Ergodicity and Chaos

Yuqing Li

March 24, 2022

Abstract

1 Markov Operator

Definition 1. *A Markov operator over a Polish space $\mathcal{X}$ is a bounded linear operator on $\mathcal{B}_b(\mathcal{X})$, i.e., $\mathcal{P} : \mathcal{L}^1(\mathcal{X}) \to \mathcal{L}^1(\mathcal{X})$, s.t.*

1. *$f(\cdot) \geq 0$, $\mathcal{P}f \geq 0$;*
2. *$f(\cdot) \geq 0$, $\|\mathcal{P}f\|_1 = \|f\|_1$.*

We observe that $\mathcal{P}$ is a monotonic operator, moreover it is a contraction, i.e.,

$$\|\mathcal{P}f\|_1 \leq \|f\|_1 ,$$

for any $f \in \mathcal{L}^1(\mathcal{X})$.

Then, if there is a fixed point for the Markov operator f^*, i.e., $\mathcal{P}f^* = f^*$, we may define a measure $\mu(\cdot)$, with

$$\mu(A) = \int_A f^* \mathrm{d}\boldsymbol{x},$$

then heuristically, $\mathcal{P}$ is also a contraction on $\mathcal{L}^2(X, \mu)$, by using (M3) property in Proposition 3.1.1 [**?**]

$$\int |\mathcal{P}g|^2 \,\mathrm{d}\mu \leq \int (\mathcal{P}\,|g|)^2 \,\mathrm{d}\mu = \int (\mathcal{P}\,|g|)^2 \, f^* \mathrm{d}\boldsymbol{x}$$

$$\leq \int |g| \left(\mathcal{P} |g|\right) \mathrm{d}\mu \leq \left(\int |g|^2 \mathrm{d}\mu\right)^{1/2} \left(\int \left(\mathcal{P} |g|\right)^2 \mathrm{d}\mu\right)^{1/2},$$

$$\int \left(\mathcal{P} |g|\right)^2 \mathrm{d}\mu \leq \int |g|^2 \mathrm{d}\mu.$$

2 The Frobenius-Perron Operator

We start with a definition

Definition 2. *A measurable transformation $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ on a Polish space $\mathcal{X}$ is nonsingular if $\pi(\mathcal{S}^{-1}(A)) = 0$ for all $A \in \mathcal{B}$ satisfying $\pi(A) = 0$.*

Assume that a nonsingular transform $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ is given, we define an operator $\mathcal{P} : \mathcal{L}^1(\mathcal{X}) \to \mathcal{L}^1(\mathcal{X})$ in two steps

1. Let $f \in \mathcal{L}^1(\mathcal{X})$ and $f \geq 0$, write

$$\int_A \mathcal{P} f \mathrm{d}\pi := \int_{\mathcal{S}^{-1}(A)} f \mathrm{d}\pi,$$

2. Let $f \in \mathcal{L}^1(\mathcal{X})$, write

$$\int_A \mathcal{P} f \mathrm{d}\pi := \int_{\mathcal{S}^{-1}(A)} f^+ \mathrm{d}\pi - \int_{\mathcal{S}^{-1}(A)} f^- \mathrm{d}\pi = \int_{\mathcal{S}^{-1}(A)} f \mathrm{d}\pi,$$

One might ask,

Question 1. *What is the Frobenius-Perron operator for the ODE*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x \tag{2.1}$$

Proof. Take the Polish space as $\mathcal{X} := [-5, 5]$, then for an interval $A := [a, x]$, the nonsingular transformation is defined as $\mathcal{S}_t : \mathcal{X} \to \mathcal{X}$, then we have $\mathcal{S}_t^{-1}(A) = [a \exp(t), x \exp(t)]$, thus we obtain that

$$\int_a^x \mathcal{P} f(z) \mathrm{d}z = \int_{a \exp(t)}^{x \exp(t)} f(u) \mathrm{d}u.$$

$\square$

References

[1] Andrzej Lasota and Michael C Mackey. *Chaos, fractals, and noise: stochastic aspects of dynamics*, volume 97. Springer Science & Business Media, 1998.