-ISS) with parameters (κ, ρ, β) if, for any $x_0 \in \mathbb{R}^n$ and $\|w_t\|_2 \leq w_{\max}$ for all $t \geq 0$, the trajectory $x_{t+1} = f(x_t, \pi(x_t), w_t)$ satisfies $\|x_t\|_2 \leq \kappa\rho^t\|x_0\|_2 + \beta w_{\max}$.*[2]

In addition, we consider incremental stability (δ-S), which is commonly adopted to rigorously quantify the dependence of the current states on the history (see e.g., Angeli (2002); Rüffer et al. (2013)). For the purpose of non-asymptotic analysis, we consider exponentially decaying dependence, i.e., incremental exponential stability (δ-ES).

Definition 2 (δ-ES) *A controller π is called incrementally exponentially stable (δ-ES) with parameters (κ, ρ) if we have $\|x_t - y_t\|_2 \leq \kappa\rho^t\|x_0 - y_0\|_2$ for two trajectories $x_{t+1} = f(x_t, \pi(x_t), w_t)$ and $y_{t+1} = f(y_t, \pi(y_t), w_t)$ with any $x_0, y_0 \in \mathbb{R}^n$ and any $\|w_s\|_2 \leq w_{\max}$, $s \leq t-1$.*

Assumption 1 (On candidate controllers) *There exists at least one candidate controller π_k for $k \in \mathcal{P}_0$ to satisfy Definitions 1 and 2 with parameters (κ, ρ, β), which are known a priori.*[3] *Further, $\pi_i(x)$ for all $i \in \mathcal{P}_0$ are L_π-Lipschitz continuous. We define $\bar{\pi}_0$ as $\max_{i\in\mathcal{P}_0} \|\pi_i(0)\| \leq \bar{\pi}_0$.*

Notice that there are several important controller designs that satisfy Definitions 1 and 2. For example, it is straightforward to verify that stabilizing linear controllers on linear systems satisfy Definitions 1 and 2. Similarly, feedback linearization controllers on nonlinear systems also satisfy the two definitions above because the resulting closed-loop system is linear. Furthermore, Definitions 1 and 2 can be implied by exponentially incremental ISS (Eδ-ISS), which is commonly adopted in the online nonlinear control literature (Boffi et al., 2021; Tsukamoto et al., 2021). Besides, one can design the controller based on one stability property and verify the other stability, e.g., min-norm policy by an E-ISS control Lyapunov function can also satisfy δ-ES in some cases (see (Li et al., 2022)).

The candidate controllers can be constructed by e.g., (i) domain knowledge of potentially well-performing policies, (ii) different control designs with a finite list of possible policy parameters associated with each control design, (iii) listing a finite set of possible system dynamics $\mathcal{D}$ and designing controllers for this set, (iv) a combination of the methods above, etc. (see e.g., (Hespanha et al., 2003) for more discussions). For method (iii), if the true system belongs to $\mathcal{D}$ and if the

2. Strictly speaking, this is a relaxed version of E-ISS since we do not require exponentially decaying dependence on history disturbances as in (Shi et al., 2021).
3. For simplicity, we assume Definition 1 and 2 share the same κ, ρ, but our results can still hold for different parameters.

controllers designed for each possible system satisfy the desirable stability properties and the Lipschitz continuity when the corresponding system is the true system, then Assumption 1 is satisfied. In practice, when the true system does not belong to $\mathcal{D}$ but is close to $\mathcal{D}$, and if the control design enjoys some robustness, our algorithm can still generate desirable numerical performance as shown in Section 5. Assumption 1 is mostly needed for theoretical analysis (see Remarks 5-6 in Section 3 for more discussions).

Lastly, Assumption 1 assumes to know the parameters (κ, β, ρ) a priori, which is for simplicity and was similarly assumed in the online linear control literature (Agarwal et al., 2019; Minasyan et al., 2021). Remark 6 briefly discusses how to address the case with unknown parameters.

2) Performance metric. We measure the optimality performance of our online algorithm by policy regret, which compares with the optimal policy that satisfies Definitions 1 and 2.

Definition 3 (Policy regret) *We define* $\text{PolicyRegret}(\mathcal{A}) = \mathbb{E}_{(I_t)_{t\geq 0}} J_T(\mathcal{A}) - \min_{i\in\mathcal{B}} J_T(\pi_i)$, *where the expectation is over the potentially random controller selection* I_t *generated by algorithm* $\mathcal{A}$ *and* $\mathcal{B} = \{i \in \mathcal{P}_0 \mid \pi_i$ *satisfies Definitions 1 and 2 with parameters* $(\kappa, \rho, \beta).\}$.

In addition, we adopt the finite-gain stability, which is a commonly used stability measure for nonlinear systems with process noise (Sastry, 2013).

Definition 4 (Finite-gain stability) *For any* $1 \leq p \leq +\infty$, *a system* $x_{t+1} = f(x_t, w_t)$ *is called finite-gain* l_p *stable if there exists* $0 \leq M_1, M_2 < +\infty$ *for any* x_0, T *and any* $w_t \in \mathcal{W}$ *such that*

$$(\sum_{t=0}^{T} \|x_t\|_2^p)^{1/p} \leq M_1(\sum_{t=0}^{T} \|w_t\|_2^p)^{1/p} + M_2.$$

3) Assumptions on the dynamics and costs. We consider Lipschitz continuous nonlinear dynamics with 0 as the equilibrium point below.

Assumption 2 (On dynamics) f *is* L_f*-Lipschitz continous with respect to* (x, u, w), *i.e., for any* $x, u, w, x', u', w' \in \mathbb{R}^n$ *(*w *can be in the bounded region), i.e.,* $|f(x,u,w) - f(x',u',w')| \leq L_f(\|x - x'\| + \|u - u'\| + \|w - w'\|)$. *Further,* $f(0,0,0) = 0$.

We consider locally Lipschitz continuous cost functions below, which include quadratic tracking cost $(x_t - \hat{x}_t)^\top Q(x_t - \hat{x}_t) + (u_t - \hat{u}_t)^\top R(u_t - \hat{u}_t)$ with bounded $\{\hat{x}_t, \hat{u}_t\}$ as special cases.

Assumption 3 (On cost functions) *There exists* L_{c1}, L_{c2} *such that* $c_t(x, u)$ *satisfies the following inequality for any* t, x, x', u, u': $|c_t(x,u) - c_t(x',u')| \leq (L_{c1}(\max(\|x\|, \|x'\|) + \max(\|u\|, \|u'\|)) + L_{c2})(\|x - x'\| + \|u - u'\|)$. *Further, for all* $c_t(x, u)$, *there exists* $c_0 \geq 0$ *such that* $0 \leq c_t(0,0) \leq c_0$.

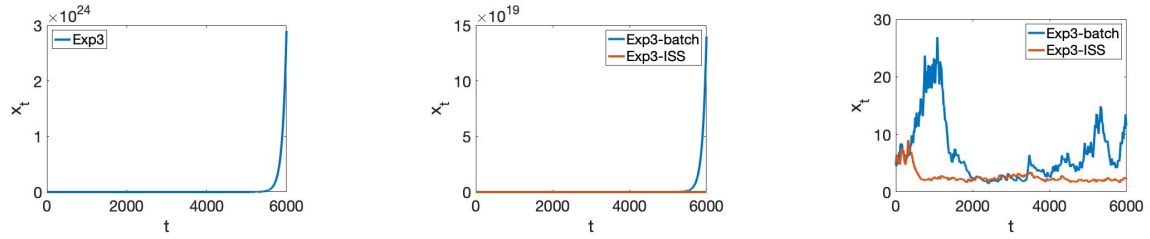
For the rest of this paper, we consider $\kappa \geq 1$, $\beta \geq 1$, $L_f \geq 1$, $L_\pi \geq 1$ for analytical simplicity.[4]

3. Algorithm design

In this section, we introduce our online algorithm for selecting candidate controllers from a controller pool that may contain unstable controllers.

This problem is closely related to multi-armed bandit (MAB) with memory, by viewing each candidate controller as one arm and noticing that the cost of the current controller depends on the history of the controllers. Thus, it is tempting to apply MAB (with memory) algorithms to our problem, such as Exp3 (Auer et al., 2002) and Exp3-batch (Lin et al., 2022; Arora et al., 2012). However, it is easy to construct examples where Exp3(-batch) fails in this setting.

4. This is without loss of generality because, if $\kappa < 1$ as an example, we can define $\kappa' = \max(\kappa, 1)$.



(a) $K_i \in [-1, -0.3, 1]$ (b) $K_i \in [-1, 0, 1]$ (c) $K_i \in [-1, -0.3, 1]$

Figure 1: Examples where Exp3 in (Auer et al., 2002) and Exp3-batch in Arora et al. (2012); Lin et al. (2022) fail to stabilize the system, in comparison to our Exp3-ISS, which stabilizes the system. Consider a system $x_{t+1} = x_t + 0.01u_t + w_t$ with w_t i.i.d. generated from Uniform$[-0.3, 0.7]$. Consider candidate controllers $u_t = K_i x_t$, where K_i are specified in the subfigure captions.

Example 1 (When Exp3(-batch) fails.) In Figure 1, we view each controller as an arm and implement Exp3 (Auer et al., 2002) and Exp3-batch (Lin et al., 2022; Arora et al., 2012), where Exp3-batch is a classical method for MAB with memory.

Figure 1(a) shows that Exp3 fails to stabilize the system even when a majority of candidate controllers are stabilizing, which is expected due to the memory-dependence of our problem. However, even with batches, Exp3 may still perform poorly, as shown in Figure 1(b-c). First, when a majority of candidate controllers do not enjoy desirable stability properties (which is exponential stability in this case), Figure 1(b) shows Exp3-batch can result in an exponential growth of states. This is because Exp3-batch is only guaranteed to work under *bounded costs* and *short memory*. However, unstabilizing candidate controllers' costs are *unbounded*, when the unstabilizing candidate controllers already steered the state x_t to be very large, the stabilizing controller will also generate a large cost when implemented at stages $t, \dots, t + \tau - 1$. In other words, the problem has *long memory* under large states. Consequently, Exp-batch may fail when there are many unstabilizing candidate controllers. Second, even when the number of unstabilizing candidate controllers is small, Exp3-batch may still perform poorly, as shown in Figure 1(c), where Exp3-batch generates large spikes in the state trajectory. This is due to explorations of unstabilizing candidates and is not rare because the cost of an unstabilizing candidate controller in one batch may not be forbiddingly large when it starts from a small initial state of this batch thanks to the stabilizing policies implemented previously. In conclusion, only adding batches to Exp3 is not enough to provide desirable stability performance for online switching control.

To handle the unstable candidate controllers, we design Exp3-ISS in Algorithm 1. In particular, we utilize Definition 1 to construct an ISS stability certificate (see Line 6 of Algorithm 1). We de-activate the controllers that fail the certificate (Line 6-8), update the controller selection probabilities p_{j+1} for the active controller pool $\mathcal{P}_{j+1}$ (Line 10-12), and select a controller from the active controller pool at the start of each batch (Line 3). In this way, Exp3-ISS can stabilize the system, which is reflected in Figure 1(b-c) and will be formally proved in Theorem 9.

Remark 5 *Notice that Algorithm 1 can be implemented as long as there exists an E-ISS candidate controller. We do not need the controller to also satisfy δ-ES for implementation and for finite-gain stability in Theorem 9. This can be helpful in practice when δ-ES is difficult to satisfy or verify.*

Algorithm 1 Exp3-ISS

1: **Input:** $(\eta_j)_{j\geq 0}$ where η_j is non-increasing. τ. κ, ρ, β. $\tilde{G}_{-1}(i) = 0$ for any $i \in \mathcal{P}_0$. A uniform distribution p_0 defined on $\mathcal{P}_0$. $t_0 = 0$.
2: **for** Batch $j = 0, 1, 2, \ldots,$ **do**
3: Initialize $\mathcal{P}_{j+1} = \mathcal{P}_j$. Select I_j from distribution p_j. Terminate the algorithm if $\mathcal{P}_j$ is empty.
4: **for** $t = t_j, \ldots, \min(t_j + \tau - 1, T)$ **do**
5: Implement π_{I_j}, observe x_{t+1}.
6: **if** $\|x_{t+1}\|_2 > \kappa\rho^{t+1-t_j}\|x_{t_j}\|_2 + \beta w_{\max}$ **then**
7: Set $\mathcal{P}_{j+1} = \mathcal{P}_j - \{I_j\}$.
8: **Break**
9: Let $t_{j+1} = t + 1$.
10: Let $g_j(I_j; I_{j-1:0}) = \frac{1}{\tau}\sum_{j=t_j}^{t_{j+1}-1} c_t(x_t, u_t)$ and $\tilde{g}_j(i; I_{j:0}) = \frac{g_j(i; I_{j-1:0})}{p_j(i)}\mathbb{1}_{(I_j=i)}$ for $i \in \mathcal{P}_{j+1}$.
11: Let $\tilde{G}_j(i; I_{j:0}) = \tilde{G}_{j-1}(i; I_{j:0}) + \tilde{g}_j(i; I_{j:0})$ for all $i \in \mathcal{P}_{j+1}$.
12: Define

$$p_{j+1}(i) = \frac{\exp(-\eta_j \tilde{G}_j(i; I_{j:0}))}{\sum_{k\in\mathcal{P}_{j+1}} \exp(-\eta_j \tilde{G}_j(k; I_{j:0}))}, \quad \forall i \in \mathcal{P}_{j+1}.$$

Though Assumption 1 requires global stability properties for theoretical analysis, since our Exp3-ISS can guarantee x_t to stay in a relatively small region, local stability properties within this region are already enough for successful implementation of our algorithms. This greatly extends the applicability of our algorithm and is reflected in our numerical experiments in Section 5.

Remark 6 *If none of the controllers in $\mathcal{P}_0$ is E-ISS, Algorithm 1 may terminate (Line 3) during implementation since it may de-activate all the controllers. If some controllers in $\mathcal{P}_0$ are E-ISS, theoretically, we can select large enough κ, β and ρ close to 1 to ensure at least some controllers can pass the ISS-stability certificate in Line 6 of Algorithm 1, thus avoiding early termination of the algorithm. In practice, we can also start with reasonably large κ, β, ρ. If all the controllers are de-activated under the current parameters, we can increase the parameters by, e.g., $\kappa \leftarrow \kappa + \Delta\kappa$, $\beta \leftarrow \beta + \Delta\beta$ and $\rho \leftarrow \frac{1+\rho}{2}$, then re-start Algorithm 1. If there exists an E-ISS controller in $\mathcal{P}_0$, Algorithm 1 can still guarantee stability since there will only be finite times of parameter updates. In practice, if we do not know whether there exists an E-ISS candidate controller, we can adopt additional termination rules, e.g., terminate the algorithm if the updated κ, β, ρ exceed certain thresholds.*

4. Theoretical results

In this section, we discuss our main results, which provide stability and regret bounds for our online algorithm. For ease of reference, we introduce two useful notations below. First, we define $\mathbb{M}$ as the number of candidate controllers that do not satisfy Definition 1.

Definition 7 *Define $\mathcal{B}_0$ as the set of controllers that do not satisfy Definition 1 under the κ, β, ρ used in Algorithm 1. Let $\mathbb{M}$ denote the number of controllers in $\mathcal{B}_0$. Notice that $\mathcal{B}_0 \subseteq \mathcal{B}^c$.*

Second, we let J denote the number of batches in Algorithm 1 for T stages.[5] It is shown in our online supplementary material (Li et al., 2022) that J is upper bounded by the following:

5. The last batch's index is $J - 1$.

Lemma 8 (Number of batches) *In horizon T, the number of batches satisfies $J \leq \lceil \frac{T-\mathbb{M}}{\tau} \rceil + \mathbb{M}$.*

We are now ready to present our stability results.

Theorem 9 (Finite-gain stability) *When $\tau \geq \frac{\log(2\sqrt{2}\kappa)}{-\log\rho}$, Algorithm 1 is finite-gain l_1 stable:*

$$\sum_{t=0}^{T} \|x_t\| \leq \beta w_{\max}(T + \alpha_1 J) + \alpha_2 (L_f(1+L_\pi)\kappa)^{\mathbb{M}} \|x_0\| + \alpha_3 (L_f(1+L_\pi)\kappa)^{\mathbb{M}} (\beta w_{\max} + \bar{\pi}_0),$$

where $\alpha_1 = \frac{\kappa}{1-\rho}\frac{1}{1-\kappa\rho^\tau}$, $\alpha_2 = \alpha_1 \frac{L_f(1+L_\pi)\kappa}{L_f(1+L_\pi)\kappa - 1}$, $\alpha_3 = \alpha_2(\frac{L_f(1+L_\pi)\kappa}{1-\kappa\rho^\tau} + L_f(2+L_\pi))$. Similarly, Algorithm 1 also achieves finite gain l_2 stability:

$$\sum_{t=0}^{T} \|x_t\|^2 = O((L_f(1+L_\pi)\kappa)^{2\mathbb{M}} \|x_0\|^2 + \beta^2 w_{\max}^2 (T+J) + (L_f(1+L_\pi)\kappa)^{2\mathbb{M}} (\beta^2 w_{\max}^2 + \bar{\pi}_0^2)).$$

Theorem 9 indicates that Algorithm 1 can guarantee bounded states despite unstabilizing controllers in the initial controller pool, which is in contrast with (batch-based) Exp3.

The bound in Theorem 9 scales as $O((L_f(1+L_\pi)\kappa)^{\mathbb{M}} + T)$. The exponential dependence on $\mathbb{M}$ can be intuitively explained as follows: since the candidate controllers are black boxes, we must try each controller in $\mathcal{P}_0 - \mathbb{B}_0$ at least once to de-activate them. This may result in exponential growth if we try the controllers in $\mathcal{P}_0 - \mathbb{B}_0$ consecutively and these controllers are unstable. Further, since $\mathbb{M}$ does not depend on the horizon T, the dependence of $\frac{1}{T}\sum_{t=0}^{T} \|x_t\|$ on $\mathbb{M}$ will diminish for large enough T. It is future work to consider non-black-box candidate controllers and leverage the controller structures to reduce the exponential term.

More specifically, when the number of batches $J = o(T)$, and when T goes to infinity, the average l_1 norm of the state converges to $\frac{1}{T}\sum_{t=0}^{T} \|x_t\| \to \beta w_{\max}$. Notice that this is the same state bound achieved by implementing an E-ISS stabilizing controller defined in Definition 1 from the beginning. This suggests that, in the long run, our algorithm can almost recover the performance of the E-ISS stabilizing controllers despite testing unstabilizing controllers at the beginning.

Next, we provide a regret guarantee for our algorithm.

Theorem 10 (Policy regret bound) *When $\tau \geq \frac{\log(2\sqrt{2}\kappa)}{-\log\rho}$ and $\eta_j = \eta$, Exp3-ISS's regret satisfies*

$$\text{PolicyRegret} \leq \alpha_4 \eta N T + (\alpha_5 \eta N \gamma^{4\mathbb{M}} + \alpha_6 \gamma^{2\mathbb{M}}) \text{poly}(\|x_0\|, \bar{\pi}_0) + \tau \log N/\eta + \alpha_7 J$$

where $\gamma = L_f(1+L_\pi)\kappa$, $\alpha_4, \ldots, \alpha_7$ are polynomials of $L_f, L_{c1}, L_{c2}, c_0, L_\pi, \kappa, \beta w_{\max}, \frac{1}{1-\rho}, \frac{1}{1-2^{3/4}\kappa\rho^\tau}$.

Corollary 11 (Regret bound order) *Let $\eta = O(\frac{1}{N^{2/3}T^{1/3}})$ and $\tau = \max(T^{1/3}N^{-1/3}, \frac{\log(2\sqrt{2}\kappa)}{-\log\rho})$. When $T \geq N$, we have*

$$\text{PolicyRegret} \leq \tilde{O}(N^{1/3}T^{2/3}) + \exp(O(\mathbb{M})),$$

where $\tilde{O}(\cdot)$ hides a $\log(N)$ factor.

Corollary 11 shows the order of our regret bound under proper conditions. The first term $\tilde{O}(N^{1/3}T^{2/3})$ is common in the policy regret bound of online bandit learning with memory and has been shown to be the optimal regret order (Dekel et al., 2014). Since online control is closely related to online learning with memory, $\tilde{O}(N^{1/3}T^{2/3})$ is likely to also be the optimal regret order for our online control setting. Obtaining a formal lower bound is our ongoing work.

Notice that the exponential term $\exp(O(\mathbb{M}))$ does not depend on the horizon T, so for large enough T, our average regret bound PolicyRegret$/T$ scales as $O(1/T^{1/3})$, which diminishes to 0. This indicates that our algorithm can almost recover the optimal performance of the controllers in $\mathbb{B}$

after learning long enough. It is also worth mentioning that such an exponential term appears in other online control settings without a stabilization assumption. For example, in (Chen and Hazan, 2021), the exponential term depends on system dimensionality in a setting with linear systems and *linear* controllers, while our exponential term depends on the number of unstabilizing controllers since we do not have knowledge or restrictions on the controller structures. It is our future work to also consider controller structures to improve the exponential term for nonlinear systems.

Proof sketch for Theorem 10. Our proof consists of two parts: we first bound an "auxiliary regret" of our algorithm, and then bound the difference between the auxiliary regret and the policy regret.

Lemma 12 (Auxiliary regret bound) *Define the* auxiliary regret *of Algorithm 1 as*

$$\text{AuxRegret}(\mathcal{A}) = \tau\, \mathbb{E}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} g_j(I_j; I_{j-1:0}) - \min_{k\in\mathcal{B}} \tau\, \mathbb{E}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} g_j(k; I_{j-1:0}),$$

where I_j is seleted by algorithm $\mathcal{A}$. Under the conditions in Theorem 10, we have

$$\text{AuxRegret} \leq \alpha_4 \eta N T + \alpha_5 \eta N (L_f(1+L_\pi)\kappa)^{4\mathbb{M}} \text{poly}(\|x_0\|, \bar{\pi}_0) + \tau \log N/\eta.$$

Lemma 13 (Difference between auxiliary regret and policy regret) *Under the conditions in Theorem 10, we have*

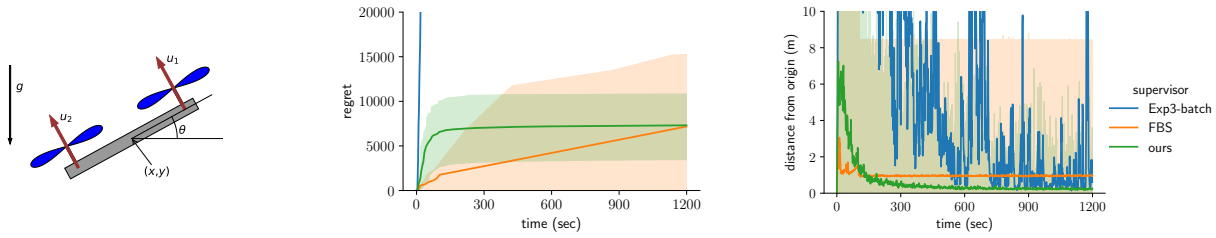
$$\text{PolicyRegret} \leq \text{AuxRegret} + \alpha_6 (L_f(1+L_\pi)\kappa)^{2\mathbb{M}} \text{poly}(\|x_0\|, \bar{\pi}_0) + \alpha_7 J.$$

The proof of Theorem 10 follows by combining the bounds in Lemma 12 and 13. The detailed proofs of the lemmas are deferred to (Li et al., 2022). We only discuss some high-level ideas below. First, auxiliary regret allows the regret benchmark to depend on the same history as that of our algorithm. It is simply called "regret" in the classical online learning setting when the cost does not depend on the history decisions. Therefore, we can borrow ideas from the regret bound proof for standard Exp3 to prove Lemma 12. However, standard Exp3 assumes uniformly bounded costs, while our problem suffers unbounded costs. To address this issue, we leverage the state bounds in Theorem 9. One technical contribution is that we bound the auxiliary regret by the bound on the total cost, $\sum_j g_j(I_j, I_{j-1:0})$, instead of the uniform bound on $g_j(I_j, I_{j-1:0})$ as in the literature (Lin et al., 2022; Arora et al., 2012). This is because the uniform bound on the cost scales as $\exp(O(\mathbb{M}))$, so directly applying this uniform bound will lead to a regret bound of order $\exp(O(\mathbb{M}))T^{2/3}$, which is much worse than our current bound $\tilde{O}(T^{2/3}) + \exp(O(\mathbb{M}))$. In fact, the uniform bound is not ideal in our case because we only suffer large states during the transient phase and enjoy small states after unstabilizing controllers are de-activated, which is also reflected in our numerical results.

Second, Lemma 13 is the only lemma that utilizes Definition 2, which establishes how fast the current state 'forgets' the history. When the current state does not depend on the history, the auxiliary regret and the policy regret are identical. Under Definition 2, the current state forgets the history exponentially fast, so by having a long enough batch size, we can bound the difference between the auxiliary regret and the policy regret. Details are in the supplementary (Li et al., 2022).

5. Numerical experiments

This section provides simulation results on a planar "quadrotor" illustrated in Figure 2(a) (Tedrake, 2022). We consider state $(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta})$, where (x, y) denotes the position and θ denotes the angle, and control inputs (u_1, u_2) from the two propellers. The dynamics are $m\ddot{x} = -(u_1 + u_2)\sin\theta$, $m\ddot{y} = -(u_1+u_2)\cos\theta - mg$, $I\ddot{\theta} = r(u_1 - u_2)$, where m is the mass, I is the moment



(a) Planar quadrotor (b) Policy regret (c) Distance trajectories

Figure 2: Comparison of Algorithm 1 with Exp3-batch in (Lin et al., 2022) and Falsification-Based Switching (FBS) in (Al-Shyoukh and Shamma, 2009) on a simulated planar quadrotor. The solid lines represent the mean value over 100 trials. The shaded regions in (b) and (c) represent the 75% percentile and the min/max over every trial, respectively.

of inertia, and r is the arm length. Our task is to fly the quadrotor towards a target. We consider 81 proportional–derivative candidate controllers as in (Lee et al., 2010), whose parameters include gains $(k_p, k_d, k_p^\theta, k_d^\theta)$ on the position and attitude, and estimations of m, I, r. We consider inaccurate estimation of m to test the robustness of our algorithm. More details on the setting are deferred to (Li et al., 2022) due to space limits.

Figure 2(b-c) compare our Exp3-ISS with Exp3-batch in (Lin et al., 2022) and Falsification-based Switching (FBS), which focuses on the stability and does not optimize the cost (Al-Shyoukh and Shamma, 2009). When comparing our algorithm with Exp3-batch, we can observe that Exp3-batch performs much worse than our algorithm in terms of both policy regret and the trajectories, with large spikes and fluctuations in the trajectory plot. When comparing our algorithm with FBS, we observe that, although FBS performs better than Exp3-ISS at the beginning, FBS generates a linearly increasing regret in expectation, which while Exp3-ISS enjoys regret sublinear in T. This is because FBS "settles" on the first stabilizing controller it identifies and does not explore to find better controllers. Therefore, unless nearly all of the controllers are unstabilizing, FBS avoids high cost of exploration at the beginning. However, since FBS essentially selects one stabilizing controller at random, linear regret is unavoidable unless FBS selects the optimal stabilizing controller by random chance. Figure 2(c) shows similar trends: though our algorithm generates larger distances at the beginning, our distances quickly diminishes to be smaller than FBS after enough exploration.

6. Conclusion and future directions

This paper proposes an online switching control algorithm by integrating the adversarial bandit algorithm Exp3 with a stability certification. Our algorithm stabilizes the system and provides sublinear policy regret despite the existence of unstabilizing candidate controllers. There are many interesting future directions, e.g., (i) discussing output feedback, where the stability certification in (Al-Shyoukh and Shamma, 2009) might be useful, (ii) considering an infinite or continuous policy pool by leveraging problem structure and continuity, (iii) fundamental regret and stability lower bounds for online switching control, (iv) time-varying dynamics where switching policies is necessary for stabilizing the system, (v) relaxing the global exponential stability assumptions to local and/or asymptotic stability, and (vi) combining switching-based control with estimation-based control as in multi-model adaptive control, etc.

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Appendices

Notations for the appendices: Denote $t_J = T+1$. Let $\mathbb{1}_S$ denote an indicator function on set S, i.e., $\mathbb{1}_S(x) = 1$ if and only if $x \in S$. In addition, let $1 \le t_{j_1}, \dots, t_{j_M} \le T$ denote the time indices when Line 6 of Algorithm 1 is activated, i.e., $\|x_{t_{j_s}}\| > \kappa\rho^{t_{j_s}-t_{j_s-1}}\|x_{t_{j_s-1}}\| + \beta w_{\max}$ for $1 \le s \le M$. Notice that $1 \le j_1, \dots, j_M \le J-1$. Also notice that j_s indicates that the previous episode $j_s - 1$ terminates by the **Break** statement. For simplicity, we denote $j_0 = 0$ and $j_{M+1} = J$, thus, $t_{j_0} = 0$ and $t_{j_{M+1}} = T+1$.

Appendix A. Proof of Lemma 8

Proof Suppose there are M episodes that terminate when the condition in Line 6 is true in Algorithm 1, then the number of episodes satisfies $J \le \lceil \frac{T-M}{\tau} \rceil + M$.

Notice that the upper bound $\lceil \frac{T-M}{\tau} \rceil + M$ increases with M when $\tau \ge 1$. Further, notice that $M \le \mathbb{M}$ by Definition 7 and Line 6 of Algorithm 1. Therefore, we obtain $J \le \lceil \frac{T-\mathbb{M}}{\tau} \rceil + \mathbb{M}$. ■

Appendix B. Stability analysis: proof of Theorem 9 and supportive lemmas

In the following, we are going to prove not only Theorem 9 but also finite-gain l_4 stability, that is,

$$\sum_{t=0}^{T} \|x_t\|^4 = O((L_f(1+L_\pi)\kappa)^{4\mathbb{M}}\|x_0\|^4 + \beta^4 w_{\max}^4 (T+J) + (L_f(1+L_\pi)\kappa)^{4\mathbb{M}}(\beta^4 w_{\max}^4 + \bar{\pi}_0^4)), \quad (2)$$

which will be useful for our regret analysis.

To prove these finite-gain stability properties, we will first provide a sequence of supportive lemmas on the bounds of the states, which will also be useful for the regret analysis.

B.1. Supportive lemmas on the bounds of states in Algorithm 1

In this subsection, we provide supportive lemmas on the bounds of the states generated by Algorithm 1. We will discuss the bounds in the l_2 norm, l_2 norm squared, and l_2 norm quartic, which will be used to prove finite-gain l_1, l_2, and l_4 stability, as well as the regret bounds.

Lemma 14 (Bounds on states in a single episode) *In Algorithm 1, at each episode $0 \le j \le J-1$, for $t_j \le t \le t_{j+1} - 1$, we have*

$$\|x_t\| \le \kappa\rho^{t-t_j}\|x_{t_j}\| + \beta w_{\max}\mathbb{1}_{(t>t_j)}.$$

Consequently, we have

$$\sum_{t=t_j}^{t_{j+1}-1} \|x_t\| \le \frac{\kappa}{1-\rho}\|x_{t_j}\| + \beta w_{\max}(t_{j+1} - t_j - 1)$$

$$\sum_{t=t_j}^{t_{j+1}-1} \|x_t\|^2 \le \frac{2\kappa^2}{1-\rho^2}\|x_{t_j}\|^2 + 2\beta^2 w_{\max}^2(t_{j+1} - t_j - 1)$$

$$\sum_{t=t_j}^{t_{j+1}-1} \|x_t\|^4 \le \frac{8\kappa^4}{1-\rho^4}\|x_{t_j}\|^4 + 8\beta^4 w_{\max}^4(t_{j+1} - t_j - 1)$$

Proof At episode $0 \le j \le J-1$, no matter whether Algorithm 1 breaks at t_{j+1} or not, for $t_j \le t \le t_{j+1}-1$, we have

$$\|x_t\| \le \kappa\rho^{t-t_j}\|x_{t_j}\| + \beta w_{\max}\mathbb{1}_{(t>t_j)},$$

which is the first statement of this lemma. By Hölder's inequality, we obtain the following inequalities.

$$\|x_t\|^2 \le 2\kappa^2(\rho^2)^{t-t_j}\|x_{t_j}\|^2 + 2\beta^2 w_{\max}^2\mathbb{1}_{(t>t_j)}$$
$$\|x_t\|^4 \le 8\kappa^4(\rho^4)^{t-t_j}\|x_{t_j}\|^4 + 8\beta^4 w_{\max}^4\mathbb{1}_{(t>t_j)}$$

Consequently, by summing the three inequalities above over $t = t_j, \dots, t_{j+1}-1$, we obtain the following.

$$\sum_{t=t_j}^{t_{j+1}-1}\|x_t\| \le \frac{\kappa}{1-\rho}\|x_{t_j}\| + \beta w_{\max}(t_{j+1}-t_j-1)$$
$$\sum_{t=t_j}^{t_{j+1}-1}\|x_t\|^2 \le \frac{2\kappa^2}{1-\rho^2}\|x_{t_j}\|^2 + 2\beta^2 w_{\max}^2(t_{j+1}-t_j-1)$$
$$\sum_{t=t_j}^{t_{j+1}-1}\|x_t\|^4 \le \frac{8\kappa^4}{1-\rho^4}\|x_{t_j}\|^4 + 8\beta^4 w_{\max}^4(t_{j+1}-t_j-1)$$

■

Lemma 15 (Relation of states in two consecutive episodes) *For $0 \le j \le J-2$, if Algorithm 1 activates the **Break** statement at $x_{t_{j+1}}$, then*

$$\|x_{t_{j+1}}\| \le L_f(1+L_\pi)\kappa\rho^{t_{j+1}-t_j-1}\|x_{t_j}\| + L_f((1+L_\pi)\beta w_{\max} + w_{\max} + \bar{\pi}_0)$$
$$\|x_{t_{j+1}}\|^2 \le 2L_f^2(1+L_\pi)^2\kappa^2(\rho^2)^{t_{j+1}-t_j-1}\|x_{t_j}\|^2 + 2L_f^2((1+L_\pi)\beta w_{\max} + w_{\max} + \bar{\pi}_0)^2$$
$$\|x_{t_{j+1}}\|^4 \le 8L_f^4(1+L_\pi)^4\kappa^4(\rho^4)^{t_{j+1}-t_j-1}\|x_{t_j}\|^4 + 8L_f^4((1+L_\pi)\beta w_{\max} + w_{\max} + \bar{\pi}_0)^4$$

*If Algorithm 1 does not activate the **Break** statement at $x_{t_{j+1}}$, then*

$$\|x_{t_{j+1}}\| \le \kappa\rho^{t_{j+1}-t_j}\|x_{t_j}\| + \beta w_{\max}$$
$$\|x_{t_{j+1}}\|^2 \le 2\kappa^2(\rho^2)^{t_{j+1}-t_j}\|x_{t_j}\|^2 + 2\beta^2 w_{\max}^2$$
$$\|x_{t_{j+1}}\|^4 \le 8\kappa^4(\rho^4)^{t_{j+1}-t_j}\|x_{t_j}\|^4 + 8\beta^4 w_{\max}^4$$

Proof Firstly, we consider the scenario where Algorithm 1 activates the **Break** statement at $x_{t_{j+1}}$. By Assumption 2, we have

$$\|x_{t+1}\| = \|f(x_t,u_t,w_t) - f(0,0,0)\| \le L_f(\|x_t\| + \|u_t\| + \|w_t\|).$$

Further, by Assumption 1, we have

$$\|u_t\| \le \|u_t - \pi_i(0)\| + \|\pi_i(0)\| \le L_\pi\|x_t\| + \bar{\pi}_0.$$

Combining the two inequalities above yield the following.

$$\|x_{t+1}\| \le L_f(1+L_\pi)\|x_t\| + L_f w_{\max} + L_f \bar{\pi}_0.$$

Consequently, together with Lemma 14, we have

$$\begin{aligned}\|x_{t_{j+1}}\| &\le L_f(1+L_\pi)\|x_{t_{j+1}-1}\| + L_f w_{\max} + L_f\bar{\pi}_0 \\ &\le L_f(1+L_\pi)\kappa\rho^{t_{j+1}-t_j-1}\|x_{t_j}\| + L_f(1+L_\pi)\beta w_{\max} + L_f w_{\max} + L_f\bar{\pi}_0.\end{aligned}$$

Therefore,

$$\begin{aligned}\|x_{t_{j+1}}\|^2 &\le 2L_f^2(1+L_\pi)^2\kappa^2(\rho^2)^{t_{j+1}-t_j-1}\|x_{t_j}\|^2 + 2L_f^2((1+L_\pi)\beta w_{\max} + w_{\max} + \bar{\pi}_0)^2, \\ \|x_{t_{j+1}}\|^4 &\le 8L_f^4(1+L_\pi)^4\kappa^4(\rho^4)^{t_{j+1}-t_j-1}\|x_{t_j}\|^4 + 8L_f^4((1+L_\pi)\beta w_{\max} + w_{\max} + \bar{\pi}_0)^4.\end{aligned}$$

Secondly, we consider the scenario where Algorithm 1 does not activate the **Break** statement at $x_{t_{j+1}}$. Similarly to Lemma 14, we have

$$\begin{aligned}\|x_{t_{j+1}}\| &\le \kappa\rho^{t_{j+1}-t_j}\|x_{t_j}\| + \beta w_{\max}, \\ \|x_{t_{j+1}}\|^2 &\le 2\kappa^2(\rho^2)^{t_{j+1}-t_j}\|x_{t_j}\|^2 + 2\beta^2 w_{\max}^2, \\ \|x_{t_{j+1}}\|^4 &\le 8\kappa^4(\rho^4)^{t_{j+1}-t_j}\|x_{t_j}\|^4 + 8\beta^4 w_{\max}^4.\end{aligned}$$

■

Now, we are ready to bound the states by discussing the **Break** activation stages $t_{j_1}, \dots, t_{j_M}$, which were initially defined in the **Additional notations** at the beginning of the appendices.

Lemma 16 (Bounds on states between two Break activations) *Denote* $\gamma_0 = 2\kappa^2\rho^{2\tau}$. *Suppose* $2\gamma_0^2 < 1$, *i.e.,* $\tau \ge \frac{\log(2\sqrt{2}\kappa)}{-\log\rho}$. *For* $1 \le s \le M+1$, *for* $j_{s-1}+1 \le j \le j_s - 1$, *we have*

$$\begin{aligned}\|x_{t_j}\| &\le \left(\sqrt{\gamma_0/2}\right)^{j-j_s-1}\|x_{t_{j_{s-1}}}\| + \frac{\beta w_{\max}}{1-\sqrt{\gamma_0/2}} \\ \|x_{t_j}\|^2 &\le \gamma_0^{j-j_{s-1}}\|x_{t_{j_{s-1}}}\|^2 + \frac{2\beta^2 w_{\max}^2}{1-\gamma_0} \\ \|x_{t_j}\|^4 &\le (2\gamma_0^2)^{j-j_{s-1}}\|x_{t_{j_{s-1}}}\|^4 + \frac{8\beta^4 w_{\max}^4}{1-2\gamma_0^2}\end{aligned}$$

Consequently,

$$\begin{aligned}\sum_{j=j_{s-1}}^{j_s-1}\|x_{t_j}\| &\le \frac{1}{1-\sqrt{\gamma_0/2}}\|x_{t_{j_{s-1}}}\| + \frac{\beta w_{\max}}{1-\sqrt{\gamma_0/2}}(j_s - j_{s-1} - 1) \\ \sum_{j=j_{s-1}}^{j_s-1}\|x_{t_j}\|^2 &\le \frac{1}{1-\gamma_0}\|x_{t_{j_{s-1}}}\|^2 + \frac{2\beta^2 w_{\max}^2}{1-\gamma_0}(j_s - j_{s-1} - 1) \\ \sum_{j=j_{s-1}}^{j_s-1}\|x_{t_j}\|^4 &\le \frac{1}{1-2\gamma_0^2}\|x_{t_{j_{s-1}}}\|^4 + \frac{8\beta^4 w_{\max}^4}{1-2\gamma_0^2}(j_s - j_{s-1} - 1)\end{aligned}$$

Proof For $j_{s-1}+1 \le j \le j_s - 1$, x_{t_j} does not activate **Break**, hence, we can apply the second scenario in Lemma 15 and obtain

$$\|x_{t_j}\| \le \sqrt{\frac{\gamma_0}{2}}\|x_{t_{j-1}}\| + \beta w_{\max} \le \left(\sqrt{\frac{\gamma_0}{2}}\right)^{j-j_{s-1}} \|x_{t_{j_{s-1}}}\| + \frac{\beta w_{\max}}{1-\sqrt{\gamma_0/2}}$$

Similarly, we have $\|x_{t_j}\|^2 \le 2\kappa^2\rho^{2\tau}\|x_{t_{j-1}}\|^2 + 2\beta^2 w_{\max}^2 = \gamma_0\|x_{t_{j-1}}\|^2 + 2\beta^2 w_{\max}^2 \le \gamma_0^{j-j_{s-1}}\|x_{t_{j_{s-1}}}\|^2 + \frac{2\beta^2 w_{\max}^2}{1-\gamma_0}$, and $\|x_{t_j}\|^4 \le 8\kappa^4\rho^{4\tau}\|x_{t_{j-1}}\|^4 + 8\beta^4 w_{\max}^4 = 2\gamma_0^2\|x_{t_{j-1}}\|^4 + 8\beta^4 w_{\max}^4 \le (2\gamma_0^2)^{j-j_{s-1}}\|x_{t_{j_{s-1}}}\|^4 + \frac{8\beta^4 w_{\max}^4}{1-2\gamma_0^2}$. Then, by summing over j, we complete the proof. ■

Lemma 17 (Relation of states at two consecutive Break activations) *Define* $\gamma_1 = 2L_f^2(1+L_\pi)^2\kappa^2$. *For* $1 \le s \le M$,

$$\begin{aligned}
\|x_{t_{j_s}}\| &\le \sqrt{\frac{\gamma_1}{2}}\sqrt{\frac{\gamma_0}{2}}^{j_s - j_{s-1}-1}\|x_{t_{j_{s-1}}}\| + \alpha_8 \\
\|x_{t_{j_s}}\|^2 &\le \gamma_1\gamma_0^{j_s-j_{s-1}-1}\|x_{t_{j_{s-1}}}\|^2 + \alpha_9 \\
\|x_{t_{j_s}}\|^4 &\le 2\gamma_1^2(2\gamma_0^2)^{j_s-j_{s-1}-1}\|x_{t_{j_{s-1}}}\|^4 + \alpha_{10}
\end{aligned}$$

where $\alpha_8 = \sqrt{\frac{\gamma_1}{2}}\frac{\beta w_{\max}}{1-\sqrt{\gamma_0/2}} + L_f((1+L_\pi)\beta w_{\max} + w_{\max} + \bar{\pi}_0)$, $\alpha_9 = \frac{2\gamma_1}{1-\gamma_0}\beta^2 w_{\max}^2 + 2L_f^2((1+L_\pi)\beta w_{\max} + \bar{\pi}_0 + w_{\max})^2$, *and* $\alpha_{10} = 8L_f^4((1+L_\pi)\beta w_{\max} + \bar{\pi}_0 + w_{\max})^4 + \frac{16\gamma_1^2}{1-2\gamma_0^2}\beta^4 w_{\max}^4$.

Proof By Lemma 15 and Lemma 16, at t_{j_s}, we have

$$\begin{aligned}
\|x_{t_{j_s}}\| &\le L_f(1+L_\pi)\kappa\rho^{t_{j_s}-t_{j_s-1}}\|x_{t_{j_s-1}}\| + L_f((1+L_\pi)\beta w_{\max} + w_{\max} + \bar{\pi}_0) \\
&\le \sqrt{\frac{\gamma_1}{2}}\|x_{t_{j_s-1}}\| + L_f((1+L_\pi)\beta w_{\max} + w_{\max} + \bar{\pi}_0) \\
&\le \sqrt{\frac{\gamma_1}{2}}\sqrt{\frac{\gamma_0}{2}}^{j_s-j_{s-1}-1}\|x_{t_{j_{s-1}}}\| + \sqrt{\frac{\gamma_1}{2}}\frac{\beta w_{\max}}{1-\sqrt{\gamma_0/2}} + L_f((1+L_\pi)\beta w_{\max} + w_{\max} + \bar{\pi}_0)
\end{aligned}$$

Similarly, we can complete the proof by the following.

$$\begin{aligned}
\|x_{t_{j_s}}\|^2 &\le 2L_f^2(1+L_\pi)^2\kappa^2(\rho^2)^{t_{j_s}-t_{j_s-1}}\|x_{t_{j_s-1}}\|^2 + 2L_f^2((1+L_\pi)\beta w_{\max} + w_{\max} + \bar{\pi}_0)^2 \\
&\le \gamma_1\|x_{t_{j_s-1}}\|^2 + 2L_f^2((1+L_\pi)\beta w_{\max} + w_{\max} + \bar{\pi}_0)^2 \\
&\le \gamma_1\gamma_0^{j_s-1-j_{s-1}}\|x_{t_{j_{s-1}}}\|^2 + \gamma_1\frac{2\beta^2 w_{\max}^2}{1-\gamma_0} + 2L_f^2((1+L_\pi)\beta w_{\max} + w_{\max} + \bar{\pi}_0)^2 \\
\|x_{t_{j_s}}\|^4 &\le 8L_f^4(1+L_\pi)^4\kappa^4(\rho^4)^{t_{j+1}-t_j-1}\|x_{t_{j_s-1}}\|^4 + 8L_f^4((1+L_\pi)\beta w_{\max} + w_{\max} + \bar{\pi}_0)^4 \\
&\le 2\gamma_1^2\|x_{t_{j_s-1}}\|^4 + 8L_f^4((1+L_\pi)\beta w_{\max} + w_{\max} + \bar{\pi}_0)^4 \\
&\le 2\gamma_1^2(2\gamma_0^2)^{j_s-1-j_{s-1}}\|x_{t_{j_{s-1}}}\|^4 + 2\gamma_1^2\frac{8\beta^4 w_{\max}^4}{1-2\gamma_0^2} + 8L_f^4((1+L_\pi)\beta w_{\max} + w_{\max} + \bar{\pi}_0)^4
\end{aligned}$$

■

Next, we can bound the starts of episodes by the following.

Lemma 18 (Bounds on the initial states of episodes)

$$\sum_{j=0}^{J-1}\|x_{t_j}\| \le \frac{1}{1-\sqrt{\gamma_0/2}}\frac{\sqrt{\frac{\gamma_1}{2}}^{\mathbb{M}+1}-1}{\sqrt{\frac{\gamma_1}{2}}-1}(\|x_0\|+\frac{\alpha_8}{\sqrt{\frac{\gamma_1}{2}}-1})+\frac{\beta w_{\max}}{1-\sqrt{\frac{\gamma_0}{2}}}J$$
$$\sum_{j=0}^{J-1}\|x_{t_j}\|^2 \le \frac{1}{1-\gamma_0}\frac{\gamma_1^{\mathbb{M}+1}-1}{\gamma_1-1}(\|x_0\|^2+\frac{\alpha_9}{\gamma_1-1})+\frac{2\beta^2 w_{\max}^2}{1-\gamma_0}J$$
$$\sum_{j=0}^{J-1}\|x_{t_j}\|^4 \le \frac{1}{1-2\gamma_0^2}\frac{(2\gamma_1^2)^{+1}-1}{2\gamma_1^2-1}(\|x_0\|^4+\frac{\alpha_{10}}{2\gamma_1^2-1})+\frac{8\beta^4 w_{\max}^4}{1-2\gamma_0^2}J$$

Proof Let's first focus on $\sum_{j=0}^{J-1}\|x_{t_j}\|$. By Lemma 16, we have the following.

$$\begin{aligned}\sum_{j=0}^{J-1}\|x_{t_j}\| &= \sum_{s=0}^{M}\sum_{j=j_s}^{j_{s+1}-1}\|x_{t_j}\| \\ &\le \sum_{s=0}^{M}\left[\frac{1}{1-\sqrt{\gamma_0/2}}\|x_{t_{j_s}}\|+\frac{\beta w_{\max}}{1-\sqrt{\gamma_0/2}}(j_{s+1}-j_s-1)\right] \\ &= \frac{1}{1-\sqrt{\gamma_0/2}}\sum_{s=0}^{M}\|x_{t_{j_s}}\|+\frac{\beta w_{\max}}{1-\sqrt{\gamma_0/2}}J \\ &\le \frac{1}{1-\sqrt{\gamma_0/2}}\frac{\sqrt{\frac{\gamma_1}{2}}^{\mathbb{M}+1}-1}{\sqrt{\frac{\gamma_1}{2}}-1}(\|x_0\|+\frac{\alpha_8}{\sqrt{\frac{\gamma_1}{2}}-1})+\frac{\beta w_{\max}}{1-\sqrt{\gamma_0/2}}J\end{aligned}$$

where the last inequality is because of the following. For $1 \le s \le M$, by Lemma 17, we have

$$\|x_{t_{j_s}}\| \le \sqrt{\frac{\gamma_1}{2}}\sqrt{\frac{\gamma_0}{2}}^{j_s-j_{s-1}-1}\|x_{t_{j_{s-1}}}\|+\alpha_8 \le \sqrt{\frac{\gamma_1}{2}}\|x_{t_{j_{s-1}}}\|+\alpha_8,$$

where the equalities hold for all $1 \le s \le M$ when $j_s = s$ for $1 \le s \le M$, i.e., the first M episodes all activate **Break**. Consequently, we have $\|x_{t_{j_s}}\| \le \sqrt{\frac{\gamma_1}{2}}^{s}\|x_0\|+\alpha_8\frac{\sqrt{\frac{\gamma_1}{2}}^{s}}{\sqrt{\frac{\gamma_1}{2}}-1}$ and

$$\sum_{s=0}^{M}\|x_{t_{j_s}}\| \le \frac{\sqrt{\frac{\gamma_1}{2}}^{M+1}-1}{\sqrt{\frac{\gamma_1}{2}}-1}(\|x_0\|+\frac{\alpha_8}{\sqrt{\frac{\gamma_1}{2}}-1}) \le \frac{\sqrt{\frac{\gamma_1}{2}}^{\mathbb{M}+1}-1}{\sqrt{\frac{\gamma_1}{2}}-1}(\|x_0\|+\frac{\alpha_8}{\sqrt{\frac{\gamma_1}{2}}-1}),$$

where we used $M \le \mathbb{M}$ by Definition 7 and by Algorithm 1, and $\gamma_1/2 > 1$ because we assumed $\kappa \ge 1$ and $L_f \ge 1$ for simplicity.

Similarly, by Lemma 16 and Lemma 17, we can complete the proof by the following.

$$\sum_{j=0}^{J-1}\|x_{t_j}\|^2 \le \frac{1}{1-\gamma_0}\sum_{s=0}^{M}\|x_{t_{j_s}}\|^2+\frac{2\beta^2 w_{\max}^2}{1-\gamma_0}J$$

$$\leq \frac{1}{1-\gamma_0}\frac{\gamma_1^{\mathbb{M}+1}-1}{\gamma_1-1}(\|x_0\|^2+\frac{\alpha_9}{\gamma_1-1})+\frac{2\beta^2 w_{\max}^2}{1-\gamma_0}J$$

$$\sum_{j=0}^{J-1}\|x_{t_j}\|^4 \leq \frac{1}{1-2\gamma_0^2}\sum_{s=0}^{M}\|x_{t_{j_s}}\|^4+\frac{8\beta^4 w_{\max}^4}{1-2\gamma_0^2}J$$

$$\leq \frac{1}{1-2\gamma_0^2}\frac{(2\gamma_1^2)^{\mathbb{M}+1}-1}{2\gamma_1^2-1}(\|x_0\|^4+\frac{\alpha_{10}}{2\gamma_1^2-1})+\frac{8\beta^4 w_{\max}^4}{1-2\gamma_0^2}J$$

■

B.2. Proof of Theorem 9 and proof of (2)

The proof is straightforward from Lemma 14 and Lemma 18. Let's first consider $\sum_{t=0}^{T}\|x_t\|$.

$$\sum_{t=0}^{T}\|x_t\| \leq \sum_{j=0}^{J-1}\sum_{t=t_j}^{t_{j+1}-1}\|x_t\|$$

$$\leq \sum_{j=0}^{J-1}\left[\frac{\kappa}{1-\rho}\|x_{t_j}\|+\beta w_{\max}(t_{j+1}-t_j-1)\right]$$

$$\leq \frac{\kappa}{1-\rho}\sum_{j=0}^{J-1}\|x_{t_j}\|+\beta w_{\max}T$$

$$\leq \beta w_{\max}T+\frac{\kappa}{1-\rho}\left(\frac{1}{1-\sqrt{\gamma_0/2}}\frac{\sqrt{\frac{\gamma_1}{2}}^{\mathbb{M}+1}-1}{\sqrt{\frac{\gamma_1}{2}}-1}(\|x_0\|+\frac{\alpha_8}{\sqrt{\frac{\gamma_1}{2}}-1})+\frac{\beta w_{\max}}{1-\sqrt{\frac{\gamma_0}{2}}}J\right)$$

$$= \beta w_{\max}(T+\alpha_1 J)+\alpha_2(L_f(1+L_\pi)\kappa)^{\mathbb{M}}\|x_0\|+\alpha_3(L_f(1+L_\pi)\kappa)^{\mathbb{M}}(\beta w_{\max}+\bar{\pi}_0)$$

where the last equality is because we defined $\gamma_0 = 2\kappa^2\rho^{2\tau}$, $\gamma_1 = 2L_f^2(1+L_\pi)^2\kappa^2$, $\alpha_8 \leq (L_f(1+L_\pi)\kappa/(1-\kappa\rho^\tau)+L_f(2+L_\pi))(\beta w_{\max}+\bar{\pi}_0)$, and $\frac{1}{\sqrt{\frac{\gamma_1}{2}}-1}\leq 1$.

Similarly, we can complete the proof by the following.

$$\sum_{t=0}^{T}\|x_t\|^2 \leq \sum_{j=0}^{J-1}\sum_{t=t_j}^{t_{j+1}-1}\|x_t\|^2$$

$$\leq \sum_{j=0}^{J-1}\left[\frac{2\kappa^2}{1-\rho^2}\|x_{t_j}\|^2+2\beta^2 w_{\max}^2(t_{j+1}-t_j-1)\right]$$

$$\leq \frac{2\kappa^2}{1-\rho^2}\sum_{j=0}^{J-1}\|x_{t_j}\|^2+2\beta^2 w_{\max}^2 T$$

$$\leq \frac{2\kappa^2}{1-\rho^2}\left(\frac{1}{1-\gamma_0}\frac{\gamma_1^{\mathbb{M}+1}-1}{\gamma_1-1}(\|x_0\|^2+\frac{\alpha_9}{\gamma_1-1})+\frac{2\beta^2 w_{\max}^2}{1-\gamma_0}J\right)+2\beta^2 w_{\max}^2 T$$

$$\leq 2\beta^2 w_{\max}^2(T+\alpha_{11}J)+\alpha_{12}(L_f(1+L_\pi)\kappa)^{2\mathbb{M}}\|x_0\|^2+\alpha_{13}(L_f(1+L_\pi)\kappa)^{2\mathbb{M}}(\beta^2 w_{\max}^2+\bar{\pi}_0^2) \tag{3}$$

$$
\begin{aligned}
\sum_{t=0}^{T}\|x_t\|^4 &\le \sum_{j=0}^{J-1}\sum_{t=t_j}^{t_{j+1}-1}\|x_t\|^4 \\
&\le \sum_{j=0}^{J-1}\left[\frac{8\kappa^4}{1-\rho^4}\|x_{t_j}\|^4 + 8\beta^4 w_{\max}^4(t_{j+1}-t_j-1)\right] \\
&\le \frac{8\kappa^4}{1-\rho^4}\sum_{j=0}^{J-1}\|x_{t_j}\|^4 + 8\beta^4 w_{\max}^4 T \\
&\le \frac{8\kappa^4}{1-\rho^4}\frac{1}{1-2\gamma_0^2}\frac{(2\gamma_1^2)^{\mathbb{M}+1}-1}{2\gamma_1^2-1}(\|x_0\|^4+\frac{\alpha_{10}}{2\gamma_1^2-1}) + \frac{8\kappa^4}{1-\rho^4}\frac{8\beta^4 w_{\max}^4}{1-2\gamma_0^2}J + 8\beta^4 w_{\max}^4 T \\
&\le 8\beta^4 w_{\max}^4(T+\alpha_{14}J) + \alpha_{15}(L_f(1+L_\pi)\kappa)^{4\mathbb{M}}\|x_0\|^4 + \alpha_{16}(L_f(1+L_\pi)\kappa)^{4\mathbb{M}}(\beta^4 w_{\max}^4+\bar{\pi}_0^4)
\end{aligned} \tag{4}
$$

where $\alpha_{11},\dots,\alpha_{16}$ are polynomials of κ, $\frac{1}{1-\rho}$, $\frac{1}{1-2^{3/4}\kappa\rho^\tau}$, L_f, L_π.

Appendix C. Regret analysis: proofs of Theorem 10, Lemma 12, Lemma 13, and Corollary 11

Notice that the proof of Theorem 10 is straightforward from combining Lemma 12 and Lemma 13. Further, notice that the proof of Corollary 11 is straightforward by plugging in the choices of algorithm parameters in Corollary 11 and by using Lemma 8 to obtain $J \le O(T/\tau+\mathbb{M})$. Therefore, it suffices to prove Lemma 12 and Lemma 13, which is detailed below.

C.1. Proof of Lemma 12

In this proof, we first introduce a supportive lemma, then divide AuxRegret into separate terms, and provide upper bounds on each terms, which will be combined to prove Lemma 12.

Firstly, we introduce the supportive lemma below.

Lemma 19 (Supportive lemma) *Conditioning on the natural filtration $\mathcal{F}(I_0,\dots,I_{J-1})$, we have*

$$
g_j(I_j; I_{j-1:0}) = \mathop{\mathbb{E}}_{i\sim p_j}\tilde{g}_j(i; I_{j:0}), \qquad \mathop{\mathbb{E}}_{I_{j:0}}\tilde{g}_j(k; I_{j:0}) = \mathop{\mathbb{E}}_{I_{j-1:0}} g_j(k; I_{j-1:0}),
$$

for any $k \in \mathcal{P}_j$.

Proof The first equality is proved by the following.

$$
\begin{aligned}
\mathop{\mathbb{E}}_{i\sim p_j}\tilde{g}_j(i; I_{j:0}) &= \sum_{i\in\mathcal{P}_j} p_j(i)\tilde{g}_j(i; I_{j:0}) \\
&= \sum_{i\in\mathcal{P}_j} p_j(i)\frac{g_j(i; I_{j-1:0})}{p_j(i)}\mathbb{1}_{(I_j=i)} \\
&= g_j(I_j; I_{j-1:0})
\end{aligned}
$$

The second equality is proved by the following.

$$
\mathop{\mathbb{E}}_{I_{j:0}}\tilde{g}_j(k; I_{j:0}) = \mathop{\mathbb{E}}_{I_{j-1:0}}\mathop{\mathbb{E}}_{I_j}[\tilde{g}_j(k; I_{j:0}) \mid I_{j-1:0}]
$$

$$= \mathop{\mathbb{E}}_{I_{j-1:0}} \sum_{I_j \in \mathcal{P}_j} p_j(I_j) \frac{g_j(k; I_{j-1:0})}{p_j(k)} \mathbb{1}_{(I_j=k)}$$
$$= \mathop{\mathbb{E}}_{I_{j-1:0}} g_j(k; I_{j-1:0})$$

■

Secondly, we divide AuxRegret into several terms that are convenient for proving upper bounds. For any $k \in \mathcal{B}$, we introduce

$$\text{AuxRegret}_k = \tau \mathop{\mathbb{E}}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} g_j(I_j; I_{j-1:0}) - \tau \mathop{\mathbb{E}}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} g_j(k; I_{j-1:0}). \tag{5}$$

Notice that it suffices to provide a uniform upper bound on AuxRegret_k for all $k \in \mathcal{B}$ in order to upper bound AuxRegret. Therefore, we will focus on AuxRegret_k in the rest of this proof. Notice that we can divide AuxRegret_k into several terms below by leveraging Lemma 19.

$$\text{AuxRegret}_k = \tau \mathop{\mathbb{E}}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} \mathop{\mathbb{E}}_{i\sim p_j} \tilde{g}_j(i; I_{j:0}) - \tau \mathop{\mathbb{E}}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} g_j(k; I_{j-1:0})$$

Further, by adding and subtracting a same term, we have

$$\mathop{\mathbb{E}}_{i\sim p_j} \tilde{g}_j(i; I_{j:0}) = \underbrace{\frac{1}{\eta} \log\left(\mathop{\mathbb{E}}_{i\sim p_j} \exp(-\eta \tilde{g}_j(i; I_{j:0}))\right) + \mathop{\mathbb{E}}_{i\sim p_j} \tilde{g}_j(i; I_{j-1:0})}_{\text{Term } 1_j}$$
$$\underbrace{-\frac{1}{\eta} \log\left(\mathop{\mathbb{E}}_{i\sim p_j} \exp(-\eta \tilde{g}_j(i; I_{j-1:0}))\right)}_{\text{Term } 2_j}.$$

Therefore, we can rewrite AuxRegret_k as the following.

$$\text{AuxRegret}_k = \tau \mathop{\mathbb{E}}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} \text{Term } 1_j + \tau \mathop{\mathbb{E}}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} \text{Term } 2_j - \tau \mathop{\mathbb{E}}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} g_j(k; I_{j-1:0}) \tag{6}$$

In the following lemmas, we provide upper bounds on $\mathbb{E}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} \text{Term } 1_j$ and $\mathbb{E}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} \text{Term } 2_j$. The upper bound on AuxRegret_k is straightforward by combining the upper bounds in Lemma 20 and Lemma 21, which completes the proof.

Lemma 20 (Bounds on the sum of Term 1_j)

$$\mathop{\mathbb{E}}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} \text{Term } 1_j \leq \frac{1}{\tau} \left(\alpha_4 \eta N T \text{poly}(\beta w_{\max}, \bar{\pi}_0) + \alpha_5 \eta N (L_f(1+L_\pi)\kappa)^{4\mathbb{M}} \text{poly}(\|x_0\|, \beta w_{\max}, \bar{\pi}_0)\right)$$

Lemma 21 (Bounds on the sum of Term 2_j)

$$\mathop{\mathbb{E}}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} \text{Term } 2_j \leq \sum_{j=0}^{J-1} \mathop{\mathbb{E}}_{I_{0:j-1}} g_j(k; I_{j-1:0}) + \frac{\log N}{\eta}$$

The proofs of Lemma 20 and Lemma 21 are provided in the following.

C.1.1. PROOF OF LEMMA 20

Firstly, we bound Term 1_j by the following.

$$\begin{aligned}
\text{Term } 1_j &\le \frac{1}{\eta}\left(\mathop{\mathbb{E}}_{i\sim p_j}\exp(-\eta\tilde{g}_j(i;I_{j-1:0}))-1\right)+\mathop{\mathbb{E}}_{i\sim p_j}\tilde{g}_j(i;I_{j-1:0})\\
&\le \frac{1}{\eta}\left(\mathop{\mathbb{E}}_{i\sim p_j}\frac{\eta^2}{2}\tilde{g}_j^2(i;I_{j-1:0})-\eta\tilde{g}_j(i;I_{j-1:0}))\right)+\mathop{\mathbb{E}}_{i\sim p_j}\tilde{g}_j(i;I_{j-1:0})\\
&= \frac{\eta}{2}\mathop{\mathbb{E}}_{i\sim p_j}\tilde{g}_j^2(i;I_{j-1:0})\\
&= \frac{\eta}{2}\sum_{i\in\mathcal{P}_j}p_j(i)\frac{g_j^2(i;I_{j-1:0})}{p_j^2(i)}\mathbb{1}_{(I_j=i)}\\
&= \frac{\eta}{2}\frac{g_j^2(I_j;I_{j-1:0})}{p_j(I_j)} \qquad (7)
\end{aligned}$$

Secondly, we bound $\mathbb{E}_{(I_j)_{j\ge0}}$ Term 1_j by the following.

$$\begin{aligned}
\mathop{\mathbb{E}}_{(I_j)_{j\ge0}}\text{Term } 1_j &\le \mathop{\mathbb{E}}_{(I_j)_{j\ge0}}\frac{\eta}{2}\frac{g_j^2(I_j;I_{j-1:0})}{p_j(I_j)}\\
&= \mathop{\mathbb{E}}_{I_{j-1:0}}\mathop{\mathbb{E}}_{I_j}\left(\frac{\eta}{2}\frac{g_j^2(I_j;I_{j-1:0})}{p_j(I_j)}\mid I_{j-1:0}\right)\\
&= \mathop{\mathbb{E}}_{I_{j-1:0}}\sum_{I_j\in\mathcal{P}_j}\left(p_j(I_j)\frac{\eta}{2}\frac{g_j^2(I_j;I_{j-1:0})}{p_j(I_j)}\right)\\
&= \mathop{\mathbb{E}}_{I_{j-1:0}}\sum_{I_j\in\mathcal{P}_j}\frac{\eta}{2}g_j^2(I_j;I_{j-1:0}) \qquad (8)
\end{aligned}$$

where the first inequality is by (7).

Next, we bound $g_j^2(i;I_{j-1:0})$ for any $i\in\mathcal{P}_j$. For any $i\in\mathcal{P}_j$, we let $\mathring{x}_{t_j},\dots,\mathring{x}_{t_{j+1}-1}$ denote the state trajectory generated by implementing policy π_i at episode j and implementing policy $\pi_{I_{j'}}$ at episode $0\le j'\le j-1$. Notice that $\mathring{x}_{t_j}=x_{t_j}$, where x_{t_j} is the state trajectory generated by Algorithm 1. We can bound $g_j^2(i;I_{j-1:0})$ by the following.

$$\begin{aligned}
g_j^2(i;I_{j-1:0}) &= \frac{1}{\tau^2}\left(\sum_{t=t_j}^{t_{j+1}-1}c_t(\mathring{x}_t,\mathring{u}_t)\right)^2\\
&\le \frac{1}{\tau^2}\sum_{t=t_j}^{t_{j+1}-1}c_t^2(\mathring{x}_t,\mathring{u}_t)(t_{j+1}-t_j)\\
&\le \frac{1}{\tau}\sum_{t=t_j}^{t_{j+1}-1}\left(2L_{c1}\|\mathring{x}_t\|^2+2L_{c1}\|\mathring{u}_t\|^2+L_{c2}\|\mathring{x}_t\|+L_{c2}\|\mathring{u}_t\|+c_0\right)^2\\
&\le \frac{5}{\tau}\sum_{t=t_j}^{t_{j+1}-1}\left(4L_{c1}^2(\|\mathring{x}_t\|^4+\|\mathring{u}_t\|^4)+L_{c2}^2(\|\mathring{x}_t\|^2+\|\mathring{u}_t\|^2)+c_0^2\right)
\end{aligned}$$

$$= \frac{20L_{c1}^2}{\tau} \sum_{t=t_j}^{t_{j+1}-1} (\|\mathring{x}_t\|^4 + \|\mathring{u}_t\|^4) + \frac{5L_{c2}^2}{\tau} \sum_{t=t_j}^{t_{j+1}-1} (\|\mathring{x}_t\|^2 + \|\mathring{u}_t\|^2) + \frac{5c_0^2}{\tau}(t_{j+1} - t_j),$$

where the second inequality is by

$$\begin{aligned}
|c_t(x,u)| &= |c_t(x,u) - c_t(0,0) + c_t(0,0)| \\
&\le |c_t(x,u) - c_t(0,0)| + c_0 \\
&\le L_{c1}(\|x\| + \|u\|)^2 + L_{c2}(\|x\| + \|u\|) + c_0 \\
&\le 2L_{c1}\|x\|^2 + 2L_{c1}\|u\|^2 + L_{c2}\|x\| + L_{c2}\|u\| + c_0.
\end{aligned}$$

By Assumption 1, we have $\|\mathring{u}_t\|^4 \le 8L_\pi^4\|\mathring{x}_t\|^4 + 8\bar{\pi}_0^4$ and $\|\mathring{u}_t\|^2 \le 2L_\pi^2\|\mathring{x}_t\|^2 + 2\bar{\pi}_0^2$. Hence, together with Lemma 14, we have

$$\begin{aligned}
\sum_{t=t_j}^{t_{j+1}-1} (\|\mathring{x}_t\|^4 + \|\mathring{u}_t\|^4) &\le \sum_{t=t_j}^{t_{j+1}-1} (\|\mathring{x}_t\|^4 + 8L_\pi^4\|\mathring{x}_t\|^4 + 8\bar{\pi}_0^4) \\
&= (1 + 8L_\pi^4) \sum_{t=t_j}^{t_{j+1}-1} \|\mathring{x}_t\|^4 + 8\bar{\pi}_0^4(t_{j+1} - t_j) \\
&\le (1 + 8L_\pi^4)\frac{8\kappa^4}{1-\rho^4}\|x_{t_j}\|^4 + (1 + 8L_\pi^4)8\beta^4 w_{\max}^4(t_{j+1} - t_j) + 8\bar{\pi}_0^4(t_{j+1} - t_j) \\
\sum_{t=t_j}^{t_{j+1}-1} (\|\mathring{x}_t\|^2 + \|\mathring{u}_t\|^2) &\le \sum_{t=t_j}^{t_{j+1}-1} (\|\mathring{x}_t\|^2 + 2L_\pi^2\|\mathring{x}_t\|^2 + 2\bar{\pi}_0^2) \\
&= (1 + 2L_\pi^2) \sum_{t=t_j}^{t_{j+1}-1} \|\mathring{x}_t\|^2 + 2\bar{\pi}_0^2(t_{j+1} - t_j) \\
&\le (1 + 2L_\pi^2)\frac{2\kappa^2}{1-\rho^2}\|x_{t_j}\|^2 + (1 + 2L_\pi^2)2\beta^2 w_{\max}^2(t_{j+1} - t_j) + 2\bar{\pi}_0^2(t_{j+1} - t_j)
\end{aligned}$$

By combining the inequalities above, we obtain the following uniform upper bound on $g_j^2(i; I_{j-1:0})$ for any $i \in \mathcal{P}_j$.

$$\begin{aligned}
g_j^2(i; I_{j-1:0}) &\le \frac{20L_{c1}^2}{\tau}(1 + 8L_\pi^4)\frac{8\kappa^4}{1-\rho^4}\|x_{t_j}\|^4 + \frac{20L_{c1}^2}{\tau}((1 + 8L_\pi^4)8\beta^4 w_{\max}^4 + 8\bar{\pi}_0^4)(t_{j+1} - t_j) \\
&\quad + \frac{5L_{c2}^2}{\tau}(1 + 2L_\pi^2)\frac{2\kappa^2}{1-\rho^2}\|x_{t_j}\|^2 + \frac{5L_{c2}^2}{\tau}((1 + 2L_\pi^2)2\beta^2 w_{\max}^2 + 2\bar{\pi}_0^2)(t_{j+1} - t_j) \\
&\quad + \frac{5c_0^2}{\tau}(t_{j+1} - t_j) \\
&= \frac{160L_{c1}^2}{\tau}(1 + 8L_\pi^4)\frac{\kappa^4}{1-\rho^4}\|x_{t_j}\|^4 + \frac{10L_{c2}^2}{\tau}(1 + 2L_\pi^2)\frac{\kappa^2}{1-\rho^2}\|x_{t_j}\|^2 \\
&\quad + \left(\frac{160L_{c1}^2}{\tau}((1 + 8L_\pi^4)\beta^4 w_{\max}^4 + \bar{\pi}_0^4) + \frac{10L_{c2}^2}{\tau}((1 + 2L_\pi^2)\beta^2 w_{\max}^2 + \bar{\pi}_0^2) + \frac{5c_0^2}{\tau}\right)(t_{j+1} - t_j) \\
&= \frac{1}{\tau}\left(\alpha_{17}\|x_{t_j}\|^4 + \alpha_{18}\|x_{t_j}\|^2 + \alpha_{19}\mathrm{poly}(\beta w_{\max}, \bar{\pi}_0)(t_{j+1} - t_j)\right)
\end{aligned}$$

where $\alpha_{17}, \dots, \alpha_{19}$ are polynomials of $\kappa, L_{c1}, L_{c2}, \frac{1}{1-\rho}, L_\pi, c_0$. By plugging the uniform upper bound on $g_j^2(i; I_{j-1:0})$ above to (8), we obtain the following.

$$\mathop{\mathbb{E}}_{(I_j)_{j\geq 0}} \text{Term } 1_j \leq \mathop{\mathbb{E}}_{I_{j-1:0}} \frac{\eta N}{2\tau} \left(\alpha_{17}\|x_{t_j}\|^4 + \alpha_{18}\|x_{t_j}\|^2 + \alpha_{19}\text{poly}(\beta w_{\max}, \bar{\pi}_0)(t_{j+1} - t_j)\right) \tag{9}$$

Therefore, by summing over $j = 0, \dots, J-1$, we can prove Lemma 20 as follows.

$$\begin{aligned}
\mathop{\mathbb{E}}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} \text{Term } 1_j &\leq \sum_{j=0}^{J-1} \mathop{\mathbb{E}}_{I_{j-1:0}} \frac{\eta N}{2\tau} \left(\alpha_{17}\|x_{t_j}\|^4 + \alpha_{18}\|x_{t_j}\|^2 + \alpha_{19}\text{poly}(\beta w_{\max}, \bar{\pi}_0)(t_{j+1} - t_j)\right) \\
&= \mathop{\mathbb{E}}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} \frac{\eta N}{2\tau} \left(\alpha_{17}\|x_{t_j}\|^4 + \alpha_{18}\|x_{t_j}\|^2 + \alpha_{19}\text{poly}(\beta w_{\max}, \bar{\pi}_0)(t_{j+1} - t_j)\right) \\
&\leq \frac{\eta N}{2\tau}\alpha_{17}\left(\frac{1}{1-\gamma_0}\frac{\gamma_1^{\mathbb{M}+1}-1}{\gamma_1 - 1}(\|x_0\|^2 + \frac{\alpha_9}{\gamma_1 - 1}) + \frac{2\beta^2 w_{\max}^2}{1-\gamma_0}J\right) \\
&+ \frac{\eta N}{2\tau}\alpha_{18}\left(\frac{1}{1-2\gamma_0^2}\frac{(2\gamma_1^2)^{\mathbb{M}+1}-1}{2\gamma_1^2 - 1}(\|x_0\|^4 + \frac{\alpha_{10}}{2\gamma_1^2 - 1}) + \frac{8\beta^4 w_{\max}^4}{1-2\gamma_0^2}J\right) \\
&+ \alpha_{19}\text{poly}(\beta w_{\max}, \bar{\pi}_0)T\frac{\eta N}{2\tau} \\
&= \frac{1}{\tau}\left(\alpha_4 \eta N T \text{poly}(\beta w_{\max}, \bar{\pi}_0) + \alpha_5 \eta N (L_f(1+L_\pi)\kappa)^{4\mathbb{M}}\text{poly}(\|x_0\|, \beta w_{\max}, \bar{\pi}_0)\right)
\end{aligned}$$

where the last equality is because we defined $\gamma_0 = 2\kappa^2\rho^{2\tau}$, $\gamma_1 = 2L_f^2(1+L_\pi)^2\kappa^2$, $\alpha_9 = \frac{2\gamma_1}{1-\gamma_0}\beta^2 w_{\max}^2 + 2L_f^2((1+L_\pi)\beta w_{\max} + \bar{\pi}_0 + w_{\max})^2$, and $\alpha_{10} = 8L_f^4((1+L_\pi)\beta w_{\max} + \bar{\pi}_0 + w_{\max})^4 + \frac{16\gamma_1^2}{1-2\gamma_0^2}\beta^4 w_{\max}^4$.

C.1.2. PROOF OF LEMMA 21

$$\begin{aligned}
\text{Term } 2_j &= -\frac{1}{\eta}\log\left(\mathop{\mathbb{E}}_{i\sim p_j} \exp(-\eta \tilde{g}_j(i; I_{j-1:0}))\right) \\
&= -\frac{1}{\eta}\log\left(\sum_{i\in\mathcal{P}_j} p_j(i)\exp(-\eta\tilde{g}_j(i; I_{j-1:0}))\right) \\
&= -\frac{1}{\eta}\log\left(\frac{\sum_{i\in\mathcal{P}_j}\exp(-\eta\tilde{G}_{j-1}(i; I_{j-1:0}))\exp(-\eta\tilde{g}_j(i; I_{j-1:0}))}{\sum_{i\in\mathcal{P}_j}\exp(-\eta\tilde{G}_{j-1}(i; I_{j-1:0}))}\right) \\
&= -\frac{1}{\eta}\log\left(\frac{\sum_{i\in\mathcal{P}_j}\exp(-\eta\tilde{G}_j(i; I_{j-1:0}))}{\sum_{i\in\mathcal{P}_j}\exp(-\eta\tilde{G}_{j-1}(i; I_{j-1:0}))}\right) \\
&= -\frac{1}{\eta}\log\left(\frac{1}{N}\sum_{i\in\mathcal{P}_j}\exp(-\eta\tilde{G}_j(i; I_{j-1:0}))\right) + \frac{1}{\eta}\log\left(\frac{1}{N}\sum_{i\in\mathcal{P}_j}\exp(-\eta\tilde{G}_{j-1}(i; I_{j-1:0}))\right)
\end{aligned}$$

where we define $\tilde{G}_j(i; I_{j-1:0}) = \tilde{g}_j(i; I_{j-1:0}) + \tilde{G}_{j-1}(i; I_{j-1:0})$ for $i \in \mathcal{P}_j - \mathcal{P}_{j+1}$ when $j \geq 0$.

For further ease of notations, let's define $\mathcal{P}_{-1} = \{1, \dots, N\}$ and $\Phi_j(\eta) = \frac{1}{\eta} \log \left(\frac{1}{N} \sum_{i \in \mathcal{P}_j} \exp(-\eta \tilde{F}_j(i; I_{j-1:0})) \right)$ for $j \geq -1$. Notice that $\Phi_{-1}(\eta) = 0$ because $\tilde{F}_{-1}(\cdot) = 0$. Then, for $j \geq 0$, we have

$$\begin{aligned}
\text{Term } 2_j &= -\Phi_j(\eta) + \Phi_{j-1}(\eta) + \frac{1}{\eta} \log \left(\frac{\sum_{i \in \mathcal{P}_j} \exp(-\eta \tilde{G}_{j-1}(i; I_{j-1:0}))}{\sum_{i \in \mathcal{P}_{j-1}} \exp(-\eta \tilde{G}_{j-1}(i; I_{j-1:0}))} \right) \\
&\leq -\Phi_j(\eta) + \Phi_{j-1}(\eta)
\end{aligned}$$

since $\mathcal{P}_{j-1} \supseteq \mathcal{P}_j$ for $j \geq 0$.

By summing Term 2_j over j, we obtain

$$\begin{aligned}
\mathop{\mathbb{E}}_{(I_j)_{j \geq 0}} \sum_{j=0}^{J-1} \text{Term } 2_j &= \mathop{\mathbb{E}}_{(I_j)_{j \geq 0}} \sum_{j=0}^{J-1} (-\Phi_j(\eta) + \Phi_{j-1}(\eta)) \\
&= \mathop{\mathbb{E}}_{(I_j)_{j \geq 0}} \Phi_{-1}(\eta) - \Phi_{J-1}(\eta) = \mathop{\mathbb{E}}_{(I_j)_{j \geq 0}} (-\Phi_{J-1}(\eta)) \\
&= \mathop{\mathbb{E}}_{(I_j)_{j \geq 0}} \frac{-1}{\eta} \log \left(\frac{1}{N} \sum_{i \in \mathcal{P}_{J-1}} \exp(-\eta \tilde{G}_{J-1}(i; I_{J-1:0})) \right) \\
&\overset{(a)}{\leq} \mathop{\mathbb{E}}_{(I_j)_{j \geq 0}} \frac{-1}{\eta} \log \left(\frac{1}{N} \exp(-\eta \tilde{G}_{J-1}(k; I_{J-1:0})) \right) \\
&= \mathop{\mathbb{E}}_{(I_j)_{j \geq 0}} \frac{-1}{\eta} \log \left(\exp(-\eta \tilde{G}_{J-1}(k; I_{J-1:0})) \right) + \frac{\log N}{\eta} \\
&= \mathop{\mathbb{E}}_{(I_j)_{j \geq 0}} \tilde{G}_{J-1}(k; I_{J-1:0}) + \frac{\log N}{\eta} \\
&= \mathop{\mathbb{E}}_{(I_j)_{j \geq 0}} \sum_{j=0}^{J-1} \tilde{g}_j(k; I_{j:0}) + \frac{\log N}{\eta} \\
&= \sum_{j=0}^{J-1} \mathop{\mathbb{E}}_{(I_j)_{j \geq 0}} \tilde{g}_j(k; I_{j:0}) + \frac{\log N}{\eta} \\
&= \sum_{j=0}^{J-1} \mathop{\mathbb{E}}_{I_{0:j}} \mathop{\mathbb{E}}_{I_{j+1:J-1}} [\tilde{g}_j(k; I_{j:0}) \mid I_{0:j}] + \frac{\log N}{\eta} \\
&= \sum_{j=0}^{J-1} \mathop{\mathbb{E}}_{I_{0:j}} \tilde{g}_j(k; I_{j:0}) + \frac{\log N}{\eta} \\
&\overset{(b)}{=} \sum_{j=0}^{J-1} \mathop{\mathbb{E}}_{I_{0:j-1}} g_j(k; I_{j-1:0}) + \frac{\log N}{\eta}
\end{aligned}$$

where k in the inequality (a) is the same k in the definition of AuxRegret_k and this inequality (a) holds because $k \in \mathcal{B} \subseteq \mathcal{P}_{J-1}$; besides, the equality (b) is because Lemma 19 and $k \in \mathcal{B} \subseteq \mathcal{P}_j$ for any $0 \leq j \leq J-1$. This completes the proof.

C.2. Proof of Lemma 13

For ease of notations, we introduce the following definitions. Let $i^* \in \arg\min_{i\in\mathcal{B}} J_T(\pi_i)$. Let $(\hat{x}_t, \hat{u}_t)$ for $t \geq 0$ denote the state and action trajectories generated by policy π_{i^*}, where $\tilde{x}_0 = x_0$. Let (x_t, u_t) for $t \geq 0$ denote the state and action trajectories generated by our Algorithm 1. Further, for $0 \leq j \leq J-1$, define $(\tilde{x}_t, \tilde{u}_t)$ for $t_j \leq t \leq t_{j+1}-1$ by the state and action trajectories generated by implementing policy π_{i^*} in episode j and implementing the same policies as Algorithm 1 in episode $0, \ldots, j-1$, where $\tilde{x}_0 = x_0$. Notice that $\tilde{x}_{t_j} = x_{t_j}$ for all j.

Then, we have the following relation between the policy regret and the auxiliary regret.

$$
\begin{aligned}
\text{PolicyRegret} &= \mathop{\mathbb{E}}_{(I_j)_{j\geq 0}} \sum_{t=0}^{T} c_t(x_t, u_t) - \sum_{t=0}^{T} c_t(\hat{x}_t, \hat{u}_t) \\
&= \mathop{\mathbb{E}}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} \sum_{t=t_j}^{t_{j+1}-1} c_t(x_t, u_t) - \mathop{\mathbb{E}}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} \sum_{t=t_j}^{t_{j+1}-1} c_t(\tilde{x}_t, \tilde{u}_t) \\
&\quad + \mathop{\mathbb{E}}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} \sum_{t=t_j}^{t_{j+1}-1} c_t(\tilde{x}_t, \tilde{u}_t) - \sum_{j=0}^{J-1} \sum_{t=t_j}^{t_{j+1}-1} c_t(\hat{x}_t, \hat{u}_t) \\
&= \mathop{\mathbb{E}}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} \tau g_j(I_j; I_{j-1:0}) - \mathop{\mathbb{E}}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} \tau g_j(i^*; I_{j-1:0}) \\
&\quad + \mathop{\mathbb{E}}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} \sum_{t=t_j}^{t_{j+1}-1} c_t(\tilde{x}_t, \tilde{u}_t) - \sum_{j=0}^{J-1} \sum_{t=t_j}^{t_{j+1}-1} c_t(\hat{x}_t, \hat{u}_t) \\
&\leq \text{AuxRegret} + \mathop{\mathbb{E}}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} \sum_{t=t_j}^{t_{j+1}-1} c_t(\tilde{x}_t, \tilde{u}_t) - \sum_{j=0}^{J-1} \sum_{t=t_j}^{t_{j+1}-1} c_t(\hat{x}_t, \hat{u}_t)
\end{aligned}
$$

Therefore, it suffices to bound $\mathbb{E}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} \sum_{t=t_j}^{t_{j+1}-1} c_t(\tilde{x}_t, \tilde{u}_t) - \sum_{j=0}^{J-1} \sum_{t=t_j}^{t_{j+1}-1} c_t(\hat{x}_t, \hat{u}_t)$. By Assumption 1, Assumption 3, Definition 1, and Definition 2, we have the following.

$$
\begin{aligned}
&|c_t(\tilde{x}_t, \tilde{u}_t) - c_t(\hat{x}_t, \hat{u}_t)| \leq L_{c1}[\max(\|\tilde{x}_t\|, \|\hat{x}_t\|) + \max(\|\tilde{u}_t\|, \|\hat{u}_t\|) + L_{c2}](\|\tilde{x}_t - \hat{x}_t\| + \|\tilde{u}_t - \hat{u}_t\|) \\
&\leq L_{c1}[\kappa\rho^{t-t_j}(1+L_\pi)\max(\|\tilde{x}_{t_j}\|, \|\hat{x}_{t_j}\|) + (1+L_\pi)\beta w_{\max} + L_{c2}](1+L_\pi)\kappa\rho^{t-t_j}\|\tilde{x}_{t_j} - \hat{x}_{t_j}\| \\
&\leq L_{c1}\kappa^2(\rho^2)^{t-t_j}(1+L_\pi)^2(\|\tilde{x}_{t_j}\| + \|\hat{x}_{t_j}\|)^2 \\
&\quad + L_{c1}[(1+L_\pi)\beta w_{\max} + L_{c2}](1+L_\pi)\kappa\rho^{t-t_j}(\|\tilde{x}_{t_j}\| + \|\hat{x}_{t_j}\|) \\
&\leq L_{c1}\kappa^2(\rho^2)^{t-t_j}(1+L_\pi)^2(2\|\tilde{x}_{t_j}\|^2 + 2\|\hat{x}_{t_j}\|^2) \\
&\quad + L_{c1}[(1+L_\pi)\beta w_{\max} + L_{c2}](1+L_\pi)\kappa\rho^{t-t_j}(\|\tilde{x}_{t_j}\| + \|\hat{x}_{t_j}\|)
\end{aligned}
$$

Therefore, we have the following bound.

$$
\begin{aligned}
&\text{PolicyRegret} \leq \text{AuxRegret} + \mathop{\mathbb{E}}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} \sum_{t=t_j}^{t_{j+1}-1} (c_t(\tilde{x}_t, \tilde{u}_t) - c_t(\hat{x}_t, \hat{u}_t)) \\
&\leq \text{AuxRegret} + \mathop{\mathbb{E}}_{(I_j)_{j\geq 0}} \sum_{j=0}^{J-1} \left[L_{c1} \frac{\kappa^2}{1-\rho^2}(2\|\tilde{x}_{t_j}\|^2 + 2\|\hat{x}_{t_j}\|^2)(L_\pi + 1)^2 \right.
\end{aligned}
$$

$$
\begin{aligned}
&+L_{c1}\frac{\kappa}{1-\rho}(\|\tilde{x}_{t_j}\|+\|\hat{x}_{t_j}\|)(L_\pi+1)(\beta w_{\max}(L_\pi+1)+L_{c2})\Big] \\
&\le \text{AuxRegret}+L_{c1}\frac{\kappa^2}{1-\rho^2}(L_\pi+1)^2(2\mathop{\mathbb{E}}_{(I_j)_{j\ge 0}}\sum_{j=0}^{J-1}\|\tilde{x}_{t_j}\|^2+2\mathop{\mathbb{E}}_{(I_j)_{j\ge 0}}\sum_{j=0}^{J-1}\|\hat{x}_{t_j}\|^2) \\
&\quad+L_{c1}\frac{\kappa}{1-\rho}(L_\pi+1)(\beta w_{\max}(L_\pi+1)+L_{c2})(\mathop{\mathbb{E}}_{(I_j)_{j\ge 0}}\sum_{j=0}^{J-1}\|\tilde{x}_{t_j}\|+\mathop{\mathbb{E}}_{(I_j)_{j\ge 0}}\sum_{j=0}^{J-1}\|\hat{x}_{t_j}\|) \\
&\overset{(a)}{\le} \text{AuxRegret}+2L_{c1}\frac{\kappa^2}{1-\rho^2}(L_\pi+1)^2\left(\frac{1}{1-\gamma_0}\frac{\gamma_1^{\mathbb{M}+1}-1}{\gamma_1-1}(\|x_0\|^2+\frac{\alpha_9}{\gamma_1-1})+\frac{2\beta^2 w_{\max}^2}{1-\gamma_0}J\right. \\
&\quad\left.+\frac{\kappa^2}{1-\rho^2}\|x_0\|^2+2\beta^2 w_{\max}^2 J\right) \\
&\quad+L_{c1}\frac{\kappa}{1-\rho}(L_\pi+1)(\beta w_{\max}(L_\pi+1)+L_{c2})\left(\frac{1}{1-\sqrt{\gamma_0/2}}\frac{\sqrt{\frac{\gamma_1}{2}}^{\mathbb{M}+1}-1}{\sqrt{\frac{\gamma_1}{2}}-1}(\|x_0\|+\frac{\alpha_8}{\sqrt{\frac{\gamma_1}{2}}-1})\right. \\
&\quad\left.+\frac{\beta w_{\max}}{1-\sqrt{\frac{\gamma_1}{2}}}J+\frac{\kappa}{1-\rho}\|x_0\|+\beta w_{\max}J\right) \\
&\le \text{AuxRegret}+\alpha_6(L_f(1+L_\pi)\kappa)^{2\mathbb{M}}\text{poly}(\|x_0\|,\beta w_{\max},\bar{\pi}_0)+\alpha_7\beta^2 w_{\max}^2 J.
\end{aligned}
$$

where (a) uses $\tilde{x}_{t_j}=x_{t_j}$, Lemma 18, and $\sum_{j=0}^{J-1}\|\hat{x}_{t_j}\|\le\frac{\kappa}{1-\rho}\|x_0\|+\beta w_{\max}J$, $\sum_{j=0}^{J-1}\|\hat{x}_{t_j}\|^2\le\frac{\kappa^2}{1-\rho^2}\|x_0\|^2+2\beta^2 w_{\max}^2 J$ by the definition of $\hat{x}_t$. This completes the proof of Lemma 13.

Appendix D. Additional examples

Consider system $\dot{x}=-x^3+u$. Consider a point-wise min-norm policy

$$
\pi(x)=\begin{cases}x^3-2x & \text{if } x^2<2\\ 0 & \text{if } x^2\ge 2.\end{cases}
$$

It is straigthtforward to verify that this policy satisfies Definitions 1 and 2.